THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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VOLUME 74
1967

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND BUFFALO, N. Y.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 74 NUMBER 1

PART I

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JANUARY 1967

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

R. A. ROSENBAUM, *Editor*

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ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

General Office: SUNY at Buffalo, Buffalo, N. Y. 14214. *Executive Director:* H. M. GEHMAN

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $6.00. For nonmembers the subscription price is $10.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

AWARD FOR DISTINGUISHED SERVICE TO PROFESSOR WILLIAM LARKIN DUREN, JR.

Service to mathematics, whether distinguished or not, is a metaphor. Mathematics is an abstraction, and when we say that something is done for the benefit of mathematics we are speaking figuratively and giving an abstraction and interpretation of the motives for the action. But service to people who want to learn mathematics, or to teach mathematics, or to advance mathematics, or to use mathematics, is no metaphor. Each of these services stands as a responsibility before all those who have the capacity to perform them, and the degree to which the responsibility is accepted depends heavily on the degree to which one feels himself to be a part of a community of human beings in whose lives mathematics plays an important role.

William Larkin Duren, Jr. is one who has always accepted the responsibility for such services, not with resignation but willingly. And when this drive toward helpfulness to the mathematical community is coupled with ability and tenacity such as his, the natural outcome is truly distinguished service to mathematics.

After receiving his B.A. and M.S. degrees from Tulane University, and in 1930 his doctorate from the University of Chicago, Duren taught for a year at what is now Wayne State University and then rejoined the faculty at Tulane University for twenty-four years, but not uninterruptedly. From 1943 to 1945 he served as Operations Analyst with the U. S. Army Air Force. In 1948 he became chairman of the Department of Mathematics at Tulane, and by his good management and his consciousness of the opportunities present in the rapidly changing scene of the early 1950's he greatly raised the stature of the department.

He had already served some users of mathematics by his work during the war; in 1953 he served the advancers of mathematics by becoming the first Program Director in the Mathematical Sciences in the National Science Foundation. This he followed by a sort of Toynbee pattern of "withdrawal and return"; in the year 1953–54, holding a Ford Foundation Faculty Fellowship, he visited a number of universities to try to diagnose the troubles that were clearly besetting mathematical education, and to decide how he could most effectively help in the remedying of them.

One serious trouble, immediately apparent to any active mathematician who considered the matter, was that undergraduate instruction had stagnated for about half a century, unaffected by the radical changes at the research level. As a matter of fact, there had been several efforts to improve the undergraduate program, but being local these could not overcome the inertia of the vast bulk of institutions. Effective reform demanded a broader base. In January of 1953 Duren convinced the then President of the Mathematical Association of Amer-



WILLIAM LARKIN DUREN, JR.

ica of the need of appointing a committee to consider the problems of making available in our society the values of modern mathematics. This was the beginning of the Committee on the Undergraduate Program, better known as the CUP or as the Duren committee, since appropriately Duren was its chairman. With a less dynamic chairman the CUP might have turned out a description of a desirable curriculum, with a proposed list of topics for each course, to be commended in principle and ignored in practice. But Duren wanted none of that. Instead, the committee outlined a system of courses in mathematics involving some rather radical changes, and then showed that the proposals were not merely utopian by producing a collection of text-books on which the proposed courses could be based—all of this in a time when liberal support for course-improvement projects was merely an optimist's dream. For four of the six years Duren was chairman of CUP; for the other two he was ineligible, being President of the Association, but he did most of the chairman's work anyhow.

After six years it was clear that many aspects of mathematical education, such for example as programs for small colleges and programs for training teachers, needed careful study, so at the request of CUP it was dissolved and a larger group, the CUPM, with a broader charge, was constituted. This gave Duren a five-year respite from this particular activity, until he returned to serve as chairman of CUPM during 1963–65.

A complete list of the committees and panels on which Duren has served, if such a list could be compiled, would be long indeed. Frequently he has been asked to speak at conferences on mathematical education, and not in this country alone; he was one of the four American speakers at the Japan-U. S. Conference on Collegiate Mathematical Education held at Katada (near Kyoto) in 1964, and was the only American mathematician at the US-AID-NSF-Gov't. of India Conference on Science Education in India, held in Srinagar in the summer of 1966. He has served as a National Science Foundation College Visitor-Lecturer. He served as Dean of the College of Arts and Sciences of the University of Virginia from 1955 to 1962, and since then, as University Professor, he has helped with the creation of the Department of Applied Mathematics and Computer Science in the School of Engineering and Applied Science of the University of Virginia. But besides this he has been unstinting in good advice to many departments, large and small, that have known of his capability and experience and have wanted his help in building themselves up.

So, today, the Association extends to Bill Duren its recognition of his services, past and present, confident that by his very nature, those distinguished services are not about to stop.

E. J. McSHANE

AWARD OF THE 1967 CHAUVENET PRIZE TO PROFESSOR GUIDO L. WEISS

The Board of Governors of the Mathematical Association of America voted to award the 1967 Chauvenet Prize to Professor Guido L. Weiss of Washington University for his paper "Harmonic Analysis," published in *MAA Studies in Mathematics*, Volume 3 (1965) 124–178.

A certificate and monetary award in the amount of five hundred dollars was presented to Professor Weiss at the time of the Business Meeting of the Association on January 27, 1967, at Houston.

The Chauvenet Prize is awarded for a noteworthy expository paper published in English, such as will come within the range of profitable reading for members of the Association. The purpose of the prize is to stimulate the writing of expository works by American scholars. The 1967 Prize, awarded for a paper published in 1965 by a member of the Association, is the sixteenth award of the Chauvenet Prize since its institution by the MAA in 1925. For a list of the names of the previous winners, see this MONTHLY, 71 (1964), page 589 and 72 (1965), pp. 2–3.

Professor Weiss was born on December 29, 1928, in Trieste, Italy. He received his Ph.B. degree in 1949, his M.S. in 1951, and his Ph.D. in 1956, all from the University of Chicago. During 1953 and part of 1954, he was employed as a mathematician in the Advisory Board on Simulation (a project at the University of Chicago). In 1955, Professor Weiss was appointed as an instructor at De Paul University; he was promoted to Assistant Professor in 1956 and to Associate Professor in 1959. From May to September 1960 he was a Visiting Professor at the University of Buenos Aires. During 1960–61, he spent a year as National Science Foundation Postdoctoral Fellow at the Institute Poincaré at the Sorbonne. In 1961, he was appointed Associate Professor at Washington University and was promoted to Professor in 1963. He spent the Summer Quarter of 1963 as Visiting Professor at the University of Chicago. During the 1964–65 academic year, he was associated as Senior Postdoctoral National Science Foundation Fellow with the University of Geneva.

Professor Weiss' many substantial contributions in the field of analysis, and specifically to the theory of harmonic functions and harmonic analysis, are contained in his numerous papers which have appeared in both domestic and foreign publications.

In accepting the 1967 Chauvenet Prize, Professor Weiss expressed his strong feeling that the writing of expository papers should be encouraged and that, for this reason, he was very happy to write the chapter on "Harmonic Analysis" in *MAA Studies in Mathematics*. He found it most encouraging that, while never expecting this, he had received such a high honor for writing this article.

STARLIKE GRAPHS

GARY CHARTRAND, Western Michigan University and
S. F. KAPOOR, Michigan State University

1. Introduction. A connected ordinary graph $G(V, X)$ with vertex set V and edge set X will be called *starlike at a vertex* $p \in V$ (or simply *starlike*) provided (i) there exists a spanning tree $T(V, Y)$ in G which is homeomorphic to a star graph S, with deg p equal to the number of edges in S and (ii) for all $a \in V$, the distance from p to a is the same in T as in G. Such a point p will be called a *center*.

Examples of starlike graphs include (see Fig. 1 where the dotted lines indicate the edges in T) star graphs, complete graphs, paths, cycles and complete bipartite graphs. (For definitions of basic terms see [2] and [3].)

The principal result of this paper is a characterization of starlike graphs. Several results concerning the structure of such graphs are discussed.

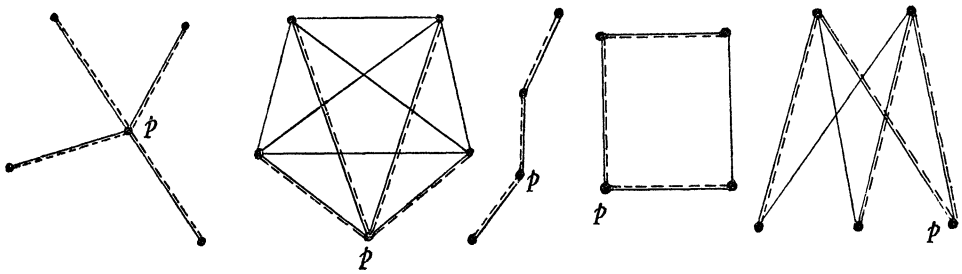


FIG. 1

2. Basic terminology. Let G be an *ordinary graph*, i.e. finite, undirected, and without loops or multiple edges. It is well known that if the *distance* $d(u, v)$ between two vertices u and v of a connected ordinary graph $G(V, X)$ is defined to be zero if $u = v$ and as the length of a shortest path joining u and v if $u \neq v$, then the vertex set V of G is a metric space with this distance function.

A *tree* is a connected graph without cycles. A *spanning tree* in a connected graph G is a tree which contains all the vertices of G. A *star graph* S is a tree having a point p adjacent to all the other points of S.

A graph G with vertex set V is said to be *isomorphic* to a graph G' with vertex set V' if there exists a one-to-one correspondence between V and V' such that two vertices are adjacent in G if and only if the corresponding vertices in G' are adjacent. Two graphs G and G' are said to be *homeomorphic* if it is possible to insert new vertices of degree 2 into their edges in such a way that the resulting graphs are isomorphic.

A vertex v of a graph G is a *cut point* if the removal of v (and all edges incident with v) from G results in a subgraph G' having more components than G.

In particular, the removal of a cut point from a connected graph results in a disconnected graph. A *block* of a graph is a maximal connected subgraph which contains no cut points of itself. A block will be called *trivial* if it contains exactly one edge, and otherwise it will be called *nontrivial*. A cut point p is called *k-block* if it belongs to k nontrivial blocks. In particular for $k=0$, p is called *trivial*; for $k=1$, p is called *mono-block* and for $k\geqq 2$, p is called *poly-block*. The number of blocks to which a vertex v belongs is called the *connective index* of v and is denoted by $i(v)$. It follows immediately that v is a cut point of G if and only if $i(v)\geqq 2$ and if v is a k-block cut point, then $i(v)\geqq k$.

For each point a of a graph $G(V, X)$ and for each $i=0, 1, 2, \cdots$, we define $A_i(a)=\{v\in V \mid d(a, v)=i\}$. For example, $A_0(a)=\{a\}$ and $A_1(a)$ consists of all those vertices adjacent to a. Consider the (multiple-valued) function f defined on $A_{i-1}(a)$, $i\geqq 1$, by

$$f(u) = \{v \in A_i(a) \mid (u, v) \in X\}.$$

Hence, for $w\in A_i(a)$, $i\geqq 1$,

$$f^{-1}(w) = \{u \in A_{i-1}(a) \mid (u, w) \in X\}.$$

Then if W is a nonempty subset of $A_i(a)$,

$$f^{-1}(W) = \{f^{-1}(w) \mid w \in W\}.$$

A set $Z\subseteq X$ in a graph $G(V, X)$ is called *independent* if no two elements of Z have a vertex in common. If V_1 and V_2 are nonempty disjoint subsets of V then V_1 is said to be *matched into* V_2 if there exists an independent set $Z\subseteq X$ which joins all elements of V_1 with elements of V_2. For $v_1\in V_1$ and $W\subseteq V_1$ define

$$g(v_1) = \{v_2 \in V_2 \mid (v_1, v_2) \in X\}$$

and

$$g(W) = \{g(w) \mid w \in W\}.$$

3. A characterization of starlike graphs. The following theorem (see [1], p. 92) gives a necessary and sufficient condition under which V_1 can be matched into V_2.

THEOREM (D. KÖNIG AND P. HALL). *V_1 can be matched into V_2 if and only if for all $W\subseteq V_1$, $|g(W)|\geqq|W|$.*

($|C|$ denotes the cardinal number of the set C.)

THEOREM 1. *A connected graph $G(V, X)$ is starlike if and only if there exists a point $p\in V$ such that for every nonempty subset W of $A_i(p)$, $i\geqq 2$, $|f^{-1}(W)|\geqq|W|$.*

Proof. Suppose $G(V, X)$ is starlike with a spanning tree T and center p as defined above. Let $W=\{w_1, w_2, \cdots, w_k\}$ be a nonempty subset of $A_i(p)$, $i\geqq 2$. For each $w_j\in W$, $j=1, 2, \cdots, k$, let u_j be the vertex in $A_{i-1}(p)$ which lies on

the path in T from p to w_j. Then $u_j \in f^{-1}(w_j)$ so that $\{u_1, u_2, \cdots, u_k\} \subseteq f^{-1}(W)$. This implies that $|f^{-1}(W)| \geqq |W|$.

Conversely, suppose there exists a point p such that if W is any nonempty subset of $A_i(p)$, $i \geqq 2$, then $|f^{-1}(W)| \geqq |W|$. We show that $G(V, X)$ is starlike at p by defining a spanning tree $T(V, Y)$ with the desired properties.

Consider the bipartite subgraphs B_i of $G(V, X)$, $i \geqq 2$, where the vertex set of B_i is $A_{i-1}(p) \cup A_i(p)$ and whose edge set X_i consists of all elements of X which join a vertex of $A_i(p)$ with a vertex of $A_{i-1}(p)$.

With $A_i(p)$ and $A_{i-1}(p)$ playing the roles of V_1 and V_2, respectively, we note that for $W \subseteq A_i(p)$, $f^{-1}(W) = g(W)$. Hence the theorem of König and Hall applies to give a matching of $A_i(p)$ into $A_{i-1}(p)$, $i \geqq 2$. This produces an independent set $Y_i \subseteq X_i$ such that $|Y_i| = |A_i(p)|$. Defining Y_1 to be the set of all edges incident with p, we clearly have the desired tree $T(V, \bigcup Y_i)$, (see Fig. 2).

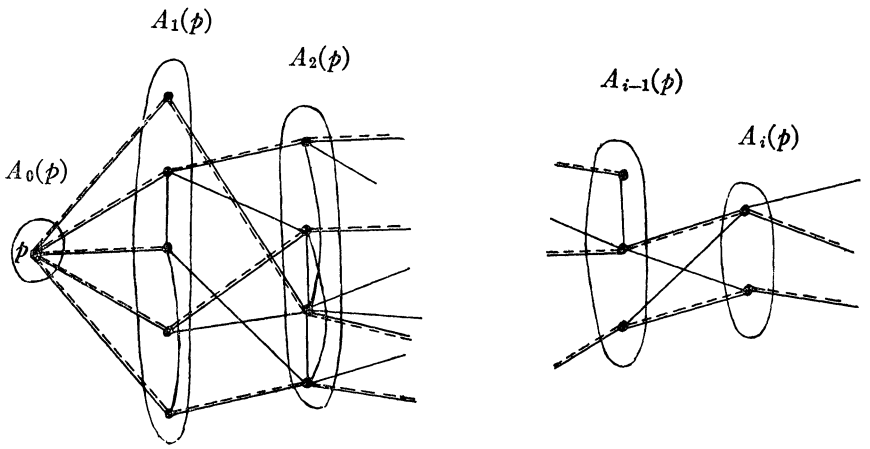


FIG. 2

4. Some properties of starlike graphs.

THEOREM 2. *Let $G(V, X)$ be a connected graph for which $|V| = n$. If there exists a vertex $p \in V$ with* deg $p \geqq n-2$, *then G is starlike and p is a center.*

Proof. Since deg $p \geqq n-2$, $A_1(p) = \{a_1, a_2, \cdots, a_k\}$, where $k = n-1$ or $n-2$. We define a spanning tree $T(V, Y)$ for each of the two cases as follows. For $k = n-1$, let $Y = \{(p, a_j) | j = 1, 2, \cdots, n-1\}$.

If $k = n-2$, then there exist vertices a' and b with $a' \in A_1(p)$ and $A_2(p) = \{b\}$ such that $(a', b) \in X$. In this case, take $Y = \{(p, a_j) | j = 1, 2, \cdots, n-2\} \cup \{(a', b)\}$. Hence, G is starlike and p is a center.

THEOREM 3. *Every block of a starlike graph is starlike.*

Proof. Let $G(V, X)$ be a graph which is starlike at p, and let $T(V, Y)$ be a corresponding spanning tree in G. If B is a block of G, and p is in B, then B is

starlike at p since we can take as the edge set for an appropriate spanning tree T' of B all those edges belonging to both B and T. If $p \notin B$, then there exists a smallest value r of i for which a point $p' \in B \cap A_i(p)$. Clearly, p' is a cut point of G, and no other point of B belongs to $A_r(p)$. B is starlike at p', and we take for the edge set of an appropriate spanning tree T' of B those edges belonging to $B \cap T$.

The converse of the preceding theorem does not follow as can be seen by the graph given in Figure 3.



FIG. 3

THEOREM 4. *If a starlike graph G contains a nontrivial block B, then any center of G belongs to B.*

Proof. If G is starlike at p and $p \notin B$, then there is a minimum value r of i (where $r \geqq 1$) such that a point v of B belongs to $A_i(p)$. As in the proof of Theorem 3, v is a cut point of G, and no other vertex of B belongs to $A_r(p)$. Since B is nontrivial there are at least two vertices w_1 and w_2 in B adjacent to v. Clearly, $\{w_1, w_2\} \subseteq A_{r+1}(p)$. If we let $W = \{w_1, w_2\}$, then $f^{-1}(W) = \{v\}$, which contradicts Theorem 1; hence, $p \in B$.

COROLLARY. *If a graph G containing only one poly-block cut point p is starlike, then p is the unique center of G.*

Proof. Since p is a poly-block cut point, G contains at least two nontrivial blocks. But by Theorem 4, any center must belong to all these nontrivial blocks and p is the only point with this property.

REMARK. From the preceding theorem and corollary, it then follows that starlike graphs have at most one poly-block cut point.

THEOREM 5. *If a starlike graph G has a vertex p with $i(p) \geqq 3$, then p is the unique center of G.*

Proof. Suppose G is starlike at v and $v \neq p$. If $d(v, p) = r$, then $p \in A_r(v)$. Any path from v to p contains points of only one block to which p belongs; hence, the distance from v to points in the other (two or more) blocks to which p belongs exceeds r. In particular, there are at least two points w_1 and w_2 (one in each of two such blocks) which are adjacent to p such that $w_1 \in A_{r+1}(v)$ and $w_2 \in A_{r+1}(v)$. Now if $W = \{w_1, w_2\}$, we see that $f^{-1}(W) = \{p\}$, and this contradicts Theorem 1. Thus, G is starlike at p.

COROLLARY. *A tree is starlike if and only if it has at most one vertex of degree three or more.*

Proof. If a tree has no vertex of degree three or more, it is a path and is starlike. On the other hand, if it has precisely one vertex of degree three or more, then it is homeomorphic to a star graph and is starlike.

If a tree T has two vertices of degree three or more, then since $i(v) = \deg v$ in a tree, T cannot be starlike, for otherwise Theorem 5 gives an immediate contradiction.

It therefore follows that if a starlike tree has more than one center, it is a path.

5. Summary. In a connected graph G, let

$c =$ the number of nontrivial blocks,
$\tau =$ the number of vertices of degree three or more, and
$N =$ the intersection of all the nontrivial blocks of G.

Then,

(a) G is starlike if $c = 0$ and $\tau \leqq 1$.
(b) G is not starlike if $c = 0$ and $\tau \geqq 2$ or $c \geqq 2$ and $N = \phi$.
(c) G may be starlike at p if $c \geqq 2$ and $N = \{p\}$.
(d) If $c = 1$, any center of G must lie in the nontrivial block if the graph is starlike.

The authors are most grateful to Professor E. A. Nordhaus for his suggestions and comments.

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A PROBLEM ON RANDOM POINTS IN A TRIANGLE

ULRICH KRENGEL, University of California, Berkeley

1. Problem. Let $\triangle$ be the triangle in R^2, which is determined by the points (0, 0), (2, 0), and (1, 1). We denote the Lebesgue measure in the plane by m. Clearly $m(\triangle) = 1$. We say that n points $A_1, \cdots, A_n$ are chosen (independently) at random in $\triangle$, if their coordinates (x_i, y_i) are determined at random in such a way that for any n measurable subsets $D_1, \cdots, D_n$ of $\triangle$ the probability $P(A_1 \in D_1, \cdots, A_n \in D_n)$ is $m(D_1) \cdots m(D_n)$.

Now call $(x, y) \in \triangle$ *covered* if for some k the inequalities $|x - x_k| < y - y_k$ and $y - y_k > 0$ are both satisfied. Call A_i a *corner-point* if it is not covered. Let

N_n be the number of corner-points. We calculate the expectation of this random variable and derive recursion-formulas for the variance and for the exact distribution, which allow easy calculations. A closed expression for the variance and for the distribution has not been obtained.

We further discuss some related problems, which are of interest for the applications, and derive results in the much easier case in which $\triangle$ is replaced by the unit-square. This case corresponds to a combinatorial problem.

2. Application. The described problem is a variant of a question suggested by crystallographic investigations. A wire of length l is exposed to such conditions that at random locations and random times crystals are born along the wire. These are small points at their birth but grow with constant speed u on both sides. When they touch their neighbour the growth stops on the side where contact is made. New crystals can be born only on uncovered wire. What will be the expected number of crystals when the wire is covered or after time t? To simplify this question we assume that at time $t=0$ there are already two crystals at the endpoints of the wire. This may be assumed if the wire carries crystals already and a specified uncovered piece between two crystals is observed.

In the case $u=0$ one obtains a two-dimensional Poisson-distribution, the coordinates being location and time. In the case $u>0$ the set of possible birth-points is an isosceles triangle. Normalizing the time-scale we may assume it to be rectangular.

The reduction to the problem on random points is done as follows: we imagine that the crystals don't grow except for the two at the ends of the wire. Then we obtain random-points in a triangle. The number R of random-points is Poisson-distributed:

$$\text{Prob}\{R = n\} = e^{-\lambda}\frac{\lambda^n}{n!} \tag{1}$$

(Computation of λ: For a wire of length l the limit

$$\gamma = \lim_{t\to 0}\frac{1}{tl} \times \text{Prob}\{\text{at least one crystal until time } t\}$$

exists and is an empirical constant which measures the tendency to form crystals. $\lambda=\gamma l^2/4u$.)

If the crystals grow, the distribution is essentially the same except that some of the n crystals do not come into being; these are the "covered" random points. Hence, if N_∞ is the number of crystals born,

$$\text{Prob}(N_\infty = k) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\,\text{Prob}\,(N_n = k). \tag{2}$$

This reduces as well the problem of calculating the expectation and variance

of N_∞ to the corresponding problem for N_n. In practice λ is small, so that the series converges rapidly and the distribution of N_n is needed only for small n.

3. Expectation. We first calculate the expectation of N_n and of N_∞.

THEOREM 1. *The expected number of corner-points out of n random points in* $\triangle$ *is given by*

$$E_n = E(N_n) = \frac{(2^n n!)^2}{(2n)!} - 1. \tag{3}$$

Proof. Let $A_i = (x_i, y_i)$, $i = 1, \cdots, n$ be the random points. A_n is a corner-point if and only if the triangle between (x_n, y_n), $(x_n - y_n, 0)$, and $(x_n + y_n, 0)$ contains none of the points $A_1, \cdots, A_{n-1}$. Its area is y_n^2; hence this has probability $(1-y_n^2)^{n-1}$. Integrating this in $\triangle$, we find the probability that A_n is a corner-point to be

$$\int_0^1 (1 - y_n^2)^{n-1}(2 - 2y_n)dy_n. \tag{4}$$

But the points being chosen independently, this probability is the same for all A_i so that we find the expectation of N_n to be

$$E_n = 2n \int_0^1 (1 - z^2)^{n-1}(1 - z)dz. \tag{5}$$

The evaluation of (5) yields (3).

If one counts only the corner-points with y-coordinates smaller or equal to $s \leqq 1$, the integral in (4) runs from 0 to s. Now assuming the number n Poisson-distributed as in (1), the expectation of the number of corner-points (x_k, y_k) with $y_k \leqq s$ is

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} 2n \int_0^s (1 - z^2)^{n-1}(1 - z)dz &= \exp(-\lambda)2\lambda \int_0^s \exp(\lambda(1 - z^2))(1 - z)dz \\ &= (4\lambda\pi)^{1/2}\{\Phi(s(2\lambda)^{1/2}) - \tfrac{1}{2}\} \\ &\quad + \exp(-\lambda s^2) - 1, \end{aligned} \tag{6}$$

where $\Phi(x)$ is the normalized normal distribution. For $s = 2ut/l \leqq 1$ this is the expected number of crystals formed up until time $t \leqq l/2u$. Especially this proves:

THEOREM 2. *If* N_∞ *is the number of corner-points among Poisson-distributed random points,* $E(N_\infty)$ *is given by* (6) *with* $s = 1$.

This method of derivation may be applied in the case that there are no crystals at the endpoints of the wire in the beginning, but the integrals in that case get far more complicated and must be split into many parts.

4. Recursion formulas. For the derivation of recursion formulas for the distribution of N_n and its variance choose independently n points in $\triangle$ and let (x_0, y_0) be the point with the smallest y-coordinate. It is unique with probability 1. $\Delta' = \triangle \cap \{y \geqq y_0\}$ splits into a covered rectangle of area say F_1 and two uncovered triangles of area F_2 and F_3. The ratios $F_1 : F_2 : F_3$ determine the joint distribution of the numbers h, i, j of points in F_1, F_2, and F_3. Without loss of generality, assume $y_0 = 0$. We obtain $F_1 = x_0(2 - x_0)/2$, $F_2 = x_0^2/4$ and $F_3 = (2-x_0)^2/4$. Put $\text{Prob}\{N_n = k\} = P_{nk}$. $N_n = k$ holds if and only if for some k_2 and k_3 with $k_2 + k_3 = k-1$ there are k_2 corner-points in F_2 and k_3 corner-points in F_3. Using the multinomial distribution for $n-1$ points and integrating over all possible values of x_0 with uniform density $1/2$, we obtain

$$P_{nk} = \frac{1}{2}\int_0^2 \sum_{h+i+j=n-1} \frac{(n-1)!}{h!i!j!} F_1^h F_2^i F_3^j \sum_{k_2+k_3=k-1} P_{ik_2} P_{jk_3} dx_0. \tag{7}$$

First we calculate the integral

$$\int_0^2 F_1^h F_2^i F_3^j dx_0 = 2^{h+1} \frac{(2i+h)!(2j+h)!}{(2h+2i+2j+1)!}.$$

We introduce the generating functions $P_n(s) = \sum_{k=0}^{\infty} P_{nk} s^k$. Clearly $P_0(s) = 1$. For $n \geqq 0$ replacing n by $n+1$ in (7) we get

$$P_{n+1}(s) = \frac{s}{2} \sum_{h+i+j=n} P_i(s) P_j(s) \frac{n!}{h!i!j!} 2^{h+1} \frac{(2i+h)!(2j+h)!}{(2n+1)!}. \tag{8}$$

This is already the recursion for the probabilities, but (8) is simplified if we introduce $Q_{nk} = (2n)!(2^n n!)^{-1} P_{nk}$ and $Q_n(s) = (2n)!(2^n n!)^{-1} P_n(s)$. With these changes (8) is equivalent to

$$Q_{n+1}(s) = s \sum_{i+j=0}^{n} Q_i(s) Q_j(s) \frac{(n+i-j)!(n-i+j)!}{(2i)!(2j)!(n-i-j)!}. \tag{9}$$

Computation starts from $Q_0(s) = 1$. The coefficients

$$c(n, i, j) = (n+i-j)!(n-i+j)!((2i)!(2j)!(n-i-j)!)^{-1}$$

turn out to be integers.

Here are the first few polynomials $Q_k(s)$: $Q_0(s) = 1$, $Q_1(s) = s$, $Q_2(s) = s + 2s^2$, $Q_3(s) = 2s + 8s^2 + 5s^3$, $Q_4(s) = 6s + 38s^2 + 47s^3 + 14s^4$, $Q_5(s) = 24s + 220s^2 + 415s^3 + 244s^4 + 42s^5$. Let us state the result:

THEOREM 3. *The generating functions* $Q_n(s)$ *of* $Q_{nk} = (2n)!(2^n n!)^{-1} \text{Prob}\{N_n = k\}$ *are recursively determined by* $Q_0(1) = 1$ *and* (9).

We now start to derive a recursion formula for the variances by taking derivatives in (9). For short we write $Q_n = Q_n(1)$, $Q_n' = Q_n'(1)$ and $Q_n'' = Q_n''(1)$. We obtain the equations:

$$Q'_{n+1} = Q_{n+1} + 2\sum_{k+j=0}^{n} Q'_k Q_j c(n, k, j) \tag{10}$$

and

$$Q''_{n+1} = 4\sum_{k+j=0}^{n} Q'_k Q_j c(n, k, j) + 2\sum_{k+j=0}^{n} Q''_k Q_j c(n, k, j) + 2\sum_{k+j=0}^{n} Q'_k Q'_j c(n, k, j). \tag{11}$$

Clearly $Q_n = (2n)!(2^n n!)^{-1}$ and $Q'_n = Q_n E_n = 2^n n! - Q_n$ by Theorem 1. Together with (10) this implies the identity

$$\begin{aligned} 2^{n+1}(n+1)! = Q'_{n+1} + Q_{n+1} &= 2\sum_{k+j=0}^{n} (Q'_k + Q_k) Q_j c(n, k, j) \\ &= 2\sum_{k+j=0}^{n} 2^k k! Q_j c(n, k, j). \end{aligned} \tag{12}$$

These equations now allow us to simplify (11) to

$$Q''_{n+1} = \sum_{k+j=0}^{n} Q''_k Q_j c(n, k, j) + 2\sum_{k+j=0}^{n} 2^k k! 2^j j! c(n, k, j) - 2Q_{n+1}. \tag{13}$$

We put $V_n = \mathrm{Var}(N_n)$. From $V_n = P_n'' + P_n' - P_n'^2$ and using Theorem 1 it is shown that

$$(2n)!(2^n n!)^{-1} V_n = Q_n'' - 2Q_n + 3(2^n n!) - (2^n n!)^3((2n)!)^{-1}.$$

With $W_n = Q_n'' - 2Q_n + 3(2^n n!)$ from (12) and (13) the formula

$$W_{n+1} = 2\sum_{k+j=0}^{n} (W_k Q_j + 2^{k+j} k! j!) c(n, k, j) \tag{14}$$

is obtained by straight-forward computation. We have proved the

THEOREM 4. *$V_n = \mathrm{Var}(N_n)$ may be calculated from*

$$V_n = 2^n n!((2n)!)^{-1} W_n - (2^n n!)^4((2n)!)^{-2},$$

where $W_0 = 1$ and W_n is recursively determined by (14).

REMARK. We mention that an explicit expression for P_{nk} and $P_{\infty k} = \mathrm{Prob}\{N_\infty = k\}$ may be derived as an iterated integral. Take (0, 0), ($\sqrt{2}$, 0) and ($\sqrt{2}$,$\sqrt{2}$) as corners of $\triangle$. Then a point (x, y) will be called covered if for some k both of the inequalities $x > x_k$, $y < y_k$ are satisfied. P_{nk} is $(n)_k$ times the probability that just $A_1, \cdots, A_k$ are corner-points and $x_1 < \cdots < x_k$ holds. Given $A_1, \cdots, A_k$ the points $A_{k+1}, \cdots, A_n$ must lie in the "covered" domain which has area $F_{(k)} = \sum_{j=1}^{k} (\sqrt{2} - x_j)(y_j - y_{j-1})$ with $y_0 = 0$. We obtain for $n \geqq 1$ and $1 \leqq k \leqq n$:

$$P_{nk} = (n)_k \int_0^{\sqrt{2}} dx_1 \int_0^{x_1} dy_1 \int_{x_1}^{\sqrt{2}} dx_2 \int_{y_1}^{x_2} dy_2 \cdots \int_{x_{k-1}}^{\sqrt{2}} dx_k \int_{y_{k-1}}^{x_k} dy_k F_{(k)}^{n-k}.$$

Formula (2) now implies:

$$P_{\infty k} = e^{-\lambda}\lambda^k \int_0^{\sqrt{2}} dx_1 \int_0^{x_1} dy_1 \cdots \int_{x_{k-1}}^{\sqrt{2}} dx_k \int_{y_{k-1}}^{x_k} dy_k e^{\lambda F(k)}.$$

The second integration, however, already leads to nonelementary functions, so that the recursion-formulas seem to be more useful.

5. Ladder-points. A far more natural mathematical question arises if we replace the triangle $\triangle$ by the unit-square, which shall be noted by $\triangle^*$. Here again covered points shall be defined as in the remark after Theorem 4. For this problem we use the same notation adding a star * to every symbol. In this case the P_{nk}^* remain unchanged if the uniform distributions of the x- and y-coordinates are replaced by any two continuous distributions F_x and F_y as long as all coordinates x_i, y_j are chosen independently.

In fact the problem for $\triangle^*$ is a combinatorial one rather than one of geometric probability. Given a sequence $n_1, n_2, \cdots$, of real numbers, call (k, n_k) a *ladder-point* if $n_k > n_i$ holds for all $i<k$ (analogous to the definition in the theory of random walk). P_{nk}^* is the probability of k ladder-points in a random-permutation $(n_1, \cdots, n_n)$ of the numbers $1, \cdots, n$ with all permutations equally probable.

THEOREM 5. *N_n^* is the sum of n independent random-variables $X_1, \cdots, X_n$ with* Prob $\{X_i=1\} = 1/i$ *and* Prob $\{X_i=0\} = (i-1)/i$. *Especially*:

$$E(N_n^*) = \sum_{i=1}^{n} 1/i \quad \textit{and} \quad \operatorname{Var}(N_n^*) = \sum_{i=1}^{n} (i-1)/i^2.$$

Proof. For $n=1$ the statement is obvious. Let it be true for n and consider permutations of $0, 1, \cdots, n$. These have the same distribution of ladder-points as those of $1, \cdots, n+1$. 0 yields a ladder-point if and only if it occurs at the first place, that is, with probability $1/n+1$, and the number of other ladder-points is not changed by putting 0 at any place.

THEOREM 6. *The exact distribution P_{nk}^* of N_n^* for $n \geqq 1$ is given by*

$$P_{nk}^* = \sum_{1 \leqq i_1 < i_2 \cdots < i_k = n} (i_1 \cdots i_k)^{-1}. \tag{15}$$

It satisfies the recursion formula

$$P_{nk}^* = (1/n) \sum_{m=0}^{n-1} P_{m(k-1)}^*. \tag{16}$$

Proof. From Theorem 5 it follows that P_{nk}^* is the sum of all terms $(n!)^{-1}(j_1-1) \cdots (j_{n-k}-1)$ with $1 \leqq j_1 < \cdots < j_{n-k} \leqq n$. This is equivalent to (15). To prove (16) observe that n yields a ladder-point wherever it occurs in the permutation $(n_1, \cdots, n_n)$. For every j the probability that $n_j = n$ is $1/n$, and if $n_j = n$ the

other $k-1$ ladder-points must occur among the first $m=j-1$ places. There m different numbers are ordered at random which may be given the new names $1', \cdots, m'$ in an order-preserving way.

The exact distribution as well could be derived from the recursion formula by induction. Calculation of P^*_{nk} is done faster using the recursion formula than using the explicit expression.

This work was prepared with partial support of the National Science Foundation, GP-2593.

Acknowledgment. I wish to thank Dr. H. Klinger for bringing the problem, as described in the beginning of Section 2, to my attention. Dr. Klinger (jointly with the author) was the first to obtain the expectation of the number of crystals born up until some time t by heavy use of integral-equations, Laplace transforms, etc. I also thank Professors J. Karush, M. Loève, P. K. Sen and W. Whitman for their interest and remarks, and especially Professor P. A. Moran for pointing out to us the related paper of J. L. Meijering: Interface area, edge length, and number of vertices in crystal aggregates with random nucleation; Philips Res. Rep. 8, 270–290, 1953. Meijering, among other questions in higher dimensions, considers a wire of infinite length and derives the distribution of the *length* of the crystals. Infinite length of the wire is a natural assumption if λ is large.

FUNCTIONS ON SEMI-SIMPLE ALGEBRAS

C. G. CULLEN, University of Pittsburgh, and C. A. HALL, Westinghouse Electric Corp., Bettis Atomic Laboratory

1. Introduction. Let $\mathfrak{A}$ be a finite dimensional linear associative algebra over the field $\mathfrak{F}$. We will consider only the case where $\mathfrak{A}$ is semi-simple (zero radical) over the real field R or the complex field C. It is known that

$$\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_k$$

where the $\mathfrak{A}_i$ are simple algebras over $\mathfrak{F}$ [1].

Let F be a function from a domain $\mathfrak{D} \subseteq \mathfrak{A}$ to $\mathfrak{A}$. For each $\alpha \in \mathfrak{D}$ we have unique α_i and β_i in $\mathfrak{A}_i$, $i=1, 2, \cdots, k$, such that

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k \quad \text{and} \quad F(\alpha) = \beta_1 + \beta_2 + \cdots + \beta_k.$$

If F is Hausdorff-differentiable then F_i, the restriction of F to $\mathfrak{A}_i$, satisfies

$$F_i(\alpha_i) = \beta_i, \qquad i = 1, 2, \cdots, k[7].$$

In general F_i need not map $\mathfrak{A}_i$ into $\mathfrak{A}_i$ and in fact the correspondence $\alpha_i \to \beta_i$ may not even define a function (single-valued) from $\mathfrak{A}_i$ to $\mathfrak{A}_i$.

If we start with functions F_i, from $\mathfrak{A}_i$ to $\mathfrak{A}_i$, then we can construct a function $F = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ on $\mathfrak{A}$ by defining

$$F(\alpha) = F(\alpha_1 + \alpha_2 + \cdots + \alpha_k) = F_1(\alpha_1) + F_2(\alpha_2) + \cdots + F_k(\alpha_k).$$

The domain of F is $\mathfrak{D}=\mathfrak{D}_1\oplus\mathfrak{D}_2\oplus\cdots\oplus\mathfrak{D}_k$, where $\mathfrak{D}_i\subseteq\mathfrak{A}_i$ is the domain of F_i, and in this case F_i is clearly the restriction of F to $\mathfrak{A}_i$.

In this paper we will discuss how the nature of the F_i determines the nature of the function F.

2. Primary functions. In this section we will discuss functions on $\mathfrak{A}$ which are extensions of functions from C to C. Such functions are called primary functions and can be defined in several different, but essentially equivalent ways [3]. We choose the following definition of Sylvester and Buchheim.

DEFINITION 2.1. *Let f be a single valued function from C to C and let $\mathfrak{A}$ be an algebra over C. Consider $\alpha\in\mathfrak{A}$ with minimum polynomial $m(z)=\prod_{i=1}^{t}(z-\lambda_i)^{s_i}$ and assume that f is defined at each λ_i and analytic at λ_i if $s_i>1$. Now let $L_{f,m}(z)$ be the unique polynomial of degree less than the degree of $m(z)$ satisfying*

$$\left.\frac{d^jL_{f,m}(z)}{dz^j}\right|_{z=\lambda_i}=f^{(j)}(\lambda_i),\qquad \begin{matrix} j=0,1,\cdots,s_i-1,\\ i=1,2,\cdots,t.\end{matrix}$$

The function F, defined by $F(\alpha)=L_{f,m}(\alpha)$, is called the primary function with stem function f.

$L_{f,m}(z)$ is the Lagrange-Hermite interpolation polynomial and is given explicitly in [2] or [3].

The above definition can be applied to algebras over R by extending the ground field to C and computing $F(\alpha)$ as in Definition 1.1. If $F(\alpha)$ is an element of the original algebra then F is said to be defined at α with functional value $F(\alpha)$. A necessary and sufficient condition for this to occur is that $f(\bar{z})=\overline{f(z)}$ at the zeros of the minimum polynomial of α [5]. In this case it is known that $L_{f,m}(z)$ is a polynomial with real coefficients [2].

Suppose now that $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{A}_2\oplus\cdots\oplus\mathfrak{A}_k$ and that F_i is primary on $\mathfrak{D}_i\subseteq\mathfrak{A}_i$ with stem function $f, i=1, 2, \cdots, k$. If the minimum polynomial of $\alpha_i\in\mathfrak{A}_i$ is $m_i(z)$ then from Definition 2.1 we have

$$F_i(\alpha_i)=L_{f,m_i}(\alpha_i),\qquad i=1,2,\cdots,k.$$

We will show that the function F, from $\mathfrak{A}$ to $\mathfrak{A}$, defined by

$$F(\alpha)=F(\alpha_1+\alpha_2+\cdots+\alpha_k)=F_1(\alpha_1)+F_2(\alpha_2)+\cdots+F_k(\alpha_k)$$

is a primary function on $\mathfrak{A}$ with stem function f.

We first prove a preliminary result which is of interest in its own right. We show that the minimum polynomial, in Definition 2.1, may be replaced by any polynomial multiple of the minimum polynomial without changing the function on $\mathfrak{A}$ at all.

THEOREM 2.1. *Let F be a primary function on a domain $\mathfrak{D}\subseteq\mathfrak{A}$ with stem function f. Let $\alpha\in\mathfrak{D}$ have minimum polynomial $m(z)$ and let $p(z)$ be any polynomial divisible by $m(z)$. If f is defined at the roots of $p(z)$ and analytic at the repeated roots of $p(z)$ then $F(\alpha)=L_{f,m}(\alpha)=L_{f,p}(\alpha)$.*

Proof. Since $\mathfrak{A}$ is isomorphic to a subalgebra of a total matrix algebra it suffices to prove the theorem for $\alpha = A_1 \in C_n = \{n \times n \text{ matrices over } C\}$.

Let A_2 be a matrix such that the minimum polynomial of

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

is $p(z)$ and let $F(A)$, $F(A_1)$ and $F(A_2)$ be the functional values computed using Georgi's definition of a matrix function which is known to be equivalent to Definition 2.1 [3]. Thus we have $F(A) = L_{f,p}(A)$ and $F(A_1) = L_{f,m}(A_1)$. Now we have, since

$$F(A) = \begin{bmatrix} F(A_1) & 0 \\ 0 & F(A_2) \end{bmatrix},$$

$$F(A) = L_{f,p}(A) = \begin{bmatrix} L_{f,p}(A_1) & 0 \\ 0 & L_{f,p}(A_2) \end{bmatrix} = \begin{bmatrix} F(A_1) & 0 \\ 0 & F(A_2) \end{bmatrix}$$

$$= \begin{bmatrix} L_{f,m}(A_1) & 0 \\ 0 & L_{f,p}(A_2) \end{bmatrix}$$

and it follows that $L_{f,p}(A_1) = L_{f,m}(A_1)$ as claimed.

THEOREM 2.2. *Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_k$ be semi-simple over $\mathfrak{F}$ with the $\mathfrak{A}_i$ as its simple direct summands. If F_i is the primary function on $\mathfrak{A}_i$ with stem function f, $i = 1, 2, \cdots, k$, then the function $F = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ on $\mathfrak{A}$ defined by*

$$F(\alpha) = F(\alpha_1 + \alpha_2 + \cdots + \alpha_k) = F_1(\alpha_1) + F_2(\alpha_2) + \cdots + F_k(\alpha_k)$$

is the primary function on $\mathfrak{A}$ with stem function f.

Proof. Consider any $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ in the domain of F where $\alpha_i \in \mathfrak{A}_i$ has minimum polynomial $m_i(z)$. The minimum polynomial of α is then $m(z) = \text{l.c.m.}\{m_1(z), m_2(z), \cdots, m_k(z)\}$. Now from Definition 2.1 we have

$$F(\alpha) = F_1(\alpha_1) + F_2(\alpha_2) + \cdots + F_k(\alpha_k)$$

$$= L_{f,m_1}(\alpha_1) + L_{f,m_2}(\alpha_2) + \cdots + L_{f,m_k}(\alpha_k)$$

which by Theorem 2.1 reduces to $F(\alpha) = L_{f,m}(\alpha_1) + L_{f,m}(\alpha_2) + \cdots + L_{f,m}(\alpha_k)$ and by elementary properties of direct sums

$$F(\alpha) = L_{f,m}(\alpha_1 + \alpha_2 + \cdots + \alpha_k) = L_{f,m}(\alpha).$$

Thus F is the primary function on $\mathfrak{A}$ with stem function f.

The converse of Theorem 2.2 was proved in [6].

3. Intrinsic functions. In this section we discuss the intrinsic functions on $\mathfrak{A}$. The concept of an intrinsic function on $\mathfrak{A}$ was introduced in 1960 by Rinehart [4].

DEFINITION 3.1. *Let $\mathfrak{G}$ be the group of all automorphisms and anti-automorphisms of $\mathfrak{A}$ which leave $\mathfrak{F}$ element-wise invariant. A function F is said to be intrinsic on a domain $\mathfrak{D}\subseteq\mathfrak{A}$ if for every $\Omega\in\mathfrak{G}$ and for every $\alpha\in\mathfrak{D}$ we have $\Omega\alpha\in\mathfrak{D}$ and $F(\Omega\alpha)=\Omega F(\alpha)$.*

Every primary function on $\mathfrak{A}$ is known to be intrinsic but in general the class of intrinsic functions on $\mathfrak{A}$ is larger than the class of primary functions on $\mathfrak{A}$ [4, 2]. We will have need of a slightly less restrictive class of functions than those described in Definition 3.1.

DEFINITION 3.2. *Let $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{A}_2\oplus\cdots\oplus\mathfrak{A}_k$ be semi-simple with the $\mathfrak{A}_i$ as its simple direct summands and let $\mathfrak{H}$ be the subgroup of $\mathfrak{G}$ such that if $\Omega\in\mathfrak{H}$ then $\Omega\mathfrak{A}_i=\mathfrak{A}_i$, $i=1, 2, \cdots, k$. F is $\mathfrak{H}$-intrinsic if $F(\Omega\alpha)=\Omega F(\alpha)$ for all $\Omega\in\mathfrak{H}$.*

THEOREM 3.1. *Let $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{A}_2\oplus\cdots\oplus\mathfrak{A}_k$ be semi-simple with the $\mathfrak{A}_i$ as simple direct summands. Let F_i be a function on $\mathfrak{A}_i$, $i=1, 2, \cdots, k$, and define $F=F_1\oplus F_2\oplus\cdots\oplus F_k$ on $\mathfrak{A}$ by*

$$F(\alpha) = F(\alpha_1+\alpha_2+\cdots+\alpha_k) = F_1(\alpha_1)+F_2(\alpha_2)+\cdots+F_k(\alpha_k).$$

Then F is H-intrinsic on $\mathfrak{A}$ if and only if the F_i are intrinsic on the $\mathfrak{A}_i$.

Proof. Let F be $\mathfrak{H}$-intrinsic and consider Ω_i any automorphism or anti-automorphism of $\mathfrak{A}_i$. Let $\Omega\in\mathfrak{H}$ be such that its restriction to $\mathfrak{A}_i$ is Ω_i. (Consider $\alpha_j\in\mathfrak{A}_j$ where $j\neq i$. If Ω_i is an automorphism define $\Omega\alpha_j=\alpha_j$. If Ω_i is an anti-automorphism define $\Omega\alpha_j=\phi^{-1}(\phi(\alpha_j)^t)=\alpha_j'$ where ϕ is an isomorphism of $\mathfrak{A}_j$ into a matrix algebra.) Now, since F is intrinsic on $\mathfrak{A}$,

$$\begin{aligned}F(\Omega\alpha) &= F_1(\Omega\alpha_1)+F_2(\Omega\alpha_2)+\cdots+F_i(\Omega_i\alpha_i)+\cdots+F_k(\Omega\alpha_k) = \Omega F(\alpha)\\ &= \Omega F_1(\alpha_1)+\Omega F_2(\alpha_2)+\cdots+\Omega_i F_i(\alpha_i)+\cdots+\Omega F_k(\alpha_k)\end{aligned}$$

and since $\Omega\in\mathfrak{H}$ we have $F_j(\Omega\alpha_j)$, $\Omega F_j(\alpha_j)\in\mathfrak{A}_j$; $j=1, 2, \cdots, k$; so it follows, from the uniqueness of the $\mathfrak{A}_i$ component of $\Omega F(\alpha)=F(\Omega\alpha)$, that $F_i(\Omega_i\alpha_i)=\Omega_i F_i(\alpha_i)$ and hence that F_i is intrinsic on $\mathfrak{A}_i$.

Now suppose that each F_i is intrinsic and consider $\Omega\in\mathfrak{H}$. Since Ω, restricted to $\mathfrak{A}_i$, is an automorphism or anti-automorphism of $\mathfrak{A}_i$ it follows that

$$\begin{aligned}F(\Omega\alpha) &= F(\Omega\alpha_1+\Omega\alpha_2+\cdots+\Omega\alpha_k)\\ &= F_1(\Omega\alpha_1)+F_2(\Omega\alpha_2)+\cdots+F_k(\Omega\alpha_k)\\ &= \Omega F_1(\alpha_1)+\Omega F_2(\alpha_2)+\cdots+\Omega F_k(\alpha_k)\\ &= \Omega(F_1(\alpha_1)+F_2(\alpha_2)+\cdots+F_k(\alpha_k))\\ &= \Omega F(\alpha).\end{aligned}$$

Thus F is $\mathfrak{H}$-intrinsic as claimed, completing the proof.

It is well known [1] that the simple direct summands of $\mathfrak{A}$ are the minimal left ideals of $\mathfrak{A}$ and that they are unique up to isomorphism. For any $\Omega\in\mathfrak{G}$, $\Omega\mathfrak{A}_i$ is again a minimal left ideal of $\mathfrak{A}$ and hence is a direct summand of $\mathfrak{A}$. If

$\Omega\mathfrak{A}_i$ is isomorphic to $\mathfrak{A}_j (i \neq j)$ then we have $\dim(\mathfrak{A}_i) = \dim(\mathfrak{A}_j)$. If no pair of direct summands are isomorphic ($\dim(\mathfrak{A}_i) \neq \dim(\mathfrak{A}_j)$ for $i \neq j$) then, as a direct corollary of Theorem 3.1, we have

THEOREM 3.2. *Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_k$ be semi-simple with the $\mathfrak{A}_i$ as simple direct summands where $\mathfrak{A}_i$ is not isomorphic to $\mathfrak{A}_j$ if $i \neq j$. Let F_i be a function on $\mathfrak{A}_i$, $i = 1, 2, \cdots, k$, and define $F = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ on $\mathfrak{A}$ by*

$$F(\alpha) = F(\alpha_1 + \alpha_2 + \cdots + \alpha_k) = F_1(\alpha_1) + F_2(\alpha_2) + \cdots + F_k(\alpha_k).$$

Under these conditions F is intrinsic on $\mathfrak{A}$ if and only if the F_i are intrinsic on the A_i.

We now give an example to show the necessity of the added hypothesis in Theorem 3.2. Let

$$\mathfrak{A} = R_2 \oplus R_2 = \left\{ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \middle| A_1, A_2 \in R_2 \right\},$$

$R_2 = \{2 \times 2 \text{ matrices over } R\}$, and consider the functions $F_1(A_1) = A_1$ and $F_2(A_2) = A_2^2$ which are clearly intrinsic on the direct summands. It is easy to see that the mapping Ω, defined by

$$\Omega \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} A_2 & 0 \\ 0 & A_1 \end{bmatrix},$$

is an automorphism of $\mathfrak{A}$. Now for $F = F_1 \oplus F_2$ and $A = \operatorname{dg}\{2, 2, 3, 3\}$ we have

$$\Omega F(A) = \Omega F(\operatorname{dg}\{2, 2, 3, 3\}) = \Omega \operatorname{dg}\{2, 2, 9, 9\} = \operatorname{dg}\{9, 9, 2, 2\}$$

$$\text{while } F(\Omega A) = F(\operatorname{dg}\{3, 3, 2, 2\}) = \operatorname{dg}\{3, 3, 4, 4\} \neq \Omega F(A).$$

Thus $F = F_1 \oplus F_2$ is not intrinsic on $\mathfrak{A}$ even though F_1 and F_2 are intrinsic on the respective direct summands.

As the final result of this section we give conditions on the functions F_i which are necessary and sufficient for $F = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ to be intrinsic on $\mathfrak{A}$.

THEOREM 3.3. *Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_k$ be semi-simple with the $\mathfrak{A}_i$ as simple direct summands. Let F_i be a function on $\mathfrak{A}_i$, $i = 1, 2, \cdots, k$, and define $F = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ on $\mathfrak{A}$ by*

$$F(\alpha) = F(\alpha_1 + \alpha_2 + \cdots + \alpha_k) = F_1(\alpha_1) + F_2(\alpha_2) + \cdots + F_k(\alpha_k).$$

If there exists an automorphism or anti-automorphism Ω' of $\mathfrak{A}$ such that $\Omega'\mathfrak{A}_i = \mathfrak{A}_j (i \neq j)$, then require that $F_j(\Omega'\alpha_i) = \Omega' F_i(\alpha_i)$. Under these conditions F is intrinsic on $\mathfrak{A}$ if and only if the F_i are all intrinsic on the $\mathfrak{A}_i$.

Proof. If F is intrinsic then it is $\mathfrak{H}$-intrinsic and Theorem 3.1 assures us that the F_i are intrinsic.

Now let F_i be intrinsic on $\mathfrak{A}_i$, $i = 1, 2, \cdots, k$, and consider any automorphism or anti-automorphism Ω of $\mathfrak{A}$. Let ρ be the permutation on k-letters such

that $\Omega\mathfrak{A}_i=\mathfrak{A}_{\rho(i)}$, $i=1, 2, \cdots, k$. Our hypotheses yield

$$\Omega F_i(\alpha_i) = F_{\rho(i)}(\Omega\alpha_i) = F_{\rho(i)}((\Omega\alpha)_{\rho(i)}) \qquad \text{for } i = 1, 2, \cdots, k,$$

where $(\Omega\alpha)_{\rho(i)}$ is the unique $\mathfrak{A}_{\rho(i)}$ component of $\Omega\alpha$. Now for any $\alpha=\alpha_1+\alpha_2+\cdots+\alpha_k$ in $\mathfrak{A}$ we have

$$\Omega F(\alpha) = \Omega \sum_i F_i(\alpha_i) = \sum_i \Omega F_i(\alpha_i) = \sum_i F_{\rho(i)}((\Omega\alpha)_{\rho(i)}) = F(\Omega\alpha).$$

Thus F is intrinsic on $\mathfrak{A}$ as desired.

Presented to the Amer. Math. Soc., on January 24, 1966.

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ONE MORE "MANY-MORE" ASSERTION

J. T. RYAN, Research Triangle Institute, Durham, N. C.

1. Introduction. Since 1859 mathematicians have spent thousands of hours counting prime numbers in their separate linear forms. Chebyshev suggested then that separate enumeration of the primes into the $4k+1$ and $4k+3$ arithmetic progressions might shed some light on the properties of these primes. His suggestion led Glaisher [1] actually to count by hand the number of such primes in the first hundred thousand numbers of the first through the ninth million numbers. This work by Glaisher was the first in a virtual parade of such efforts. From such counts as these and other mathematical results, a number of assertions have been made during the past 100 years about the relative numbers of integers in different arithmetic progressions. This paper describes the results of one of the latest of these counting efforts carried out by the author on a high speed digital computer. On the basis of the tallies obtained and some heuristic arguments by Shanks [2], a very general "many-more" assertion is made (Conjecture 3 below).

2. "Many-more" assertions. A "many-more" assertion is any statement (either of fact or of strong conjecture) that describes some sense in which one type of number occurs "more frequently" than another type in the sequence of positive integers. A theorem first proved by Phragmen [3], in 1891 and later, in a simpler way, by Landau, [4], [5], will provide a good example of such an assertion:

THEOREM 1. *Let $\pi_-(n)$ be the number of primes of the form $4k-1$ that are less than or equal to n. Similarly, let $\pi_+(n)$ be the number of primes of the form $4k+1$ that are less than or equal to n. If we let*

$$\Delta(n) = \pi_-(n) - \pi_+(n) \quad \text{and} \quad t(n) = \frac{\Delta(n)}{\sqrt{n}/\ln n} \tag{1, 2}$$

then (as Chebyshev, in fact, first conjectured but did not prove) for a certain subsequence, n_i, of the integers,

$$\lim_{i\to\infty} t(n_i) = 1. \tag{3}$$

Although this theorem certainly does not state that there are "many-more" primes of the form $4k-1$ than of the form $4k+1$, it shows a "sense" for which $4k-1$ primes predominate over $4k+1$ primes. (It is interesting to note that Glaisher incorrectly surmised the opposite from his early counts.)

Shanks provides us with a *quantitative statistical* interpretation of Chebyshev's "many-more" primes assertion. He suggests [2]:

CONJECTURE 1. If we let

$$\tau(n) = \frac{\Delta(n)\sqrt{n}}{\pi(n)}, \tag{4}$$

where $\pi(n)$ is the number of primes less than or equal to n, then the mean value of $\tau(n)$,

$$\bar{\tau}(n) = \frac{1}{n-1}\sum_{i=2}^{n} \tau(i) \tag{5}$$

has a limit as $n\to\infty$, and this limit equals $+1$; i.e.,

$$\lim_{n\to\infty} \bar{\tau}(n) = 1. \tag{6}$$

If this were true, we might say that in *an average sense* there are many more primes of the form $4m-1$ than of the form $4m+1$. More formally this can be stated: There exists an N such that $\bar{\tau}(n)>0$ for all $n>N$. Further, for any $\rho<1$, there exists an N such that $\bar{\tau}(n)>\rho$ for $n>N$.

Although Shanks presented some numerical evidence and heuristic arguments for Conjecture 1, he did not prove it. Nor is it likely to be proved or

disproved in the near future because of its known association with the famous Riemann Hypothesis about the zeros of the zeta function. In connection with his aforementioned heuristic arguments, Shanks was led to the following generalization of Conjecture 1:

CONJECTURE 2. Let a be a positive integer and let $\pi_{\pm}^{(a)}(n)$ be the number of positive integers of the form $4k \pm 1$ which are $\leqq n$ and which are the products of a (not necessarily distinct) primes; and further, let

$$\Delta^{(a)}(n) = \pi_{-}^{(a)}(n) - \pi_{+}^{(a)}(n), \tag{7}$$

$$\sum^{(a)}(n) = \pi_{-}^{(a)}(n) + \pi_{+}^{(a)}(n), \tag{8}$$

and

$$\tau^{(a)}(n) = \frac{\Delta^{(a)}(n)\sqrt{n}}{\sum^{(a)}(n)}. \tag{9}$$

Here we define $\tau^{(a)}(n) = 0$ if $\sum^{(a)}(n) = 0$. Then the average value:

$$\bar{\tau}^{(a)}(n) = \frac{1}{n}\sum_{k=1}^{n} \tau^{(a)}(k) \tag{10}$$

will tend to $(-1)^{a+1}$ in the limit as $n \to \infty$ for $a = 1, 2, 3, \cdots$.

This phenomenon will be called the "Shanks Flip-Flop Phenomenon." It is this conjecture and the heuristic arguments of Shanks that led the author to an even more general assertion regarding numbers in *any* arithmetic progression. This conjecture and the supporting evidence is presented in the next sections.

3. Another "many-more" assertion. Shanks' argument for the flip-flop behavior of the τ-function shown by Conjecture 2 was based on the assumption that the mean values of $\tau^{(1)}(x)$ (i.e., $\bar{\tau}^{(1)}(x) = 1/(x-1)\sum_{n=2}^{x} \tau(n)$) for $x = n^{1/a} (a = 2, 3, \cdots)$ are all equal to 1. This would imply that

$$\frac{\pi_{-}^{(1)}(x) - \pi_{+}^{(1)}(x)}{\pi_{-}^{(1)}(x) + \pi_{+}^{(1)}(x)}\sqrt{x} = 1 \tag{11}$$

in the mean, or that

$$\frac{\pi_{-}^{(1)}(x)}{\pi_{-}^{(1)}(x) + \pi^{(1)}(x)} = \frac{1}{\sqrt{x}} + \frac{\pi_{+}^{(1)}(x)}{\pi_{-}^{(1)}(x) + \pi^{(1)}(x)} = \frac{1}{\sqrt{x}} + 1 - \frac{\pi_{-}^{(1)}(x)}{\pi_{-}^{(1)}(x) + \pi_{+}^{(1)}(x)} \tag{12}$$

or simply

$$\frac{\pi_{-}^{(1)}(x)}{\pi_{-}^{(1)}(x) + \pi_{+}^{(1)}(x)} = \frac{1}{2}[x^{-1/2} + 1]. \tag{13}$$

It therefore follows that of all the odd primes of (geometric) mean order $n^{1/a}$, a fraction equal to $\frac{1}{2}(1+n^{-1/2a})$ is of the form $4m-1$ and similarly a fraction equal to $\frac{1}{2}(1-n^{-1/2a})$ is of the form $4m+1$. In this paper the order of an integer or group of integers is simply some number (usually an integer) which is "close to" the given integer or group of integers. That is, it is the approximate magnitude of the integer or group of integers. The geometric mean order of the prime factors of a number of order n and belonging to C^a is $n^{1/a}$ where C^a is the class of a-fold composite numbers.

We might now inquire as to whether this result or at least the Shank's Flip-Flop Phenomenon can be extended to the distribution of a-fold composites for any modulus.

The extension to the modulus 6 is simple because the multiplication table for primes of the form $6m-1$ and $6m+1$ is the same as for the modulus 4 primes:

	$6m+1$	$6m-1$
$6m+1$	$6m+1$	$6m-1$
$6m-1$	$6m-1$	$6m+1$

If the linear forms $4m-1$ and $4m+1$ are divided into the arithmetic progressions of modulus 8, we will have the four classes $8m+1$, $8m+3$, $8m+5$, and $8m+7$. We see that the classes $8m+1$ and $8m+5$ belong to the class of $4m+1$ forms. Landau's generalization of Chebyshev's assertion (for primes) to any modulus, [5, p. 704–711], suggests that the deficiency exhibited by the $4m+1$ primes will be confined entirely to $8m+1$. Shanks [2] provided numerical evidence to support this extension. He further showed that among the three remaining forms $8m+3$, $8m+5$, and $8m+7$, there are no significant differences in the "Chebyshev" sense.

The multiplication table modulo 10

1	3	7	9
3	9	1	7
7	1	9	3
9	7	3	1

provides us with two classes falling on the diagonal, the forms $10m+1$ and $10m+9$. These residue classes are the quadratic residues. Shanks also discussed the primes modulo 10. Here, evidence was shown which supports the conjecture

that the fluctuations in the $10m+3$ or $10m+7$ primes diminish the $10m+1$ primes. Extending his argument to numbers in C^a, we see that the residue classes 3 and 7 share the lead for odd-fold composites and the residue classes 1 and 9 share the lead for even-fold composites. In general, we might conjecture that the quadratic residue classes will always share the lead for even-fold composites, and that the quadratic nonresidue classes will always share the lead for odd-fold composites. Formally, this can be stated as

CONJECTURE 3. In the mean, among all C^a numbers of order n, a fraction equal to

$$\frac{1}{\Phi(m)}\left[1-(-1)^{a+1}(\Phi(m)-R(m))n^{-1/2}\right] \tag{14}$$

is of the form $mk+b$, where b is a quadratic residue of m, and a fraction equal to

$$\frac{1}{\Phi(m)}\left[1+(-1)^{a+1}R(m)n^{-1/2}\right] \tag{15}$$

is of the form $mk+c$, where c is a quadratic nonresidue of m and where $R(m)$ is the number of quadratic residues of m and $\Phi(m)$ is Euler's Function.

4. Some numerical evidence. Computer programs were written to provide data that might strengthen the case for Conjecture 3. Table I shows the average relative positions of the four modulo 8 linear forms. The numbers in this table were calculated in the following manner:

The position of each form was recorded at intervals of 100 up to 2,000,000. If a form was in the lead, it received a 1; if it was second, it received a 2, etc. If two forms shared a position, then the number of the position plus one-half was recorded for both forms and forms in all subsequent position would receive numbers accordingly, that is, as if there had been no ties. For example, if form A were in first place, forms B and C tied for second place, and form D in third place, form A would have a 1 recorded for it, forms B and C would have a $2\frac{1}{2}$ recorded for them, and form D would have a 4 recorded for it. These position values were then averaged over the stated values.

TABLE I

Average Relative Position of $\pi_{8,b}^{(a)}(n)$

a	$b=1$	3	5	7
1	4	2.02	1.97	2.01
2	1	3.03	2.99	2.98
3	4	1.96	1.98	2.06
4	1.01	2.96	3.07	2.96
5	3.99	2.00	2.02	1.99
6	1.01	3.14	2.95	2.90
7	3.94	2.02	2.04	2.00

Tables II and III provide similar data for the modulo 12 and modulo 10 linear forms.

TABLE II

Average Relative Position of $\pi_{12,b}^{(a)}(n)$

a	$b=1$	5	7	11
1	4	1.97	2.03	2.00
2	1.02	3.00	2.95	3.03
3	3.92	2.02	2.05	2.01
4	1.15	2.89	2.97	2.99
5	3.77	2.06	2.10	2.07
6	1.19	2.93	2.91	2.97
7	3.51	2.16	2.20	2.13

TABLE III

Average Relative Position of $\pi_{10,b}^{(a)}(n)$

a	$b=1$	3	7	9
1	3.51	1.51	1.48	3.50
2	1.51	3.53	3.47	1.49
3	3.48	1.46	1.55	3.51
4	1.57	3.49	3.45	1.49
5	3.54	1.43	1.51	3.52
6	1.53	3.52	3.50	1.45
7	3.50	1.54	1.47	3.49

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MATHEMATICAL NOTES

A TOTIENT FUNCTION ON THE RESIDUE CLASSES MOD m AND OTHER SEMIGROUPS

ROBERT SPIRA, University of Tennessee

Is it possible to define an Euler totient function on an arbitrary commutative semigroup? In this paper we take up first the special case of defining such a ϕ-function on the semigroup of the integers mod m, and then study the conditions under which an analogously defined function would be multiplicative in an arbitrary commutative semigroup. The definition of the ϕ-function on the residue classes arose from trying to characterize the ordinary ϕ-function on the semigroup of rational integers using the residue classes themselves rather than by counting. This characterization was successful, as is shown by the corollary to Theorem 3. The point of view of using residue classes is the proper one, since for semigroups one would have to start with an assumed compatible residue partition, rather than a residue structure determined by an additive group.

We start with some definitions. Let the set of integers be partitioned into a set M of residue classes M_j.

$$M_j = \{n \mid n \equiv j \pmod m\} = \{n \mid n = sm + j,\ s = 0, \pm 1, \pm 2, \pm 3, \cdots\}$$
$$M = \{M_j \mid j = 1, \cdots, m\}.$$

DEFINITIONS:

(a) $(M_i, M_j) = 1$ *if and only if there is an* a *in* M_i *and a* b *in* M_j, *such that* $(a, b) = 1$,

(b) $\phi_m(j) =$ *the number of* M_i *such that* $(M_i, M_j) = 1$,

(c) $\phi(M) = \min_j \phi_m(j)$.

$\phi_m(j)$ is the totient function of the title, and is later shown to be multiplicative in m. We now start on a series of theorems giving the properties of $\phi_m(j)$. Hancock [1] also came upon some of these properties, starting from the idea of counting lattice points in a cube such that $(i, j, k) = 1$.

LEMMA. *If* $(a, b) = 1$ *and* n *is an integer, then there is an* x *such that* $(ax + b, n) = 1$.

Proof. While this lemma follows from Dirichlet's theorem, the following elementary proof can be given. x can be found $\leqq n$. Let $(a, n) = d$, $a = a_1 d$, $n = n_1 d$. Then $(a_1, n_1) = 1$. Let $n = N \cdot D$, where D contains all the primes dividing d to the highest possible power. Thus, $d \mid D$, $N \mid n_1$, $(N, d) = 1$. Also, since $(a_1, n_1) = 1$, $(a_1, N) = 1$, so $(a, N) = 1$. Thus, $ka + b (k = 1, \cdots, N)$ is a complete residue system mod N. Choose k so that $(ka + b, N) = 1$. Then $(ka + b, D) = 1$, for if p is a prime and $p \mid (ka + b, D)$, $p \mid d$. Hence, $p \mid a$, so $p \mid b$, but $(a, b) = 1$. Hence $(ka + b, n) = 1$.

THEOREM 1. $(M_i, M_j)=1$ *if and only if* $(m, i, j)=1$.

Proof. $(M_i, M_j)=1$ if and only if $(sm+i, tm+j)=1$ is solvable in integers s and t. Let $(m, i)=d_i$, $(m, j)=d_j$, $d=(d_i, d_j)$. If $d>1$, $d \mid sm+i$, $d \mid tm+j$, and we have $(sm+i, tm+j) \geqq d>1$ for all s and t, so $(M_i, M_j) \neq 1$. If $d=1$, using the lemma, choose s so that $s(m/d_i)+i/d_i=N_1$, $(N_1, d_j)=1$. Then choose t so that $t(m/d_j)+j/d_j=N_2$, $(N_2, N_1 d_i)=1$. Then $(sm+i, tm+j)=(d_i N_1, d_j N_2)=1$.

We remark that the definition of relative primality of M_i and M_j coincides with the notion derived from the theorem of unique factorization mod m (Vandiver and Weaver [2]). As shown there, each residue j is uniquely factorable (in a special sense), $j \equiv u p_1^{j_1} \cdots p_k^{j_k}$, where the p_i's are prime divisors of m and $(u, m)=1$. From our Theorem 1, it follows that $(M_i, M_j)=1$ if and only if $(m, i, j)=1$, and this is easily seen to be equivalent to i and j having no common primes also in common with m.

THEOREM 2. $\phi_m(m)=\phi(m)$.

Proof. In Theorem 1, set $j=m$, and count the number of times $(m, i, j)=(m, i, m)=(i, m)=1$. This number is clearly $\phi(m)$.

THEOREM 3. $\phi_m(j) \geqq \phi_m(m)$, *equality occurring only when* j *has all the prime factors of* m.

Proof. The case $m=1$ is clear. If $(j, m)=1$, $(i, j, m)=1$ for all i, so $(M_i, M_j)=1$ for all i so $\phi_m(j)=m>\phi(m)$ for $m>1$. Let now $(j, m)=d$, $1<d \leqq m$. Then $(i, j, m)=(i, d)$. Since $d \mid m$, $(m, i)=1$ implies $(d, i)=1$, so $\phi_m(j) \geqq \phi_m(m)$. If p is a prime dividing m, and $p \nmid j$, then $(p, j)=1$, so $\phi_m(j)>\phi_m(m)$. But if all primes dividing m divide j, then there are no more than $\phi(m)$ i's relatively prime to j, so Theorem 3 is proved.

COROLLARY. $\phi(M)=\phi(m)$.

THEOREM 4.

$$\phi_m(j) = m \frac{\phi((j, m))}{(j, m)}.$$

Proof. Let $(j, m)=d$. Then $(i, j, m)=(i, d)$, and $\phi_m(j)=$ the number of i's, $1 \leqq i \leqq m$, for which $(i, d)=1$. This is clearly $(m/d) \cdot \phi(d)$, and the theorem follows.

COROLLARY. $\phi_m(j)$ *is multiplicative in* m.

Proof. If $(m, n)=1$, then $(j, mn)=(j, m) \cdot (j, n)$, and also $((j, m), (j, n))=1$. Hence

$$\phi_{mn}(j) = mn \frac{\phi((j, mn))}{(j, mn)} = mn \frac{\phi((j, m) \cdot (j, n))}{(j, m) \cdot (j, n)}$$

$$= m \frac{\phi((j, m))}{(j, m)} \cdot n \frac{\phi((j, n))}{(j, n)} = \phi_m(j) \cdot \phi_n(j).$$

Let N_j $(j=1, \cdots, n)$ be the residue classes mod n. The following two theorems give the properties of residue classes which are used to show multiplicativeness for commutative semigroups.

THEOREM 5. $(m, n)=1$ *if and only if for all* i *and* j, $M_i \cap N_j$ *is nonempty.*

Proof. If $(m, n)=1$, we can solve $xm+i \equiv j \pmod{n}$, so $M_i \cap N_j$ is not empty. If $(m, n)=d>1$, take $j=d$, $i=1$, and the equation $xm+1 \equiv d \pmod{n}$ is impossible, so $M_1 \cap N_d$ is empty.

Of course, the classes $M_i \cap N_j$ form the residue classes of $m \cdot n$, when $(m, n)=1$. The following theorem shows the nature of relative primality of the classes of mn in relation to the m and n classes.

THEOREM 6. *If* $(m, n)=1$, *then* $(M_i \cap N_s, M_j \cap N_t)=1$ *if and only if* $(M_i, M_j)=(N_s, N_t)=1$.

Proof. Let r_1, r_2 be the least positive residues (mod mn) of $M_i \cap N_s$ and $M_j \cap N_t$ respectively. By Theorem 1, we need only show $(r_1, r_2, mn)=1$. Let p be a prime dividing (r_1, r_2, mn), and let p divide say m. Now $r_1 \equiv i \pmod{m}$ and $r_2 \equiv j \pmod{m}$, so p would divide both i and j, along with m, a contradiction to $(i, j, m)=1$. For the converse, if $M_i \cap N_s$ and $M_j \cap N_t$ are relatively prime, then one can find a in $M_i \cap N_s$ and b in $M_j \cap N_t$ such that $(a, b)=1$, and this a and b will serve to show the relative primality of M_i and M_j as well as N_s and N_t.

We now study the application of these ideas to a commutative semigroup S. As usual, we define $a \mid b$ if and only if there is a c in S such that $b=ca$. To say that a is a *unit* means, that $a \mid b$ for all b in S. Let S be partitioned into a finite number of residue classes compatible with multiplication, $S=\bigcup_j M_j$, $M=\{M_j\}$. Let $(a, b)=1$ mean that the only common divisors of a and b are units. Then define, as before, $(M_i, M_j)=1$ as the existence of a in M_i and b in M_j such that $(a, b)=1$; $\phi(M_i)$ = the number of M_j such that $(M_i, M_j)=1$, and $\phi(M)=\min_i \phi(M_i)$.

Now let $N=\{N_j\}$ be another such finite compatible partition. Let $M \cdot N=\{M_i \cap N_j\}$ be the common refinement of all intersections. It is easy to see that this is always compatible, and it is clearly finite. Using the conditions from Theorems 5 and 6 we obtain the following multiplicativeness theorem for semigroups.

THEOREM 7. *If* M *and* N *are two compatible partitions of the semigroup* S *such that* (i) *for all* i *and* j $M_i \cap N_j$ *is not empty*, (ii) $(M_i \cap N_s, M_j \cap N_t)=1$ *if and only if* $(M_i, M_j)=(N_s, N_t)=1$, *then* $\phi(M \cdot N)=\phi(M) \cdot \phi(N)$.

Proof. By condition (i) $M_i \cap N_j$ is a nonempty class of the partition $M \cdot N$. By condition (ii) $\phi(M_i \cap N_s)=$ the number of $M_j \cap N_t$ relatively prime to $M_i \cap N_s=\phi(M_i) \cdot \phi(N_s)$. Hence $\phi(M_i \cap N_s)$ will be minimized when $\phi(M_i)$ and $\phi(N_s)$ are minimized, so $\phi(M \cdot N)=\phi(M) \cdot \phi(N)$.

To sum up, we have defined a ϕ-function which can be thought of as a totient function on the integers mod m, and also as a multiplicative function on the

integers with j fixed. In addition, we have studied the definition of such a function as a multiplicative function on commutative semigroups.

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n AND $n+1$ CONSECUTIVE INTEGERS WITH EQUAL SUMS OF SQUARES

W. H. SIMONS, University of California, Davis, and University of British Columbia and
H. L. ALDER, University of California, Davis

In [1] Alder gave a complete set of solutions in positive integers of the equation

$$x^2 + (x+1)^2 + (x+2)^2 + \cdots + (x+n-1)^2 = y^2 + (y+1)^2 + (y+2)^2 + \cdots + (y+n)^2, \tag{1}$$

i.e., the case where the sum of n consecutive squares equals the sum of $n+1$ consecutive squares, for all positive integers n considered in [2], namely where n and $n+1$ are squarefree.

It is the purpose of this paper to give the complete set of solutions of (1) for all n.

As in [1], we rewrite (1) as

$$nx^2 + n(n-1)x - n^2 = (n+1)y^2 + n(n+1)y. \tag{2}$$

Letting $x=y+z$, we can rewrite (2) as $y^2-2n(z-1)y-n(n+z)(z-1)=0$ or $[y-n(z-1)]^2=n^2(z-1)^2+n(n+z)(z-1)$ or

$$[y - n(z-1)]^2 = n(n+1)z(z-1). \tag{3}$$

We now distinguish two cases:

CASE I. $n\equiv 1$ or $2 \pmod 4$.

If $n\equiv 2 \pmod 4$, let $n=2m$ and determine positive integers a and b, with b squarefree, so that

$$m(2m+1) = a^2b. \tag{4}$$

If $n\equiv 1 \pmod 4$, let $n+1=2m$ and determine positive integers a and b, with b squarefree, so that

$$(2m-1)m = a^2b. \tag{5}$$

Substituting either (4) or (5) into (3), we obtain

$$[y - n(z-1)]^2 = 2a^2bz(z-1). \tag{6}$$

Since the left hand side is divisible by a^2, it follows that $y-n(z-1)$ is divisible by a; moreover, since the left hand side is divisible by b and b is square-

free, $y-n(z-1)$ is also divisible by b. We now conclude that $y-n(z-1)$ is divisible by ab. This is obvious if $(a, b)=1$. If, on the other hand, $(a, b)>1$, let p be a prime dividing both a and b and p^{α} the largest power of p dividing a, then the left hand side of (6) is divisible by $p^{2\alpha+1}$ and consequently by $p^{2\alpha+2}$, so that $y-n(z-1)$ is divisible by $p^{\alpha+1}$ and, using the same argument for all primes dividing both a and b, our conclusion follows. Therefore, we can let

$$y - n(z - 1) = abw \tag{7}$$

and, by substitution into (6), obtain $2z(z-1)=bw^2$ or

$$(2z - 1)^2 - 2bw^2 = 1, \tag{8}$$

which is Pell's equation, usually written

$$x^2 - Dy^2 = 1,$$

with $x=2z-1$, $y=w$, $D=2b$. Note that D is not a perfect square, since b is squarefree and not equal to 2. If b were 2, it would follow from (4) or (5) that the product of two consecutive integers is a perfect square which is impossible.

The Pell equation (8) can be most conveniently solved, as in [1], by use of the infinite continued fraction expansion of $\sqrt{2b}$. Since the appropriate numerators and denominators, respectively, of the convergents give all solutions of (8), we have obtained a complete set of solutions of (1) for Case I.

CASE II. $n\equiv 0$ or 3 (mod 4).

If $n\equiv 0$ (mod 4), we let $n=4m$ and determine positive integers a and b, with b squarefree, so that

$$m(4m + 1) = a^2b. \tag{9}$$

If $n\equiv 3$ (mod 4), we let $n+1=4m$ and determine positive integers a and b, with b squarefree, so that

$$(4m - 1)m = a^2b. \tag{10}$$

Substituting either (9) or (10) into (3) we obtain

$$[y - n(z - 1)]^2 = 4a^2bz(z - 1). \tag{11}$$

Since the left-hand side must be divisible by both a and b, we let

$$y - n(z - 1) = abw \tag{12}$$

and obtain $4z(z-1)=bw^2$ or

$$(2z - 1)^2 - bw^2 = 1, \tag{13}$$

which again is Pell's equation. If b is even, all solutions of (1) are obtained by use of the continued fraction expansion of $\sqrt{b}$. If b is odd, however, we obtain all solutions of (1) by using only those solutions of (13) obtained from the continued fraction expansion of $\sqrt{b}$ for which w is even, thus guaranteeing that $2z-1$ is odd.

Example. $n=100$.

Since this is Case II, we let $25\cdot 101=a^2b$, so that $a=5$, $b=101$. Now

$$\sqrt{101} = (10, \overline{20}), \tag{14}$$

so that, if we denote the ith convergent of $\sqrt{101}$ by

$$C_i = \frac{p_i}{q_i}, \tag{15}$$

we find all solutions of (13) for which w is even by $2z-1=p_{2k}$, $w=q_{2k}$, where $k=1, 2, 3, \cdots$.

The first solution, therefore, would be obtained for $k=1$ and can easily be found to be $2z-1=p_2=201$, $w=q_2=20$.

Since the convergents in a continued fraction expansion are related recursively by

$$p_i = a_ip_{i-1} + p_{i-2}$$
$$q_i = a_iq_{i-1} + q_{i-2},$$

and since from (14) the partial quotients a_i (for $i\geqq 2$) are all 20, we have for $i=2k$

$$p_{2k} = 20p_{2k-1} + p_{2k-2} = 20(20p_{2k-2} + p_{2k-3}) + p_{2k-2} = 401p_{2k-2} + 20p_{2k-3}$$
$$= 401p_{2k-2} + (p_{2k-2} - p_{2k-4}) = 402p_{2k-2} - p_{2k-4}.$$

Completely analogously we find $q_{2k}=402q_{2k-2}-q_{2k-4}$. This shows that the complete set of solutions of equation (13) with $b=101$ is given by the following recursion formulae:

$$\begin{aligned} 2z - 1 &= p_{2k} = 402p_{2k-2} - p_{2k-4}, \\ w &= q_{2k} = 402q_{2k-2} - q_{2k-4}, \end{aligned} \tag{16}$$

with $p_2=201$, $q_2=20$.

The solutions for x and y are then given by

$$\begin{aligned} y &= 100(z - 1) + 505w, \\ x &= y + z, \end{aligned} \tag{17}$$

and the solution with smallest x and y is given by $y=20100$, $x=20201$.

The authors wish to acknowledge the help of their students, Mr. S. P. Galovich and Mr. R. D. Hall, in the work on this problem.

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PROOF OF SCHERK'S CONJECTURE ON THE REPRESENTATION OF PRIMES

J. L. BROWN, JR., Pennsylvania State University

In Dickson's History ([1], p. 436), this paragraph appears: "H. F. Scherk [2] stated the following empirical theorems: *Every prime of odd rank (the n-th prime* 1, 2, 3, 5, $\cdots$ *being of rank n) can be composed by addition and subtraction of all the smaller primes, each taken once*; *thus*

$$13 = 1 + 2 - 3 - 5 + 7 + 11 = -1 + 2 + 3 + 5 - 7 + 11.$$

Every prime of even rank can be composed similarly, except that the next earlier prime is doubled; *thus*

$$\begin{aligned} 17 &= 1 + 2 - 3 - 5 + 7 - 11 + 2\cdot 13 \\ &= -1 - 2 + 3 - 5 + 7 - 11 + 2\cdot 13." \end{aligned}$$

The purpose of this note is to give a simple proof of Scherk's observations by making use of Bertrand's postulate and a lemma developed earlier by the author [3].

LEMMA. *Let* $f_1=1$, f_2, f_3, $\cdots$ *be a sequence of positive integers satisfying* $f_{m+1} \leqq 1+\sum_1^m f_i$ *for* $m \geqq 1$. *Then, corresponding to any integer n satisfying the inequality* $0 \leqq n < 1+\sum_1^k f_i$, *there exist binary coefficients* $\alpha_i=\alpha_i(n)$ *such that* $n=\sum_1^k \alpha_i f_i$ ($\alpha_i=0$ *or* 1 *for each i*).

A proof of the lemma will be found in [3].

If the sequence of primes (with unity adjoined) is denoted by $p_1=1$, $p_2=2$, $p_3=3$, $p_4=5$, $\cdots$ then the subject theorem has the form

THEOREM (SCHERK). *For each integer* $n \geqq 1$, *there exist coefficients* $\epsilon_i=\epsilon_i(n)$ *such that* $\epsilon_i=+1$ *or* -1 *for each i and*

$$\text{(a)}\quad p_{2n+1} = p_{2n} + \sum_1^{2n-1} \epsilon_i p_i, \qquad \text{(b)}\quad p_{2n} = 2p_{2n-1} + \sum_1^{2n-2} \epsilon_i p_i.$$

Proof. By Bertrand's postulate, $p_{k+1} \leqq 2p_k$ for $k \geqq 1$ with strict inequality for $k>1$. Making use of this fact, it is easily established by induction that

$$p_{k+1} \leqq 1 + \sum_1^k p_i \qquad (k \geqq 1). \tag{1}$$

For $n \geqq 1$, we have

$$0 < p_{2n+1} - p_{2n} \leqq p_{2n} \leqq 1 + \sum_1^{2n-1} p_i,$$

so that

$$0 < \tfrac{1}{2}\left(p_{2n+1} - p_{2n} + \sum_1^{2n-1} p_i\right) < 1 + \sum_1^{2n-1} p_i.$$

It follows from the lemma that the integer

$$\tfrac{1}{2}\left(p_{2n+1} - p_{2n} + \sum_{1}^{2n-1} p_i\right)$$

may be expanded in the form

$$\tfrac{1}{2}\left(p_{2n+1} - p_{2n} + \sum_{1}^{2n-1} p_i\right) = \sum_{1}^{2n-1} \gamma_i p_i \qquad (\gamma_i \text{ binary}). \tag{2}$$

Equivalently,

$$p_{2n+1} = p_{2n} + \sum_{1}^{2n-1} (2\gamma_i - 1)p_i \qquad (n \geqq 1), \tag{3}$$

which, on noting that $2\gamma_i - 1 = \pm 1$ for each i, proves (a).

To establish (b), we observe that for $n \geqq 1$,

$$0 \leqq 2p_{2n-1} - p_{2n} = p_{2n-1} + (p_{2n-1} - p_{2n}) < p_{2n-1} \leqq 1 + \sum_{1}^{2n-2} p_i,$$

so that $0 \leqq \frac{1}{2}(2p_{2n-1} - p_{2n} + \sum_1^{2n-2} p_i) < 1 + \sum_1^{2n-2} p_i$. Again, the lemma guarantees binary coefficients γ_i such that

$$\tfrac{1}{2}\left(2p_{2n-1} - p_{2n} + \sum_{1}^{2n-2} p_i\right) = \sum_{1}^{2n-2} \gamma_i p_i,$$

or equivalently,

$$p_{2n} = 2p_{2n-1} + \sum_{1}^{2n-2} (1 - 2\gamma_i)p_i \qquad (n \geqq 1). \tag{4}$$

Since $1 - 2\gamma_i = \pm 1$ for each i, the proof of (b) is complete.

Lastly, we prove a related result concerning the expansion of arbitrary positive integers in terms of certain complete sequences. (A sequence $\{f_i\}$ of positive integers is said to be *complete* if and only if every positive integer n can be represented in the form $n = \sum_1^\infty \alpha_i f_i$, where each $\alpha_i = \alpha_i(n)$ is a binary coefficient.)

THEOREM. *Let $\{u_i\}_1^\infty$ be an increasing sequence of positive integers which is complete and such that the equation $u_i \equiv 1 \pmod 2$ is satisfied for infinitely many distinct values of i. Then every integer $n > 0$ has a representation in the form*

$$n = \sum_{1}^{k(n)} \epsilon_i u_i, \qquad \textit{with } \epsilon_i = \pm 1 \textit{ for each } i. \tag{5}$$

Proof. Given an integer $n > 0$, choose k_0 as the smallest integer such that

$$\text{(i) } n \leqq \sum_{1}^{k_0} u_i \quad \text{and} \quad \text{(ii) } n + \sum_{1}^{k_0} u_i \equiv 0 \pmod 2.$$

Since $\{u_i\}$ is an increasing complete sequence, it is known ([3], Theorem 1, p. 558) that $u_1=1$ and $u_{n+1}\leqq 1+\sum_1^n u_i$ for $n\geqq 1$. Applying the lemma to the integer $\frac{1}{2}(n+\sum_1^{k_0} u_i)$ yields

$$\tfrac{1}{2}\left(n+\sum_1^{k_0} u_i\right)=\sum_1^{k_0}\gamma_i u_i$$

with binary coefficients γ_i, or equivalently,

$$n=\sum_1^{k_0}(2\gamma_i-1)u_i, \tag{6}$$

which is in the required form (5).

As a special case of this theorem, it follows that every integer n can be written in the form

$$n=\sum_1^{k(n)}\epsilon_i p_i, \qquad \epsilon_i=\pm 1,$$

where the sequence $\{p_i\}$ is the sequence of primes (plus unity) used previously.

If the equation $u_i\equiv 1 \pmod 2$ is not satisfied for infinitely many i, the conclusion of the latter theorem is not necessarily valid. For, the sequence $u_i=2^{i-1}$ $(i=1, 2, \cdots)$ is certainly increasing and complete, but any sum of the form $\sum_1^{k(n)}\epsilon_i u_i=\sum_1^{k(n)}\epsilon_i 2^{i-1}$ has odd parity, and hence no even integers can be represented in the form (5) for this particular sequence.

An alternate treatment of Scherk's theorem and additional references may be found in Sierpiński ([4] pp. 140–142).

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ON THE NILPOTENCY OF ALGEBRAS

ALEXANDER ABIAN, Ohio State University

In [1] it is established that if every basis element of a finite dimensional linear associative algebra over a field of *characteristic zero* is nilpotent then the algebra is nilpotent.

Below we establish the same result without requiring that the characteristic of the field be zero.

THEOREM. *A finite dimensional linear associative algebra A (over any field) is nilpotent if and only if A has a basis every element of which is nilpotent.*

Proof. The necessity follows trivially since every element of a nilpotent algebra is nilpotent.

To prove the sufficiency, let $\{b_1, \cdots, b_n\}$ be a basis of the algebra A such that for every $i=1, \cdots, n$

(1) b_i *is nilpotent.*

Assume, on the contrary, that A is not nilpotent and let N be the radical of A. Thus, by our assumption $A \neq N$. Clearly, A/N is a semi-simple algebra. Without loss of generality, for some $k \leqq n$, we may take $\{b_1+N, \cdots, b_k+N\}$ as a basis of A/N.

Since A/N is semi-simple, for some $h \leqq k$, it is a direct sum of simple algebras $S_1, \cdots, S_h$, where for every $i=1, \cdots, h$ a subset $\mathfrak{s}_i$ of $\{b_1+N, \cdots, b_k+N\}$ serves as a basis of S_i.

In view of (1), for every $i=1, \cdots, h$, every element of $\mathfrak{s}_i$ is nilpotent (in the algebra S_i). In particular

(2) *every element of* $\mathfrak{s}_1$ *is nilpotent.*

Since S_1 is a simple algebra, S_1 is a direct product of a total matric algebra M and a division algebra D. Let $\mathfrak{M}=\{m_{11}, \cdots, m_{tt}\}$ be a basis of M with

(3) $m_{ij}m_{hk} = 0$ *if* $j \neq h$ *and* $m_{ij}m_{jk} = m_{ik}$ *for* $i, j, h, k = 1, \cdots, t$.

Moreover, let $\mathfrak{D}=\{d_1, \cdots, d_p\}$ be a basis of D such that

(4) $$\mathfrak{s}_1 = \{m_{uv}d_w \mid m_{uv} \in \mathfrak{M} \quad \text{and} \quad d_w \in \mathfrak{D}\}$$

and where

(5) $m_{uv}d_w = d_w m_{uv}$ *for every* $m_{uv} \in \mathfrak{M}$ *and every* $d_w \in \mathfrak{D}$.

In view of (3), however, each of the elements m_{ii} with $i=1, \cdots, t$ is idempotent and hence not nilpotent. Also, since D is a division algebra none of the elements d_w with $w=1, \cdots, p$ is a divisor of zero. Therefore, in view of (5) we conclude

(6) *there exists an element of* $\{m_{uv}d_w \mid m_{uv} \in \mathfrak{M}$ *and* $d_w \in \mathfrak{D}\}$ *which is not nilpotent.*

Comparing (3), (4) and (6) we conclude that our assumption leads to a contradiction. Hence our assumption is false and $A=N$. Thus, indeed A is a nilpotent algebra, as desired.

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ON PSEUDOPRIMES WHICH ARE PRODUCTS OF DISTINCT PRIMES

K. SZYMICZEK, Katowice, Poland

A composite number n is said to be pseudoprime if $n \mid 2^n - 2$. Let $P(x)$ denote the number of pseudoprimes $\leqq x$ and let $P_k(x)$ denote the number of square-free pseudoprimes $\leqq x$ having k distinct prime factors. P. Erdös [1] proved that for x sufficiently large,

$$P(x) < 2x \exp\{-\tfrac{1}{3}(\log x)^{1/4}\}, \tag{1}$$

and stated that there is an estimation of $P(x)$ from below $P(x) > c \log x$, which is due to D. H. Lehmer.

In the present paper I prove the inequality $P_2(x) > \frac{1}{4} \log x$, and also estimations of $P_k(x)$ and $P(x)$ from below. As a consequence of (1) I shall prove that the series $\sum 1/P_n$, where P_n is the nth pseudoprime, is convergent.

Now we prove the following lemma.

LEMMA. *If k is a natural number $\geqq 2$ and x is sufficiently large, then*

$$P_{k+1}(x) \geqq P_k(\log x). \tag{2}$$

Proof. Let n be a pseudoprime which is product of $k \geqq 2$ distinct odd primes. In view of a theorem of Zsigmondy [3], there exists a prime $p > n$ such that $p \mid 2^{n-1} - 1$ and $n - 1 \mid p - 1$. Thus,

$$np \mid 2^{n-1} - 1. \tag{3}$$

On the other hand, $np - 1$ is divisible by $n - 1$, since $n - 1 \mid p - 1$ and $np - 1 = n(p-1) + n - 1$. Then by (3) we get

$$np \mid 2^{np-1} - 1,$$

i.e. np is a pseudoprime which is product of $k+1$ distinct odd primes. We observe that if n and m are natural numbers, $n \neq m$, and p, q are primes such that $p > n$, $q > m$, then $np \neq mq$. If $np = mq$ and $p > n$ then m is divisible by p, hence $m \geqq p$, and we get $m > n$. In view of symmetry $m < n$, which is contradictory. Consequently $np \neq mq$. Thus, if n, m are distinct pseudoprimes having $k \geqq 2$ distinct prime factors, the adequate pseudoprimes np and mq are distinct, too.

From (3) it follows that

$$p \mid (2^{(n-1)/2} - 1)(2^{(n-1)/2} + 1),$$

and therefore

$$p \leqq 2^{(n-1)/2} + 1 < e^{n/2}.$$

Thus, if $n \leqq \log x$ then $pn < e^{1/2 \log x} \log x = x^{1/2} \log x < x$. Hence, for every pseudoprime $n = p_1 \cdots p_k \leqq \log x$ there exists at least one pseudoprime $p_1 \cdots p_k p < x$. Thus, by the above remark, we obtain (2).

THEOREM 1. *If* $x \geqq 2^{2^2}-1$, *then*

$$P_2(x) > \tfrac{1}{4} \log x. \tag{4}$$

Proof. Let m be an odd number >3. In view of Zsigmondy's [3] theorem there exist prime numbers p and q such that

$$p \mid 2^m - 1,\ q \mid 2^m + 1,\ m \mid p - 1,\ 2m \mid q - 1. \tag{5}$$

Since p and m are odd, $2m \mid p-1$. Further,

$$p \mid 2^m - 1 \mid 2^{q-1} - 1,\ q \mid 2^{2m} - 1 \mid 2^{p-1} - 1:$$

hence, by a theorem of J. H. Jeans [2], pq is pseudoprime. From (5) we get

$$pq < 2^{2m} - 1.$$

Thus, for every odd number $m>3$ there exists a pseudoprime of the form pq which is less than $2^{2m}-1$.

Let x be sufficiently large and m be the greatest odd number for which

$$2^{2m} - 1 \leqq x. \tag{6}$$

By the above argument, there are at least $(m-3)/2$ of pseudoprimes of the form pq less than x, i.e.

$$P_2(x) \geqq \frac{m-3}{2}.$$

We see that there are at least $(m-3)/2$ pseudoprimes pq, where p, q are primes satisfying (5), whereas there exist pseudoprimes pq not satisfying (5), for example:

$$17 \cdot 257,\ (17 \mid 2^8 - 1,\ 257 \mid 2^8 + 1),$$

$$23 \cdot 89,\ (23 \cdot 89 = 2^{11} - 1).$$

We also remark that for $m=11$ there are two pseudoprimes satisfying (5), namely $23 \cdot 683$ and $89 \cdot 683$. Thus, if $x \geqq 2^{2^2}-1$, we may write

$$P_2(x) \geqq \frac{m-3}{2} + 3 > \frac{m+2}{2}. \tag{7}$$

From the definition of m in (6) it follows that

$$x < 2^{2(m+2)} - 1 < e^{2(m+2)},$$

whence $m+2 > \frac{1}{2} \log x$, which, together with (7) gives (4), and the theorem is proved.

REMARK. It may be easily shown that the inequality (4) holds for $x \geqq 1387$, but not for any other x: $1 < x < 1387$.

THEOREM 2. *If k is a natural number $\geqq 2$ and x is sufficiently large, then $P_k(x) > \frac{1}{4}\log_{k-1} x$, where $\log_k x$ denote the k times iterated logarithm.*

Proof. The statement can be easily proved by induction on k if one applies Theorem 1 and our lemma.

THEOREM 3. *If k is a natural number and x is sufficiently large, then*

$$P(x) > \tfrac{1}{4}\log\left\{x\prod_{n=1}^{k}\log_n x\right\}.$$

Proof. For sufficiently large x, $P(x) > P_2(x) + P_3(x) + \cdots + P_{k+2}(x)$, whence, by Theorem 2,

$$\begin{aligned} P(x) &> \tfrac{1}{4}\{\log x + \log_2 x + \cdots + \log_{k+1} x\} \\ &= \tfrac{1}{4}\log\left\{x\prod_{n=1}^{k}\log_n x\right\}. \end{aligned}$$

Now we prove another result.

THEOREM 4. *The series $\sum 1/P_n$, where P_n is the n-th pseudoprime, is convergent.*

Proof. If we put $x = P_n$ then the right hand side of (1) becomes

$$n < 2P_n \exp\{-\tfrac{1}{3}(\log P_n)^{1/4}\}.$$

Since $n < P_n$, we have

$$\frac{1}{P_n} < \frac{2}{n\exp\{\frac{1}{3}(\log n)^{1/4}\}}. \tag{8}$$

On the other hand, for large m, $m^{1/4} > 4\log m$, and thus for sufficiently large n,

$$(\log n)^{1/4} > 4\log\log n.$$

Hence $\frac{1}{3}(\log n)^{1/4} > \log(\log n)^{4/3}$, and

$$\exp\{\tfrac{1}{3}(\log n)^{1/4}\} > (\log n)^{4/3}. \tag{9}$$

From (8) and (9) we get

$$\frac{1}{P_n} < \frac{2}{n(\log n)^{4/3}},$$

and Theorem 4 follows from the well-known convergence of $\sum 2/\{n(\log n)^{4/3}\}$.

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It is easy to add restrictions on K, f_k, X_k, Y_k such that (B) alone is a necessary and sufficient condition in Theorem 2. We have, for example, the following.

COROLLARY. *Suppose that X_k, Y_k are nonempty, connected, metrizable spaces for all $k \in K$, and that f_k is nonconstant for at least two values of k. Then f is closed iff f_k is closed and $f_k^{-1}(y_k)$ is compact for all $y_k \in Y_k$, $k \in K$.*

This work was supported by the National Science Foundation Grant GP-2557.

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INTEGRAL PART AND FACTORIAL IN CERTAIN EUCLIDEAN DOMAINS

J. CHIDAMBARASWAMY, University of California, Berkeley (Now at University of Toledo, Ohio)

The result "The exponent of the highest power of a prime p that divides $n!$ is $\sum_{k=1}^{\infty} [n/p^k]$" is well known. The purpose of this note is to define integral part and factorial in certain Euclidean domains and establish the above result in those domains.

For simplicity, we select the domain G of Gaussian integers as a prototype of the domains (see the remark at the end) in which the above result can be stated. Let us recall that G consists of all complex numbers $a+bi$, where a and b are rational integers, that the multiplicative group of units of G consists of the numbers ± 1, $\pm i$, and that the quotient field F of G is the set of all complex numbers $r+si$, where r and s are rational numbers. We extend the norm function N usually defined on the nonzero numbers of G by $N(a+bi)=a^2+b^2$ to the nonzero numbers of F by saying $N(r+si)=r^2+s^2$; this is equivalent to saying that if

$$r + si = (a + bi)/(c + di), \quad 0 \neq a + bi \in G, \quad 0 \neq c + di \in G,$$

then $N(r+si)=N(a+bi)/N(c+di)$. We also extend the usual relation of associativity in G to F by saying that $r+si$ is associated to $r'+s'i$ if there exists a unit η of G such that $r+si=\eta(r'+s'i)$. This is an equivalence relation in F and so it induces a partition of F into various equivalence classes, each class other than the zero class (the class whose only member is zero) containing 4 associated numbers i.e. $\pm(a+bi)$, $\pm i(a+bi)$. We pick one number from each of these nonzero classes and denote by P the set of numbers so picked, so that no two numbers of P are associated and each nonzero number of F is associated to exactly one number of P. In fact, it is readily verified that we can take for P the set of all those numbers $r+si$ of F with $r>0$, $s \geqq 0$ and this we shall do for definiteness. We call Gaussian integers of P the P-Gaussian integers.

DEFINITION 1. *For $r+si\in F$, the integral part of $r+si$, denoted by $[r+si]$, is the number of P-Gaussian integers $x+yi$ with $N(x+yi)\leqq N(r+si)$ in the case $N(r+si)\geqq 1$, and is 0 otherwise.*

We observe that associated numbers have the same integral part. Also, since the number of representations of a positive integer n as a sum two squares of integers is $4\sum_{d^2|n} V(n/d^2)$, where $V(n)$ is the number of solutions (mod n) of the congruence $x^2\equiv -1 \pmod{n}$, and since with $x+yi$ all its associates have the same norm, it follows that for any $r+si\in F$,

$$[r+si]=\sum_{j=1}^{[r^2+s^2]}\sum_{d^2|j} V(j/d^2).$$

DEFINITION 2. *For $r+si\in F$, the factorial of $r+si$, denoted by $(r+si)!!$ is the product of all P-Gaussian integers $x+yi$ with $N(x+yi)\leqq N(r+si)$ in case $N(r+si)\geqq 1$, and is 1 otherwise.*

For any $0\neq a+bi\in G$, and any $r+si\in F$, let $M(a+bi, r+si)$ denote the number of multiples of $a+bi$ which are in P and whose norm is less than or equal to that of $r+si$.

THEOREM 1. (1) $M(a+bi, r+si)=[(r+si)/(a+bi)]$.

Proof. Let $m=[r+si]$, and $l=[(r+si)/(a+bi)]$. If $m=0$, both sides of (1) are zero. If $m\geqq 1$, let x_n+y_ni, $1\leqq n\leqq m$ be all the P-Gaussian integers with $N(x_n+y_ni)\leqq N(r+si)$, and a_k+b_ki, $1\leqq k\leqq l$, be all the P-Gaussian integers with $N(a_k+b_ki)\leqq N((r+si)/(a+bi))$.

Clearly, by the definition of P and the fact that the norm of the product of two numbers of F is equal to the product of their norms, there is a 1-1 correspondence between the numbers x_j+y_ji which are divisible by $a+bi$ and the numbers a_k+b_ki. The correspondence can be expressed as

$$(a_k+b_ki)(a+bi)\eta_{kj}=x_j+y_ji,$$

where η_{kj} is a unit. The theorem is clear.

THEOREM 2. *For any $r+si\in F$, and any prime $a+bi\in G$, the exponent of the highest power of $a+bi$ that divides $(r+si)!!$ is*

$$\sum_{k=1}^{\infty}[(r+si)/(a+bi)^k]. \tag{2}$$

Proof. Since $N(a+bi)>1$, (2) terminates. Using the same notation as in the proof of Theorem 1, the result is clear if $m=0$ or $m=1$. If $m\geqq 2$, by Definition 2 and the meaning of the notation x_n+y_ni,

$$(r+si)!!=(x_1+y_1i)(x_2+y_2i)\cdots(x_m+y_mi). \tag{3}$$

Now, the right hand side of (3), by the proof of Theorem 1, can be written as

$$\beta(a+bi)^{[(r+si)/(a+bi)]}((r+si)/(a+bi))!!,$$

where β is an associate of the product of all numbers in the right hand side of (3) which are not divisible by $a+bi$. The rest of the proof is obvious.

REMARK. Clearly what we have done can be done in any Euclidean domain E in which the following conditions hold.

(i) The norm function is multiplicative, i.e., $N(ab)=N(a)N(b)$.

(ii) For each positive integer j, $\{x\in E: N(x)=j\}$ is finite.

(iii) $N(e)>0$ (i.e., $N(e)=1$ by (i)), e being the identity of E.

AN OBSERVATION ON THE DISTRIBUTION OF PRIMES

M. STEIN AND S. M. ULAM, University of California, Los Alamos Scientific Laboratory

Primes up to 100,000,000 have been stored on a rather fast access memory disc connected with a computer, MANIAC II, in our Laboratory. We have looked at some properties of the distribution of primes when the sequence is "compressed" by dividing each prime by its logarithm and taking the nearest integer to this quotient. Of several properties which this new sequence seems to show, the following is perhaps most striking.

It can be stated as follows: For each integer k consider the interval between k and $k+1+E(\log k)$. For some k's there will be no prime in this interval. For some others there will be exactly one prime in this range. For other k's there will be exactly 2 primes, etc. We consider the frequencies of these classes. Let $\gamma_0(n)$ be, therefore, the ratio to n of the number of all k's from 1 to n, such that there is no prime in the range between k and $k+E(\log k)+1$. Actually the interval examined by the machine was from k to $k+E(\log^2 k/\log k-1)$. Similarly we define $\gamma_1(n)$, $\gamma_2(n)$, etc. We have tabulated the γ's in intervals of 10,000 up to $n=100{,}000{,}000$.

It seems that they approach definite limits.

n	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7
10,000	.2502..	.4681..	.2415..	.0385..	.0017..	0.0000	0.0000	0
50,000	.2693..	.4520..	.2350..	.0421..	.0014..	0.0000	0.0000	0
500,000	.2780..	.4456..	.2300..	.0443..	.0025..	0.0000	0.0000	0
1,000,000	.2851..	.4407..	.2249..	.0459..	.0033..	0.0000	0.0000	0
2,000,000	.2853..	.4379..	.2269..	.0466..	.0033..	0.0000	0.0000	0
5,000,000	.2902..	.4335..	.2242..	.0484..	.0037..	0.0001..	0.0000	0
10,000,000	.2960..	.4293..	.2209..	.0492..	.0045..	0.0002..	0.0000	0
20,000,000	.3017..	.4242..	.2182..	.0504..	.0052..	0.0002..	0.0000	0
40,000,000	.3023..	.4226..	.2180..	.0513..	.0056..	0.0002..	0.0000	0
80,000,000	.3048..	.4211..	.2163..	.0516..	.0059..	0.0003..	0.0000	0
100,000,000	.3052..	.4202..	.2162..	.0521..	.0060..	0.0003..	0.0000	0

The figures in the Table are rounded to four decimals. We have accumulated our statistics by counting in contiguous intervals of length given above.

These numbers do not appear, at least as yet, to approach a Poisson distribution. γ_0 is smaller and γ_1 is larger than the Poisson value and the tail of the distribution seems to go down to zero faster. There is no proof, of course, of the *existence* of the limit of the γ's, but the impression which the full data give is certainly that they do exist. The first time when six primes appear in our range is somewhere after 22579983 and before 22589996. Up to our $n=10^8$ there is not a single case of seven primes in our intervals. Presumably, for every ℓ, there will exist ℓ primes in this range for sufficiently great n, but the frequency of these in the set of all integers goes to zero extremely fast with increasing ℓ.

We are planning to obtain other statistical data about the distribution of these "compressed" or scaled primes.

This work was performed under the auspices of the U. S. Atomic Energy Commission.

A CONDITION FOR A RING TO BE BOOLEAN

D. L. BARNETT, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio

The purpose of this note is to show that on a (not necessarily associative) ring R with unit it is impossible to introduce a third nontrivial binary operation $\circ$ which is distributive on opposite sides with respect to the ring addition and multiplication, and such that R has an identity element under $\circ$. In fact, the introduction of $\circ$ on R forces R to be Boolean and $\circ$ to coincide with the ring multiplication. In what follows, the letters x, y, z will be used to denote arbitrary elements of a ring.

THEOREM. *Let R be a ring with unit* 1 *and let R be closed under a binary operation $\circ$ with identity element e. Let $\circ$ satisfy*

$$(x + y) \circ z = (x \circ z) + (y \circ z),$$
$$x \circ (yz) = (x \circ y)(x \circ z).$$

Then R is a Boolean ring, $e=1$, and $x \circ y = xy$.

The proof of the theorem will consist of a series of lemmas, where besides the usual ring axioms, the above two one-sided distributive laws are used repeatedly.

LEMMA 1. $(1 \circ 1)(x \circ y)+(x \circ 1)(1 \circ y)=0$.

Proof. $(1 \circ 1)(1 \circ y) + (x \circ 1)(x \circ y) = 1 \circ (1y) + x \circ (1y) = (1 + x) \circ (1y)$ $=((1+x) \circ 1)((1+x) \circ y)=((1 \circ 1)+(x \circ 1))((1 \circ y)+(x \circ y))=(1 \circ 1)(1 \circ y)$ $+(1 \circ 1)(x \circ y)+(x \circ 1)(1 \circ y)+(x \circ 1)(x \circ y)$, and the lemma is obtained by comparing the beginning and the end of this equality.

LEMMA 2. $1 \circ 1=1$.

Proof. $1 \circ 1 = 1(1 \circ 1) = (1 \circ e)(1 \circ 1) = 1 \circ (e1) = 1 \circ e = 1$.

LEMMA 3. *R has characteristic two.*

Proof. Set $x = y = 1$ in Lemma 1 and use Lemma 2 to obtain $1+1=0$.

LEMMA 4. $e = 1$.

Proof. From Lemma 1, $0 = (1 \circ 1)(e \circ e) + (e \circ 1)(1 \circ e) = 1e + 1 \cdot 1 = e + 1$. This and Lemma 3 imply $e = 1$.

LEMMA 5. *R is a Boolean ring, i.e. $x^2 = x$.*

Proof. $x = x \circ 1 = x \circ (1 \cdot 1) = (x \circ 1)(x \circ 1) = xx$.

LEMMA 6. $x \circ y = xy$.

Proof. From Lemmas 1, 2, and 4,

$$0 = (1 \circ 1)(x \circ y) + (x \circ 1)(1 \circ y) = 1(x \circ y) + xy = (x \circ y) + xy,$$

i.e. $x \circ y = xy$, because R has characteristic two. The theorem is proved.

Notice that the associative law of multiplication was not used in the above. It can be proved from what we have assumed and proved, using the fact that multiplication is distributive over multiplication (true because the operation $\circ$ turned out to be multiplication).

CLASSROOM NOTES

TRIANGLES WITH RATIONAL SIDES AND ANGLE RATIOS

D. E. DAYKIN AND A. OPPENHEIM, University of Malaya

Let m, n be arbitrary fixed coprime positive integers, and let T denote the set of all triangles ABC with angles

$$(1) \qquad A = m\theta, \quad B = n\theta, \quad C = \pi - (m+n)\theta, \quad \text{where} \quad 0 < \theta < \pi/(m+n),$$

so that $nA = mB$. An interesting problem in number theory is that of finding the set S of integer sided triangles of T whose sides have greatest common divisor 1. For $m/n = 1/3$, $2/3$ a solution of this problem due to one of us (A.O.) was given in [1], and we present here the primitive general solution in Theorem 1 below. The key to the problem is the

LEMMA. *A triangle ABC of T has rational side ratios if and only if* $\cos\theta$ *is rational.*

Proof. Suppose first that the sides a, b, c of a triangle of T are rational. Then by the cosine rule $\cos m\theta$, $\cos n\theta$, $\cos (m+n)\theta$ are rational. Also by the sine rule

there is a real number ρ such that $\rho \sin m\theta$, $\rho \sin n\theta$, $\rho \sin (m+n)\theta$ are a, b, c respectively, and hence are rational. Since $\cos^2 m\theta + \sin^2 m\theta = 1$ the number ρ^2 is rational. Now

$$\cos(m-n)\theta = \cos m\theta \cos n\theta + \rho^{-2}(\rho \sin m\theta)(\rho \sin n\theta),$$
$$\rho \sin(m-n)\theta = (\rho \sin m\theta)\cos n\theta - \cos m\theta(\rho \sin n\theta),$$

and so $\cos (m-n)\theta$, $\rho \sin (m-n)\theta$ are rational. Repeating this argument like Euclid's algorithm shows, because m, n are coprime, that $\cos \theta$ is rational. The converse result follows in a similar way, and this proves the lemma.

Next let $\phi(r/s; t)$ denote the integer

$$\phi(r/s; t) = \text{Imag}\{[r + i(s^2 - r^2)^{1/2}]^t/(s^2 - r^2)^{1/2}\} \quad \text{for } r, s, t = 1, 2, \cdots. \tag{2}$$

Then if $\cos \theta = r/s$, for $t = m$, n, $m+n$ the number

$$\begin{aligned}(s^{m+n-1} \sin t\theta)/\sin \theta &= s^{m+n}(s^2 - r^2)^{-1/2} \sin t\theta \\ &= s^{m+n}(s^2 - r^2)^{-1/2}\,\text{Imag}\{[\cos \theta + i \sin \theta]^t\} \\ &= s^{m+n-t}\phi(r/s; t),\end{aligned} \tag{3}$$

is an integer. The lemma shows that all rational sided triangles of T may be determined by taking in (1) those angles θ which have rational cosines. Moreover the sides of these triangles will then be proportional to $\sin m\theta$, $\sin n\theta$, $\sin (m+n)\theta$, and these numbers appear in (3). Hence we have proved the first part of

THEOREM 1. *The sides a, b, c respectively of the triangles of S are obtained by dividing the integers*

$$s^n\phi(r/s; m), \qquad s^m\phi(r/s; n), \qquad \phi(r/s; m+n), \tag{4}$$

by their greatest common divisor g, when r, s range over positive integral values such that $(r, s) = 1$ and $\cos (\pi/(m+n)) < r/s < 1$. Moreover $g = 1$ if s is odd but $g = 2^{m+n-1}$ if s is even.

To complete the proof we need only discuss the greatest common divisor g of the numbers (4) for fixed r, s. Let $\cos \theta = r/s$ and

$$\psi(t) = \psi(r/s; t) = s^t \cos t\theta \qquad \text{for } t = 1, 2, \cdots,$$

then $\psi(1) = r$, $\psi(2) = 2r^2 - s^2$ and $\psi(t+1) = 2r\psi(t) - s^2\psi(t-1)$ for $t > 1$, and so by induction $\psi(t)$ is an integer for $t \geqq 1$.

For convenience we suppose that $m > n$ and write $\phi(t)$ for $\phi(r/s; t)$. Then by (3),

$$\phi(t) = (s^{t-1} \sin t\theta)/\sin \theta \qquad \text{for } t = 1, 2, \cdots$$

and, if M is any integer,

$$s^M\phi(m-n) = s^{M-3n}(s^n\phi(m))\psi(n) - s^{M-m-2n}(s^m\phi(n))\psi(m). \tag{5}$$

Now suppose that p is a prime which divides $s^n\phi(m)$ and $s^m\phi(n)$. Then, since $\psi(t)$ is an integer, (5) shows that p divides $s^M\phi(m-n)$ for M sufficiently large, $M \geqq m+2n$. Repeating this argument like Euclid's algorithm, since m, n are coprime, we find that p divides $s^N\phi(1) = s^N$ for N sufficiently large, and so p divides s. This result also holds if $m=n$ for then $m=1$. Now

$$\phi(1) = 1, \quad \phi(2) = 2r \quad \text{and} \quad \phi(t+1) = 2r\phi(t) - s^2\phi(t-1) \quad \text{for } t > 1, \tag{6}$$

and by induction on these relations, or directly from (2), we see that for $t \geqq 1$,

$$\phi(t) = qs + (2r)^{t-1}, \tag{7}$$

for some integer $q=q(t)$. Hence if p also divides $\phi(m+n)$, since it divides s and $(r, s)=1$, it follows by (7) that $p=2$. Thus we have shown that the greatest common divisor g of the numbers (4) is $g=2^u$ for some integer u, and $u=0$ when s is odd.

If 4 divides s, then r is odd and for $t \geqq 1$, by induction on (6), the highest power of 2 which divides $\phi(t)$ is 2^{t-1}. If s is even but not divisible by 4, then the highest power of 2 which divides $\phi(t)$ is 2^{t-1} when t is not divisible by 3, but 2^t divides $\phi(t)$ when 3 divides t. Since $(m, n)=1$ it follows that $g=2^{m+n-1}$ whenever s is even, and the theorem is proved.

Reference

1. R. K. Guy, Triangles with $B=3A$, $2B=3A$, Bull. Malay Math. Soc., 1 (1954) 15, 48, 56–60.

A SIMPLE UNIQUENESS THEORY FOR ORDINARY LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

L. E. PURSELL, Grinnell College

In the following we outline a development of the uniqueness of a solution to an initial value problem in ordinary linear homogeneous differential equations which is suitable for a brief introduction to ordinary differential equations (such as a unit in a first year calculus course). Our main theorem is obtained by the simple artifice of applying the method of variation of parameters to a *homogeneous* equation. To simplify exposition we will consider only second order linear differential equations, but our theorems and proofs can be extended easily to higher order equations.

We assume that the student is familiar with the elementary theory of determinants and systems of linear algebraic equations; he has learned how to use the characteristic polynomial $r^2+Pr+Q=0$ to find pairs ϕ_1, ϕ_2 of solutions to the differential equation $y''+Py'+Qy=0$ (P and Q constants) such that the Wronskian, $W(\phi_1, \phi_2; x)=\phi_1(x)\phi_2'(x)-\phi_1'(x)\phi_2(x)$, is different from zero for all x. (Note: The notion of linear independence does not appear explicitly in our theory.) He has also been shown that if ϕ_1 and ϕ_2 are solutions of a linear homogeneous differential equation, then so is $C_1\phi_1+C_2\phi_2$ where C_1 and C_2 are constants. We then prove:

THEOREM 1. *If P and Q are constants (or continuous functions of x on an interval I) and ϕ_1, ϕ_2, and f are solutions of the differential equation*

$$y'' + Py' + Qy = 0$$

on I such that $W(\phi_1, \phi_2; x) \neq 0$ for all x in I, then there exist constants C_1 and C_2 such that

$$f(x) = C_1\phi_1(x) + C_2\phi_2(x) \qquad \text{for all } x \text{ in } I.$$

Proof. Since $W(\phi_1, \phi_2; x) \neq 0$, the system of equations

$$f(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x) \tag{1a}$$

$$f'(x) = u_1(x)\phi_1'(x) + u_2(x)\phi_2'(x) \tag{1b}$$

has a unique pair of solutions $(u_1(x), u_2(x))$ for each x in I. From Cramer's rule:

$$u_1(x) = \frac{f(x)\phi_2'(x) - f'(x)\phi_2(x)}{W(\phi_1, \phi_2; x)}$$

and

$$u_2(x) = \frac{\phi_1(x)f'(x) - \phi_1'(x)f(x)}{W(\phi_1, \phi_2; x)};$$

we see that u_1 and u_2 are differentiable since ϕ_1, ϕ_2, and f are solutions of a second order differential equation, and hence are twice differentiable. Differentiating (1b) and then substituting into the differential equation we obtain

$$\begin{aligned} 0 &= f'' + Pf' + Qf \\ &= (u_1'\phi_1' + u_2'\phi_2' + u_1\phi_1'' + u_2\phi_2'') + P(u_1\phi_1' + u_2\phi_2') + Q(u_1\phi_1 + u_2\phi_2) \\ &= (u_1'\phi_1' + u_2'\phi_2') + u_1(\phi_1'' + P\phi_1' + Q\phi_1) + u_2(\phi_2'' + P\phi_2' + Q\phi_2). \end{aligned}$$

Hence

$$0 = u_1'(x)\phi_1'(x) + u_2'(x)\phi_2'(x) \qquad \text{for all } x \text{ in } I. \tag{2a}$$

But if we differentiate (1a) and subtract (1b) from the result we obtain:

$$0 = u_1'(x)\phi_1(x) + u_2'(x)\phi_2(x) \qquad \text{for all } x \text{ in } I. \tag{2b}$$

Since the determinant, $W(\phi_1, \phi_2; x)$, of the system of equations (2b, 2a) is different from zero for all x in I, we have $u_1'(x) = 0$ and $u_2'(x) = 0$ for all x in I. Hence $C_1 = u_1(x)$ and $C_2 = u_2(x)$ are constants.

THEOREM 2. *If there exists a pair (ϕ_1, ϕ_2) of solutions to the differential equation $y'' + Py' + Qy = 0$ on an interval I where P and Q are constants (or continuous functions of x) for which $W(\phi_1, \phi_2; x) \neq 0$ for all x in I, x_0 is any point in I, and a and b are any two real numbers, then* (i) *there is a unique solution f of the differential equation on I such that $f(x_0) = a$ and $f'(x_0) = b$. Also if g is any solution of*

the differential equation on an interval J containing x_0 which satisfies these initial conditions, then (ii) $g(x)=f(x)$ *for all* x *in* $I\cap J$.

Proof. (i) By the previous theorem every solution of the differential equation on I is of the form $f(x)=C_1\phi_1(x)+C_2\phi_2(x)$. If we substitute this linear combination into the initial conditions we have:

$$a = C_1\phi_1(x_0) + C_2\phi_2(x_0) \tag{3a}$$

$$b = C_1\phi_1'(x_0) + C_2\phi_2'(x_0). \tag{3b}$$

Since $W(\phi_1, \phi_2; x_0)=0$, this system of equations has a unique pair (C_1, C_2) of solutions. Hence f exists uniquely on I.

(ii) By Theorem 1 there exist constants k_1 and k_2 such that $g=k_1\phi_1+k_2\phi_2$ on $I\cap J$. Then

$$a = g(x_0) = k_1\phi_1(x_0) + k_2\phi_2(x_0)$$

$$b = g'(x_0) = k_1\phi_1'(x_0) + k_2\phi_2'(x_0).$$

Hence (k_1, k_2) is a solution of the system of equations (3a, b). But this system of equations has a unique pair of solutions. Therefore $k_1=C_1$, $k_2=C_2$, and $g(x)=f(x)$ for all x in $I\cap J$.

Instead of starting with the existence assumption that a pair of solutions (ϕ_1, ϕ_2) such that $W(\phi_1, \phi_2)\neq 0$ exists, we could start with the assumption: Given a number $x=x_0$, for each pair of numbers (a, b) (or for at least two pairs (a_1, b_1) and (a_2, b_2) such that $a_1b_2\neq b_1a_2$) there exists a twice differentiable function $f_{a,b}$ defined on some neighborhood $N_{a,b}(x_0)$ of x_0 which is a solution to the differential equation on $N_{a,b}(x_0)$ such that $f_{a,b}(x_0)=a$ and $f'_{a,b}(x_0)=b$. (Such an assumption seems "natural" in many applications of differential equations in which empirical evidence leads us to believe that there is a solution to our "mathematical model" but we are concerned with the question: Have we imposed enough conditions in our model to ensure a unique solution?) If we take $\phi_1=f_{1,0}$ and $\phi_2=f_{0,1}$, then $W(\phi_1, \phi_2; x_0)=1$. Since $W(\phi_1, \phi_2)$ is continuous it will be different from zero in some neighborhood $N(x_0)$ of x_0 contained in both $N_{1,0}(x_0)$ and $N_{0,1}(x_0)$. We may then take $N(x_0)$ as the interval I of Theorems 1 and 2.

Notes. These results evolved from a conversation with Professor Beryl Clotfelter of the Grinnell College physics department concerning how to present efficiently a few rudiments of linear differential equations to a freshman physics class. I first observed that if f is a solution of $y' - ay = 0$ and we set $u(x)=f(x)/\exp(ax)$, then

$$\begin{aligned} u'(x) &= [\exp(ax)f'(x) - a\exp(ax)f(x)]/\exp(2ax) \\ &= [f'(x) - af(x)]/\exp(ax) = 0 \qquad \text{for all } x. \end{aligned}$$

Hence $u(x)=$constant and $f(x)=$(constant) $\exp(ax)$. Then noting that these calculations closely resemble the method of variation of parameters, I worked out the results given above.

Most textbooks on differential equations which I have examined either do not establish uniqueness or else establish uniqueness and existence simultaneously by means of the Picard theory. One textbook, *An introduction to ordinary differential equations* by Earl A. Coddington, Prentice Hall, 1961, pp. 56–58, establishes uniqueness prior to taking up the Picard theory by first proving the theorem: *if ϕ is a solution of $y''+ay'+by=0$ on an interval I containing the point x_0, then for all x in I:*

$$\|\phi(x_0)\|\exp(-k\,|\,x-x_0\,|\,) \leqq \|\phi(x)\| \leqq \|\phi(x_0)\|\exp(k\,|\,x-x_0\,|\,),$$

where $\|\phi(x)\| = ((\phi(x))^2+(\phi'(x))^2)^{1/2}$ *and* $k=1+|\,a\,|+|\,b\,|$. (He also extends this later to higher order equations and to equations with variable coefficients.)

In *Calculus with analytic geometry*, R. E. Johnson and F. L. Kiokemeister, Allyn and Bacon, 2nd ed., 1960, pp. 639–640, uniqueness is established for second order equations with constant coefficients by first proving that if f is a solution of $y''-ky$ (k a constant) on some interval, then f is analytic on that interval.

UNIQUENESS OF MEASURES

J. B. ROBERTSON, Cornell University

Let $\mathfrak{A}$ be a collection of subsets of a set X, and let $\sigma(\mathfrak{A})$ denote the smallest σ-ring containing $\mathfrak{A}$. Throughout this paper μ and ν will denote two finite measures on $\sigma(\mathfrak{A})$ which agree on $\mathfrak{A}$, i.e. $\mu(A)=\nu(A)$ for all A in $\mathfrak{A}$. In this paper we give some sufficient conditions on the structure of $\mathfrak{A}$ to insure that μ and ν must also agree on $\sigma(\mathfrak{A})$. It is well known that if $\mathfrak{A}$ is a semi-ring (i.e. $\mathfrak{A}$ is closed under finite intersections and the difference of any two sets in $\mathfrak{A}$ is the finite disjoint union of sets in $\mathfrak{A}$) then μ and ν must agree on $\sigma(\mathfrak{A})$ (cf. [1] sec. 13, prob. 4). (Notice that this is a slightly more general concept than that of a semiring as defined in [1], but the proof of the uniqueness of measures is the same in either case.) A special case of a semiring is a collection of sets which is closed under finite intersections and differences. It is also known (cf. [1] sec. 5, prob. 2) that if $\mathfrak{A}$ is a lattice (i.e. $\mathfrak{A}$ is closed under finite intersections and finite unions) then μ and ν must agree on $\sigma(\mathfrak{A})$. Halmos (cf. [1] sec. 13, prob. 3) has given, however, an example of a class of sets $\mathfrak{A}$, which happens to be closed under complements, and of two finite measures which agree on $\mathfrak{A}$ but not on $\sigma(\mathfrak{A})$. We shall prove:

THEOREM. *Let $\mathfrak{A}$ be a collection of subsets of X and let μ and ν be two finite measures on $\sigma(\mathfrak{A})$ which agree on $\mathfrak{A}$.*

(a) *If $\mathfrak{A}$ is closed under finite intersections, then μ and ν agree on $\sigma(\mathfrak{A})$.*

(b) *If $\mathfrak{A}$ is closed under finite unions, then μ and ν agree on $\sigma(\mathfrak{A})$.*

(c) *If $\mathfrak{A}$ is closed under differences, then μ and ν agree on $\sigma(\mathfrak{A})$.*

(d) *If $\mathfrak{A}$ is closed under symmetric differences, then μ and ν agree on $\sigma(\mathfrak{A})$.*

Proof (a). The following two formulas may be established by induction.

$$(1)\qquad \mu(A(1)\cup\cdots\cup A(n)) = -\sum_{k=1}^{n}(-1)^k\sum\mu(A(i_1)\cap\cdots\cap A(i_k)),$$

$$(2)\qquad \mu(A(1)\cap\cdots\cap A(n)) = -\sum_{k=1}^{n}(-1)^k\sum\mu(A(i_1)\cup\cdots\cup A(i_k)),$$

where the second summations in the two equations extend over all $(i_1, \cdots, i_k)$ such that $1\leqq i_1<\cdots<i_k\leqq n$. The same equations, of course, hold with μ replaced by ν. Formula (1) yields the result that if $\mathfrak{A}$ is closed under intersections and if μ and ν agree on $\mathfrak{A}$, then they must also agree on the set of all finite unions of sets in $\mathfrak{A}$, and formula (2) yields the same statement with unions and intersections interchanged. Thus if we define $\mathfrak{A}_n$ inductively by $\mathfrak{A}_0=\mathfrak{A}$, $\mathfrak{A}_{2n+1}$ = the set of all finite unions of sets in $\mathfrak{A}_{2n}$, and $\mathfrak{A}_{2n+2}$ = the set of all finite intersections of sets in $\mathfrak{A}_{2n+1}$, then $\mathfrak{A}_n\subseteq\mathfrak{A}_{n+1}$, $\mathfrak{A}_{2n}$ is closed under intersections, $\mathfrak{A}_{2n+1}$ is closed under unions, and, by induction, μ and ν agree on $\mathfrak{A}_n$. Now let $\mathfrak{A}_\infty=\bigcup_n\mathfrak{A}_n$. Clearly we have $\sigma(\mathfrak{A}_\infty)=\sigma(\mathfrak{A})$, μ and ν agree on $\mathfrak{A}_\infty$, and $\mathfrak{A}_\infty$ is a lattice. Thus, by known results, μ and ν agree on $\sigma(\mathfrak{A})$.

(b) Let $\mathfrak{B}$ be the collection of all finite intersections of sets in $\mathfrak{A}$. Clearly $\sigma(\mathfrak{A})=\sigma(\mathfrak{B})$ and by (2) μ and ν agree on $\mathfrak{B}$. Thus by (a) μ and ν agree on $\sigma(\mathfrak{A})$.

(c) $\mathfrak{A}$ is in fact a semiring: $A\cap B=A-(A-B)$. Thus (c) follows from known results.

(d) As in the proof of (b) it is sufficient to show that μ and ν agree on all sets which are finite intersections of sets in $\mathfrak{A}$. Let $A(1), \cdots, A(n)$ be sets in $\mathfrak{A}$. Let Y be the set of all n-tuples $(i_1, \cdots, i_n)$ such that for all $j=1, \cdots, n$, $i_j=0$ or 1 but not all of the i_j's are 1. Y has 2^n-1 points. For $(i_1, \cdots, i_n)$ in Y let

$$A(i_1, \cdots, i_n) = \Big(\bigcap_{\{j:i_j=0\}} A(j)\Big)\cap\Big(\bigcap_{\{k:i_k=1\}} A(k)'\Big).$$

(A' denotes the complement of A.) $A(i_1, \cdots, i_n)$ is either empty or an atom of the finite ring generated by $\{A(1), \cdots, A(n)\}$. Let $\mu(i_1, \cdots, i_n)=\mu(A(i_1, \cdots, i_n))$ and $\nu(i_1, \cdots, i_n)=\nu(A(i_1, \cdots, i_n))$. It is now sufficient to show that $\mu(i_1, \cdots, i_n)=\nu(i_1, \cdots, i_n)$ for all $(i_1, \cdots, i_n)$ in Y. Let $B(k)=\{(i_1, \cdots, i_n)$ in $Y: i_k=1\}$. If A is any finite set, $|A|$ will denote the number of points in A.

LEMMA 1. *Let* $1\leqq n_1<\cdots<n_r\leqq n$, $1\leqq m_1<\cdots<m_s\leqq n$, *and suppose that* $(n_1, \cdots, n_r)\neq(m_1, \cdots, m_s)$. *Then*:

(a) $|B(n_1)\Delta\cdots\Delta B(n_r)|=2^{n-1}$,

(b) $|(B(n_1)\Delta\cdots\Delta B(n_r))\cap(B(m_1)\Delta\cdots\Delta B(m_s))|=2^{n-2}$.

Proof of Lemma 1. (a) By renumbering we may suppose that $n_k=k$ for $k=1, \cdots, r$. As may easily be seen by induction, a point x is in $B(1)\Delta\cdots\Delta B(r)$ if and only if the number of $B(i)$'s, $i=1, \cdots, r$, which contain x is odd. Thus

the number of points in $B(1)\Delta \cdots \Delta B(r)$ is the number of points $(i_1, \cdots, i_n)$ such that for some k there are exactly $2k+1$ 0's in $(i_1, \cdots, i_r)$. Since there are 2^{n-r} points in Y with the first r places fixed and not all ones, there are $\binom{r}{k}2^{n-r}$ points with exactly k 0's in the first r places. Thus

$$|B(1)\Delta \cdots \Delta B(r)| = \sum_{k \text{ odd}} \binom{r}{k} 2^{n-r}.$$

$$\sum_{k \text{ odd}} \binom{r}{k} = \sum_{k \text{ even}} \binom{r}{k} = 2^{r-1}$$

as can easily be seen by induction using the relation

$$\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k}.$$

Thus $|B(1)\Delta \cdots \Delta B(r)| = 2^{r-1}2^{n-r} = 2^{n-1}$ as desired.

(b) Again we may suppose that $n_k = k$ and also that $m_k = t+k$ for $t \geqq 1$. We may also assume that $s \leqq r \leqq s+t$. Thus $\min(r-t, 0)$ is the number of sets $\{B(n_1), \cdots, B(n_r)\}$ and $\{B(m_1), \cdots, B(m_s)\}$ have in common. Suppose first that $t < r < s+t$. As in the proof of part (a) of this lemma it is easy to see that $(i_1, \cdots, i_n)$ is in $(B(1)\Delta \cdots \Delta B(r)) \cap (B(t+1)\Delta \cdots \Delta B(t+s))$ if and only if for some k there are exactly k 0's in $(i_{t+1}, \cdots, i_r)$ and if k is even (odd), there are an odd (even) number of 0's in $(i_1, \cdots, i_t)$ and in $(i_{r+1}, \cdots, i_{t+s})$. Thus

$$|(B(1)\Delta \cdots \Delta B(r)) \cap (B(t+1)\Delta \cdots \Delta B(t+s))|$$

$$= 2^{n-s-t} \sum_{k=0}^{r-t} \binom{r-t}{k} \sum \binom{t}{i} \binom{s+t-r}{j}$$

where the second summation is over all odd i and j if k is even and over all even i and j if k is odd. Thus $|(B(1)\Delta \cdots \Delta B(r)) \cap (B(t+1)\Delta \cdots \Delta B(t+s))| = 2^{n-s-t}2^{r-t}2^{t-1}2^{s+t-r-1} = 2^{n-2}$ as desired. The special case when $s+t \leqq r$ is treated in a similar manner and yields the equation:

$$|(B(1)\Delta \cdots \Delta B(r)) \cap (B(t+1)\Delta \cdots \Delta B(t+s))|$$

$$= 2^{n-r} \sum_{i \text{ odd}} \binom{s}{i} \sum_{j \text{ even}} \binom{r-s}{j} = 2^{n-r}2^{s-1}2^{r-s-1} = 2^{n-2}$$

as before. Similarly, if $r \leqq t$, we obtain

$$|(B(1)\Delta \cdots \Delta B(r)) \cap (B(t+1)\Delta \cdots \Delta B(t+s))|$$

$$= 2^{n-r-s} \sum_{i \text{ odd}} \binom{r}{i} \sum_{j \text{ odd}} \binom{s}{j} = 2^{n-r-s}2^{r-1}2^{s-1} = 2^{n-2}$$

which completes the proof of the lemma.

Since μ is a measure, we have the equations

$$\sum \mu(i_1, \cdots, i_n) = \mu(A(n_1)\Delta \cdots \Delta A(n_k)), \tag{3}$$

where the summation is over all $(i_1, \cdots, i_n)$ in $B(n_1)\Delta \cdots \Delta B(n_k)$. Now think of the $\mu(i_1, \cdots, i_n)$'s as 2^n-1 unknowns. (3) is then a system of 2^n-1 equations (one equation for each distinct $B(n_1)\Delta \cdots \Delta B(n_k)$) with 2^n-1 unknowns. Let M denote the 2^n-1 by 2^n-1 matrix whose elements are the coefficients of the $\mu(i_1, \cdots, i_n)$'s in (3), and let $\mathfrak{y}$ and $\mathfrak{v}$ be the 2^n-1 dimensional vectors whose components (in some order) are the $\mu(i_1, \cdots, i_n)$'s and the $\nu(i_1, \cdots, i_n)$'s respectively. On replacing μ by ν in (3) we get the same system of equations since, by hypothesis $\mu(A(n_1)\Delta \cdots \Delta A(n_k)) = \nu(A(n_1)\Delta \cdots \Delta A(n_k))$. Thus we have the vector equation $\mathfrak{y}M = \mathfrak{v}M$. Therefore we shall have the desired result, namely that $\mu(i_1, \cdots, i_n) = \nu(i_1, \cdots, i_n)$ for all $(i_1, \cdots, i_n)$ in Y, if we can show that $\det M \neq 0$. M is a matrix of 0's and 1's and Lemma 1 tells us that the diagonal elements of MM^t (where M^t denotes the transpose of M) are all equal to 2^{n-1} and every other element is equal to 2^{n-2}. Let M_k be the $k \times k$ matrix with 2's in the diagonal elements and 1's everywhere else. Then

$$(\det M)^2 = (2^{n-2})^{2^n-1} \det M_{2^n-1}. \tag{4}$$

LEMMA 2. $\det M_k = k+1$.

Proof of Lemma 2. The proof is by induction on k. For $k=1$, M_k is just the number 2 and so the lemma is true. Suppose the lemma is true for k. On subtracting the second row of M_{k+1} from its first row and expanding the new determinant by cofactors of its first row, we obtain

$$\det M_{k+1} = \det M_k + \det N_k, \tag{5}$$

where N_k is a $k \times k$ matrix in which every diagonal element except the first is equal to 2 and every other element is equal to 1. On subtracting the second row of N_k from its first row and then expanding the new determinant in cofactors of its first row, we find that $\det N_k = \det N_{k-1}$. Since $\det N_1 = 1$, $\det N_k = 1$ for all k, and hence (5) gives the desired result.

Lemma 2 and equation (4) imply that $\det M \neq 0$ and, in turn, completes the proof of the theorem.

We remark that by a standard measure-theoretic argument the theorem remains true if instead of supposing that μ and ν are finite measures on $\sigma(\mathfrak{A})$, we suppose only that μ and ν are measures on $\sigma(\mathfrak{A})$ which are σ-finite on $\mathfrak{A}$. That some such finiteness conditions must be imposed on the values of the measures on $\mathfrak{A}$, as opposed to $\sigma(\mathfrak{A})$, is shown in [1], sec. 13, prob. 5.

This research was supported by the Office of Naval Research under grant No. WR-401(50) with Cornell University.

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A PROOF OF THE CAUCHY INTEGRAL THEOREM

FR. A. O. KONNULLY, St. Albert's College, Ernakulom, Kerala (India)

If $f(L)$ is analytic within, and continuous on, a simple closed curve C, then $\int_C f(z)dz=0$. The following proof of this theorem is a modification of my earlier argument [1].

It is assumed that the curve C is continuous and rectifiable and that it has a finite number of maxima and minima, that is, it can be divided into a finite number of parts, say N parts, on each of which each of the coordinates is either increasing, decreasing or constant.

From the given conditions we have:

(i) To a given $\epsilon>0$ there corresponds an η such that $|f(z)-f(z')|<\epsilon$ if $|z-z'|<\eta$, η being independent of the positions of z and z' inside or on C. (Uniform continuity.)

(ii) If C' be a closed curve lying entirely within the region bounded by C, to a given $\epsilon'>0$ there corresponds an η' such that

$$\left|\frac{f(z)-f(z')}{z-z'}-f'(z)\right|<\epsilon',$$

if $|z-z'|<\eta'$, η' being independent of the positions of z and z' inside or on C', (Goursat's Lemma).

(iii) LEMMA. *If Γ be a closed curve lying entirely within C, then*

$$\int_\Gamma [f(z)+zf'(z)]dz=0.$$

To prove this, let g be the greatest distance of the origin from C and γ be the length of the curve Γ. Then

$$\begin{aligned}\int_\Gamma f(z)dz &= \lim \sum f(z_i)(z_i-z_{i-1}), \qquad \text{points } z_i \text{ being on } \Gamma,\\ &= \lim \sum z_i[f(z_i)-f(z_{i+1})]\\ &= \lim \sum -z_i[f'(z_i)+\epsilon_i](z_{i+1}-z_i)\\ &= -\int_\Gamma zf'(z)dz, \qquad \text{since } \lim \sum \epsilon_i z_i(z_{i+1}-z_i)=0,\ |\epsilon_i z_i(z_{i+1}-z_i)|\end{aligned}$$

being ultimately less than $\epsilon' g\gamma$ by (ii).

Proof. First we prove the theorem for a curve C enclosing a region within which we can find a point O such that any radius vector from O meets the curve in a single point. We take such a point within C as origin.

$$\int_C f(z)dz=\lim \sum f(z_i)(z_i-z_{i-1}).$$

Let $I(t)$ denote $\lim \sum f(tz_i)t(z_i-z_{i-1})$, where z_i are points on C. If C_t denotes the curve traced by the point tz as z moves on C, $0<t<1$, then $I(t)=\int_{C_t} f(z)dz$. If $0<\delta<\frac{1}{2}$, then $C_{1-\delta}$ is a curve lying entirely within the region bounded by C.

Now,

$$\frac{d}{dt}\lim \sum f(tz_i)t(z_i - z_{i-1}) = \lim \sum \frac{d}{dt} f(tz_i)t(z_i - z_{i-1}), \qquad (0 < t \leqq 1 - \delta)$$

since

$$\left|\frac{I(t+h) - I(t)}{h} - \lim \sum [f(tz_i) + tz_i f'(tz_i)](z_i - z_{i-1})\right|$$

$$= \lim \left| \sum \left[\frac{f(\overline{t+h}\, z_i) - f(tz_i)}{z_i h} - f'(tz_i)\right] tz_i(z_i - z_{i-1}) \right.$$

$$\left. + \sum [f(\overline{t+h}\, z_i) - f(tz_i)](z_i - z_{i-1}) \right| \leqq \epsilon' gL + \epsilon L,$$

for $h<1/g \min(\eta, \eta')$, by (i) and by (ii) with $C'=C_{1-\delta}$, L being the length of the curve C.

Thus

$$\frac{d}{dt} I(t) = \lim \sum [f(tz_i) + tz_i f'(tz_i)](z_i - z_{i-1})$$

$$= 1/t \int_{C_t} [f(z) + zf'(z)]dz = 0 \quad \text{by (iii).}$$

Therefore

$$I(1-\delta) - I(\delta) = \int_{\eta}^{1-\delta} \frac{d}{dt} I(t)dt = 0$$

so that we have $I(1)=I(1)-I(1-\delta)+I(\delta)$. Let δ tend to 0. Then $I(\delta)$ tends to 0. Also $I(1)-I(1-\delta)$ tends to 0, since, when δ is small enough,

$$\left| \sum |f(z_i) - f(\overline{1-\delta}\, z_i)|(z_i - z_{i-1}) \right| < \epsilon L$$

by (i) and therefore

$$|I(1) - I(1-\delta)| = \lim \left| \sum [f(z_i) - f(\overline{1-\delta}\, z_i)](z_i - z_{i-1}) \right.$$

$$\left. + \delta \sum f(\overline{1-\delta}\, z_i)(z_i - z_{i-1}) \right|$$

can be made arbitrarily small. Hence $I(1) = \int_C f(z)dz = 0$.

Next, suppose that C is a curve enclosing any region D. Divide D by lines drawn parallel to the x-axis and y-axis, partially into regular subregions which are squares and partially into irregular subregions which are portions of squares

of which the remaining part lies beyond C. Then the integral $\int f(z)dz$ taken in the same sense along the boundary of each of these subregions is equal to the integral $\int f(z)dz$ along the complete boundary C. (For, each portion of the transversals separates two contiguous regions and is described twice in integration in opposite senses, so that when these integrals are added there will remain only the integrals taken along the curve C whose sum is the integral $\int_C f(z)dz$.) And, by what has been proved above, the integral taken round each of the regular subregions is zero, since $f(z)$ is analytic within and continuous on the boundary of any such region which, being a square, has always an interior point such that any radius vector from it meets the boundary in a single point. Thus $\int_C f(z)dz$ is equal to the sum of the integrals $\int f(z)dz$ taken round the irregular subregions.

Let c_k be the boundary of any of the irregular subregions and s_k be its length. If h_k be the length of the side of the square of which the irregular subregion is a portion and b_k be the length of the arc of C which forms a part of c_k, then $s_k \leqq (4h_k+b_k)$. And any two points of the region bounded by c_k will be apart by a distance not exceeding $\sqrt{2}h_k$.

Given $\epsilon>0$, the parallels may be so chosen that each $h_k<\eta/\sqrt{2}$. Then by (i), for any point z on c_k and a fixed point a_k on or within c_k we have $|f(z)-f(a_k)| \leqq \epsilon$. Thus,

$$\left| \int_{c_k} f(z)dz - \int_{c_k} f(a_k)dz \right| = \left| \int_{c_k} [f(z) - f(a_k)]dz \right| \leqq \epsilon s_k.$$

That is, $|\int_{c_k} f(z)dz| \leqq \epsilon s_k$, since $\int_{c_k} f(a_k)dz = f(a_k)\int_{c_k} dz = 0$.

Thus we have $|\int_C f(z)dz| = |\sum_k \int_{c_k} f(z)dz| \leqq \epsilon(\sum_k s_k)$, where $\sum_k$ denotes summation over all irregular subregions arising from the division of D into smaller parts by the parallels chosen when ϵ is given. And $\sum_k s_k \leqq \sum_k (4h_k+b_k) \leqq 2N(P+L)$, where P is the perimeter of a square with sides parallel to the co-ordinate axes which would contain within it the entire region D. For, the boundary c_k of each subregion has an arc of C (of length b_k) to form a part of it which belongs to it exclusively. And if we take all the boundaries whose arcs go to make up each one of the N parts of C on which each of the co-ordinates is either increasing or decreasing or constant, then the sides h_k of the corresponding squares for each part of C, the side of any square with arc b_k belonging partly to more than one part of C, such as the squares which contain a vertex (the junction of two of the N parts), being counted once in connection with each part of C, will add up to not more than two of the sides of the square containing D (all the transversals dividing D into subregions being supposed to lie in it); so that we have $\sum 4h_k \leqq 2NP$. Thus $|\int_C f(z)dz| \leqq \epsilon 2N(P+L)$. Since ϵ is arbitrary, it follows that $\int_C f(z)dz = 0$.

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MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

Send manuscripts to R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457

MODERNIZATION OF SCHOOL MATHEMATICS IN HOLLAND

W. J. BRANDENBURG, Groningen University, Holland

In Holland a reorientation of grammar school teachers in mathematics by means of courses has begun.

Four courses were given so far, the first one in September 1963 and the last one in September 1965. Each course took five days. They were given at the universities at Utrecht and Groningen and at the Technical University at Eindhoven.

In principle the content was independent of modernization, if any, of the curriculum. In all places mentioned the same subjects were treated. The subjects were: set theory; logics and linear algebra; linear analysis; topology and group theoretical aspects of elementary geometry. The courses consist of lectures and practical exercises. The lectures were generally given by professors. For practical exercises assistance was given by members of the scientific staff. For the exercises the students were divided into groups. The courses were free of charge for the participators. Hotel and travelling expenses were borne by the State. Each course was followed by about 525 grammar school teachers, i.e., about 50 percent of the number of competent mathematics teachers in this country.

Further it is important that between the courses these teachers have been working in study groups at the execution of projects resulting from these courses. At the moment two school-projects are being executed: a geometry project and an algebraic analysis project. For the geometry project a text has been made by Troelstra and Kuipers for a new treatment, based on modern insight, of geometry in the first form. The experiments were verified and coordinated by the Committee. Twelve grammar schools participate in the geometry project.

The continuation of this project for the second forms of the schools mentioned is being prepared. The concept of "representation" is put in a central place in geometry. Those who want to orient themselves somewhat can best consult *Transformatie Meetkunde* (Transformation Geometry) 1, 2 and 3 by Troelstra, Habermann, De Groot and Bulens.

The algebra project is executed at seven schools.

The project is executed in the superior forms of secondary schools and gymnasiums, consequently in the 4th and 5th forms and in the 5th and 6th forms respectively. These schools will hold a separate final examination for these subjects of instruction.

Apart from this also discussions take place about teaching analytical geometry, especially with regard to introduction of vectors.

At Groningen a course in modern algebra is given for secondary school teachers from the three Northern provinces. This course has an annex at Leeuwarden. It has been organized by the Stichting voor Pedagogisch Onderwijs aan de Rijksuniversiteit te Groningen (Foundation for Pedagogical Teaching at the State University at Groningen). The supervision has been entrusted to Gerretsen, member of the Curriculum for Mathematics. As text book was used: John L. Kelley, Introduction to Modern Algebra, New York.

During twenty weeks (ten before and ten after Christmas holidays) two hours a week, on Thursdays from 4:30 to 6:15 p.m., were devoted to teaching. In the first hour a lecture is given for all students together and in the second hour work is done by groups of twenty under the guidance of instructors. There are 120 students and consequently six groups. There are two groups at Leeuwarden and four at Groningen. The instructors are also always present at the first weekly lecture. The course may also be followed by advanced technical school teachers.

The subjects "Axioms for Numbers" and "Sets and Numbers" were treated in five afternoons. "Vectors and Lines" and "Inner Products" were treated in six afternoons. "The Complex Plane" and "Vector Geometry" were taught in seven afternoons. "Matrix Algebra" took two afternoons. The course was taught by W. J. Brandenburg.

The lecturers and instructors drew up a report about each lecture by means of a form containing various categories. This procedure resulted in a report.

In order to ascertain the result of the last mentioned course the pupils of the instructed teachers have been tested. During this test the pupils were examined with regard to their knowledge of the concept "intersection" and their skill in using it.

In the course of this examination on a very moderate scale three equal tests were applied, each dealing with different teachers and different pupils:

I—A pretest, before the teachers had acquainted themselves with the concept "intersection."

I. PRETEST

5 teachers	number of pupils	number of correct items
form	*N*	*score*
1	71	5
2	57	1
3	—	—
4	36	2.5
total	164	8.5

Number of good items per pupil: 0,05 on the average

II. TEST

22 teachers *form*	*N*	*score*
1	24	20
2	32	10
3	32	15.5
4	28	23.5
LTS*	6	3.5
	122	72.5

Number of good items per pupil: 0.6 on the average
* Primary Technical School

III. POSTTEST

10 teachers *form*	*N*	*score*
1	10	3
2	19	8
3	13	2
4	10	0
UTS**	6	0.5
	58	13.5

Number of good items per pupil: 0.24 on the average
** Secondary Technical School

II—A test, some weeks after the teachers had been introduced to the concept "intersection."

III—A posttest, after the course had ended all but for the last lesson.

The concept "intersection" was taken because modernization of school teaching generally begins by introducing "sets" and "operations of sets." The operation which is most often necessary in our school teaching of mathematics is the determination of the intersection of two sets.

The outcome of test I being 0.05 items correct on the average per pupil indicates that without instruction given to the teachers the pupils are not acquainted yet with the concept "intersection" without their school.

The outcome of test II being 0.6 items correct on the average per pupil indicates that some weeks after the instruction given to the teachers the concept "intersection" had not yet been dealt with in their teaching.

The outcome of test III being 0.24 items correct on the average per pupil indicates that immediately after the end of the course it does not influence the teaching of mathematics in schools to such an extent that the pupils are already acquainted with the concept "intersection."

Next year a second course will be organized on the same lines but then about geometry based on linear algebra. As regards some concepts the second course

will continue the first one. Out of the 120 participators 115 have already expressed their willingness to follow the second course too.

The members of the course have been working hard as could be concluded quite well from the exercises.

The course in modern algebra will be repeated on the same lines in four other places this year.

For advanced elementary education an instruction project will be prepared.

In the years to come some influence may perhaps be recorded. The question arises whether one should stimulate this without anything more. Actually we have to conclude that it is a too optimistic idea to modernize the teaching of mathematics only by instructing teachers in modern mathematics.

The instruction should at least be followed by

1. a pedagogical working up of modern mathematics into subject matter of teaching; and
2. a project in which this subject matter is tested in schools in a way that meets the requirements of educational research.

NATIONAL ASPECTS OF THE MAA-SA CONTEST IN THE DEVELOPMENT OF TALENT

NURA D. TURNER, State University of New York at Albany

This paper is relative to a study I have been conducting for eight years on the academic and professional progress of 117 students (on 14 of these I have incomplete information with respect to fields of preparation and degrees) who ranked in the top one percent in the Upstate New York Section of the MAA in the 1958–60 Contests and a group of students who ranked in the top .03 percent nationally in the 1958 Contest. The title listed implies not only the identification of high ranking students but furthering the counselling, guiding, and encouraging of these mathematically talented students. The work covered by this paper does not include the latter implication; it has to do only with a follow-up study of the group defined but does include the effect a mathematical background has had on the chosen careers of the group, a glance at home backgrounds, and recommendations to the MAA for post-contest study.

Students in the study come from a cross section of typically American families. Among their parents are farmers, ministers of the gospel, machine shop workers, warehouse foremen, secondary school teachers, college professors, administrators, carpenters, and medical doctors; some parents have not had as much as a high school training; some have college degrees. They are products of the opportunity that the USA provides to all. At least one student, who is a promising medical doctor, illustrates this advantage that living in the USA provides. His grandparents came here as immigrants. Two were illiterate. One grandfather had a push cart, later a store; but his father had eleven years of schooling and eventually became a district sales manager and his mother graduated from high school and is a secretary.

The Contest has properly identified these students as talented in mathematics. The majority of the students have received scholarships on both the undergraduate and graduate levels. Many have either borrowed money or worked during the summer to supplement scholarship aid. The average grade of college work, when the average has been taken over all subjects, is a B+.

The subject matter field attracting the greatest number of students has been mathematics. Related fields have attracted a majority of the students. The chart on the following page shows the frequency of fields to which students have been attracted and the degrees which they have obtained or are seeking. Some students are still in the academic atmosphere working on doctorates, master's degrees and even bachelor's degrees; some are in medical residencies; some are farming; some are teaching on the college or university level; some are practicing law; some are training for a religious life; some are working on staffs of commercial airplane companies, in defense work related to the Gemini, Saturn, and Apollo projects as well as work related to air force weapons, working for oil companies, for retail credit companies, for accounting firms; some are doing army duty; and some are homemakers.

No one is teaching in the secondary school. Of the five who completed certification requirements for secondary teaching, one feels he might consider the field; two had a taste and quickly left; a fourth is working in a YMCA; and the fifth said, "I always loved mathematics—couldn't say the same for teaching—couldn't stand the placid acceptance of mediocrity among most of my prospective colleagues."

Throughout the range of the careers into which these students have gone, or plan to go, from architecture to biomedical engineering, to linguistics, to medicine, to teaching at the college level, there is a recognition of the helpful effect of mathematics. Some illustrations will suffice.

An architectural student states: "In the technical courses in architecture school (acoustics, structures, heating and air-conditioning) students with strong mathematics background find a functional understanding much easier. Too many structures students give up at the sign of an integral."

A young accountant emphasizes that "many in the accounting profession, public and private, have not yet understood the applicability of mathematical approaches using statistical methods and mathematical models."

An aeronautical engineer who will obtain his doctorate in September feels he is ending up being more of a mathematician. Once he realized that most engineers know "far too little mathematics" and that those in charge of the engineers "know a lot more," most of the courses he took from then on were "disguised mathematics" courses. Now he feels well prepared to solve problems that are presently unsolvable simply because those who have attempted solutions have not known enough mathematical techniques.

A biomedical engineering student expecting a doctorate in 1967 has been applying mathematics, particularly control system theory, "to the study of physiological systems, i.e. circulatory systems, respiratory systems, etc. so that

Field of Study	Degree			
	Bachelor	*Master*	*Doctorate*	*Total*
Architecture	2			2
Business Administration	1	5		6
Chemistry			3	3
English		1		1
Engineering	9*	5	5	19
General Science	1			1
History (American)			1	1
Law	3			3
Library Science		1		1
Linguistics	1		1	2
Mathematics	6	6	21	33
Medicine			8	8
Music	1			1
Philosophy			3	3
Physics		1	11	12
Political Science		1		1
Religion	1	1		2
Sociology	1		2	3
Zoology			1	1
Total	26	21	56	103

* One a dropout who has never returned; another who has applied for admission to an engineering school.

they may be investigated in a more formal, quantitative way."

One law student explains that mathematics courses have had little direct relationship to his law studies but that the study of mathematics has helped him to develop a crisp analytical approach to legal problems. He states: "My free-lance study of symbolic logic (Copi, Quine) has been of direct use in law school; a logical formation of the problem is not a solution, but certainly is a healthy start."

Two other law students, one an attorney in the Tax Division, Department of Justice, the other Legal Counsel, Office of National Planning, Government of Liberia, stress the importance of a mathematical background. One feels it has helped develop his powers of analysis and the other that the development of such powers may have contributed to his great success in law school.

A doctoral candidate in linguistics is glad to have learned what mathematics he did because of its relevance to his field. "First, there is much computation in acoustic phonetics, which studies the duration and frequencies of the sounds of speech. Second, linguistics has the influence of probability and probabilistic thinking, especially in the related area of communication theory. More important, a strain of mathematical thinking runs through all linguistics; the entire field has been described as an effort to mathematize language."

A medical student who is doing his utmost to integrate his mathematical training with medicine has learned computer language on the side. For him it is "fascinating to even think that perhaps someday one may be able to program a computer to read X-rays."

These students have already published scholarly works. A list of known publications follows:

1. Earl B. Arnold, An Evaluation of an On-Line Computing System, Shell Development Technical Report, No. 298–65.
2. D. J. Benney and J. C. Luke, On the Interactions of Permanent Waves of Finite Amplitude, J. Math. and Phys., December, No. 4, 43 (1964) 309–313.
3. Howard A. Bodner, Active All-Pass Network, *Electronics Letters*, (London), Vol. 1, No. 4, June, 1965.
4. G. J. Grayson, and W. J. Kuhns, Hemolytic Factors Affecting the Survival of Transfused Cadaver Blood, (an abstract), *Transfusion*, September-October, No. 5, 3 (1963) 433.
5. J. C. Luke, A Perturbation Method for Non-Linear Dispersive Wave Problems, Proc. Roy. Soc. A, No. 430, May 31, 292 (1966) 403–413.
6. Alan Zame, A Combinatorial Word Problem, this MONTHLY, May, 70 (1963) 531–539.

Carl E. Baum has published 4 classified technical reports, 22 papers (one classified) for a restricted distribution list, and has contributed to 2 large summary type documents (classified). All these deal with nuclear weapons effects and instrumentation.

Three articles by Krohn Mateosian, and Rhodes are to be published: "Methods of the Algebraic Theory of Machines, I" in the *Journal of Computer and System Sciences;* "Complexity of Ideals in Finite Semigroups and Finite State Machines," in *Journal of Mathematical Systems Theory;* and "Lectures on Algebraic Theory of Finite Semigroups and Finite State Machines," in *Procedures of Conference on Algebraic Theory of Machines, Languages, and Semigroups.*

So far what you have heard might have been expected. What I say now you may or may not have expected.

Among the students making mathematics and physics their careers, 32 in the doctoral category, there are many who feel the need for guidance particularly from mathematics teachers in high school and undergraduate college work.

One student who will begin an NSF post doctoral fellowship in mathematics this fall states, "If I had not had a certain amount of experience in advance (gained from reading an algebra book in the library while in the 8th grade), I think that I might have become confused and discouraged when algebra was first presented to me in the ninth grade. . . . My enjoyable experiences in mathematics have been almost entirely outside the classroom."

Another who is on his last lap of a doctorate in theoretical physics reveals, "When I was in elementary and high school, I was unaware of the existence of many important branches of mathematics in which I have since taken courses. I was also unaware of the extent of the importance of mathematics in modern science and engineering."

Still another who will obtain his doctorate in mathematics in January and who completed a four year high school mathematics sequence in two years and scored A grades in exemption examinations his junior year in college states, "At no time, even in special classes or in mathematics team practice sessions did

any teacher indicate what mathematicians do or what mathematics is all about. In my first two years of college, I was not enlightened further. As a junior, my mathematics adviser merely asked which courses I was interested in and signed her approval. Nonscience majors, on the other hand, were given an eye-opening introductory course in foundations of modern mathematics. As an upper junior, I registered for an advanced version of this course and finally learned what mathematicians call mathematics. I would say that the school system has wasted about ten years of my mathematical life."

The meaning of what is said by another student, who will receive a doctorate in mathematics in September, is veiled, but still evident. At first he thought in order to study mathematics one would have to become a physicist or an engineer and only later was surprised to discover that one could pursue a career studying pure mathematics.

A student now working toward a doctorate in sociology has this advice: "The need for better precollege guidance is critical. The student should not rely solely on parents or high school counselors, but should seek out more objective and experienced sources of information." Apparently, the young man has little respect for guidance counselors.

While wanting guidance and encouragement, there is indication that many students studying on the doctoral level resent and resist pressure to study mathematics.

A student working on a doctorate in sociology feels that encouraging any and all students to study mathematics just because it is such a good field represents "a somewhat unjustified academic imperialism." He would rather give students a clear picture of what each field is and can be, providing them with a background that will give them self-confidence, and let them decide what they want to do.

A student well on his way to becoming a medical doctor implies he objected to pressure when he said, "I received nothing but encouragement everywhere I turned. Too much, I'm afraid."

Others indicated an appreciation of a lack of pressure. A statement will suffice. "My family always encouraged me without, however, forcing me into any particular direction."

In general, students resent pressure. They want to be free to make the choice of a career, and they want a good introduction to the field of their choice.

About one-fourth of the students studying or working in mathematics or in related fields have lost interest to some extent in mathematics.

Of three on the doctoral level in mathematics, one has found the subject "too dry–emotionless" but the time too late for changing fields. A second, interested in linguistics, is following the advice of one of his professors that one can more easily be a professor of mathematics with a side interest in linguistics than vice versa. A third, studying for a doctorate, who had mathematical talent compared to those around her in high school, is finding, along with other girls in the graduate school where she is studying, that her interest in higher mathe-

matics is limited. She adds, "If I were less far along, I would consider a career in one of the social sciences—anthropology or psychology. I believe my mathematical background would be useful in these fields in terms of a rigor of thought and ability to state what a problem is. The existence of areas such as 'mathematical psychology' confirms my belief in the overlap of such areas."

Several continued with mathematics to a bachelor's degree and switched to another field on the graduate level. Pure mathematics was not the "cup of tea" for one student. While continuing to major in mathematics, he took courses stressing practical applications (e.g. numerical analysis, computer simulations). For another, pure mathematics became too abstract for his "less than rigorous personality"; he found "operations research as potentially a more attractive blend of quantitative techniques and practical applications." Both these young men obtained master's degrees in business administration. Another found that when he reached the "more advanced and theoretical courses, such as advanced calculus," he "did not enjoy them nearly as much as earlier courses." He switched to medicine. Still another found that most of advanced mathematics did not interest him as much as working with numbers. He decided to try actuarial work for a year. He has passed Part 8 of the actuarial examinations and must be in the work to stay. Two others switched to library science and political science.

The switch for others came after their contact with calculus or advanced calculus—some, good students on the 'A' level in mathematics. One felt that "mathematics was just a 'game', having little relevance to everyday life." He switched to English. For another student, mathematics courses at the level of advanced calculus did not hold his interest. "They seemed to be more and more a matter of theorem proof, theorem proof." He changed his major to physics, studied on the master's level, and then entered the field of computer programming.

Four students are known to have become so disinterested that they became dropouts. Three are making comebacks, however. One is well on his way to a bachelor's degree in accounting while working for an accounting firm; his interest in mathematics waned in his sophomore year but he now realizes its importance in the sophisticated direction in which accounting is turning. A second who left engineering, again in the sophomore year, has been working as an inspector for General Electric, attending night school, and has applied for admission to an engineering college. The third is back studying architecture after a checkered career. The fourth dropped out at the end of his sophomore year and returned to the farm where he is happy.

Improvement in mathematics instruction and counselling could help students to stay with mathematics and keep from floundering. Attention has been called to the change in interest in mathematics after introduction to calculus courses and to dropouts at about the time or after the time a student is introduced to such courses.

What a student who is working on a doctorate in mathematics says may

shed light on the situation. His reference is to a freshman honors course in calculus. "My mother, father, brother, and sister had each done very well in high school and each of them had fallen down when introduced to calculus in college. I was determined that the curtain wasn't going to descend on me as it had on them and fortunately I ran into one of the best teachers I have ever had. . . . He was very youthful and dynamic in the classroom and expected a great deal from his students. I have always felt very fortunate to have had him as a teacher in that course, especially as I have seen so many other students who have done poorly in freshman calculus principally because of lack of interest. The material is not dull at all—it is only that it is often presented in a noninteresting way. And I feel this course is the most important for a prospective student of mathematics; it occurs at the time he is deciding what field to enter."

What another student who is working on a doctorate in philosophy says may shed more light on the situation. His reference is to his freshman calculus course. He reached the point of "hating" mathematics. "A discouraging and unsuccessful bout with the calculus my freshman year helped to direct me away from the scientific and engineering fields. Whether I would have become interested in liberal arts and especially in philosophy if I had not conceived a dislike for mathematics is questionable. My course in Symbolic and Mathematical Logic (taught by a superb teacher) has just this year (first year of graduate work) reawakened my appreciation for and delight in mathematical thought. I wish now I had persevered in mathematical studies. My interest in mathematics (and my skill) were in problem solving and manipulation rather than theory which probably had a lot to do with my difficulties with the theoretical mathematics I took in college."

While one could not expect all students who have been identified as mathematically talented to pursue a career in pure mathematics, some of the students covered by this paper seem to have been literally driven out of mathematics. While students do not want to be pressured and indicate resistance to pressure to take mathematics, they also indicate that in the educational system there exists a lack of counselling, guidance, and encouragement to continue with mathematics.

The MAA-SA Contest has so far only identified mathematically talented students. The 1966 report of "The Seventeenth Annual H.S. Mathematics Contest" states the purpose as going no further than seeking to sharpen certain mathematical concepts and skills.

A great amount of money has been spent on this contest. From my experience as contest chairman in the Upstate New York Section, the total cost for administering the examination internationally, including cost of use of building facilities, cost of teacher time in preparing students and proctoring and grading the examination papers, and cost of section chairmen and national committee members time cannot be less than a half million dollars per year. One might consider that the buildings would still exist and the teachers would still be paid if the Contest were not held. But any business, industry, or organization evalu-

ates the cost of a project in terms of employee time and facilities used even though the employees would have been paid and the facilities kept up had the project not been undertaken. The expenditure of such a sum of money each year should result in more than mere identification of talent or sharpening of concepts and skills.

There are certain weaknesses in mathematics education. This paper has identified at least three of them.

1. There is a weakness in the area of introducing students to a general knowledge of what mathematics is about.

2. There is a weakness in mathematics counselling on both the secondary and college levels.

3. There is a weakness in teaching mathematics at the calculus level, both elementary and advanced. Too many mathematics students are turning to other fields at this point. Students have been frank in saying this has been where they have lost interest.

This evaluation of the results of the Annual High School Mathematics Contest involving, primarily, students in one section of the USA but, as well, a group of students on a national basis, points out definitively the need for strengthening the Contest program by using it as more than a tool merely for identification of talent and sharpening of skills. As in all good educational programs, it should be used as a tool to improve the educational process.

As a result of the evaluation contained in this paper, I recommend to the Mathematical Association of America that a committee be appointed for the purpose of finding ways in which the MAA can contribute to advising, guiding and encouraging mathematically talented students identified by the Annual High School Mathematics Contest and to investigate reasons for mathematics dropouts at the calculus level.

The opportunity of using this testing program as an educational tool is important enough to demand study and the thinking of several people.

International Congress of Mathematicians, Moscow, USSR, August 18, 1966.

MATHEMATICS IN NEPAL

T. J. LARDNER, Massachusetts Institute of Technology

I recently returned from Kathmandu, Nepal, where I spent the academic year 1965–1966 as a Fulbright Lecturer in Mathematics at Tribhuvan University. While in Nepal, I often thought of how trivial our problems in education are compared to those existing in underdeveloped countries such as Nepal. At times, one feels that to make a minor improvement in educational policy in such countries requires a major change in the economic and social structure of the country.

The purpose of this article is to acquaint the reader briefly with Nepal, its history, [1] and its system of education, especially in mathematics.

Nepal is a Hindu kingdom of a size approximately 100 miles by 500 miles, situated in the Himalayan region north of India and south of Tibet, and with a population of approximately 10 million people. In the east it is bordered by the Indian protectorate of Sikkim, and in the west by the state of Uttar Pradesh in India. Nepal is well-known as the birthplace of Buddha, for Mount Everest, and as the home of the Gurka soldiers. It contains a wider range of physical diversity than probably any other country of comparable size. The mountain mass of the Himalayas in the north contain some of the world's highest peaks, and to the south, less than 100 miles away, the cultivated fields and jungles of the northern rim of the Gangetic plain are 600 feet above sea level.

The country can be divided into three main geographic regions: the Mountain Region, which constitutes almost three-fourths of the total area; the Kathmandu Valley; and the Tarai Region, a narrow belt which extends along the boundary with India in the northern part of the Ganges River plain. Nepal is so mountainous and hilly, that, for example, a 35-minute flight by DC-3 from Kathmandu to Pokhra, a distance of 90 miles, takes 8 days by foot, the only other means of travel.

Kathmandu, the capital, is the center of almost all the cultural, educational and intellectual activity in Nepal, and the history of Nepal is often only the history of the valley of Kathmandu which contains, besides Kathmandu, the towns of Patan and Bhaktapur. The three towns came under a common king in 1769 when Prithvi Narayan Shah, the ruler of the Kingdom of Gorkha, conquered the valley and established the Shah Dynasty of which the present king, Mahendra Bir Bikram Shah Deva, is a member. However, in 1846, Jang Bahadur Rana became Premier, suppressed the authority of the king, and from 1846 to 1950 the Rana family ruled Nepal in a feudal manner. The country, during this time, was closed to foreign visitors with only a few exceptions.

There were no schools in Nepal when Jang Bahadur Rana became Premier, and an English tutor was engaged for his children in 1854. It was not until about 1880 that the first high school opened in Nepal, and this was restricted to children of the Rana family. Other than a few Sanskrit institutions in Kathmandu and some village literacy classes, there were no other schools in Nepal in the late nineteenth century.

A brief shift in government educational policy came in 1901 when Prime Minister Deva Shamsher Rana proposed the establishment of a system of universal public primary education. His plans were not successful, and only a few primary schools were opened. Before World War II, several new middle and high schools were founded in Patan and outside the valley, and a girls' high school was opened in Kathmandu.

In the early twentieth century, education beyond the high school level could be obtained only outside the country. As a consequence, less than a dozen Nepalese received higher academic or technical training during this period. Of these, most attended either Patna University or Benares Hindu University in India. In 1918 Prime Minister Chandra Shamsher Rana established a college

for Nepalese students in Kathmandu named Tri-Chandra College, which offered a two-year course in history, mathematics, logic, and languages. In 1919, science was added, and in 1924 with an enrollment of thirty, it became a full four-year institution, the first college in Nepal. It remained affiliated with Patna University until the establishment of the national university in Nepal in 1959.

After World War II, more or less forced to do so by the example of growing educational facilities in China and India, the Ranas made a modest effort to provide more schools. Their principal gesture was the establishment of a number of schools modeled on those in India. Just before 1951, Nepal had 310 primary and middle schools, 11 high schools and two colleges (the Sanskrit College founded in 1948 and the Tri-Chandra College).

In 1950, King Tribhuvan was able to gain control of the government with the help of India, and removed the Ranas from power. Nepal was then opened to the outside world.

Although King Tribhuvan had declared at the time of the formation of the new government that a parliamentary democratic system under a constitutional monarchy would be established, the creation of such a system was delayed for 8 years by the turmoil which followed the overthrow of the Ranas, until finally elections were held in 1959. However, in 1960 King Mahendra, who became king after his father's death in 1955, abolished the short-lived parliamentary regime and took personal control of the government.

With the overthrow of the Ranas, educational facilities expanded. Today Nepal has in operation 5000 primary schools with about 12,000 teachers and an estimated enrollment of 350,000 pupils which is about 25 per cent of the potential enrollment. There are 57,000 students in grades 6–10 enrolled in 645 secondary schools. However, educational standards are poor. It is reported that at least one-third of the primary school teachers are completely untrained—many are barely literate. Approximately 45 per cent of the pupils entering grade one do not enroll in grade two, and not more than 25 per cent of a first grade class can be expected to complete grade five, the terminal grade of the primary school. About fifty per cent of the students who enter high school drop out, and less than 50 per cent of those who take the final school leaving certificate examination (the SLC examination) are successful. Emphasis throughout primary and secondary school is on memory and rote learning and the situation is not different in the colleges and in the university.

There are thirty colleges affiliated with Tribhuvan University, the national university. Fourteen of the colleges are located in Kathmandu valley with the largest and oldest college being Tri-Chandra College, with a student population of about 1850 and a staff of 90.

Tribhuvan University, located in Kathmandu, was founded in 1959 and has the dual task of administering the undergraduate college examinations and the teachings of graduate courses in history, political science, economics, commerce, culture, Sanskrit, Nepali, Hindi, English, geography, physics, botany, zoology, and mathematics. There are 423 students and 67 members of the teaching staff.

Approximately half the teachers are Indians in Nepal with the Colombo plan or with the Indian Aid Mission. At present, conditions at the University are poor with classes held in old inadequate government rest houses. There are no departmental offices, no quiet areas for working, and no academic community or spirit as such among the faculty. Lectures in many cases are read from textbooks unavailable to the students, and homework or additional reading for the students is usually not assigned, or if assigned, usually not done by the students. With the exception of a few departments, nothing is done by the faculty members in the way of research or in improving or updating course material. Unfortunately, the presence of Western or modern Indian professors does little to change the system because of local politics.

An undergraduate student cannot major in mathematics in the colleges. Instead, he takes first the Intermediate Science degree (I.Sc.), then the B.Sc. degree, each degree requiring two years of study in mathematics, physics, and chemistry. It is also possible to take mathematics as part of the arts program, but this is not common.

For the I.Sc. degree which is offered at five colleges throughout Nepal, the course in mathematics includes algebra, plane geometry, solid geometry, coordinate geometry, trigonometry, differential and integral calculus, statics and dynamics. The level of knowledge reached by the students is about equal to that reached by students in a good high school in the United States. It is interesting that the mathematics courses are given concurrently without too much effort put into building on previous knowledge. At the end of two years, the student takes the university I.Sc. exam which is an external examination set by a committee of University and College professors from India and Nepal on the basis of the published syllabi. In this exam approximately 65 per cent of the students fail. There is no appeal once the results are published, and a student cannot examine his graded paper; it is destroyed soon after the test results are published. He can, however, for a fee, request immediately after the test scores are published a regrading of his paper, but only in rare instances have published results been changed upon regrading.

After the I.Sc. degree, the student who does not go abroad for his B.Sc. degree or for other studies, goes to Tri-Chandra College, which is the only college in Nepal which grants the B.Sc. degree. The program in mathematics for the bachelor's degree is a continuation of the I.Sc. program mentioned earlier. Here the level reached by the best student is equivalent to that of a second year college student in science in the United States. At the end of the two years of study, the student again takes a set of external university examinations. In these B.Sc. examinations, the number of students passing is greater than that in the I.Sc. examination for almost 50 per cent pass.

In preparing for the examinations at the end of each two year period, a student often will not study a given subject in order to concentrate on what he considers an easier subject, for example, dynamics is considered difficult as compared to statics and hydrostatics. As a consequence, at the end of the bachelor's

studies many students find it difficult to solve ordinary differential equations even of the first order, have no familiarity with Fourier Series, and do not understand partial differentiation, to name only a few topics. Emphasis in the B.Sc. course as well as in the I.Sc. course is on memorization of details just before examination time in the hope of being able to answer enough questions on the examination for a passing grade. There is much effort on the part of students and college teachers who tutor students for additional income to guess the questions which will appear on the examinations, and "guess" examination papers are found in the bookshops before examination time.

If the examinations are too difficult or if the students feel—rightly or wrongly—that the syllabus was not followed, demonstrations or strikes take place during examination time.

During trouble at the examination centers in 1966, the following comment appeared in one of the local papers:

"An inquiry should be conducted to probe the causes which led to the suspension of the Tribhuvan University examinations at two examination centers. It should be the duty of the University authorities to keep the students satisfied with their behavior. The administrative and other affairs of the University should be so planned as not to give any cause to students to feel agitated."

This comment shows, I think, the attitude of the public and students toward the examinations. Of course, there is much basis for criticism of the examination system, but it is surprising how the students are able to dictate conditions to University officials under the threat of a student strike.

After obtaining a B.Sc. degree from Tri-Chandra College, a student can major in mathematics at Tribhuvan University for two years for the M.Sc. degree. In the first year, a student takes three required courses: (i) algebra-determinants, matrices, theory of equations, vector analysis and inequalities; (ii) analysis, real variables; and (iii) infinite series, analytical geometry of three dimensions and differential equations. In addition, he must take one optional course from the listing of courses found in the Appendix. At the end of this first year, the student takes a set of four external examinations. He must pass all four examinations before starting the second year course of study. Again in the second year, there are three required courses plus one optional course. The required courses are (i) statics and dynamics; (ii) spherical harmonics and complex variables; and (iii) partial differential equations, analytical geometry and differential geometry.

Again, the student takes four examinations. If he passes these examinations, he is then awarded the M.Sc. degree. It is necessary to go abroad for any advanced study, as the Masters Degree is the highest offered by the University. At present (1966) no Nepalese holds a Ph.D. degree in mathematics.

The mathematics faculty at the University consists of three full-time members, two of whom are Indian Colombo Plan Professors, and three part-time members who also teach at Tri-Chandra College.

At present there are three students in mathematics, one in the first year who

is considered "not serious" and two in the second year. Approximately twenty students have received an M.Sc. degree in mathematics since 1959, and of these, two work in Government ministries, while the rest teach in different colleges. Mathematics is not a popular field of study because there are few jobs available to those with a degree in mathematics. One can always teach in the colleges outside of Kathmandu valley, but this is not attractive. The future of mathematics in Nepal is uncertain as almost all the students wish to go into sciences. In fact, I have been told by students that the mathematics professors encourage students to study science rather than mathematics. But the future of students in science, i.e., physics, chemistry, zoology, botany, is even more uncertain because there are few small industries and no research facilities in the country.

Appendix—Mathematics Optional Courses

(a) Spherical Trigonometry and Astronomy
(b) Statistics I*
(c) Quantum Mechanics*
(d) Hydro Mechanics*
(e) Theory of Elasticity*
(f) Theory of Numbers
(g) Projective Geometry
(h) Theory of Relativity
(i) Statistics II
(j) Functions of Complex Variable
(k) Differential Geometry
(l) Functions of a Real Variable
(m) Modern Algebra
(n) Advanced Statics and Dynamics
(o) Measure and Integration

Those marked with asterisks have been given once.

Reference

1. The best source of information on Nepal is "The U. S. Army Area Handbook for Nepal," DA Pamphlet 550-35, U. S. Govt. Printing Office, which contains an extensive bibliography.

A NEW DOCTORAL PROGRAM IN MATHEMATICS EDUCATION RESEARCH

J. M. SCANDURA, University of Pennsylvania

The recent revolution in mathematics education has raised many fundamental questions and given rebirth to others. What are the objectives of mathematics education—and, how can they be used as a basis for curriculum development? What is really learned when mathematics is discovered? What does the master teacher do which makes him a master teacher—can others be taught to do these things? How does one measure mathematical knowledge and ability? To what extent can mathematical creativity be taught? These are just some of the questions about which we have only partial answers and about which more complete answers must be attained if mathematical education is to reach its potential. Unfortunately, with the immediate and pressing needs for more and better instructional materials, in-service teacher training, and the like, few con-

temporary mathematics educators have had the time to acquire the specialized training needed to conduct definitive research on such questions. They generally have too much to do already.

In recognition of the need for more research specialists in mathematics education and with support from the U. S. Office of Education, the Graduate School of Education, University of Pennsylvania has instituted a new doctoral program in mathematics education research leading to the Ed.D. and Ph.D. degrees.

This program has been especially designed for recent graduates with distinguished undergraduate records in mathematics or mathematics education. Advanced students with a master's degree in mathematics, mathematics education, statistics, or any area of behavioral science research with a strong undergraduate preparation in mathematics may be admitted into the program with advanced standing. The program is primarily intended to prepare students for positions of leadership in mathematics education research in universities, research laboratories and organizations, and federal, state, and local research agencies.

This program is interdisciplinary in nature and involves study in four core areas: *mathematics*, *recent mathematical curricula* (including elementary, secondary, and college levels), *psychology* and *educational psychology*, and *research methodology* (including measurement, statistics, experimental design, computer programming, operations research, etc.). In addition to minimal requirements in these areas, each student is expected to specialize in one or more of the core areas. Trainees will also be encouraged to elect appropriate courses in *logic*, *operations research*, *statistics*, *psycho-linguistics*, and the sciences. Independent study and research is encouraged and extensive opportunities are provided for participation in research seminars and a variety of research activities in mathematics education.

Programs may be tailored to the needs and objectives of individual students but most students will select one of two general options. The first option (Mathematics Emphasis) is weighted heavily towards mathematics whereas the second, an option which might appropriately be called Psycho-Mathematics, or the Psychology of Mathematics Learning, requires more intensive study in psychology and educational psychology along with supporting work in such areas as logic, operations research, statistics, and linguistics. The minimal mathematics requirement in the second option (Psycho-Mathematics) is the equivalent of a (steeply graded) University of Pennsylvania undergraduate major and in the first (Mathematics Emphasis) is the equivalent of a Pennsylvania master's degree. Participation in advanced seminars in mathematics education and independent study devoted to research in mathematics education is required of all candidates. For this purpose, four new courses have been instituted, the last three being devoted exclusively to research: Research in Mathematics Education, Seminar on Innovation in Mathematics Curriculum and Instruction, Research Seminar on the Psychology of Mathematics Learning.

Stipends, for qualified candidates, vary from tuition scholarships to approxi-

mately $4,000 per year plus full tuition for those with three dependents. In addition to fellowships, for full-time study, a number of research assistantships and summer research assistantships are available for $\frac{3}{4}$ time students.

Qualified college seniors and recent graduates, with an undergraduate major or master's degree in mathematics or mathematics education, are especially urged to apply. Those interested in the Ph.D. program should apply for admission to the Graduate School of Arts and Sciences; Ed.D. applicants should apply to the Graduate School of Education.

Further information on the program may be obtained by consulting the University of Pennsylvania Graduate Studies Bulletin and Undergraduate Catalogue for course descriptions and by writing Dr. Joseph M. Scandura, Associate Professor of Mathematics Education, Graduate School of Education, University of Pennsylvania, Philadelphia, Pennsylvania 19104.

THE ROLE OF AXIOMATICS AND PROBLEM SOLVING IN MATHEMATICS

E. G. BEGLE, Stanford University

For the International Congress of Mathematicians, scheduled for the summer of 1966 in Moscow, U.S.S.R., the International Commission on Mathematical Instruction chose three topics as being of particular interest at the time. These three topics were:

A. University programs in mathematics for physicists;
B. The use of the axiomatic method in the teaching of mathematics in the secondary school;
C. The role of problems in the development of mathematical activity in students.

The national commission of each of the countries adhering to the International Commission on Mathematical Instruction was asked to report on the state of affairs in its country with respect to each of these three topics. When this charge was discussed by the United States Commission on Mathematical Instruction, it was agreed that the first topic had already been thoroughly studied by the Panel on Physical Sciences and Engineering of the Committee on the Undergraduate Program in Mathematics. On the other hand, it was agreed that there is a wide diversity of opinion in this country with respect to the other two topics, in particular that concerning the axiomatic method.

The United States Commission felt that any report to the International Commission should reflect this wide range of opinions and that an effective way of doing this would be to ask individual mathematicians and mathematics teachers, representing various points of view, to prepare statements of their own opinions on these matters.

The United States Commission felt that the set of these statements could not only form the report to the International Commission but also, if made widely available in the United States, encourage discussion of these important topics and, hopefully, reduce misunderstandings in the mathematical community.

E 1953. *Proposed by D. J. Newman, Yeshiva University*

What is the maximum possible number of real solutions for the system $y=e^x$, $P(x, y)=0$, if $P(x, y)$ is required to be a polynomial of degree n?

E 1954. *Proposed by D. J. Newman, Yeshiva University*

Prove that there is no position in which an $n\times n$ square can cover more than $(n+1)^2$ integral lattice points.

SOLUTIONS OF ELEMENTARY PROBLEMS

Cutting Up a Chain

E 1798 [1965, 665]. *Proposed by Pete Said, Fresno State College, California*

A chain has n links and each link weighs one unit. p cuts are to be made in the chain in such a manner that the resulting subchains, together with the p cut links and a balance scale, can be used to weigh every multiple of the unit weight from 1 to n. Find p in terms of n.

I. *Solution by C. B. A. Peck, State College, Pa.* Assume the chain is open (not endless) and allow any number of links to be placed in either pan of the scale. With p cut links one can weigh from 1 to p. Add a chain of k links and one can weigh from $k-p$ to $k+p$. If $p+1\geqq k-p$, there is no gap. If we let $k=2p+1$, we can weigh from 1 to $3p+1$. Proceeding by induction, if we can weigh from 1 to $3^{k+1}p+(3^{k+1}-1)/2$ by p cut links and chains with $2p+1$, $3(2p+1)$, $\cdots$, $3^k(2p+1)$ links, then we can weigh from 1 to $3^{k+1}p+(3^{k+1}-1)/2+3^{k+1}(2p+1)$ $=3^{k+2}p+(3^{k+2}-1)/2$ by adding a chain with $3^{k+1}(2p+1)$ links, providing the number of chains (the exponent of the 3's) does not exceed $p+1$. Then p cut links are necessary and (if distributed as described) sufficient for a chain of n links to furnish the desired weights if

$$3^p p + (3^p + 1)/2 < n \leqq 3^{p+1}p + (3^{p+1} - 1)/2.$$

Thus p is any integer k ($\leqq n$) such that $3^{k+1}\geqq(2n+1)/(2k+1)$. For the smallest such p there is no explicit formula.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* If the links must be placed in one scale pan, this is essentially problem E 636 [1945, 276]. If, however, links may be placed in either pan of a two-pan balance, then, by analogous reasoning, p is determined by

$$3^p(p - \tfrac{1}{2}) - \tfrac{1}{2} < n \leqq 3^{p+1}(p + \tfrac{1}{2}) - \tfrac{1}{2}.$$

Also solved by W. Bluger, R. B. Eggleton (Australia), Michael Goldberg, R. F. Jackson, Walter Penney, Robin Sibson (England), and D. L. Silverman.

A New Line Associated with a Given Triangle

E 1799 [1965, 665]. *Proposed by Joseph Langr, Prague, Czechoslovakia*

On the sides BC, CA, AB of the triangle $ABC \equiv T$ are chosen points A_1, A_2; B_1, B_2; C_1, C_2 in such a way that $AC_1 = AB_2 = BC$, $BA_1 = BC_2 = CA$, $CB_1 = CA_2 = AB$. (1) Show that the orthocenters H_1, H_2, H_3, H of the triangles AB_2C_1, BC_2A_1, CA_2B_1 and T lie on the Euler line of the triangle formed by the lines B_2C_1, C_2A_1, A_2B_1. (This Euler line is parallel to the line joining the incenter and circumcenter of T.) (2) Show also that $HH_1 : HH_2 : HH_3 = \cos A : \cos B : \cos C$.

Solution by J. M. Quonian, Saint-Etienne, France. Denote the circumcenter and the incenter of triangle T by O and I. Employing a conjugate coordinate system having I as origin and the inradius r of T as unity, we may let

$$(a, 1/a), \qquad (b, 1/b), \qquad (c, 1/c)$$

denote the coordinates of the points A'', B'', C'' of contact of the incircle of T with the sides BC, CA, AB. For later convenience, set

$$s = a + b + c, \qquad t = bc + ca + ab, \qquad p = abc.$$

The coordinates of A are $x_A = 2bc/(b+c)$, $y_A = 2/(b+c)$, with similar expressions for the coordinates of B and C. The coordinates of B_2 are

$$x_{B_2} = 2bc/(b + c) - 2ab(b - c)/(c + a)(a + b),$$
$$y_{B_2} = 2/(b + c) + 2a(b - c)/b(c + a)(a + b),$$

and those of C_1 are

$$x_{C_1} = 2bc/(b + c) + 2ca(b - c)/(c + a)(a + b),$$
$$y_{C_1} = 2/(b + c) - 2a(b - c)/c(c + a)(a + b).$$

The coordinates of C_2 and A_1, and of A_2 and B_1, are given by similar expressions. Denoting by $A'B'C'$ the triangle with sides along the lines B_2C_1, C_2A_1, A_2B_1, the coordinates of vertex A' are

$$x_{A'} = 2bc/(b + c) - 2a(a^2 - bc)/(c + a)(a + b),$$
$$y_{A'} = 2/(b + c) + 2(a^2 - bc)/a(c + a)(a + b).$$

There are similar expressions for the coordinates of B' and C'. The circumcenter O' of triangle $A'B'C'$ has coordinates

$$x_{O'} = 2(t^2 + ps)/(ts - p), \qquad y_{O'} = 2(s^2 + t)/(ts - p),$$

and one sees that

$$\overrightarrow{IO'} = \overrightarrow{IA} + \overrightarrow{IB} + \overrightarrow{IC}.$$

The differences $x_{O'} - x_{A'}$ and $y_{O'} - y_{A'}$ are equal to $2a$ and $2/a$ respectively, so that *the circumradius of triangle $A'B'C'$ is twice the inradius of triangle T.*

The coordinates of the orthocenter H' of triangle $A'B'C'$ are

$$x_{H'} = 2(t^2 + ps)/(ts - p) - 2s = x_{O'} - 2x_{H''},$$
$$y_{H'} = 2(s^2 + t)/(ts - p) - 2t/p = y_{O'} - 2y_{H''},$$

where H'' is the orthocenter of triangle $A''B''C''$. The coordinates of the orthocenter H of triangle T are

$$x_H = 2(t^2 - ps)/(ts - p), \qquad y_H = 2(s^2 - t)/(ts - p),$$

and those of the orthocenter H_1 of triangle AB_2C_1 are

$$x_{H_1} = 2bc/(b + c) - 2a(b^2 + c^2)/(c + a)(a + b),$$
$$y_{H_1} = 2/(b + c) - 2a(b^2 + c^2)/bc(c + a)(a + b).$$

Similar expressions give the coordinates of the orthocenter H_2 of triangle BC_2A_1 and the orthocenter H_3 of triangle CA_2B_1. The coordinates of the circumcenter O of triangle T are

$$x_O = 2ps/(ts - p), \qquad y_O = 2t/(ps - t).$$

The equation of the line H_1H_2 is

$$psy - tx + 2(t^3 - ps^3)/(ts - p) = 0,$$

which is parallel to the line OI and has the points O', H_3, H, H' on it.

Finally, we have

$$\begin{aligned}(HH_1)^2 &= (x_H - x_{H_1})(y_H - y_{H_1}) \\ &= [2a(b^2 + c^2)s/(ts - p)][2(b^2 + c^2)t/bc(ts - p)] \\ &= [(b^2 + c^2)^2/b^2c^2][4pts/(ts - p)^2],\end{aligned}$$

whence

$$HH_1 = [-(b^2 + c^2)/2bc][4\sqrt{pts}/(p - ts)].$$

Since $-(b^2+c^2)/2bc = \cos A$, we have

$$HH_1/\cos A = 4\sqrt{pts}/(p - ts) = 4r(IH'')/[r^2 - (IH'')^2].$$

A Construction Related to a Given Circle

E 1800 [1965, 666]. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Given a circle (O), a point C on it, and two points A, B not on the circle. Find a point S on circle (O) such that SC bisects the angle ASB.

I. *Solution by Joseph Basile, Brussels, Belgium.* Let SA, SB cut circle (O) in X and Y. Then X and Y are symmetric in the diameter d of (O) through C. Choose on (O) any three pairs of points X_i, Y_i $(i=1, 2, 3)$ such that each pair

is symmetric in d. Let AX_i and BY_i cut (O) in X_i' and Y_i'. Then X_i' and Y_i' form two homographic ranges on (O) whose double points (constructed by the classical method) are the possible positions for S.

II. *Solution by R. H. F. Denniston, University of Leicester, England.* Find inverses, with respect to C as center, of (O), A, B; these will be a straight line o' and points A' and B'. Bisect $A'B'$ at M and draw the internal and external bisectors of angle $A'CB'$. Parallel to these bisectors, draw MP and MQ through M to meet o' in P and Q. Through C draw a line parallel to o' to meet MP and MQ in D and E. Find a point S' on o' such that $(S'P)(S'Q)=(CD)(CE)$. Then S, the inverse of S', is the required point.

Proof. Take MP and MQ as x and y coordinate axes; then the product xy has the same value at S' and at C. Now consider the locus of a point whose joins to A' and B' are equally inclined to the axes; the equation of this locus is easily seen to be of the form $xy=$constant. But C is on the locus; so S' is on the locus too, and the angles $CA'S'$, $S'B'C$ are equal. Now angle $CA'S'$ is equal to angle ASC, by similar triangles; similarly, angles $S'B'C$, CSB are equal. Thus SC bisects angle ASB.

Also solved in a manner similar to solution I by R. H. F. Denniston, and analytically by P. A. Scheinok.

A Zeta-function Summation

E 1801 [1965, 666]. *Proposed by Ralph Greenberg, University of Pennsylvania*
Prove that

$$\sum_{k=1}^{\infty} \zeta(2k)/2^{2k}(2k^2+k) = \log(\pi/e),$$

where $\zeta(n)=\sum_{j=1}^{\infty} j^{-n}$, the Riemann zeta function.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* We have

$$\sum_{k=1}^{\infty} \zeta(2k)/2^{2k}(2k^2+k) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} (2j)^{-2k}/(2k^2+k),$$

replacing the ζ-function by its definition. We reverse the order of summation, decompose the denominator into partial fractions and sum, obtaining

$$\sum_{j=1}^{\infty} \{2\log(2j) - (2j+1)\log(2j+1) + (2j-1)\log(2j-1) + 2\}.$$

We consider the corresponding finite sum and use Stirling's formula, to get

$$\begin{aligned}\sum_{j=1}^{n} \{2\log(2j) - (2j+1)\log(2j+1) + (2j-1)\log(2j-1) + 2\} &= 2n\log(2e) + 2\log(n!) - (2n+1)\log(2n+1)\\ &\approx (2n+1)\log\frac{2n}{2n+1} + \log\pi.\end{aligned}$$

As $n\to\infty$, this last quantity approaches $-1+\log\pi=\log(\pi/e)$.

II. *Solution by A. E. Danese, State University of New York at Buffalo.* The result is immediate if we let $z=1$ in

$$\sum_{k=1}^{\infty}\zeta(2k, z)/2^{2k}(2k^2+k) = (2z-1)\log(z-\tfrac{1}{2}) - 2z + 1 + \log 2\pi - 2\log\Gamma(z),$$

where $\zeta(s, z)=\sum_{n=0}^{\infty}(z+n)^{-s}$, $z\neq 0, -1, -2, \cdots$. (Burnside's formula, *Higher Transcendental Functions*, vol. I, Erdelyi, et al., p. 48, (11).)

Also solved by L. F. Epstein, Edward Garelis, S. H. Greene, Sidney Heller, Stephen Hoffman, R. S. Jackson, M. S. Klamkin, David Klarner & Martin Muldoon, E. S. Langford, F. C. Smith, David Zeitlin and the proposer.

A Consequence of the Mean Value Theorem

E 1802 [1965, 666]. *Proposed by Dov Avishalom, Tel-Aviv, Israel*

For functions of class C_n prove that

$$f^{(n)}(a) = \lim_{h\to 0}\left\{\frac{1}{h^n}\sum_{k=0}^{n}\left[(-1)^{n-k}\binom{n}{k}f(a+kh)\right]\right\}.$$

I. *Solution by M. S. Klamkin, Ford Scientific Laboratory.* First, it is well known that

$$\sum_{k=0}^{n}(-1)^k k^r\binom{n}{k} = \begin{cases}0 & (r=0, 1, \cdots, n-1)\\ n! & (r=n)\end{cases}$$

(Easily proved by differentiating $(1-x)^n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k$ r times ($r\leqq n$) and evaluating at $x=1$.) Consequently, it follows from L'Hospital's rule that the desired limit equals

$$\frac{1}{n!}\left\{D^n\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F(a+kh)\right\}_{h=0} = F^n(a).$$

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* A proof is given in Wilhelm Maak, *An Introduction to Modern Calculus*, pp. 85–86, as an immediate consequence of the Mean Value Theorem for higher differential quotients for functions of class C_n.

Also solved by J. M. Ash, R. J. Cormier, S. H. Greene, D. M. Hancasky, Stephen Hoffman, J. E. Humes, Jr., E. S. Langford, Norman Miller, M. J. Pascual, B. E. Rhoades, P. A. Scheinok, Al Somayajulu, Sidney Spital, Julius Vogel, Benjamin Volk, P. H. Young, David Zeitlin, and the proposer.

Ash notes the generalization: Let $a_0, a_1, \cdots, a_n$ be any $n+1$ distinct real numbers, and let f be any function such that $f^{(n)}(a)$ exists, then there is a unique set of numbers $b_0, \cdots, b_n$ such that

$$\lim_{h\to 0}\left\{\frac{1}{h^n}\sum_{k=0}^{n}b_k f(a+a_k h)\right\} = f^{(n)}(a).$$

A Mean Value Property

E 1803 [1965, 666]. *Proposed by F. P. Callahan, Blue Bell, Pa.*

Find the most general form of $f(x)$, defined for all real x, such that

$$\frac{f(v)-f(u)}{v-u}=f'(pu+qv) \tag{1}$$

holds for all u, v; where p, q are positive with $p+q=1$.

Solution by Robert Coen, Brooklyn, N. Y. First we note the symmetry of (1) in u, v so that

$$f'(pv+qu)=f'(pu+qv), \tag{2}$$

whence, if $p\neq q$, f' is constant and f is linear.

On the other hand, if $p=q=\frac{1}{2}$, we may differentiate (1) with respect to u and to v and add the results, obtaining

$$\frac{f'(v)+f'(u)}{v-u}-2\frac{f(v)-f(u)}{(v-u)^2}=0$$

which, because of (1), gives

$$\frac{f'(v)+f'(u)}{2}=f'\left(\frac{u+v}{2}\right),$$

and this implies that f' is linear, so that f is quadratic. Finally, $f(x)=a+bx$, and $f(x)=a+bx+cx^2$ with $p=q=\frac{1}{2}$ do satisfy (1).

Also solved by R. A. Avelsgaard, S. H. Greene, D. M. Hancasky, R. F. Jackson, M. S. Klamkin, B. G. Klein, D. C. B. Marsh, R. F. Rossa & R. J. Gazik, and Richard Sinkhorn.

Virtually the same problem is in Pi Mu Epsilon Journal, vol. 3, No. 1, 1959, p. 18, prob. 103.

The Taylor Operator

E 1804 [1965, 666]. *Proposed by Carmen Artino, John Carroll University*

Let $\mathfrak{V}$ be the vector space (over the real field) of all polynomials of degree less than n. Let $\mathbf{D}$ denote the differentiation operator on $\mathfrak{V}$ and $\mathbf{I}$ the identity transform on $\mathfrak{V}$; i.e., $f(x)\in\mathfrak{V}$, $f(x)\mathbf{D}=f'(x)$ and $f(x)\mathbf{I}=f(x)$. Let $\mathbf{T}$ be a linear transformation on $\mathfrak{V}$ given by $f(x)\mathbf{T}=f(x+1)$. Show that

$$\mathbf{T}=\mathbf{I}+\frac{\mathbf{D}}{1!}+\frac{\mathbf{D}^2}{2!}+\cdots+\frac{\mathbf{D}^{n-1}}{(n-1)!}.$$

Solution by Simon Vatriquant, Brussels, Belgium. Since $f(x)$ is a polynomial, the Taylor expansion exists and terminates. The degree of the polynomial being

less than n, we have $f(x)\mathbf{D}^{n+k} \equiv 0$ for $k \geqq 0$. Consequently

$$f(x+1) = f(x) + \frac{1}{1!}f'(x) + \frac{1^2}{2!}f''(x) + \cdots + \frac{1^{n-1}}{(n-1)!}f^{(n-1)}(x)$$

or

$$f(x)\mathbf{T} = f(x)\mathbf{I} + f(x)\mathbf{D}/1! + f(x)\mathbf{D}^2/2! + \cdots + f(x)\mathbf{D}^{n-1}/(n-1)!.$$

According to the distributivity property of the operators we have

$$f(x)\mathbf{T} = f(x)[\mathbf{I} + \mathbf{D}/1! + \mathbf{D}^2/2! + \cdots + \mathbf{D}^{n-1}/(n-1)!],$$

as desired.

Also solved by S. S. Bartnikowski, D. R. Beuerman, Béla Bollobás (Hungary), W. G. Dotson, Jr., J. L. Ercolano, E. W. Ewing, G. G. Ford, E. A. Franz, R. B. Grayless, S. H. Greene, D. M. Hancasky, Stephen Hoffman, Edward Hook, J. F. Horrell, M. S. Klamkin, B. G. Klein, E. S. Langford, R. E. Maas, D. C. B. Marsh, A. P. Mella, J. W. Mentele, C. J. Parry & D. H. Faust, M. J. Pascual, B. E. Rhoades, Henry Ricardo, R. F. Rossa & R. J. Gazil, P. A. Scheinok, G. F. Schumm, N. T. Sheth, Robin Sibson (England), Sidney Spital, Julius Vogel, Benjamin Volk, David Vosholz, Nolan Wallach, Max Weinryb, D. R. Wilder, O. H. Young, D. S. Zave, David Zeitlin, and the proposer.

Ercolano remarks that this is a special case of a well-known result valid in the space of all analytic functions, viz.,

$$\mathbf{T}_h = \exp[h\mathbf{D}] = \mathbf{I} + h\mathbf{D} + h^2\mathbf{D}^2/2! + \cdots + h^n\mathbf{D}^n/n! + \cdots,$$

where $T_h f(x) = f(x+h)$.

The problem is found in many texts. See Birkhoff and MacLane *Survey of Modern Algebra*, p. 214, prob. 17(b); Halmos, *Finite Dimensional Vector Spaces*, p. 41, ex. 3; Jacobson, *Lectures on Abstract Algebra*, vol. II, p. 38, prob. 2.

Inscribed Triangle in a Minkowski Plane

E 1805 [1965, 780]. *Proposed by I. J. Schoenberg, U. S. Army Research Center, University of Wisconsin*

We wish to inscribe a triangle ABC, having given sides a, b, c, in the circle $\|x\| = r$ of the Minkowski (Banach) plane M_2 of points $x = (x_1, x_2)$ with the norm $\|x\| = \max(|x_1|, |x_2|)$. Since M_2 is a metric space, it is necessary for the possibility of this construction that (i) a, b, c satisfy the triangle inequality, and (ii) a, b, and c do not exceed the diameter $2r$. Show that these conditions are also sufficient.

Solution by A. B. Carter, Cornell College, Mt. Vernon, Iowa. We may take $0 < a \leqq b \leqq c \leqq 2r$ and $c \leqq a+b$. For P, Q in M_2 denote the normed distance between P and Q by $|PQ|$. Let $A = (c-r, r)$, $B = (-r, r-a)$, and $C = (c-b-r, r)$. Clearly, triangle ABC is inscribed in the "circle" $\|x\| = r$ and $|AB| = \max(|-r-c+r|, |r-a-r|) = \max(c, a) = c$, $|AC| = \max(|c-b-r-c+r|, 0)$

$=\max(b, 0)=b$, and $|BC|=\max(|c-b-r+r|, |r-r+a|)=\max(c-b, a)$. Since $c\leqq a+b$, $c-b\leqq a$ and $|BC|=a$. This proves sufficiency.

Also solved by R. G. Albert, Béla Bollobás (Hungary), E. O. Buchman, D. I. A. Cohen, D. C. B. Marsh, Wayne Roberts, and the proposer.

Related Inequalities in a Triangle

E 1806 [1965, 781]. *Proposed by Kaidy Tan, Fukien Normal College, China*

On two sides AB, AC of triangle ABC describe equilateral triangles ABD, ACE outside the given triangle. Let G, H be the centroids of these two triangles respectively, and let M be the midpoint of BC. Prove that $AC\geqq AB$ if and only if $MG\geqq MH$.

Solution by C. S. Venkataraman, Trichur, India. Let $AB=c$, $AC=b$, and let L, N be their respective midpoints. Since triangle ABD is equilateral, DG passes through N and is perpendicular to AB, and also $GL=c/2\sqrt{3}$. Further, LM is parallel to AC. Therefore $LM=b/2$ and $\measuredangle BLM=\measuredangle A$. Similarly $HN=b/2\sqrt{3}$, $NM=c/2$ and $\measuredangle CNM=\measuredangle A$. Note that $\measuredangle GLM=\pi/2+\measuredangle A$. The cosine formula now gives us

$$MG^2 = c^2/12 + b^2/4 + bc\sin A/2\sqrt{3},$$
$$MH^2 = b^2/12 + c^2/4 + bc\sin A/2\sqrt{3}.$$

Then $MG^2-MH^2=(b^2-c^2)/6$. Therefore $AC^2\gtreqless AB^2$ according as $MG^2\gtreqless MH^2$ and conversely. Hence $AC\gtreqless AB$ if and only if $MG\gtreqless MH$.

Also solved by K. V. Achuthan & R. Sivaramakrishnan (India), R. G. Albert, P. N. Bajaj, V. K. Batra (India), A. N. Behr, Béla Bollobás (Hungary), D. I. A. Cohen, R. H. Denniston (England), Ragnar Dybvik (Norway), R. B. Eggleton (Australia), Hyman Gabai & Steven Szabo, Michael Goldberg, D. M. Hancasky, Ned Harrell, W. E. Hoff, Stephen Hoffman, John Isbell, Mrs. M. R. Iyer (India), M. S. Klamkin, J. D. E. Konhauser, F. Lauenberger (Switzerland), D. C. B. Marsh, Norman Miller, Barbara W. Nason, C. B. A. Peck, J. M. Quoniam (France), Simeon Reich (Israel), L. A. Ringenberg, P. A. Scheinok, R. R. Schryer, Robin Sibson (England), Al Somayajulu, Sidney Spital, Janet Tang, C. Van de Vyle (Belgium), Hazel S. Wilson, J. W. Wilson & Ray Carry, Dale Woods, Paul Yang, and K. L. Yocom.

An Identity in Derivatives

E 1807 [1965, 781]. *Proposed by Peter Rolland, Jr., University of Illinois*

Let p be a polynomial of degree n. Prove that

$$\sum_{k=0}^{n} \frac{p^{(k)}(0)}{(k+1)!}x^{k+1} = \sum_{k=0}^{n} (-1)^k \frac{p^{(k)}(x)}{(k+1)!}x^{k+1}, \qquad p^{(0)}(x)\equiv p(x).$$

I. *Solution by C. Van de Vyle, Evergem, Belgium.* We put $M(x)$ and $N(x)$ for the left and right members, respectively, of the proposed identity. Because $M(x)$ and $N(x)$ are polynomials and $M(0)=N(0)=0$, it is sufficient to prove that $dM/dx=dN/dx$.

$$\frac{dM}{dx} = \sum_{k=0}^{n} \frac{p^{(k)}(0)}{k!} x^k = p(x), \qquad \text{(Maclaurin)}$$

$$\frac{dN}{dx} = \sum_{k=0}^{n} (-1)^k \frac{p^{(k)}(x)}{k!} x^k + \sum_{k=0}^{n} (-1)^k \frac{p^{(k+1)}(x)}{(p+1)!} x^{k+1}$$

$$= p(x) + (-1)^n \frac{p^{(n+1)}(x)}{(n+1)!} x^{n+1} = p(x),$$

from which it is evident that $M(x) \equiv N(x)$.

II. *Solution by A. W. Roberts, Macalester College.* For any t, s, the Taylor expansion gives

$$p(t) = \sum_{k=0}^{n} \frac{p^{(k)}(s)}{k!} (t-s)^k = p(s) + \sum_{k=0}^{n} \frac{p^{(k+1)}(s)}{(k+1)!} (t-s)^{k+1},$$

the second summation extended to n since $p^{(n+1)}(x) \equiv 0$. If we set $t = x$, $s = 0$,

$$p(x) - p(0) = \sum_{k=0}^{n} \frac{p^{(k+1)}(0)}{(k+1)!} x^{k+1}. \tag{1}$$

If we set $t = 0$, $s = x$,

$$p(0) - p(x) = \sum_{k=0}^{n} \frac{p^{(k+1)}(x)}{(k+1)!} (-x)^{k+1}. \tag{2}$$

Multiplication of (2) by -1 and comparison with (1) gives the desired result.

III. *Solution by N. J. Fine, Pennsylvania State University.* Let

$$P(a, t) = \sum_{k=0}^{n} \frac{p^{(k)}(a)}{(k+1)!} t^{k+1}.$$

Then

$$D_t P(a, t) = \sum_{k=0}^{n} p^{(k)}(a) t^k / k! = p(a + t).$$

Since $P(a, 0) = 0$, we have

$$P(a, t) = \int_0^t p(a + u)\,du = \int_a^{a+t} p(u)\,du.$$

In the problem, the first sum is $P(0, x) = \int_0^x p(u)\,du$, and the second is

$$-P(x, -x) = -\int_x^0 p(u)\,du = \int_0^x p(u)\,du.$$

Also solved by R. G. Albert, A. D. Anderson, Robert Baillie, R. A. Bell, Béla Bollobás (Hungary), J. L. Brown, Jr., E. O. Buchman, John Burslem, C. A. Church, Jr., D. I. A. Cohen, R. C. Cormier, G. C. Dodds, Ragnar Dybvik (Norway), R. B. Eggleton (Australia), Phillip Fung, P. K. Garlick, Frederick Gray & R. I. Price, Richard Guilfoyle, D. M. Hancasky, Bill Helliwell, Stephen Hoffman, David Huestis, Mrs. M. R. Iyer (India), R. F. Jackson, G. G. Killough, P. G. Kirmser, M. S. Klamkin, Kenneth Kramer, E. S. Langford, R. E. Maas, J. E. MacDonald, Jr., B. D. McLemore, Beatrice Margolis, C. F. Marion, D. C. B. Marsh, Norman Miller, Q. G. Mohammed (India), Alexander Morgan, R. E. Moyer, P. V. O'Neil, C. B. A. Peck, Harsh Pittie, J. M. Quoniam (France), Simeon Reich (Israel), Henry Ricardo, P. A. Scheinok, P. S. Schnare, Robin Sibson (England), Richard Sinkhorn, David A. Smith, Franklin C. Smith, Al Somayajulu, G. P. Speck, M. R. Spiegel, Sidney Spital, Preston Stein, Julius Vogel, Benjamin Volk, A. W. Warrick, Gerald Werner, B. D. Wick, H. H. Wong, K. L. Yocom, P. H. Young, David Zeitlin, and the proposer.

Bell notes the following more general result: For any positive integer r and polynomial p of degree n, we have

$$\sum_{k=0}^{n} p^{(k)}(0)x^{k+r}/(k+r)! = \sum_{k=0}^{n} (-1)^k x^{k+r} p^{(k)}(x)\binom{k+r-1}{r-1} \Big/ (k+r)!.$$

A Max-min Problem

E 1808 [1965, 781]. *Proposed by H. T. Croft, Cambridge University, England*

C is a configuration of 5 points P_i $(i=1, \cdots, 5)$ in a closed domain D. Find $M=\max_C \min_{i\neq j} P_iP_j$ in the following cases:

(i) D is the unit square;

(ii) D is the unit equilateral triangle;

(iii) D is the unit circle;

(iv) D is the surface of the unit sphere.

In (iv) P_iP_j may be interpreted either as Euclidean or great-circle distances; essentially the same problem results.

Solution by D. C. B. Marsh, Colorado School of Mines. (i) $M=2^{-1/2}$. Consider four closed $\frac{1}{2}$-unit squares whose union is the given unit square. At least one of these must contain at least two of the five points. The maximum spacing of two such points is $2^{-1/2}$, occurring when the points are diagonally opposite. This value is also attainable for the unit square when the five points are placed at the vertices and center of the square.

(ii) $M=\frac{1}{2}$. As in (i), partition the triangle into four $\frac{1}{2}$-unit equilateral triangles and locate the points at any five of the six vertices of the $\frac{1}{2}$-unit triangles.

(iii) $M=2 \sin (\pi/5)$. Partition the circle into five congruent sectors and space the points equidistant about the circumference. Were two points to lie within a circular sector of arc $\phi(\phi \leqq 2\pi/5)$ their greatest separation would be $\max(1, 2 \sin \frac{1}{2}\phi) \leqq 2 \sin (\pi/5)$.

(iv) $M=\frac{1}{2}\pi$. This is possible for one point at the north pole and the other four 90° apart along the equator. Were one point arbitrarily labelled the north pole and the other four located farther than $\pi/2$ from it, then these four would have to lie in the southern hemisphere proper. Partitioning the hemisphere into

four congruent regions, we observe that the max-min for these four points cannot exceed $\pi/2$.

This is the great-circle interpretation. When Euclidean distance is used, the same argument leads to $M=\sqrt{2}$.

Also solved by Michael Goldberg, Norman Miller, C. B. A. Peck, and the proposer.

Goldberg refers to a paper by K. Schütte and B. L. van der Waerden, *Auf welcher Kugel haben, 5, 6, 7, 8, oder 9 Punkte mit Mindesabstand Eins Platz?*, Math. Annalen, 123 (1951) 96–124.

Farmer Jones Rides Again

E 1809 [1965, 781]. *Proposed by Pete Said, Fresno State College*

A square has sides of length 2. It is to be rolled along a curve without slipping and so that the center of the square moves in the horizontal line $\{(x, y) \mid y=2^{1/2}\}$. What is the equation of the principal period of the curve?

Solution by A. P. Boblett, U. S. Naval Ordnance Laboratory, Corona, Calif. When $x=0$: $y=\sqrt{2}-1$; $y'=0$; and $y''<0$. Assume $s=-y'$, where s is the distance measured along the curve.

$$s=-y'=\int [1+(y')^2]^{1/2}dx: \qquad -y''=[1+(y')^2]^{1/2}: \qquad (y'')^2=1+(y')^2.$$

By inspection a solution is $y=A\pm\cosh(x)$, and a solution which meets the initial conditions is

$$y=\sqrt{2}-\cosh(x).$$

It is easily verified that the center of the square lies on $y=\sqrt{2}$ for $-\sinh^{-1}(1) \leqq x \leqq \sinh^{-1}(1)$.

Also solved by T. J. Bishop, D. I. A. Cohen, Richard Gisselquist, R. F. Jackson, Sidney Spital, the proposer, and the solvers of E 1668.

This is essentially a repeat of E 1668 [1965, 82], where the result agrees with the above except for a shift of the y-axis.

Sum of the Cubes of the Digits

E 1810 [1965, 781]. *Proposed by John Harvey and John E. Wetzel, University of Illinois*

For each natural number n let $C(n)$ be the sum of the cubes of the decimal digits of n. Show that for every n, iterating C eventually leads to one of the following repeating cycles: 1, 55-250-133, 136-244, 153, 160-217-352, 370, 371, 407, 919-1459.

Solution by Stephen Hoffman, Trinity College, Hartford, Conn. Let $B=\{1, 55, 133, 136, 153, 160, 217, 244, 250, 352, 370, 371, 407, 919, 1459\}$. Then it is easily shown that $C(B)=B$. If $S=\{n \mid n\in I^+$ and there exists $p\in I^+$ such that $C^p(n) \in B\}$, examination shows that all one digit positive integers are in S. If all k-

digit integers are in S and if n has $k+1$ digits, then $C(n)=q+d^3$, where $q\in S$, and d is one digit of n, so $0\leqq d\leqq 9$. If $d=0$ then $C(n)\in S$. Examination of the 126 numbers given by $1\leqq d\leqq 9$ shows that each has an iterate, under C, which is in B.

Also solved by D. C. B. Marsh, P. K. Subramanian, Robin Sibson (England), and the proposer.

C. B. A. Peck quotes Sierpinski (*Elementary Theory of Numbers*, Warszawa, 1964, p. 268) that this result was shown by Iseki, Proc. Japan. Acad., 36 (1960), pp. 578–583. See also B. M. Stewart, *Sums of functions of digits*, Can. J. Math., 12 (1960) 374–389.

A Problem of Partitions

E 1811 [1965, 781]. *Proposed by Irving Adler, North Bennington, Vermont*

If $a_1, \cdots, a_n$ are distinct positive integers, the positive integer s is said to be attainable through $\{a_1, \cdots, a_n\}$ if there exist nonnegative integers $x_1, \cdots, x_n$ such that $x_1a_1+\cdots+x_na_n=s$. The critical number for a set is the least positive integer N such that every integer $n\geqq N$ is attainable through that set.

If a and d are relatively prime, prove that the critical number for $\{a, a+d, a+2d\}$ is $(m+d-1)a-d+1$, where m is the least positive integer $\geqq\frac{1}{2}(a-1)$.

Solution by Alfred Brauer, University of North Carolina and Wake Forest College. Let $a_1, a_2, \cdots, a_k$ be relatively prime positive integers and $k>2$. We consider the Diophantine equation

$$x_1a_1 + x_2a_2 + \cdots + x_ka_k = n. \tag{1}$$

Let $f(a_1, \cdots, a_k)$ be the largest integer for which (1) has no solution in positive integers $x_1, \cdots, x_k$. Then

$$g(a_1, \cdots, a_k) = f(a_1, \cdots, a_k) - a_1 - a_2 - \cdots - a_k$$

is the largest integer not representable by (1) with nonnegative integers $x_1, \cdots, x_k$.

G. Frobenius, in his lectures, proposed the following problem: Find upper bounds for $f(a_1, \cdots, a_k)$. Such bounds were obtained by I. Schur (see [2]) and in my papers [2], [3], [4] with B. M. Seelbinder and J. E. Shockley, and by Schockley [7]. In [2] the exact bound was obtained for the case that $a_1, a_2, \cdots, a_k$ are consecutive integers. This was generalized by J. B. Roberts [6] for the case of an arithmetic progression $a, a+d, \cdots, a+(k-1)d$, with $d>1$. Simpler proofs of this generalization were given in my review [5] of [6] and by P. T. Bateman [1]. Problem E 1811 is the case $k=3$.

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1. P. T. Bateman, *Remark on a recent note on linear forms*, this MONTHLY, 65 (1958) 517–518.

2. Alfred Brauer, *On a problem of partitions*, Amer. J. of Math., 64 (1942) 299–312.

3. Alfred Brauer and B. M. Seelbinder, *On a problem of partitions* II, Amer. J. of Math., 76 (1954) 343–346.

4. Alfred Brauer and J. E. Shockley, *On a problem of Frobenius*, J. reine u. angewandte Math., 211 (1962) 215–220.

5. Alfred Brauer, Math. Rev., 19 (1958) 1038.
6. J. B. Roberts, *Note on linear forms*, Proc. Amer. Math. Soc., 7 (1956) 465–469.
7. J. E. Shockley, *On the best bound for the solvability of a linear Diophantine equation by positive integers*, Ph.D. diss., Univ. of North Carolina, Chapel Hill, 1962.

Also solved by the proposer.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate signed sheets and should be mailed before July 31, 1967. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5450. *Proposed by H. Kestelman, University College, London, England*

A real-valued function f defined on a vector space V is called a pseudo-norm if, for all x, y in V,

$$f(x + y) \leqq f(x) + f(y) \quad \text{and} \quad f(-x) = f(x).$$

Suppose $f_1, f_2, \cdots$ are pseudo-norms in V, and E_0 is the set of those v in V with $\sum_n f_n(v) < \infty$; show that E_0 is either empty or is a group under addition.

If $\epsilon_1, \epsilon_2, \cdots$ is a sequence of elements of V and S_ϵ is the set of all v with $\sum_n f_n(v+\epsilon_n) < \infty$, show that if S_ϵ includes at least two elements then E_0 is infinite and S_ϵ is a translate of E_0. Deduce when $V = R_n$, that if E_0 has positive interior Lebesgue measure then S_ϵ is either empty or else equal to R_n.

5451. *Proposed by J. P. Williams and Thomas Crimmins, University of Michigan*

It is well known that if T is a bounded linear operator from a Hilbert space H into itself, then T maps H *onto* itself if and only if its adjoint is bounded below: $\|T^*x\| \geqq C\|x\|$, $(x \in H)$ for some positive constant C (see Dunford and Schwartz, Vol. 1, p. 488, for example).

Show that the weaker condition $\|Tx\| + \|T^*x\| \geqq C\|x\|$ $(x \in H)$ does not imply that T is one-to-one or that the range of T is dense. Does this condition imply that the range of T is closed?

5452. *Proposed by E. S. Barnes, University of Malaya, Kuala Lumpur*

Let $n = 2m+1$, $n > 3$. Show that n real numbers $x_1, x_2, \cdots, x_n$ have the property that the sum of any m of them is less than or equal to the sum of the remaining $m+1$ if and only if $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is a nonnegative linear combination of the $2n$ vectors $\mathbf{E} \pm \mathbf{e}_i$, $(i = 1, \cdots, n)$, where $\mathbf{e}_i$ are the unit vectors and $\mathbf{E} = \sum_1^n \mathbf{e}_i$.

5453. *Proposed by R. S. Doran, University of Washington*

Prove or disprove: If X is a countably compact, locally compact uniform space with unique uniform structure, then X is compact.

5454. *Proposed by G. Szász, Nyiregyháza, Hungary*

What are the elements of the semigroup generated by the free generators a, b and subject to the generating relations $a^2b=a$, $b^2a=b$? Has this semigroup some idempotent elements?

5455. *Proposed by G. Szász, Nyiregyháza, Hungary*

Let S be a semigroup having the following property: if a and b are any elements of S such that the principal ideal generated by a does not contain the element b, then there exists a prime ideal P in S containing a, but not containing b. Prove that S is intra-regular.

5456. *Proposed by F. D. Faulkner, U. S. Naval Postgraduate School*

Let $\mathfrak{A}=\{A\}$ be the set of all real, third-order, circulant, orthogonal matrices:

$$A=\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}=A(a,b,c) \qquad \text{with } A^{-1}=A^T.$$

1. Show that there is one A for each value of a, $1\geqq a\geqq 2/3$, with $2/3\geqq b\geqq 0\geqq c\geqq -1/3$.

2. Show that all matrices of $\mathfrak{A}$ are obtained by permuting the elements obtained in problem 1, and the negatives of such matrices.

3. Show that there is an infinite number of matrices in $\mathfrak{A}$ whose elements are all rational.

4. Show that if two adjacent elements of A are rational, then A has all rational elements.

5457. *Proposed by T. J. Grilliot, Duke University*

Let R be a ring with identity. Prove that the sets

$$A=\{x\in R\colon x \text{ is not invertible}\},$$
$$B=\{x\in R\colon x \text{ is not left invertible}\},$$
$$C=\{x\in R\colon x \text{ is not right invertible}\}$$

are all equal if any one of them is closed under addition.

5458. *Proposed by C. C. Lindner, Coker College, Hartsville, S. C.*

Let G be a finite nonabelian group in which every Sylow subgroup of G is maximal in G. Prove that G has no center.

5459. *Proposed by Thomas C. Brown, Reed College and Kiev State University, U.S.S.R.*

Let S be a semigroup with three generators g_1, g_2, g_3, in which $x^2=x^3$ and $x^2y=yx^2$ for all x, y, in S. Show that S can be written as the union of seven pairwise disjoint semigroups and that seven is maximal.

SOLUTIONS OF ADVANCED PROBLEMS

An Arithmetic Inequality

5326 [1965, 915; 1966, 902]. *Proposed by C. S. Venkataraman and R. Sivaramakrishnan, Trichur, India*

If $\sigma_k(n)$ denotes the sum of the kth powers of the positive divisors of a positive integer n and $\tau(n)$ is the number of positive divisors of n, prove that

$$\frac{\sigma_k(n)}{\tau(n)} \geqq \sqrt{n^k}.$$

III. *Solution by E. S. Langford, U. S. Naval Postgraduate School.* Using the methods developed in our solution of E 1749 [1966, 84], we can show that

$$\sigma_k(n)/\tau(n) \geqq n^{k/2} + (n^{k/2} - 1)^2/\tau(n),$$

with equality if and only if n is a prime or the square of a prime. Since $\tau(n) \leqq n$ with equality only when $n=2$, it follows that

$$\sigma_k(n)/\tau(n) \geqq n^{k/2} + (n^{k/2} - 1)^2/n,$$

with equality if and only if $n=2$.

Generators and Relations for Cyclic Groups

5327 [1965, 915]. *Proposed by J. H. Conway, Gonville and Caius College, Cambridge, England*

Show that the group generated by five generators a, b, c, d, e, subject only to the relations $ab=c$, $bc=d$, $cd=e$, $de=a$, $ea=b$, is cyclic of order 11.

I. *Solution by J. A. Wenzel, University of Kansas.* All generators can be expressed as words in a and b. Thus $a=babab^2ab$ and $b=ab^2aba=ab^2ab^2abab^2ab$. From the last equation we get the identity $i=ab^2(ab^2aba)b^2a=ab^5a$. Now $a^2=babab^2aba=bab\cdot b$ and $a=ai=a^2b(bab^2)a^2(bab^2)a=a^2ba^7$. Thus $i=aba^7=a^2ba^6$ and $ab=ba$. Therefore $b=a^5b^8=a^3b^3$ and $i=a^2b^5$. Hence $a^2b^5=a^3b^2$ and $a=b^3$. From this we have $i=b^{11}$.

II. *Solution by R. C. Lyndon, University of Michigan.* Eliminating successively $c=ab$, $d=bab$, and $e=ab^2ab$, we find that the group, G_5, is generated by a and b, with two defining relations, $babab^2ab=a$ and $ab^2aba=b$. We find that $a^2=(babab^2ab)a=bab(ab^2aba)=bab^2$, and hence that $a^5=(bab^2)a(bab^2)=b(ab^2aba)b^2=b^4$. Therefore the element $a^5=b^4$ is central. In view of the automorphism permuting a, b, c, d, and e cyclically, the element $b^5=c^4$ is central. Since b is central, G_5, generated by a and b, is abelian, and it is routine to calculate that its order is 11.

Let G_n denote the corresponding group for general $n=1, 2, \cdots$, and $A_n=G_n/G_n'$. As a module over the integers, with basis $\langle a, b\rangle$, A_n is defined by the relation matrix M^n-I, where $M=\left(\begin{smallmatrix}0 & 1\\ 1 & 1\end{smallmatrix}\right)$. M has eigenvalues λ_1, $\lambda_2=(1\pm\sqrt{5})/2$, whence M^n-I has eigenvalues λ_i^n-1, and A_n has order

$$a_n = |\det(M^n - 1)| = |(\lambda_1^n - 1)(\lambda_2^n - 1)| = |(-1)^n - (\lambda_1^n + \lambda_2^n) + 1|.$$

If $\{f_n\}$ is the Fibonacci sequence 1, 3, 4, 7, 11, 18, $\cdots$, then $a_n = f_n - 1 - (-1)^n$, and $\{a_n\} = 1, 1, 4, 5, 11, 16, \cdots$. We note that, although G_1, G_2, G_4, and G_5 are finite cyclic groups, G_3 is not abelian.

III. *Comment by Harley Flanders, University College, London.* For $n=3$, we have G is $\langle a, b, c\rangle$ with $ab=c$, $bc=a$, $ca=b$, and so G is the (non-abelian) Hamiltonian group of order 8 consisting of the quaternions ± 1, $\pm i$, $\pm j$, $\pm k$. With six generators, one finds, after some computation, that $G=\langle a, b\rangle$ with $a^2ba^2=b$, $b^2ab^2=a$. The homomorphism

$$a \to \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad b \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

maps G onto an infinite subgroup of the linear group, hence G is infinite in this case.

Also solved by Shair Ahmad, Gerd Baron (Austria), D. A. Blaeuer, J. L. Brennan, Warren Brisley (Australia), R. T. Bumby, C. M. Campbell (Scotland), H. S. M. Coxeter, M. S. Demos, J. F. Dillon, R. R. Erskine, E. W. Ewing, Steven Feigelstock, N. J. Fine, A. W. Fuller (England), M. G. Greening (Australia), E. F. Haeussler, Jr., D. A. Hejhal, W. E. Hoff, J. M. Hood, L. N. Howard, Wilfried Imrich (Austria), Roman Kaluzniacki, Tetsuzo Kimura (Japan), Kenneth Kramer, R. C. Lyness (England), J. G. Mauldon (England), H. M. MacNeille, N. S. Mendelsohn, William Moser, Moshe Mole lekoa (Basutoland), D. K. Pease, Donald Quiring, Ramjilal (India), P. S. Schnare, Donna Seaman, L. W. Shapiro, D. L. Silverman, A. Sinkov, M. F. Smiley, S. A. Smith & D. A. Blaeuer, D. A. Smith, Al Somayajulu, R. G. Stanton, Preston Stein, C. Van de Vyle (Belgium), M. J. Wicks (England), R. S. C. Wong & J. S. W. Wong, John Zolnowsky, and the proposer.

Editorial Notes. (1) In connection with II above, Lyness poses the question as to what additions to the relations $ab=c$, $bc=d$, $\cdots$, $hi=a$, $ia=b$ are necessary in order to generate the cyclic group of order 19 and notes that commutativity is inadequate.

R. T. Bumby observes that the matrix M^n-I actually contains a full description of G_n/G_n'. The theory of elementary divisors yields the fact that $G_n/G_n' \cong Z_a \oplus Z_b$ where a=g.c.d. (entries of M^n-I), $a \mid b$ and $ab=\det(M^n-I)$. A summary of relevant results is:

$$\begin{aligned} G_n/G_n &\cong Z_k \oplus Z_{5k}, && n \equiv 0 \pmod 4 \\ &\cong Z_k \oplus Z_k, && n \equiv 0 \pmod 2 \\ &\cong Z_2 \oplus Z_{2k}, && n \equiv 3 \pmod 6 \\ &\text{is cyclic,} && (n, 6) = 1. \end{aligned}$$

(2) Proofs that G_6 is infinite were provided also by Coxeter and Mendelsohn. Mendelsohn has proved G_7 to be infinite and conjectures, as do Lyndon and Moser, that G_n is infinite and non-abelian for $n \geqq 6$. The papers: (i) N. S. Mendelsohn, *An algorithmic solution for a word problem in group theory*, Can. J. Math., 14 (1964) 509–516 and 15 (1965) 505; (ii) C. T. Benson and N. S. Mendelsohn, *A calculus for a certain class of word problems in groups*, Univ. of Manitoba preprint—give an algorithmic procedure for solving all problems of the given type.

The alternative procedure is to use the Todd-Coxeter enumeration method, counting the cosets of the subgroup generated by b and finding there is only one coset—a method used by Campbell in his solution for $n=5$. In this connection see J. A. Todd and H. S. M. Coxeter, *A practical method for enumerating cosets of a finite central abstract group*, Proc. Edin. Math. Soc., (2), 5 (1936) 34–36; H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*

(Springer, 1964), (see page 8 of this reference for a discussion of G_3); C. M. Campbell, 1965 Dissertation, McGill University.

(3) In Wicks solution ($n=5$), no use is made of inverses or of cancellation laws, and he observes that the semigroup with the given generators and relations is cyclic and of order 11.

Convexity of Function Spaces

5350 [1966, 87]. *Proposed by S. A. Naimpally, Iowa State University*

Menger (Math. Ann., 100 (1928) 75–163) defined a convex metric space (X, d) to be one for which, corresponding to each pair of distinct points $p, q \in X$, there exists a point $r \in X$ different from p, q such that $d(p, q) = d(p, r) + d(r, q)$, and showed that if X is complete then p, q are joined by a metric segment $\overline{pq}$, i.e. by a subset of X which is isometric to a segment of the real line of length $d(p, q)$. Further, if this metric segment is unique we call X a *strongly convex* space.

If (X, d) is a strongly convex complete metric space with a bounded metric d, and F is the function space of all continuous self-mappings of X with metric ρ, where for $f, g \in F$, $\rho(f, g) = \sup_{p \in X} d(f(p), g(p))$, then show that F is convex. Is F convex when X is merely convex?

Solution by M. D. Mavinkurve, Siddharth College, Bombay, India. We assume compactness of the space (X, d). This, however, does not affect the counter-example which follows. Let f, g be in (F, ρ). Define h on X so that $h(x)$ is the point which bisects the unique metric segment $\overline{f(x)g(x)}$. h is continuous. For let $x_n \to x$ so that $f(x_n) \to f(x)$, $g(x_n) \to g(x)$. We write $a_n, b_n, c_n, a, \cdots$ for $f(x_n)$, $g(x_n)$, $h(x_n)$, $f(x)$, $\cdots$. Since distance is a continuous function of both its variables and f, g are continuous, we have $d(a_n, b_n) \to d(a, b)$. Therefore

$$d(a_n, c_n) = \tfrac{1}{2} d(a_n, b_n) \to \tfrac{1}{2} d(a, b) = d(a, c).$$

Since $|d(a, c_n) - d(a_n, c_n)| \leqq d(a, a_n) \to 0$, we have also $d(a, c_n) \to d(a, c)$, and similarly $d(b, c_n) \to d(b, c)$.

Let p be a cluster point of the sequence $\{c_n\}$ whose existence is assumed. Let a subsequence $\{c_{n_k}\}$ converge to p, so that $d(a, c_{n_k}) \to d(a, p)$. It follows that $d(a, p) = d(a, c)$. In the same way, $d(b, p) = d(b, c)$ and $d(a, p) + d(p, b) = d(a, b)$. By the uniqueness of the segment $\overline{ab}$, $p = c$. Therefore the sequence $\{c_n\}$ in the compact space X has a unique cluster point c, and it follows that $c_n \to c$.

That the hypothesis of strong convexity is essential is shown by the following example. Let X be the unit circle in the plane with the distance d measured along the minor arc. (X, d) is a complete, convex metric space but not strongly convex due to the presence of diametrically opposite points. Let f be the constant mapping of the circle on to a point a of the circle and let g be the continuous mapping which takes each point into the diametrically opposite point. Then $\rho(f, g) = \pi$. The convexity of F implies existence of continuous h such that $\rho(f, h) = \alpha$, $\rho(h, g) = \pi - \alpha$ where $0 < \alpha < \pi$. The image under h is a closed arc contained in a closed arc of length 2α with center at a. The restriction of h to this image is like a continuous map of a closed segment of the reals into itself and has

therefore a fixed point. It follows that $\rho(h, g)=\pi$ and $\rho(f, h)+\rho(h, g)=\pi+\alpha$, a contradiction.

Also solved by R. O. Davies (England), Erhard Heil (Germany), Donald Quiring, and the proposer.

All of the solutions for the first part above made use of compactness or local compactness of (X, d).

The Ring of Power Series

5351 [1966, 87]. *Proposed by R. G. Buschman, State University of New York at Buffalo*

Louis Brand (this MONTHLY, 71 (1964) 719–728) discusses a convolution ring S for sequences of complex numbers and its extension into a division algebra. It is not difficult to see that for a sequence $A \in S$ with $a_0=0$, the equation $X^2=A$ has no solution.

(i) Show that $X^2=A$ has a solution in S if and only if the index of the first nonzero element of the sequence $A \neq 0$ is even.

(ii) Hence, for the Fibonacci sequence, $Y^2=\{f_n\}$ has no solution, but $X^2=\{f_{n+1}\}$ has a solution. Extract the square roots of this "left-shifted" Fibonacci sequence.

(iii) Indeed, show that $\{i^{-n}P_n(i/2)\}^2=\{f_{n+1}\}$, where the P_n are the Legendre polynomials.

Solution by Harsh Pittie, Swarthmore College. The ring in question, S, is just the ring of formal power series around zero over the complex field. Now, if A is any such power series, with nonzero constant term, then A has a square root in the ring. For, let $A=\sum_0^\infty a_n z^n$. Define $B=\sum_0^\infty b_n z^n$ by $b_0=\sqrt{a_0}$ (take either of the two roots), $b_1=a_1/2b_0$, and recursively,

$$b_n = \left(a_n - \sum_0^{n-1} b_{n-k}b_k\right) \Big/ 2b_0.$$

It is easy to check that $B^2=A$. Suppose that P is any such power series with first nonzero term $p_n z^n$. Then

$$P = z^n(p_n + p_{n+1}z + \cdots) = z^n Q^2$$

by the above. We recall from elementary analysis that z^n has a square root in the ring if and only if n is even; hence P does if and only if n is even.

The left-shifted Fibonacci sequence ($f_{n+2}=f_{n+1}+f_n$, $f_1=1, f_2=2$) is generated by

$$F(z) = (-z^2 - z + 1)^{-1} \equiv \sum_0^\infty f_{n+1}z^n.$$

Then $\sqrt{F(z)}$ also has a Taylor expansion around zero. Letting $z=iw$, we see that

$$\sqrt{F(z)} = (w^2 - 2(i/2)\, w + 1)^{-1/2} = \sum_{n=0}^{\infty} P_n(\tfrac{1}{2})w^n = \sum_{n=0}^{\infty} (i^{-n}P_n(i/2))z^n.$$

Also solved by L. Carlitz, D. A. Hejhal, Douglas Moore & Sidney Spital, B. L. D. Thorpe, and the proposer.

Thorpe proves that $X^n = A$ has a solution in S if and only if the index of the first nonzero element of the sequence A is divisible by n.

Subdirect Sums of Ore Domains

5352 [1966, 88]. *Proposed by Kwangil Koh, North Carolina State University*

A ring is called duo if every one-sided ideal of the ring is a two-sided ideal. If R is a ring let R^{Δ} be the nonsingular ideal of R. Prove that for a duo ring R the following conditions are equivalent:

(i) $R^{\Delta} = (0)$.

(ii) R is isomorphic to a subdirect sum of Ore domains. (A ring is said to be an Ore domain if it is an integral domain and any pair of nonzero right ideals and left ideals has a nonzero intersection.)

(iii) R is isomorphic to a subring of a direct sum of division rings.

Solution by F. L. Sandomierski and V. C. Cateforis, University of Wisconsin. The following simple lemmas will be used:

LEMMA. 1. *The epimorphic image of a duo ring is a duo ring.*

Let R, S be rings, R a duo ring, f a ring homomorphism of R onto S. If J is a right (left) ideal of S, then $f^{-1}(J)$ is a two-sided ideal of R and hence so is $J = f(f^{-1}(J))$ of S.

LEMMA 2. *An integral domain R, which is a duo ring, is an Ore domain.*

Let I and J be left (right), hence two-sided, nonzero ideals of R. Then $0 \neq IJ \subseteq I \cap J$.

LEMMA 3. *If a ring R is prime and duo, then R is an Ore domain. (A ring is prime iff $AB = (0)$ for ideals A and B of R, implies $A = (0)$ or $B = (0)$.*

If $x \in R$ set

$$(x)_R = \{xr + xm \mid r \in R,\ m \text{ an integer}\},$$

$$R^{(x)} = \{tx + nx \mid t \in R,\ n \text{ an integer}\},$$

both two-sided ideals of R. Now let x and y be nonzero elements of R. If $xy = 0$ then $R^{(x)(y)}R = (0)$ and this implies $x = 0$ or $y = 0$ since R is prime. Hence $xy \neq 0$, and R is an integral domain. By Lemma 2, R is an Ore domain.

Proof of the problem. (i) implies (ii). We first show that R contains no nilpotent ideals $\neq (0)$: suppose $I^2 = (0)$ with I an ideal in R. Since $R^{\Delta} = (0)$, I is not large in R. A simple application of Zorn's lemma shows now that there exists an ideal J in R maximal with respect to the property that $I \cap J = (0)$ and further $I + J$ is large in R. Now $(I+J)I \subseteq I^2 + JI = JI \subseteq I \cap J = (0)$. Since $I + J$ is large in R and $R^{\Delta} = (0)$, it follows that $I = (0)$. We now use Theorems 4.25 and 4.27

(pp. 71–72) of [1] to conclude that R is isomorphic to a subdirect sum of prime rings. (ii) now follows from Lemmas 1 and 3.

(ii) implies (iii). By (ii) R is isomorphic to a subring of the (complete) direct sum of a family $\{S_\alpha | \alpha \in A\}$ of Ore domains. It is well known, e.g. [2], that the quotient ring (classical) of an Ore domain is a division ring. Thus if Q_α is the quotient ring of S_α for each α then R is isomorphic to a subring of the direct sum of $\{Q_\alpha | \alpha \in A\}$ and we have (iii).

(iii) implies (i). Suppose R is (isomorphic to) a subring of the direct sum of a family $\{Q_\alpha | \alpha \in A\}$ of division rings. Let I be a large ideal of R and x an element of R such that $xI = (0)$. If π_α for each α is the usual projection of the direct sum onto the component Q_α then $\pi_\alpha(I) = (0)$ for each α for which $\pi_\alpha(x) \neq 0$, since multiplication in the direct sum is component-wise. Let $[x]$ denote the ideal of R generated by the element x. Then $\pi_\alpha([x]) = (0)$ for each α for which $\pi_\alpha(x) = 0$ and it follows that $I \cap [x] = (0)$. But I is large in R, so that $x = 0$ and thus $R^\Delta = (0)$.

1. Neal H. McCoy, *The Theory of Rings*, New York, 1964.
2. O. Ore, *Linear equations in non-commutative fields*, Ann. Math., 32 (1931) 463–477.

Also solved by Harsh Pittie, Surjeet Singh (India), and the proposer.

Singh and the proposer used the fact that the ring as characterized in the problem is also semi-prime.

Semi-simple Subdirectly Irreducible Duo Rings

5353 [1966, 88]. *Proposed by Kwangil Koh, North Carolina State University*

A subdirectly irreducible duo ring which is semi-simple is a division ring.

Solution by C. C. Lindner, Coker College, Hartsville, S. C. Let R be a ring satisfying the given conditions. In view of problem 5227 [1965, 918] we need only show that R is primitive. Since R is subdirectly irreducible, the intersection of the nonzero ideals of R is not $\{0\}$. Since R is semi-simple, the radical of R is

$$\cap\{(0: M) \mid M \text{ an irreducible module}\} = \{0\},$$

so that $(0: M) = 0$ for some M, for otherwise we would have an intersection of nonzero ideals equal to $\{0\}$. Hence M is a faithful irreducible R-module and so R is primitive.

Also solved by V. C. Cateforis, R. A. Howland, Harsh Pittie, Surjeet Singh (India), and the proposer.

Elements of Nonzero Order

5354 [1966, 88]. *Proposed by B. R. Toskey, Seattle University*

An exercise in Fuchs, *Abelian Groups* (ex. 26, p. 36) is to prove that if G is an R-module, where R is a (commutative) domain of integrity, then the elements of nonzero order form a submodule. Is the statement true if R is a noncommutative integral domain?

Solution by Kwangil Koh, North Carolina State University. There exists an integral domain R and R-module G such that the elements of nonzero order of G do not form a submodule. Let F be a field and $R=F[x, y]$, the ring of polynomials over F in two indeterminates x and y such that $xy-yx\neq 0$. Then R is an integral domain and $xR\cap yR=(0)$. Hence by problem 5059 [1962, 1012] there must exist an R-module G whose elements of nonzero order do not form a submodule.

Also solved by G. A. Heuer, Harsh Pittie, and the proposer.

Elements in the Commutator Subgroup of a Group

5355 [1966, 88]. *Proposed by P. M. Weichsel, University of Illinois*

Let G be a group. Show that u is an element of the commutator subgroup of G if and only if there exist elements $g_1, \cdots, g_k$ in G such that $u=g_1g_2\cdots g_k$ and $1=g_kg_{k-1}\cdots g_1$.

I. *Solution by J. V. Ralston, Jr., Stanford University.* Let C be the commutator subgroup of G, and H be the set of all elements u of G which have the given form. C is generated by the commutators $xyx^{-1}y^{-1}$ of G, and H is obviously a subgroup of G. Hence, to show $C\subseteq H$, it suffices to show that a given commutator can be written as an element u of H. In fact, given $xyx^{-1}y^{-1}$, let $g_1=x$, $g_2=yx^{-1}$, $g_3=y^{-1}$. Then $u=g_1g_2g_3$, and $g_3g_2g_1=1$, and so forth.

Conversely, let $u\in H$. Then $u=g_1\cdots g_k\cdot g_1^{-1}\cdots g_k^{-1}$ since $g_1^{-1}\cdots g_k^{-1}=(g_1\cdots g_k)^{-1}=1$. Let ψ be the canonical homomorphism from G onto G/C. Then

$$\psi(u)=\psi(g_1)\cdots\psi(g_k)[\psi(g_1)]^{-1}\cdots[\psi(g_k)]^{-1}$$

which is the identity of G/C since G/C is Abelian. Thus $u\in C$ because C is the kernel of ψ.

II. *Solution by E. A. Schreiner and Dalton Tarwater, Western Michigan University.* Any commutator $aba^{-1}b^{-1}=a(ba^{-1})b^{-1}$ is of the given form, as is any product of commutators.

Conversely, if $u=g_1\cdots g_k$ and $g_kg_{k-1}\cdots g_1=1$, then $u=u\cdot 1=g_1\cdots g_kg_1^{-1}\cdots g_k^{-1}$ is in the commutator subgroup. This is clear if $n=1$ or $n=2$. Otherwise

$$u=(g_1g_2g_1^{-1}g_2^{-1})(g_2g_1)g_3\cdots g_n(g_2g_1)^{-1}g_3^{-1}\cdots g_n^{-1}$$

and the proof follows by induction on n.

Also solved by D. R. Arterburn, Anders Bager (Denmark), F. S. Brenneman, Warren Brisley (Australia), R. T. Bumby, A. J. D'Aristotle, R. O. Davies (England), Stan Dick & Mike Orleck & Preston Stein, W. G. Dotson, Bruce Feldmeyer, N. Ganesan (India), J. D. Gillam, M. G. Greening (Australia), D. M. Hancasky, Geoffrey Kandall, Irving Katz, J. B. Kelly, C. P. Lawes, Arthur Lieberman, Steve Ligh, C. C. Lindner, M. D. Mavinkurve (India), Ka Menehune, D. E. Myers, Stanton Philipp & Eugene Machson, Harsh Pittie, S. N. Rao (India), D. A. Robinson, Azriel Rosenfeld, P. S. Schnare, Surjet Singh (India), Indranand Sinha (India), J. F. Standish, Art Steger, R. J. Tondra, Kenneth Yanosko, Peter Yff (Lebanon), and the proposer.

"Positive" Subsets of a Real Field

5356 [1966, 88]. *Proposed by K. E. Whipple, Auburn University, Alabama*

Suppose that F is a subfield of the real numbers and P is a subset of F having the following properties:

(1) P is closed with respect to addition and multiplication.

(2) If x belongs to F, then only one of the following is true, (i) $x=0$, (ii) $x\in P$, (iii) $-x\in P$.

Must P be the set of positive numbers in F?

I. *Solution by R. B. Hardin, Jr., Wayne State University.* This is essentially example 2, p. 14, in Gelbaum and Olmstead, *Counterexamples in Analysis*, where the negative answer is obtained by observing that $P=\{a+b\sqrt{2}\,|\,a,\ b \text{ rational},\ a-b\sqrt{2}>0\}$ satisfies the requirements and $-\sqrt{2}\in P$.

II. *Solution by C. C. Lindner, Coker College, Hartsville, S. C.* The problem can be restated as follows: are there subfields of the real numbers having more than one ordering? The answer to this form of the question is yes, since any subfield of the form $R(\theta)$, where R is the field of rationals and θ any transcendental real number, has a noncountable number of distinct orderings. (See N. Jacobson, *Lectures in Abstract Algebra*, vol. III, p. 272.)

Also solved by Anders Bager (Denmark), R. D. Berlin, R. O. Davies (England), W. G. Dotson, Jr., Roy Dubisch, Charles Green, D. A. Hejhal, G. A. Heuer, J. E. Homer, Jr., R. J. Hursey, Jr., Sheila M. Kaye, Israel Kleiner, H. E. Lahmann (Germany), C. W. Leininger, Arthur Lieberman, T. H. Massengale, J. G. Mauldon (England), L. F. Meyers, A. R. Moore, Vishnu Parkash (India), Howard Schwartz, W. R. Scott, D. L. Silverman, Surjeet Singh (India), L. A. Stein, Preston Stein, J. E. Wilkins, Jr., and the proposer.

Editorial Note. Davis comments that the answer to the problem is in the affirmative if F is assumed to contain the square roots of all positive elements. See, e.g., van der Waerden, *Modern Algebra*, vol. 1, 2nd ed., pp. 225 ff (cited by Hursey). See also Dubisch, *Introduction to Abstract Algebra*, ex. 5, pp. 46–47.

A Sum of Multiplicative Elements

5357 [1966, 88]. *Proposed by David Rearick, University of Colorado*

If n is a positive integer, let $f(n)$ denote the number of ways of expressing n as a product of two relatively prime positive integers, disregarding order. Let $\nu(n)=(-1)^{r(n)}$ where $r(n)$ is the number of distinct prime divisors of n. Show that

$$\sum_{n=1}^{\infty}\frac{\nu(n)f^3(n)}{n^2}=\frac{7}{8}.$$

Solution by Addison Fischer, Student, Washington Irving High School, Clarksburg, W. Va. If $n=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$, then $f(n)=2^{k-1}=2^{r(n)-1}$ (this can be shown in many ways, e.g. by induction on k); we set $f(1)=1$. Hence,

$$\sum_{n=1}^{\infty} \frac{\nu(n)f^3(n)}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{(-1)^{r(n)}2^{3r(n)-3}}{n^2} = \frac{7}{8} + \frac{1}{8}\sum_{n=1}^{\infty} \frac{(-8)^{r(n)}}{n^2}.$$

Since $r(n)$ is an additive function, $(-8)^{r(n)}$ is a multiplicative function, thereby enabling us to transform this sum into a product (over all the primes):

$$= \frac{7}{8} + \frac{1}{8}\prod_p \sum_{k=0}^{\infty} \frac{(-8)^{r(p^k)}}{p^{2k}} = \frac{7}{8} + \frac{1}{8}\prod_p \left(1 - \frac{8}{p^2-1}\right) = \frac{7}{8},$$

since the term with $p=3$ vanishes and hence so does the product. This representation is valid since the product for $p>3$ is absolutely convergent.

Also solved by Anders Bager (Denmark), Robert Breusch, L. Carlitz, P. Gobel (Netherlands), Emil Grosswald, J. B. Kelly, Dan Marcus, Břetislav Novák (Czeckoslovakia), E. W. Trost (Switzerland), and the proposer.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Correspondence about Reviews will be welcome.

Statement of policy: *Since it is quite impossible to review in detail all the publications of interest to readers of the Monthly, and since full reviews are often delayed until their news value is greatly reduced, we intend in this department to print prompt "telegraphic reviews" of all materials within our field of coverage, which is collegiate mathematics from the junior through the second graduate year. (The first two collegiate years are now covered by the Mathematics Magazine, and reviews of materials at the research level may be found in many journals.) Selected materials will be reviewed again more extensively.*

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (Teacher Training), L (Library purchase), 15 (junior level)—18 (second graduate year).

Textbooks for beginning abstract algebra.

Elements of Abstract Algebra. By Richard A. Dean. Wiley, New York, 1966. xiv+324 pp. $7.95. Group oriented introduction, including rings, domains, fields, vector spaces, field extensions, Galois theory. Reviewed by Frank W. Anderson in *Science*, 153, 9 Sept., 1966, p. 1233. T (14–15).

University Algebra. By Richard E. Johnson. Prentice-Hall, Englewood Cliffs, N. J., 1966. xii+271 pp. $7.95. " . . . define abstract concepts in as simple a setting as possible, and then carry them over to more complex systems." Reviewed by F. W. Anderson in *Science*, 153, 9 Sept., 1966, p. 1233. T (14–15).

Modern University Algebra. By Marvin Marcus and Henryk Minc. Macmillan, New York, 1966. xii+244 pp. $6.95. Induction, sets, functions, cardinality, combinatorial analysis, convexity, rings for Freshmen! T (13).

Higher Algebra. By A. P. Mishina and I. V. Proskuryakov. Translated from the Russian by Ann Swinfen. Pergamon, Oxford, London, 1965. xii+277 pp. $9.50. Determinants, linear systems, matrices, bilinear and quadratic forms, polynomials, groups, rings, fields, lattices. T (13–14), S.

A First Course in Abstract Algebra. By Hiram Paley and Paul M. Weichsel. Holt, Rinehart and Winston, New York, 1966. xiii+334 pp. $8.95. Sets, number theory, groups, rings. Reviewed by F. W. Anderson in *Science*, 153, 9 Sept., 1966. p. 1233. T (15–16).

Elementary Abstract Algebra. By E. M. Patterson and D. E. Rutherford. Wiley, New York, 1965. viii+211 pp. $3.50. " . . . for those who are meeting abstract algebra for the first time." Binary operations, groups, rings, integral domains, fields, polynomial and Euclidean rings, vector spaces. T (13–15).

Introduction to Algebra. By Sam Perlis. Blaisdell, Waltham, Mass., 1966. xx+440 pp. $9.50. Linear equations and matrices, groups, rings, integral domains, fields, divisibility, classical algebra (polynomial roots), vector spaces, extension fields, determinants, linear transformations, forms, length and orthogonality. T (15).

Texts and monographs in analysis.

Plateau's Problem. By Frederick J. Almgren, Jr., Benjamin, New York, 1966. xii+74 pp. $7.00. "The new language for the expression of geometric concepts provided by the measure theoretic surfaces called varifolds. . . . " Assumes advanced calculus. S (calculus of variations), P.

The Elements of Integration. By Robert G. Bartle. Wiley, New York, 1966. x+129 pp. $6.95. " . . . to present a reader having only a modest mathematical background the chief results in the modern theory of integration which was initiated by Lebesgue in 1902." Advanced calculus assumed. T (16), S, P, L.

Studies in Non-Linear Stability Theory. By Wiktor Eckhaus. Springer Tracts in Natural Philosophy, Vol. VI. Springer-Verlag, New York, 1965. viii+117 pp. $5.50. P.

Introduction to the Theory of Differential Equations with Deviating Arguments. By L. E. El'sgol'ts. Translated by Robert J. McLaughlin. Holden-Day, San Francisco, 1966. ii+109 pp. $6.95. " . . . differential equation in which the unknown appears with various values of the argument." P.

An Introduction to Complex Analysis in Several Variables. By Lars Hormander. D. Van Nostrand, Princeton, N. J., 1966. x+224 pp. $6.95. "entirely from the viewpoint of partial differential equations." T (18), P.

Calculus of Finite Differences. 3rd ed. By Charles Jordan, Chelsea, New York, 1966. xxi+654 pp. $7.50. P, L.

A Short Course in Automorphic Functions. By Joseph Lehner. Holt, New York, 1966. vii+144 pp. $5.00. Assumes first courses in complex analysis, topology, abstract algebra, discontinuous groups, Fourier series, Riemann surfaces. T (17–18), S, P.

Harmonic Analysis. 1965 MAA Cooperative Summer Seminar. By Lynn H. Loomis, Notes by Ethan Bolker. Mathematical Assoc. of America, Buffalo, 1966. 350 pp. (paper) $3.00. "The principal theme . . . is . . . the heavy machinery required to

manipulate the Fourier Transform. . . . The nominal aim . . . to prove in a modern way a few theorems about linear partial differential equations." Norms, harmonic analysis, harmonic analysis on R^n, measure theory, analytic vector-valued functions, Haar measure, exercises and examples. P.

Unbounded Linear Operators. By Seymour Goldberg. McGraw-Hill, New York, 1966. viii+199 pp, $10.50. Systematic treatment of u.l.o. in normed linear spaces, applications to differential equations. Assumes metric spaces and real variable theory. S (functional analysis), P.

Elements of the Theory of Functions. By R. S. Guter, L. D. Kudryavtsev and B. M. Levitan. Translated by H. F. Cleaves. Translation edited by I. N. Sneddon. Pergamon Press, Oxford, London, 1966. xii+219 pp. $8.50. Basic ideas and theorems of functions of a real variable, interpolation and approximation, almost periodic functions. " . . . for specialists in sciences in which mathematics plays a prominent role and also for mathematicians who are not acquainted with the theory of functions." T (18), S, P, L.

Some Topics in Complex Analysis. By E. G. Phillips, Pergamon, Oxford, London, 1966. vii+141 pp. $5.50. Elliptic functions, Jacobian elliptic functions, conformal transformation, Schlicht functions, maximum modulus principle, integral function, expansion, special functions defined by contour integrals. S (complex variable).

An Introduction to Fourier Series and Integrals. By Robert Seeley. Edited by Robert Gunning and Hugo Rossi. Benjamin, New York, 1966. x+104 pp. $7.00. " . . . brings the reader to Lebesgue's door (with a glimpse inside). . . . " Brief but interesting historical comments in the editors' foreword and in the author's preface. T (for part of a course 14–16), S, P, L.

Asymptotic Expansions for Ordinary Differential Equations. By Wolfgang Wasow, Interscience, New York, 1965. ix+362 pp. $14.00. "emphasis on mathematical ideas underlying the various methods." T (17–18), P.

Books on statistics.

The Statistical Analysis of Series of Events. By D. R. Cox & P. A. W. Lewis. Methuen, London, 1966. vii+285 pp. $7.75. Poisson, stationary and renewal processes, superposition, comparison of rates. Bibliography. S (16–17), P.

Estimation Theory. By Ralph Deutsch. Prentice-Hall, Englewood Cliffs, N. J., 1966. xiv+269 pp. $10.95. Classical, modern techniques. Assumes elements of probability, statistics, noise theory. Heuristic, formal, nonrigorous. T, S, P.

Concepts and Methods of Experimental Statistics. By Holly C. Fryer. Allyn and Bacon, Boston. 1966. xiii+602 pp. $12.50.

The Analysis of Variance. By A. Huitson. Hafner, New York, 1966. viii+83 pp. (paper) $3.00. S (15–16), P.

Bernoulli, Bayes, Laplace, Anniversary Volume: Proceedings of an international research seminar, Statistical laboratory, University of California, Berkeley, 1963. Edited by Jerzy Neyman and Lucien M. LeCam. Springer-Verlag, New York, 1965. ix+262 pp. $9.00. P, L.

Statistical Methods. By Stefan Szulc. Translated by J. Stadler. Pergamon Press, L. I. City, New York, 1965. viii+666 pp. $17.50.

Applications in the biological and social sciences.

Mathematics and Politics. By Hayward R. Alker Jr. Macmillan, New York, 1965. 152 pp. $1.50. Not just a "how-to-do-it" book on statistics for political scientists, but an attempt to show how several important ideas in the history of mathematics may increase our understanding of politics. This involves a study of the historical inter-relationship of the two disciplines. S, P.

Basic Statistics with Business Applications. By Richard C. Clelland, John S. deCani, Francis E. Brown, J. Parker Bursk and Donald S. Murray. Wiley, New York, 1966. xvi+708 pp. $11.50. Inspired by recent "drastic revision" of the business education course at the University of Pennsylvania to include a one-year freshman course giving some technical proficiency in calculus and linear algebra and familiarity with simple set-theoretic notions. T (14, following a Freshman course in calculus and elementary linear algebra), S.

Applied Regression Analysis. By Norman Draper and Harry Smith. Wiley, New York, 1966. ix+407 pp. $11.75. Emphasis on "practical application" rather than mathematics. T (16), S, P.

Readings in Mathematical Social Science. Edited by Paul F. Lazarsfeld and Neil W. Henry. S.R.A., Chicago, 1966. 371 pp. $10.50. Papers selected to give a "broad introductory view of the new applications of mathematics to the social and behavioral sciences." P.

The Mathematics of Life Insurance, 2nd ed. By Walter O. Menge and Carl H. Fisher. Macmillan, New York, 1965. vi+189 pp. $6.95. Extensively revised edition of text by Menge and Glover, first published in 1935. T, S, P.

The Use of Mathematics in Economics. Edited by V. S. Nemchinov. M.I.T. Press, Cambridge, Mass., 1965. xxxi+377 pp. $12.50. Contributions by six Soviet economists and mathematicians who use mathematical methods in analyzing and planning a socialist economy. P.

Random Processes and the Growth of Firms, a Study of the Pareto Law. By Josef Steindl. Hafner, New York, 1965. 249 pp. $10.00. P.

Mathematics in the Social Sciences and Other Essays. By Richard Stone. M.I.T. Press, Mass., 1966. xiii+291 pp. $12.00. Reprints of essays 1960–1965, "all illustrations of my belief . . . that mathematical models are necessary both at the theoretical . . . and the applied level." Most examples from economics. Bibliography. S, P.

The Compleat Strategyst, rev. ed. By J. D. Williams, McGraw-Hill, New York, 1966. xvi+268 pp. $6.95. The revised edition of this widely known book, first published in 1954, contains a new chapter on "the pivot method" and a general method for solving matrix games. S, P.

The Geometry of Incidence. By Harold L. Dorwart. Prentice-Hall, Englewood Cliffs N. J., 1966. xvii+156 pp. $6.25.

This is a delightful little book written for the purpose of reviving an interest in projective geometry. Some acquaintance with analytic geometry will provide adequate background for reading it with profit. It is not a textbook, as it contains no exercises, but it should find its way to reading lists for those who have an interest in mathematics but have not taken a formal course in projective geometry.

After a carefully motivated introduction of ideal elements in the plane, the author discusses line curves and defines homogeneous point and line coordinates. He then presents an analytic formulation of the real projective plane and discusses two models of that plane and the principle of plane duality. The definition of a plane configuration leads to the study of three famous theorems of projective geometry, those of Pappus, Desargues, and Pascal. Analytic proofs are given, and applications are mentioned.

The final chapter contains a brief introduction to the general projective plane and to finite projective planes. The relationship of Latin squares and perfect difference sets to finite planes is indicated as an area of present day research.

The definition given for a perfect difference set is not the usual one, and, in fact, it is not satisfied by any of the sets listed except 1, 2, 4. There are relatively few typographical errors.

The book is written in an entertaining style, somewhat informal, but dignified. Appropriate historical material enlivens the exposition. Extensive notes and an excellent bibliography guide the interested reader to further and deeper study of many of the topics considered.

JANET MCDONALD, Vassar College

Algebra. By Serge Lang. Addison-Wesley, Reading, Mass., 1965. 508 pp. $12.75.

Columbia's Professor Lang, in this first year graduate text, manages to achieve an admirable and needed blend of several seemingly divergent facets of algebra. The book roughly follows the "classical" Artin-van der Waerden order, but the inclusion of newer results, especially in the chapter on representations of finite groups, greatly extends and amplifies the presentation. Lang spices the book liberally with homological notation. Fortunately, the use of diagrams is not forced, and lengthy diagram chasing is avoided. The early introduction of categories and functors provides a good basis for the extension of theorems to various topics. As with most modern works, the influence of Bourbaki is also strongly felt. The word "warning" even replaces Bourbaki's "Z." Lang has done a presentable job of editing from Bourbaki's massive publications on linear algebra and representations. Many of the chapters are independent and the use of key repetitions allows the instructor a good deal of flexibility.

The book, however, may be a little too sophisticated for the audience for which it is intended. Perhaps in the interest of being somewhat encyclopedic in a limited space, long lists of definitions are presented with few relevant examples or applications. Proofs are perhaps too abbreviated and often might be better motivated. A less parsimonious use of commas might make the text more readable. Nevertheless, all these peccadillos are amply offset by Professor Lang in a timely, complete, and illuminating work.

A. G. BRANDSTEIN, University of Connecticut

Lie Groups for Physicists. By R. Hermann. Benjamin, New York, 1966. vii+193 pp. $12.50 (domestic).

Homogeneous spaces of Lie groups, especially symmetric spaces, and their application to physics, is the main theme of this book which is intended for " . . . physicists interested in the possible application of Lie group theory to elementary-particle theory. . . . " It is *not* a book for beginners.

The first quarter of the book is devoted to reviewing and illustrating some of the basic ideas and facts in Lie group theory. Among the group theoretic concepts treated in the remaining portion of the book one finds: homogeneous vector bundles, induced representations, the universal enveloping algebra, and limits and contractions of Lie groups and algebras. These notions provide a backdrop for viewing such topics as: Wigner's representations of the Poincaré (i.e. the inhomogeneous Lorentz) group, the Klein-Gordon equation, the Dirac equation, and some of the classical asymptotic formulas for the special functions in mathematical physics. Additional topics treated

from a group theoretic point of view include the hydrogen atom and strong coupling theory, perturbation theory, gauge groups and supermultiplet theory, and Gell-Mann's formula.

Mathematical ideas are clearly presented, most of the proofs are omitted or sketched and a good bibliography is included. Typographical and other editorial-type errors occur on an average of one per every three and one-half pages, and the proof that every derivation of a semi-simple Lie algebra is inner (Lemma 4–5, p. 17) is in error.

D. J. STERLING, Bowdoin College

Structural Models: An Introduction to the Theory of Directed Graphs. By Frank Harary, Robert Z. Norman, and Dorwin Cartwright. Wiley, New York, 1965. viii+415 pages. $9.95.

The authorities who award prizes for attractive books should not miss this one. The authors have instilled a beguiling quality which invites browsing. The writing is readable, conversational even. Social scientists, for whom the book is principally intended, will find easy access to several basic and readily applicable sectors of digraph theory. The typography, design, and diagrams contribute admirably to the book's appeal and usefulness. Each chapter is launched by an appropriate, often waggish, quotation. How better to introduce "Limited Reachability" than by "Ill can he rule the great, that cannot reach the small."

Key concepts in this treatise are directional duality, connectedness, digraphs and boolean operations for matrices, condensations of digraphs, limited reachability, removal of lines and of points, blocks, acyclic digraphs, tournaments, balance, and networks. Definitions are usually well motivated, and conclusions are usually illustrated by examples and applications. The organization of the theory is formal: eighty-odd "principal theorems" are stated in a convenient reference list. The examples are remarkably varied, including ordering by likes and dislikes, competition of commodities, communication and rumor networks, sociometric choices, coalitions, status in an organization, games, consistency in paired comparisons, cost geodetics in networks. Exercises are numerous and varied in difficulty.

An important book which is attractive, useful, and generally accurate may have a few serious deficiencies. The reviewer's principal criticism is that so very many technical terms are included. The glossary, which helps overcome the disadvantage of a large vocabulary, contains some one hundred eighty items. It is not surprising, then, that many theorems are in the form of a list of equivalent propositions. Multiplicity of concepts may well enhance the usefulness of the book, but at times, happily not often, even the sprightly style of the authors does not entirely conceal the tediousness of theorems which contain no surprises. A minor suggestion for revision: since the preface promises illustrations from scaling theory, propositional logic, and cognitive structure, those items should appear by name in the index.

The reviewer has used this book as a source for a seminar for non-majors and for an independent reading project for a major. It has been distinctly satisfactory in both capacities.

D. E. CHRISTIE, Bowdoin College

Vectors. By Parry Moon and Domina Eberle Spencer, D. Van Nostrand, Princeton, 1965. 352 pages. $8.95

This book gives an introduction to vector and tensor analysis, primarily for engineers and physicists. After several introductory chapters which define tensors, vector and tensor algebra is introduced, finally specializing to the dot and cross product of vectors in three-space. Covariant derivative, gradient, divergence and curl are studied in connection with vector-valued functions. Some differential geometry is given, including the

Frenet formulas, Green's theorem in Euclidean three space, and Stoke's theorem in two-space. Among other physical applications, there is a chapter devoted to retarded potentials. There is a list of problems at the end of each chapter.

The distinctive feature of this book occurs in the introductory chapter, where the terms *holor* and *merate* are introduced. A *holor* is defined as a "mathematical entity built up of several independent elements." A *merate* seems to be defined as one of the independent elements of the holor. From these puzzling definitions, the authors infer that a holor can be expressed by different arrays of numbers in different coordinate systems, and the idea of transformation from one coordinate system to another is introduced. Tensors are defined to be holors which transform properly, and from this point on, the book proceeds along quite conventional lines.

This book makes it clear that tensor is a term that requires a good mathematical definition. Non-mathematicians have frequently had great success with objects such as the Dirac δ-function, which were not described as mathematical objects. But the concept of tensor as described by physicists has always been somewhat mysterious, and this book does little to dispel the mystery. It is possible that the word holor is intended to mean the same thing as *quantity*, as used by J. A. Schouten (*Ricci Calculus*, 2nd. ed. Springer-Verlag, Berlin, 1954). If so, the reader will find a fairly concise definition of the term in that book.

Two additional criticisms are local in character, but may hint at a fundamental logical difficulty. On page 20 of *Vectors*, we find a fairly reasonable definition: "Two holors are equal if and only if each corresponding pair of merates is equal." But on page 120, we find "But the equality of holors implies merely their equality in one coordinate system, . . . ". On page 81 is found the phrase "an *infinitesimal* change in space coordinates," as contrasted to "a *finite* change in space coordinates." One wonders what meaning is intended by "infinitesimal change."

It is clear that vectors and tensors are extremely difficult to present, even to students who are fairly conversant with mathematics. It will require a strong mixture of mathematical ability, physical insight and instinct for teaching to produce an ideal textbook on vectors for a non-mathematical audience.

E. B. LEACH, Case Institute of Technology

Selected Papers of Norbert Wiener. By Norbert Wiener, M.I.T. Press, Boston, Mass., 1964. 453 pp. $12.50.

On November 26, 1964, Norbert Wiener would have been 70 years old, and it was intended that this volume appear as a celebration of his birthday. Unhappily, he died in March of that year.

This book is a collection of some of Wiener's outstanding contributions to the field of mathematics and these are chosen mainly from his work prior to 1932. In particular, his classic works on Generalized Harmonic Analysis and Tauberian Theorems take up more than half the volume.

In general, the work carried out in these papers contributed significantly not only to much of Wiener's subsequent work but also to much of modern mathematics.

We are indebted to Y. W. Lee, N. Levinson, and W. T. Martin for bringing together for us in one volume this collection of Wiener's work. In reading its pages, we are once again struck by the magnitude of his vision, interests and perception. Here we see fully displayed not only his deep interest in the applications of mathematics but also his consummate mastery of mathematics.

The volume contains a foreword by Martin, an introduction in depth by Levinson and an analysis by Lee of Wiener's contributions to linear theory and nonlinear theory in engineering. These are excellent and will help the reader set Wiener's work into its proper historical context.

H. H. GOLDSTINE, IBM

Finite Graphs and Networks, An introduction with applications. By R. G. Busacker and T. L. Saaty. McGraw-Hill, New York, 1965. 294 pp. $11.50.

The tremendous renaissance of graph theory and combinatorics stems from their wide applicability. The first book ever written on graph theory, *per se*, was the classic by König [1]. This was followed after two decades by three more modern books [2, 3, 4]. A book [5] with the identical title as the English translation of Berge [2] consists of a collection of papers presented in 1963 at a symposium held in Czechoslovakia. The next book on graph theory [6] was written primarily for a readership of social scientists, and is the only book devoted entirely to directed graphs. Two books [7, 8] have been written on networks.

1. D. König, *Theorie der endlichen und unendlichen Graphen.* Leipzig, 1936, reprinted Chelsea, New York, 1950.

2. C. Berge, *Théorie des graphes et ses applications.* Dunod, Paris, 1958; English translation, Methuen, London, 1962.

3. O. Ore, *Theory of graphs.* Amer. Math. Soc. Colloq. Publ., 38 (1962).

4. ———, *Graphs and their uses.* Random House, New York, 1963.

5. M. Fiedler, editor. *Theory of graphs and its applications.* Czechoslovak Academy of Sciences, Prague and Academic Press, New York, 1964.

6. F. Harary, R. Norman, and D. Cartwright, *Structural models: an introduction to the theory of directed graphs.* Wiley, New York, 1965.

7. L. R. Ford, Jr. and D. R. Fulkerson, *Flows in networks.* Princeton Univ. Press, Princeton, 1962.

8. C. Berge and A. Ghouila-Houri, *Programmes, jeux, et réseaux de transport.* Dunod, Paris, 1962; English translation, Methuen, London, 1965.

Thus it is natural to ask: what does the book under review have that the other books don't have? The book is divided into two parts: I. Basic Theory, II. Applications. The first chapter of Part II, "A variety of interesting applications," includes material not available in any other one book and constitutes the main contribution. These applications include references to economics and operations research, combinatorial problems, puzzles and games, physical systems, communication networks, electrical networks, organic chemistry, statistical mechanics, genetics, cybernetics, sociometry, disarmament, linguistics, and Markov chains. As the authors indicate, these topics are not developed in detail, but rather are presented as indications of applications which must still be studied from original sources. Since this book provides an opportunity for scholars in the above-mentioned fields to observe examples of the use of graph theoretic methods, it will stimulate further interest in graph theory.

A few words of criticism. It seems inappropriate for the dust cover of a book on graph theory to display a spiral galaxy, when there are so many pretty graphs that could be used. The location of the references at the end of each chapter (and sometimes at the end of sections) is quite inconvenient. No two books on graph theory agree in terminology or notation; the present book is no exception. There are, of course, the usual assortment of misprints.

FRANK HARARY, The University of Michigan

Introduction to Calculus and Analysis, Vol. I. By R. Courant and F. John. Interscience-Wiley, New York, 1965, xxiii+661 pp. $10.50.

Courant's *Vorlesungen über Differential und Integralrechnung* (1927) and its modified English editions (1934, 1937), have exerted a deep and beneficial influence on the teaching of calculus in the United States. The many admirers of these books will welcome this new text written to satisfy the specific needs of American students with the same emphasis on applications and on intuitive understanding. As the authors say: "Mathematics presented as a closed, linearly ordered, system of truths without reference to

origin and purpose has its charm and satisfies a philosophic need. But the attitude of introverted science is unsuitable for students who seek intellectual independence rather than indoctrination; disregard for applications and intuition leads to isolation and atrophy of mathematics. It seems extremely important that students and instructors should be protected from smug purism." It is to be hoped that this book will help to reverse the direction in which the wind is now blowing.

The explanation of many topics previously treated has been amplified, clarified and made more precise. There is considerable new material and much of the older material has been promoted from fine print. The language of mapping has been introduced. But the authors do not hesitate to speak of dependent and independent variables or use the expression "y is a function of x." As they say, "This locution is used freely in the sciences, but some of the more pedantic texts avoid it. There is no point in hampering ourselves by an undue concern for hair-splitting 'precision' when it has no relation to the substance."

A supplementary volume "Problems in Calculus and Analysis" has been prepared by Albert A. Blank which reprints the problems in the text and adds hints and solutions. Exercises of a more routine character have also been included, with answers and solutions when needed.

This volume is concerned with functions of a single variable. "A Sketch of the Theory of Functions of Several Variables" contained in the previous text is omitted.

While the content of this book bears a marked resemblance to the previous English editions, this is indeed a new book. We confine ourselves to noting the most striking differences.

Chapter 1. Introduction. (*118 pp.*) This excellent chapter is an expanded and reorganized treatment of the Continuum of Numbers, The Concept of Function, The Elementary Functions and The Theory of Limits. It includes new material on Mathematical Induction and Lipschitz and Hölder Continuity.

Chapter 2. (*81 pp.*) *The Fundamental Ideas of the Integral and Differential Calculus.* The earlier material has been amplified and rearranged so that, for example, the mean value theorem for integrals precedes that for derivatives and the logarithmic and exponential functions are moved forward.

Chapter 3. (*122 pp.*) *The Techniques of Calculus* combines the earlier Chapters 3 and 4 with relatively little change, but there is an improved treatment of the substitution rule for integrals, a new section on the differential equation satisfied by $\sin x$ and $\cos x$, and an explanation of O and o notation.

Chapter 4. (*115 pp.*) *Applications in Physics and Geometry* contains an expanded and elegant treatment of curves and their included areas. Material on vectors in the plane is new.

Chapter 5. (*40 pp.*) *Taylor's Expansion.* The major change is the inclusion of an appendix on interpolation.

Chapter 6. (*28 pp.*) *Numerical Methods and Chapter 7.* (*60 pp.*) *Infinite Sums and Products* contain improvements in detail.

Chapter 8. (*61 pp.*) *Trigonometric Series* includes much new supplementary material (Weierstrass Approximation Theorem, Fejer's Approximation, Parseval's Relation, Gibbs' Phenomenon, Bernoulli Polynomials, Generating Function, Euler-Maclaurin Summation Formula).

Chapter 9. (*16 pp.*) *Differential Equations for the Simplest Types of Vibration.* Little change.

This book is a welcome addition to mathematical literature where the word "literature" is singularly appropriate. Volume II is eagerly awaited.

D. E. RICHMOND, Williams College

Methods of Matrix Algebra. By Marshall C. Pease, III. Academic Press, New York and London, 1965. xviii+406 pp. $13.75.

Starting from basic principles, the book presents a rather comprehensive development of matrix algebra and its use in a variety of physical applications. The concept of a matrix is introduced as a means of describing linear mappings of vector spaces. The general theory is developed from the concept of eigenvectors and chains of generalized eigenvectors, and carried through to such items as the Jordan canonical form, functions of a matrix, the matricant, and dyad expansions. Special topics include the solution of equations involving singular and rectangular matrices, projectors, and the direct product and Kronecker sum. The theory is illustrated by applying matrix methods to periodic systems of differential equations, to a discussion of the stability of such systems, to electromagnetic theory, and to stochastic processes. Special attention is drawn to a chapter on "The Improper Inner Product."

The book is written with the purpose of making available to the physical scientist and engineer some of the more sophisticated techniques of matrix algebra. Except for a few basic theorems, however, such as the existence of an inverse, most proofs are carried through in a reasonably detailed mathematical fashion. In some cases, the verbal discussions involving physical reasoning for certain situations did not enlighten me, although I must confess I am not in the intended audience. For example, in discussing transmission lines it does not help to have the Manley-Rowe relations invoked when I have never heard of them.

I have rather mixed feelings regarding the book as a text. As the preface mentions, the author has in mind someone near the graduate level in physics. I would expect that, in addition, the student who also has had an introductory course in matrix algebra would profit most from this text, and would be able to use it fruitfully as a reference book. If used as a text, I would guess that there is more than enough material for a full year course. Problems, without answers, are given at the end of each chapter. Relatively few misprints were noted and all these were of a trivial character.

A. B. FARNELL, General Dynamics/Convair

Linear Integral Equations. By S. G. Mikhlin. (Translated from the 1959 Russian edition under auspices of Hindustan Publishing Corp., Delhi). Gordon and Breach, New York, 1960. vii+223 pp. $9.50.

Mikhlin's book has been reviewed several times in its Russian, English and German versions. It is a sound and useful text along classical lines, with occasional illuminations of old topics. But the present translation is so poor as to be downright misleading and even incomprehensible. It should not be put into the hands of young people.

A. T. LONSETH, Oregon State University

A First Course in Partial Differential Equations. By Hans F. Weinberger. Blaisdell, Waltham, Mass., 1965. ix+446 pp. $12.75.

The main purpose of this book is to teach the techniques of solving partial differential equations by the method of separation of variables. Unlike most books on the subject, however, Professor Weinberger's book stresses the importance of concepts of real analysis, and he does this naturally and incisively. Any student interested in mathematics and applied mathematics should find this a valuable book.

As an introduction to partial differential equations, the book is an outstanding contribution. Included is the use of characteristics and energy methods, and

a self-contained treatment of functions of a complex variable. An unusual feature is the careful discussion of the evolution of a physical model into a mathematical model, using the vibrating string as an example.

Finally, it should be mentioned that Professor Weinberger's writing style makes the book a pleasure to read.

The chapter headings are: I. The one-dimensional wave equation; II. Linear second-order partial differential equations in two variables; III. Some properties of elliptic and parabolic equations; IV. Separation of variables and Fourier series; V. Nonhomogeneous problems; VI. Problems in higher dimensions and multiple Fourier series; VII. Sturm-Liouville theory and general Fourier expansions; VIII. Analytic functions of a complex variable; IX. Evaluation of integrals by complex variable methods; X. The Fourier transform; XI. The Laplace transform; XII. Approximation methods.

A. D. MacGillivray, State University of New York at Buffalo

Linear Representation of the Lorentz Group. By M. A. Naimark. Pergamon Press, New York, 1964. 447 pp. $14.50.

The linear representations of the Lorentz group are, at present, of great importance in physics, and the author remarks that he wrote the present work as a book for physicists. In any case, the book is a very valuable contribution to the mathematical textbook literature. The style is simple and lucid. The mathematical tools are reduced to the minimum necessary to prove the theorems concerning the Lorentz group, and some theorems from Functional Analysis are proved in nine appendices. At the same time, many generalizations of the results proved are mentioned, and appropriate references are given. Formulas are spelled out in great detail. The three dimensional orthogonal group, the proper and the complete Lorentz group are studied carefully, and their finite dimensional representations are given before the treatment of the infinite dimensional representations which takes up about one half of the book. The last 100 pages are dedicated to an investigation of the invariant equations as far as it can be carried out, and several of the important unsolved problems in this area are mentioned.

W. Magnus, New York University

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

PERSONAL ITEMS

Professor Joshua Barlaz, Rutgers—The State University, represented the Association at the inauguration of Carl G. Fjellman as President of Upsala College on October 4, 1966.

Professor E. L. Canfield, Drake University, represented the Association at the inauguration of William Terril Higdon as President of Graceland College, Iowa on October 15, 1966.

Professor Geraldine A. Coon, Goucher College, represented the Association at the Convocation and Dedication of the Lewis Hall of Science at Western Maryland College on October 15, 1966.

Professor K. W. Folley, Wayne State University, represented the Association at the inauguration of the Very Reverend Malcolm Carron as President of the University of Detroit on October 20, 1966.

Professor Erik Hemmingsen, Syracuse University, represented the Association at the inauguration of Albert E. Holland as President of Hobart and William Smith Colleges on October 8, 1966.

Professor H. V. Huneke, University of Oklahoma, represented the Association at the inauguration of Robert B. Kamm as President of Oklahoma State University on October 21, 1966.

Professor W. H. Leser, Franklin & Marshall College, represented the Association at the inauguration of Robert A. Christie as President of Millersville State College on October 15, 1966.

University of Arkansas: Dr. Kathrine Mires, Northwestern State College in Oklahoma, has been appointed Associate Professor; Dr. Allan Cochran, University of Oklahoma, has been appointed Assistant Professor; Professor J. E. Scroggs has been appointed Chairman of the Mathematics Department.

University of Hawaii: Professor M. R. Hestenes, UCLA, has been appointed Visiting Professor; Dr. L. R. McCulloh, University of Illinois, has been appointed Visiting Associate Professor.

University of Kansas: Dr. R. R. Colby, Boeing Company, Seattle, Dr. Mark Mandelker, University of Rochester, Dr. J. R. Porter, New Mexico State University, and Dr. R. E. Powell, Lehigh University, have been appointed Assistant Professors; Dr. Ronald Jacobowitz, University of Arizona, has been appointed Associate Professor; Associate Professor T. K. Boehme, University of California at Santa Barbara, has been appointed a Visiting member of the staff; Professor Carl Boyer, Brooklyn College, will be the Rose Morgan Professor for the fall semester.

University of Missouri: Dr. Keewhan Choi, Cornell University, has been appointed Assistant Professor of Statistics; Professor H. D. Brunk has been appointed Senior Lecturer in Statistics at the University College of Wales in Aberystwyth.

Northeast Louisiana State College: Mr. L. T. Bennett, Oklahoma State University, and Mr. Charles Smith, Oklahoma State University, have been appointed Assistant Professors; Assistant Professor Edward Anders has been appointed Head of the Mathematics Department at Arkansas State College; Mrs. Rosemary Hamner has been promoted to Assistant Professor; Assistant Professor D. R. Bedgood has been promoted to Associate Professor.

Associate Professor Robert Bridgman, Mansfield State College, has been promoted to Professor.

Associate Professor L. E. Claborn, Cornell College, is on leave-of-absence for one year of study at the University of Illinois.

Mr. Sen Fan, Brown University, has been appointed Associate Professor at Morningside College.

Dr. R. E. Huddleston, Texas Christian University, Fort Worth, has been appointed Research Associate at the Institute for Fluid Dynamics and Applied Mathematics at the University of Maryland.

Assistant Professor J. A. Pavelcak, Merrimack College, has been appointed Director of the Computing Center.

Assistant Professor H. J. Reschovsky, University of Connecticut, has been promoted to Associate Professor.

Dr. R. E. Roth, St. Bonaventure University, has been appointed Professor of Statistics and Director of the Computer Center, State University College, Geneseo.

Dr. F. M. Morgan, Clark School, New Hampshire, died on June 28, 1966. He was a charter member of the Association.

Professor Emeritus Tibor Rado, Ohio State University, died on December 28, 1965. He was a member of the Association for 35 years.

Mr. A. A. Warntz, Syracuse University, died on January 8, 1966. He was a member of the Association for five years.

BOOKS FOR ASIAN STUDENTS

Technical and scientific books and journals are needed in the schools of Asia. The Asia Foundation's Books for Asian Students Program will distribute your donations to these schools free of charge. Books must be dated 1955 or later and be in good condition. Journals must be in runs of five years or more dated 1946 or later. Shipments and inquiries may be made at any time to the program at 451 Sixth Street, San Francisco, California 94103. Donations to The Asia Foundation are tax deductible.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING

OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Spring meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held at the College of William and Mary, Williamsburg, Virginia, on April 30, 1966. Dr. Samuel Saslaw, Chairman of the Section, presided. There were 74 persons in attendance, including 70 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor Thomas L. Reynolds, College of William and Mary, Williamsburg, Virginia; Vice Chairmen, Professor Joseph Milkman, U. S. Naval Academy, Annapolis, Maryland and Professor Raymond W. Moller, Catholic University of America, Washington, D. C.; Secretary, Professor George N. Trytten, University of Maryland, College Park, Maryland; Treasurer, Professor Stanley B. Jackson, University of Maryland, College Park, Maryland.

The following program was presented:

1. *The bouncing ball does come to rest*, by W. J. Young, Emory and Henry College.
2. *Application of a differential technique to a difference system*, by A. L. Deal, III, Virginia Military Institute.
3. *A theorem on additive arithmetical functions*, by R. L. Robinson, U. S. Army.

4. *On trigonometric matrices and their applications*, by G. J. Etgen, Headquarters, N.A.S.A., Washington, D. C.

5. *An example of optimal control*, by G. S. Jones, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, (invited address).

6. *On the original Malfatti problem*, by Michael Goldberg, Bureau of Naval Weapons, U. S. Navy, Washington, D. C.

7. *Plane symmetry groups*, by C. M. Makepeace, National Civil Defense Computer Facility, Olney, Maryland.

8. *Statistical validation of mathematical computer routines*, by Carl Hammer, UNIVAC, Washington, D. C.

G. N. TRYTTEN, *Secretary*

APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The twenty-fifth annual meeting of the Metropolitan New York Section of the MAA was held on April 30, 1966, at the Polytechnic Institute of Brooklyn. There were 105 persons present, of whom 75 were members of the Association. Abraham Schwartz of the City University of New York was elected Governor of the Section; the terms of the other officers of the Section extend to the 1967 annual meeting. It was decided that next year, at the 1967 meeting, a Vice President for Junior Colleges would be elected, in addition to the established office of Vice President for High Schools.

The program of the meeting was as follows:

1. *Topics in topological algebra*, by Terrell Gardner, Queens College.

The continuous character groups of the additive reals, the circle group, and the additive integers (usual topologies) were derived, topologized as old friends, and their character groups acknowledged, leading to the statement of Pontrjagin's duality theorem. A brief discussion of commutative Banach algebras included a statement of the Stone-Weierstrass and Gelfand representation theorems. The existence of non-commutative analogues for C^+-algebras was mentioned. Finally, the p-adic fields were described, and their utility in number theory remarked.

2. *CUPM Panel on college teacher preparation*, by Leonard Gillman, University of Rochester.

The recently formed Panel is charged with examining all phases of college teacher preparation. It is recommending more summer institutes and the creation of several new types of institutes to help present teachers, and it is considering special graduate programs for the preparation of new teachers.

3. *Report of the Chairman of the Contest Committee and presentation to winners of the Metropolitan New York Section Contest*, by C. T. Salkind, Polytechnic Institute of Brooklyn.

4. *Report of the Section Governor and of the Director of the Speakers Bureau*, by J. N. Eastham, Queensborough Community College.

5. *Treasurer's report*, by Aaron Shapiro, Midwood High School.

6. *Round table discussion—"Transition in Mathematics from High School to College."* Chairman of the discussion was Meyer Jordan, Brooklyn College. Speakers on the topic were: Benjamin Bold, Stuyvesant High School; Freeman Cope, Queens College; George Crossman, New York City Board of Education; Harry Ruderman, Hunter High School; Jules Russell, Polytechnic Institute of Brooklyn; Abraham Schwartz, The City College.

The meeting concluded with a showing of the Calculus Film Project of the Committee on Educational Media.

MARY HAGEN, *Secretary*

APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the MAA was held at the University of New Mexico, Albuquerque, on April 1–2, 1966. There were 78 persons including 55 members of the Association in attendance. Section Chairman J. Mayer, University of New Mexico, presided.

Two speakers were invited to address the section. Professor E. Dyer, Rice University, lectured on "Quasi topology" and Professor I. N. Herstein, University of Chicago, lectured on "Rings of quotients." The keynote address was given at the Friday evening banquet by Professor R. B. Crouch, New Mexico State University.

A panel moderated by M. Norris, Sandia Corporation, discussed "The future structure of mathematics departments and related questions." Panel members were department chairmen J. R. Blum, University of New Mexico; H. Cohn, University of Arizona; R. B. Crouch, New Mexico State University; E. D. Nering, Arizona State University; L. J. Paige, University of California at Los Angeles.

A second panel on "The CUPM proposal for the training of mathematics teachers in the secondary schools" was moderated by R. J. Wisner, New Mexico State University. Panel members were W. T. Guy, Jr., University of Texas; A. P. Hillman, University of New Mexico; C. E. Little, Northern Arizona University; D. E. Myers, University of Arizona; and W. Scarborough, Albuquerque Council of Teachers of Mathematics.

The following officers were elected: Chairman, D. E. Myers, University of Arizona; Vice-Chairman, R. J. Wisner, New Mexico State University; Secretary-Treasurer, S. T. Kao, University of New Mexico.

The following papers were presented:

1. *On minimal Hausdorff spaces*, by J. R. Porter, New Mexico State University.
2. *Non-commutative local rings*, by M. Lomanitz, New Mexico Institute of Mining and Technology.
3. *A measure of axial symmetry for convex sets*, by Bro. B. A. deValcourt, St. Michael's College.
4. *New divisors of Mersenne numbers by evaluation of certain quadratic forms*, by E. Karst, University of Arizona.

E. L. WALTER, *Secretary-Treasurer*

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In order to qualify for the second class mail rate, MAA has had to append zip code numbers to all addresses to which the MONTHLY is mailed. This has required the retyping of many address plates which in turn has led to the possibility of errors.

We will be very grateful therefore if you will examine the address on the wrapper of your copy of the MONTHLY and notify the Buffalo MAA office of any errors. We will ourselves make all necessary corrections eventually, but the correcting process will be accelerated if our readers will cooperate.

HARRY M. GEHMAN, *Executive Director*

CALENDAR OF FUTURE MEETINGS

Forty-eighth Summer Meeting, University of Toronto, Toronto, Ontario, Canada, August 28–30, 1967.

Fifty-first Annual Meeting, San Francisco, California, January 25–27, 1968.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, West Virginia, May 6, 1967.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1967.

INDIANA, Wabash College, Crawfordsville, Spring 1967.

IOWA, Drake University, Des Moines, April 21, 1967.

KANSAS, Fort Hays State College, Hays, April 22, 1967.

KENTUCKY, Murray State University, Murray, Spring 1967.

LOUISIANA-MISSISSIPPI, Jung Hotel, New Orleans, Louisiana, March 4–5, 1967.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 18, 1967.

MINNESOTA

MISSOURI, Northeast Missouri State Teachers College, Kirksville, April 29, 1967.

NEBRASKA, University of South Dakota, Vermillion, May 6, 1967.

NEW JERSEY

NORTHEASTERN Mt. Allison University, Sackville, New Brunswick, June 23–24, 1967.

NORTHERN CALIFORNIA, University of California, Davis, February 4, 1967.

OHIO, Ohio State University, Columbus, April 22, 1967.

OKLAHOMA-ARKANSAS, Northeastern State College, Tahlequah, Oklahoma, March-April, 1967.

PACIFIC NORTHWEST, University of Montana, Missoula, June 16–17, 1967.

PHILADELPHIA, University of Delaware, Newark, November 18, 1967.

ROCKY MOUNTAIN, Western State College of Colorado, Gunnison, May 1967.

SOUTHEASTERN, Florida Presbyterian College, St. Petersburg, Florida, March 31-April 1, 1967.

SOUTHERN CALIFORNIA, San Diego State College, San Diego, March 11, 1967.

SOUTHWESTERN, University of Arizona, Tucson, March 31-April 1, 1967.

TEXAS, Austin College, Sherman, April 14–15. 1967.

UPPER NEW YORK STATE, State University College, Plattsburgh, May 20, 1967.

WISCONSIN, St. Norbert College, DePere, May 6, 1967.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, New York, New York, December 26–31, 1967.

AMERICAN MATHEMATICAL SOCIETY, Toronto, Ont., Canada, August 28–September 1, 1967.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Michigan State University, June 19–23, 1967.

ASSOCIATION FOR COMPUTING MACHINERY, Sheraton-Park, Washington, D. C., August 29–31, 1967.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 23–25, 1967.

INSTITUTE OF MATHEMATICAL STATISTICS

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Convention Center, Las Vegas, Nevada, April 16–20, 1967.

OPERATIONS RESEARCH SOCIETY OF AMERICA, New York Hilton Hotel, May 31-June 2, 1967.

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SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D.C. June 12–15, 1967 (Symposium on Applied Probability and Fluid Dynamics).

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CALCULUS AND ANALYTIC GEOMETRY, 2nd Edition, 1965, *also by Robert C. Fisher and Allen D. Ziebur.* The second edition of this popular text offers an accurate, understandable introduction to calculus and analytic geometry. At the end of the course, the freshman should have a good grasp of the essential nature of the subject and should be able to express himself in current notation. The text has been completely rewritten, but the overall coverage, objectives and unifying themes remain the same. *1965, 768 pp., $11.25*

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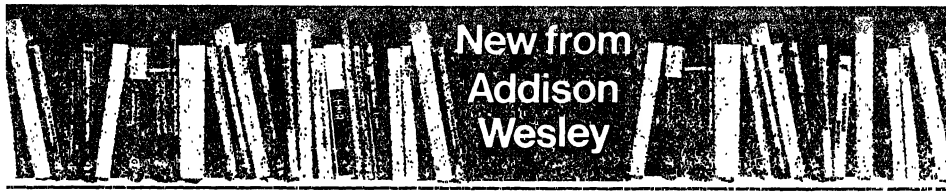
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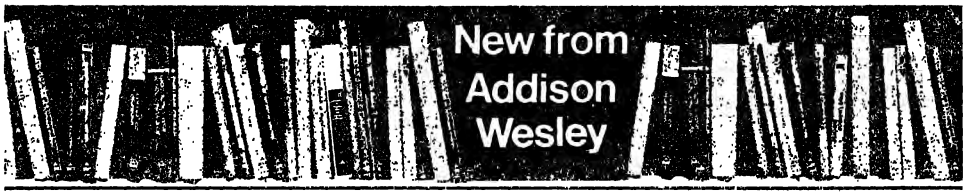
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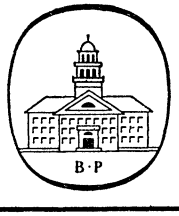
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THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 74 NUMBER 1

PART II

Fiftieth Anniversary Issue

Published in
Commemoration of the Fiftieth
Anniversary of the Founding
of the
MATHEMATICAL ASSOCIATION OF AMERICA
on December 31, 1915

JANUARY 1967

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

R. A. ROSENBAUM, *Editor*

EDITORIAL CORRESPONDENCE AND MANUSCRIPTS OF PAPERS should be addressed to the Editor, R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

General Office: SUNY at Buffalo, Buffalo, N. Y. 14214. *Executive Director:* H. M. GEHMAN

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $6.00. For nonmembers the subscription price is $10.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

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PREFACE

The fiftieth anniversary of the founding of the Mathematical Association of America was celebrated at the 1965 Summer Meeting at Cornell University. For an account of this meeting, see this MONTHLY, 72 (1965) 1053–1059 and the frontispiece of the November issue.

As part of this celebration a special program of invited addresses was presented. The texts of these addresses have been collected and are now printed in this Fiftieth Anniversary Issue and are published as a supplement to the MONTHLY.

CARL B. ALLENDOERFER
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Editorial Committee

BRIEF HISTORY OF THE MATHEMATICAL ASSOCIATION OF AMERICA BEFORE WORLD WAR II

ALBERT A. BENNETT, Brown University

1. Introduction. For a few moments you are asked to try to look backward instead of forward. To relive the past is difficult, but the recent centennial of bloody and unglamorous battles of our Civil War have partially acclimatized us to so doing. The backward glance that reduced Lot's wife to the salt of the earth seems to be almost an un-American activity.

It is not in the dull recital of statistics concerning forgotten events, that this our gathering achieves any significance. Rather, it would seem that we are here to memorialize the work of many charter members, largely here nameless, most of whom are gone, including nearly all who played a major role in the founding.

The one thing important about the Mathematical Association of America at this its semicentennial is that it is going strong. It has increased in numbers and activities and in favor with college and public.

2. The Founding. Our Association was founded under especially auspicious circumstances. The many favorable factors were not accidents, nor miracles, nor achieved through serendipity. Some were the end results of a chain of events, not always desired, nor always with the eventual outcome in view. But in large part they were secured through wise planning, tactful compromise, cajoling of the apathetic, courageous facing of pessimists in high places, and unremitting work.

The stories of the Mathematical Association of America and of the American Mathematical Monthly are closely intertwined and present also some curious aspects as to their names, as will be brought out.

Briefly cited, here are some eleven of the special items of initial good augury.

1. The Association had the (for mathematics) phenomenal initial paid individual membership of 1045 (living in 25 of our states), or one per hundred thousand of the national population at the time. It had also initially 52 institutional members.
2. The Association started with many experienced, devoted and distinguished mathematicians at the helm.
3. The Association took over a well established, favorably known journal and one with the same objectives as the Association, as its official organ, a journal with wide collegiate backing, editorially and financially, and whose staff had worked enthusiastically toward this development. This journal, the *American Mathematical Monthly* was continued without break in its general policies, although with notable enrichment.
4. Five semi-autonomous sections were part of the original plan, with generous provision for promptly setting up further future local sections.
5. The phrase "of America" was interpreted from the start to include Canada and indeed the North American Continent.

6. Not only was an annual meeting to be held each winter, but also a regular meeting of the whole Association each summer. One might note in passing that prior to 1945 only one annual meeting was cancelled, that for 1942, during the war, due to stringent travel restrictions, and only two summer meetings, those for 1924 and 1938, the latter in deference to the Semi-centennial Celebration of the American Mathematical Society.
7. Furthermore, these scheduled meetings of the Association were planned to be held successively in various parts of the country.
8. Cooperation with the American Mathematical Society was assured from the start and plans for frequent joint meetings with it and with such organizations as the American Association for the Advancement of Science, the National Council of Teachers of Mathematics, and eventually others were part of the original program.
9. One of the most serious intentions of the founders of our Association was the furtherance of publication of expository articles, monographs or books, on advanced mathematical topics. Toward success in this plan, the founders from the start maintained active cooperation with the management of the Annals of Mathematics, then sponsored by Princeton University, and secured representation by the Association upon its editorial board.
10. To avoid the danger of a self-perpetuating clique, a dual-choice slate for all Association offices was arranged for.
11. The annual dues were instituted at a sub-snob appeal level, thus inviting general support without sacrifice.

The MATHEMATICAL ASSOCIATION, now so vigorous, adopted its present name, as many of you may know, in February 1897. This Association had already been operating, and on an international basis, for several years before that time but under a longer and more restrictive title. Its official organ was established, and with the present title, in 1894 and was published in both New York and London. Later its places of publication were changed to London and Bombay. For indeed, "The Mathematical Association" is and always has been the official title of but one organization, and that, British. Its original objective was primarily the much overdue reform in the teaching of geometry. Its journal is the *Mathematical Gazette*. But enough of that.

The MATHEMATICAL ASSOCIATION OF AMERICA came into being on the last afternoon of 1915. Herbert E. Slaught, a leading, if not indeed the leading organizer of this Association, described in 1930 a dream, "a picture of the celebration in January 1966 of the Fiftieth Anniversary of the founding of the Mathematical Association, with Hedrick, Cairns, Finkel and Slaught, and many others present and in the full bloom of their years."

3. Contemporary Events. This founding was an event, like that of Einstein's announcement, that year, of the general theory of relativity, little noted by the

public. Even as might be expected the sense of mutual unconcern was unruffled. Nothing in the records of the Association might seem to hint that the Great War had been underway in deadly earnest in Europe. And not in Europe alone. Turkey was persecuting the Armenians in Asia Minor. Britain had seized Germany's rich colony in East Africa. Japan had annexed Germany's port on the China Sea. Russia, stretching across Europe and Asia to the Pacific, was committed to its last muzhik to the war effort. Canada was being drained of young man power. But the United States was at peace and with a professed neutrality. In this one felt no sham, no shame. The highest idealism saw eye-to-eye with the frankest economic profit. In 1914 even before Zeppelins were dropping bombs on a bewildered and defenseless London, indeed in the very days that the Triple Alliance and the Triple Entente were declaring mutual war, we opened the Panama Canal to neutral traffic. Although half the world was struggling for sheer existence, we celebrated throughout 1915 the Panama-Pacific International Exposition. The half million who paid admission on the last day of this Fair, had little mind for the roll of distant drums. To be sure, David Starr Jordan, Chancellor of Stanford University, after touring the country, lecturing on the impossibility of a world war, was struggling, *ex post facto*, for peace; and on December the fourth, Henry Ford, with a futile band of enthusiasts and ridiculing newsmen, had sailed to Europe on his Peace Ship, with no previous plan and no subsequent success. Most Americans greeted such rampant idealism with at best a condescending smile.

Politically, the World map of 1915 looked quite otherwise than it does now. Among the late Queen Victoria's 74 descendents (as of 1915), many were among the 78 "reigning sovereigns of Europe" (as phrased in the World Almanac of 1916). Kaiser Wilhelm of the German Empire (long self-styled, the Peace Kaiser) announced his hope of dropping a bomb on each of his two cousins, George V, Emperor (*in absentia*) of India, and Nicholas II, Emperor and absolute autocrat of Russia. And his sentiments were shared by Kaiser Franz Joseph of Austria. Emperors dotted the landscape. Bulgaria had its own Czar. Even among the six nations of Africa, Abyssinia had an emperor who called himself "King of Kings" and claimed direct descent from King Solomon and the Queen of Sheba. His sacred majesty, King George of England could do as well (but less romantically), claiming the biblical King David as an ancestor. He could glory also in the pious title of Defender of the Faith, claimed by right of descent (but hardly by right of logic) from Henry VIII, thus once ennobled by a Pope. Only Japan could do better. As averred at the time in the Encyclopaedia Britannica, the Emperor of Japan traced his lineage by unbroken line of male descent back to Jimmu Tenno, 660 B.C., the only son of the Moon Goddess. Even China, after a couple of decades as a republic, was again briefly an empire.

Later we read in the MONTHLY a few personnel notices of so and so entering the armed forces. Altogether 178 members of the MAA were eventually so listed. It is sobering to realize that already before April 1917, 10,000 German public school teachers were reported as having been killed in the war.

On New Year's Day 1916, few of those Americans, if in condition to think at all, were thinking about involvement in the war which we expected so soon would end. We had not even been informed by President Wilson that we were "too proud to fight." No one had invented the word "camouflage." The unsinkable Titanic had gone down on her maiden trip by what was called an Act of God; but the proud British liner Lusitania had for some years, until May 8, 1915, crossed and recrossed the Atlantic in presumed safety from acts of evil men. Out in the Pacific, the long uninhabited Midway Island now housed a small band of about 40 submarine telegraph cable operators and repairmen. Pearl Harbor was still part of an island Paradise. William Howard Taft, defeated in 1913 for reelection to the U. S. presidency, was president of the benign Asiatic Institute.

If at last we could be pushed into war, Americans would always volunteer in multitudes to defend the Stars and Stripes! Military draft was a European curse, not to be adopted here. The War Risk Insurance had been just established, but not to pay soldiers' widows, only ship owners' losses. The War Department could not entirely ignore the rapid phenomenal advances in the Art of War. President Teddy Roosevelt had only a few years before congratulated Kaiser Wilhelm on his fine army, the only army in Europe ready to fight, and obviously equipped to lick all Europe should such be called for. So Congress had reluctantly permitted the organization in this country of a Military Aviation Branch, but had prudently safeguarded against reckless expansion by insisting that there shall not be more than 15 military aviators. Perhaps to ensure offensive effectiveness, each of the one-man crews could carry an army pistol.

Of course, in 1916, we were immensely proud of our victorious military history. What with the semi-centennials of Civil War dates, and Teddy Roosevelt's flag-waving and the G.A.R. veterans leading every parade—all honor to our brave heroes! Pensions, restricted to those totally disabled incident to military service, amounted to $30 per month to generals down to $8 per month to sergeants and below. This seems hardly munificent; but remember, a man's wool suit, cut tailored to order, cost only from $10 to $20. For the many rich, a diamond of $1\frac{1}{2}$ carat weight could be bought for $139. Some pensioners from the American Revolution were still on the rolls. But with all our national and political enthusiasm, we were learning to be less demonstrative. Already torch light parades were a bit passé. While in 1909, on the Glorious Fourth, 215 were killed and 5092 were reported as hospitalized by fire works, in 1916 these numbers had been reduced to 30 killed and 320 injured. It took another generation with its open road and three-day holidays to send our young people wholesale to Heaven or Hell.

The general public in the United States could not, if it would, remain completely unconscious of the bitter fighting in Europe. While there were 1,216,000 immigrants in 1914, their number had dropped to 307,000 by 1915, and was to drop further. Cheap labor was becoming expensive. Canada was offering free 160-acre farms to U. S. immigrants, "our brothers from south of the border."

As neutrals, we were becoming rich, supplying arms to whomsoever could buy them—and get them. We were at peace, and fervently isolationist. We were proud of being imperialistic, of having a great fleet patrolling both the Atlantic and the Pacific. We accepted Kipling's elevated concept of the White Man's burden in regulating our few precious colonies. Prohibition and the Female Vote were topics of much running jest. But national defense was hardly mentioned in public discussion or current literature.

What were Americans thinking about in late 1915? How to get rich oneself, and yet soak the rich. How to keep labor under control and yet bust the trusts. The "malefactors of great wealth," to quote Teddy R., were yet the envy of every business man. The whole family trees of the 17 American multi-millionaires were treated better than royalty in the World Almanac for 1916. American men and women in 1915, laughing worriedly at Charles Dana Gibson's series of pictures in the old LIFE entitled "The Education of Mr. Pipp," were looking for tricks to get fat. One dieted to round out the figure. It was estimated that 70% of American homes had no bathrooms. A railroad engineer, for an 8-hour day and 6-day week, seemed happy to get $30 a week. The proudly independent American citizen was shocked by the adoption in 1914 of a constitutional amendment permitting a general income tax, especially since in that year vested interests had secured the protection of a Federal Reserve Banking System. Such American men as had, in the interest of culture, deserted the defense (and the offense) of the spittoon, and had descended to the Parisian fad for cigarettes, largely rolled their own.

Life in these United States was indeed at least superficially quite different, as 1916 opened, from what we know it today. The automobile was just coming into its own, and that with a rush. In 1915 the automobile manufacturing companies had produced on the average, twice what they did the year before. The output averaged 325 cars apiece for the year. Of course one could not expect to drive to California by car to see the Fair, even in a fine $400 roadster. The Lincoln Highway, like the Lincoln Memorial, was not yet completed. Less than a ninth of the highways of the country were surfaced. Electric starters, like electric headlights, were luxury extras available on some cars. Sound amplification could be by stethoscope, tin earhorn, or megaphone only. Household family radios were, of course, unknown. It was the heyday of vaudeville. Nickelodeons (an hour's flicks for a nickel) and jitneys at a nickel a trip kept people happily marveling. Neon signs were undreamed of. The World Almanac for 1916 mentioned 37 stars of the silent screen to 340 on the stage. But the "Birth of A Nation," with some dubbed-in color was promising new heights in entertainment, and was rehabilitating the public image of the K.K.K. Kerosene oil was the main source of household illumination, and coal was the main source of household heating. Gasoline was still largely a waste product in the effort to keep kerosene less explosive. A third of our people still lived on the farm. Even as today, so then, New York City was our chief port and our amusement center. It had 56 hotels and could be reached by 36 ferry lines. It was supplied with

almost one telephone for every ten of its inhabitants. In 1916 the World Almanac listed among known scientific facts: speed of light 184,320 mi/sec.; speed of electricity, 288,000 mi/sec. Americans were the world champions of free public education. Out of a national population of 106 million, about one per cent were enrolled in the public high schools, with few taking mathematics. The word "dropout" had not come into use. Of course college was expensive. Harvard even charged $200 for annual tuition. Family life was relatively stable. Only 3 tenths of one per cent of the men and 4 tenths of one per cent of the women had been divorced (you might figure that out!). National Baby Week had been instituted, but only later did they remember to think of Mother's Day. As for mathematics, at the close of 1915 it was undergoing a steady decline in popular prestige. The tradition of mental discipline seemed neither convincing nor attractive to young America. Prophets foresaw mathematics sharing with Latin and Greek, complete extermination from the public high school curriculum. But Pi Mu Epsilon had been founded in 1914.

4. Early Journals. Scientific societies and scientific periodicals have often been associated but not usually. And this general observation applies also specifically in the area of mathematics. However, the story of the Mathematical Association of America cannot be divorced from that of the earlier established MONTHLY.

Precisely, in this general connection, the following statistics arduously gathered at the Brown University Physical Science Library may be of some interest. There are there available 829 separate periodicals devoted entirely to pure mathematics, as well as many scientific journals with some mathematical contributions. Of these 829 journals, 368 seem unquestionably to have ceased arriving, probably due in most cases to cessation of publication, although perhaps in some cases to lapse in subscription or exchange arrangements. Of the 461 nominally still arriving, 72 are issued in the United States or Canada. Of the total continuing 461, only 73 were founded prior to 1916. Of the 72 issued in America and still continuing, only 7 antedate 1916, including *The American Journal of Mathematics* (1878), *Transactions of the American Mathematical Society* (1900), *Annals of Mathematics* (1899), and *The Mathematics Teacher* (1908).

Some years ago was founded in this country *The Mathematical Monthly*. It filled a widely felt want. As promised by its founder, it carried among other regular features, expository articles on advanced topics of mathematics and news of mathematical activities throughout the world. Of special concern to the founder and editor, it maintained a lively department devoted to Problems and Solutions at the collegiate level. An avowed objective was to raise the general level of mathematical education. It was published simultaneously at Cambridge, Massachusetts, and London, England. Although what seemed to some at the time as a high subscription rate of $3 per annum, it continued to increase in number of subscribers and number and quality of published articles. Never did

financial stringency force it to reduce an issue to emaciated thinness. This bold venture started without official sponsorship by any organization. Its founder and editor, John D. Runkle, A.M., was not in the educational field, nor was he, in his words, a professed mathematician. He was a full time government employee engaged largely in routine computing for the Nautical Almanac Office.

The founder pointed out: "All mathematicians know that there are many subjects in the higher departments of the science upon which little if anything has as yet been written among us." Again, "To the student or young mathematician, problems are indispensable to make him *sure*, and to *ground* him in the theory, as well as to exercise and develop his skill." Again, "Care should be taken not to graduate the magazine as a whole too high above the average attainments of mathematical students."

On p. 40, vol. 1, an author remarks on "the already numerous instances so frequently happening in our country of an almost useless expenditure in time and effort at the rediscovery of things brought to light centuries ago."

Although the founding of this periodical was endorsed unanimously by the Section of Mathematics and Physics at a meeting of the American Association for the Advancement of Science, it started with subscribers numbering only 62.

In accordance with the spirit of the time, Runkle established at the start two sorts of prizes: those for undergraduate problem solvers, and some more munificent for authors of expository articles.

Perhaps you will be surprised to learn the names of some of the distinguished regular contributors who helped maintain the high scientific quality of the young magazine. These included Simon Newcomb, Benjamin Peirce, Joseph Henry (of the Smithsonian Institute), William Chauvenet, George Washington Hill, and Arthur Cayley. As for the phrase "some years ago," in logic, "some" means "at least one" or "not none." A historian cringes at the phrase "many years ago," and one need not explain why this account did not start with "Once upon a time." The date was 1858. But what neither apathy nor financial crisis could do, war did. As has happened so often throughout world history to so many periodicals, in the midst of editorial work for coming issues, suddenly without prior notice to anyone, the *Mathematical Monthly* ceased forever to be. The year was 1861.

The resulting gap in mathematical periodicals lasted through the great depression following our Civil War. It was not until 1874 that America recovered enough to support *The Annalist*, which has continued, with a change of name to *The Annals of Mathematics*, through many vicissitudes to this day.

One might mention another periodical founded in 1889 in Manchester, New Hampshire, with the title ending, "et cetera"; namely, *The Monthly Journal of History, Folklore, Art, Science, Literature, Masonry, Mysticism, Mathematics, Metaphysics, Theosophy, etc.* I believe it too has passed away.

5. The *Monthly*. In 1894 was launched another American venture with almost precisely the same objectives and analogous departments as its predeces-

sor, and named *The American Mathematical Monthly: A Journal for Teachers of Mathematics in the Collegiate and Advanced Secondary Fields.* It was founded, edited and published by Benjamin F. Finkel of Drury College, Kidder, Missouri, with the assistance of a succession of associate editors.

One must confess that this new magazine, destined to become the official organ of our Association, springing into being in the heart of the Mississippi Valley, despite its high and wide aims, seems, in retrospect, on the whole, to have started off as quite provincial in comparison with its short-lived predecessor of briefer name. Let me cite some items among many instances. In a eulogy of Prof. E. B. Seitz, Prof. Finkel wrote, "He was the most distinguished mathematician of his day (1846–1883), the equal of the best, the peer of all the rest. Prof. Seitz was in Mathematics what Demosthenes was in oratory, Shakespeare in poetry, Napoleon in war." Equally laudatory eulogies, mainly of living mathematicians, appeared one each issue. Prof. Daniel Kirkwood of Indiana State University was hailed as the Kepler of America. Nor was a chance at a little moralizing to be overlooked. We are told in his biography of Prof. Hoover: "There is a valuable lesson to be learned. It is this: Energy and perseverance will bring a sure reward to earnest effort." Early issues brought much bitter argument and personal denunciation over what most of us would regard as matters of pedagogical taste. However, there was also much to respect. L. E. Dickson, while yet a graduate fellow at the University of Texas, was already launching a brilliant and prolific scientific career, as was David Eugene Smith, then of Michigan State Normal School. G. A. Miller, not yet a millionaire, also contributed regularly. Halsted dealt with charm and excitement on mathematical subjects, without intruding here his popular addresses on such topics as "Dreams," "Suicides," "The Elixir of Life," and "Mexico." E. H. Moore contributed a wise comment.

Among the current publications reviewed in early issues were Fiske's *A History of the United States for Schools*, and an issue of the *Cosmopolitan* for 1894 with a story by Kipling. In response to an inquiry originating from Brown University, the editor stated: "The Trisection of a straight line taken as the chord of a circle trisects the angle of the arc."

The largely rustic quality of early issues of the MONTHLY seems the more surprising in view of the consciousness of mathematical scholarship being developed through the recently founded University of Chicago (with E. H. Moore and his associates from Germany), already in full swing. Even in Texas, George Bruce Halsted had been spreading with singular effectiveness knowledge and enthusiasm in such lines as four-dimensional geometry and non-Euclidean geometry. But perhaps no one of these individually had the world-wide effect that did the mathematical lectures and personality in this country in 1894 of the great Felix Klein, then at the height of his powers and fame. Six years earlier had been founded the New York Mathematical Society, supported by all the leading mathematicians in this country (those in the University of Chicago, among others). In 1894, E. H. Moore secured the publication by the New York

Society in book form of the *Colloquium Lectures* that had been delivered at the International Mathematics Congress held in Chicago in 1893 in connection with the stunningly beautiful Columbian Exposition. In the summer of 1894, in connection with a summer meeting of Section A of the American Association for the Advancement of Science, the New York Society, recognizing its already achieved national status, adopted its present name, the American Mathematical Society.

It is inconceivable that this nationwide contagious interest and deep concern in mathematics did not have a share in starting the new journal at Drury College. But there seems nothing on the surface to show for it in the pages of early issues of the *American Mathematical Monthly.*

Already in 1888, close by in Chicago, mathematicians and teachers of mathematics were meeting regularly in what was to develop through the wisdom and tact of E. H. Moore into the Chicago Section of the A.M.S. Also in 1894, in California, a San Francisco Section had come into being.

The time was ripe for mathematical journals in the English language. In 1894 simultaneously with *The American Mathematical Monthly*, was founded the already mentioned British *Mathematical Gazette.* In 1895, Artemas Martin in Washington was editing and publishing both *The Mathematical Visitor* and *The Mathematics Magazine.*

In the next eighteen years, the *American Mathematical Monthly* endured many vicissitudes, acquiring and changing editors and sponsors, but always with B. F. Finkel at the helm. Even with L. E. Dickson as the associate editor from June 1905 to July 1906, and under sponsorship of both Drury College and the University of Chicago, and later from August 1906 to December 1908 sponsored by the University of Chicago, the issues became thinner as the quality of scholarship improved. The issue for March 1909 had only 16 pages. This was under the joint editorship of Finkel, Slaught and G. A. Miller, and with both the University of Chicago and the University of Illinois as sponsors. With June 1913, there was a general reorganization, with as sponsors the Universities of Chicago, Illinois, Indiana, Iowa, Kansas, Michigan, Minnesota, Missouri, and Nebraska, and with Colorado College and Oberlin College. H. E. Slaught became managing editor, and financial prospects began to brighten significantly. One cannot but note the geographical concentration of the sponsoring institutions.

Before turning to more serious history, you might consider the persistent interest in problems and their solutions at the collegiate level. This department in later years was largely directed by Otto Dunkel. Were a centennial American collection to be made of such problems, a first problem might be the choice of a title, since "Puzzle Parade," "Tomorrow's Math: Unsolved Problems for the Amateur," "Denkspiele für Kluge Köpfe" and "Pillow Problems" have already been preempted, not to mention "Recreationum Mathematicarum" (1659) and the still older "Il Nobilissimo et Antiquissimo Giuoco Rythmomachia" (1572). One might go back to Archimedes and earlier, Pythagoras. Were the title "Wrinkle" still available, what could be more descriptive of the puzzler's brow?

One might well issue as a centennial volume: "*The Wrinkle*" by Runkle, and Finkel, and Dunkel.

6. The Growth of the Association. Although the original constitution has been slightly amended, as in September, 1920, such was the wisdom and foresight of the founders that there has been no major change in plan or policy of our governing body. The four most obvious changes, apart from general growth in volume, complexity and diversity of business, are probably the following:

1. From the start as a matter of precedent desired by each of our first two presidents, Earle R. Hedrick and Florian Cajori, and followed by tacit tradition thereafter, no president has accepted nomination for a second term. Until the year 1925, the term of presidency was for twelve months; but in that year the constitution was amended to provide for a two-year presidential term while also making constitutional the single-term provision.

2. To make it legally possible for the Association to receive gifts (of which many were then and later forthcoming), the Association was incorporated in September 1920. The Council, which had functioned successfully, changed its name to the (19-member) Board of Trustees. Of these 19, seven were members *ex officio* namely the four executive officers of the Association, and the three managing editors of the MONTHLY. The remaining 12 served in groups of four, elected for three-year terms.

3. In 1941, the Board was to be replaced by the more effectively representational Board of Governors, consisting of the officers, the three latest ex-presidents, six governors-at-large and 14 regional governors, these last elected by the representative regional electorates for two- or three-year terms.

4. As a gracious and most fitting gesture, Herbert E. Slaught, fourth president of this Association was elected unanimously in 1913 as Honorary President For Life. After his decease in 1937, a similar honor was voted in 1943 to William D. Cairns, the first secretary and 19th president of the Association.

The growth and maturing of the Association may be suggested by the following items: In January 1916, there were five sections: Kansas, Ohio, Missouri, Iowa, and Indiana. In January 1944, there were 24. In 1916, there were 1045 members. In 1944, the membership amounted to 2270. In 1944 the MAA had representatives on the governing boards of the following: The National Research Council, The American Association for the Advancement of Science and the American Documentation Institute.

While even in 1941, Thornton C. Fry had described the rather limited demand, present or forseeable, for mathematicians in fields other than teaching (he put 15 mathematicians in industry as realistic for that date), and William Betz had for decades mourned and thundered against the vanishing call for mathematics in American education, the advent of World War II had changed the picture, and instantly. In 1942, the Joint War Policy Commission announced that there were about 3000 men functioning as teachers of mathematics in 4-year colleges, but that 2500 would be needed full time for teaching Army and

Navy programs alone. By 1943 there was an acute nationwide shortage of mathematicians—and of paper—and 5000 members of our armed forces were taking correspondence courses in mathematics.

The first of the distinguished succession of the Carus Monographs was that of G. A. Bliss in 1921 on Calculus of Variations. The Chauvenet Prize for "expository articles in mathematical journals" was established in 1926. Its seven recipients in the period through 1943 were G. A. Bliss, T. H. Hildebrand, G. H. Hardy, Dunham Jackson, G. T. Whyburn, Saunders MacLane and R. H. Cameron. A plan for the *Herbert Ellsworth Slaught Memorial Papers* "pamphlets of an expository value, and of from 40 to 80 pages in length," was announced in February 1941, and was promptly put into effect.

A department of Undergraduate Mathematics Clubs was inaugurated in 1919 and was for years directed by R. C. Archibald with unflagging enthusiasm. It may be worth noting that privately Archibald seriously objected to what seemed to him as bureaucracy in the very concept of the Pi Mu Epsilon. He favored strong local faculty control of all undergraduate organizations, unimpaired by national officers.

The highly successful William Lowell Putnam Prize Competition was instituted in 1938.

Now for a rueful thought, here is the advice given by E. T. Bell to the American Mathematical Society at its semi-centennial. Would that it had been heeded here. "Choose young men, preferably under thirty, to review the progress of American mathematics. For if there is anything that will give a man, young or old, a decent humility and a sane humor regarding his own efforts, it is an acquaintance with the work of his predecessors and contemporaries. The earlier this acquaintance is gained, the better for all concerned, including mathematics."

If from this historical retrospect one is to gather any constructive ideas for future development perhaps this one suggestion is worth passing along:

To have a standing provision for two recurrent decennial committees, which each ten years, reconstituted with entirely new membership, would, like the National Census Bureau, asking the same sort of questions, report fresh answers, namely:

1. A small committee on Recommended Books for College Libraries. Its timely up-to-date reports would be read and acted on.

2. A large joint commission with the National Council of Teachers of Mathematics on the Substance and Objectives of High School and First Year College Mathematics. Its reports might not be acted upon, but the Commission members would learn an awful lot.

So to take leave of you,—historically, as of during those somber days and nights of World War II when young and old sang wistfully, "When the lights go on again all over the world." A hundred million lights were about to go on in the electronic computers, blinking a new era in the continuing story of mathematics.

HISTORY OF THE MAA SINCE WORLD WAR II

R. A. ROSENBAUM, Wesleyan University

1. Preamble. Perhaps I may be permitted a personal remark which explains why I am so pleased to be given an opportunity to participate in this celebration: my father, a charter member of the Association, received his Ph.D. here at Cornell. I early came to know the mathematical names in Ithaca: Carver, Gillespie, Ranum, Silverman, Snyder, . . . and Hurwitz, who directed my father's thesis. It is a widely held opinion that mathematicians are especially attracted by two activities —music making and punning. It was Hurwitz, I have been told, who nicknamed a colleague with a reputation for making off with the magazines from the common room of Willard Straight: Procrastination, he called him, the thief of Time.

The summer meeting just 20 years ago was cancelled because of the war, and the first big post-war meeting was here at Cornell in August of 1946, with 277 MAA members in attendance. It's interesting to note some of the names: W. D. Cairns of Oberlin; Newsom, also of Oberlin; Allendoerfer of Haverford (before Winger persuaded him to defect to the West); Carver, Hurwitz, Agnew, and Walker of Cornell; Feller, Jones, Kac, LeVeque, and Rosser, also of Cornell. It looks as though fewer than half of us stay put. But there were (and are) stalwarts, like Gehman, of Buffalo; McShane, of Virginia; Meder, of Rutgers; Tucker, of Princeton.

The war was still with us in various ways: The Reverend J. T. O'Callahan of the College of the Holy Cross was awarded the Medal of Honor; the mathematics departments of several overseas colleges for servicemen were staffed by MAA members; the NRC offered fellowships to promising young people whose work had been interrupted by the war; the problems of rebuilding faculties, of allocating manpower, were pressing, especially in the face of droves of veterans returning to college under the G.I. bill.

We were on the threshold of a new era, involving dramatic changes for the rôle of mathematics in our culture and of mathematicians in our society, and the MAA was bound to be deeply affected. I shall not attempt to chronicle the details of these changes, but rather to sort out some of the dominant features of the period.

My notions of historiography have been influenced by Carlyle's "On Heroes and Hero-Worship," and I shall continue with reference to personalities. Mathematics, after all, is made by mathematicians; and our Association bears the imprint of their interests and enthusiasms, their devotion and hard work, their prudence and logic, their emotions and prejudices. I can select only some examples and will of necessity omit acknowledgement of the equally important contributions of many others.

One of our heroes is the embodiment of our virtues, although he might not appear heroic in other subcultures. I refer to Paul Erdös, whose contributions

to the MONTHLY and to Association meetings we appreciate along with his brilliant achievements in research. Erdös combines creativity and honesty with generosity and humility, extraordinary mathematical talent with exceptional human feeling. With unusual breadth in mathematics, he is able, and ready, to help anyone with a question. He can always find something good to say about the mathematics of a fellow human being, even though he may despise his politics.

It seems a long time since Erdös appeared at the Swarthmore meeting after Christmas of 1946, newly returned from the clothing store to which a group of friends had dragged him, attired in a brand new suit which somehow already looked slept in. There he stood, smiling gently at his friends' silly concern—he didn't resent the time which might have been wasted in buying the suit, for, in truth, no time had been wasted—he had thought about a mathematical problem throughout the merchandising transaction.

How different are our meetings these days! Wandering into our Association meeting at the Denver Hilton, one might think it a convention of successful account executives. "These days mathematicians can *eat*," said a personnel officer of United Aircraft, and how well-fed we are! And well-groomed too.

But it's far from accurate to imply that our ideals have changed. Our Association still harbors innocents who wouldn't notice the sneer when asked, "Have *you* ever *met a payroll*?" and who would respond, "No, I don't believe so; that's an unusual name." We also have a considerable number who have helped very substantially, one way or another, to meet some impressive payrolls. The MAA may be unique among professional organizations in having an unusually *diverse* membership *unified* by some common interests.

Indeed, I think that *unity in diversity* best characterizes our Association during the past 20 years. Let me explain what I mean. There are a number of traditional dichotomies which have become stock phrases:

(1) teaching vs. research
(2) pure vs. applied
(3) classical vs. modern
(4) intuition vs. rigor
(5) science vs. humanities
(6) elementary vs. advanced.

Like most dichotomies, these are invalid oversimplifications. (Indeed, I think all dichotomies are pernicious—except that into male and female.) Just as my caricature of our members as being either absent-minded eccentrics or sleek Madison Avenue operators is fanciful, so likewise are the various polarizations which I have just recited. In each of the relevant categories we see an entire *spectrum*.

In retrospect, it appears that during the past 2 decades the Association has encouraged a dialectic, so that we are beginning to understand and appreciate each other's positions. It is not too much to hope that we will have some reconciliations of diverse positions which will significantly enrich collegiate mathematics.

I should like to describe what I see as the Association's rôle in these dichotomies.

2. Teaching vs. Research. The rubric "teaching vs. research" is too brief to be clear; the debate that I'm thinking of is "(teaching *and* research) *vs.* (teaching *or* research)." The supporters of the first view have been eloquent: it is a perversion of our subject, they say, to separate teaching from research; our students must be exposed to scholars excited about their own creative work, for otherwise those students are likely to think of mathematics as a completed subject; on the other hand, the research worker often finds that in seeking a presentation which is really effective pedagogically, even (or perhaps, especially?) at an elementary level, he is led to a point of view from which his own research is illuminated.

All well and good, say the opponents, if you can find enough such individuals. Unfortunately, Emil Artin is rare (if not, indeed, unique!). Often creative mathematicians are simply not good expositors; and moreover, those endowed with both gifts are unlikely to have the interest, energy, and time to develop both. We are better off if we exploit to the full the talents of master teachers—individuals who are conversant with a broad range of mathematics and with current developments, even if they do not themselves contribute to research in the usual sense. Such a person should hold the position of Chairman of Undergraduate Mathematics at a university, sharing responsibility with his research-oriented colleagues for the shaping of the curriculum, and giving careful supervision to young teachers, for enthusiasm for teaching is as communicable as for research. Further, instead of conscientiously turning out papers, these mathematicians can focus their attention on the work of researchers, analyzing, synthesizing and evaluating results—and writing enlightening articles for the MONTHLY and the MATHEMATICS MAGAZINE. Indeed, we should encourage the *development* of such useful mathematical citizens by offering an appropriate degree, rather than the Ph.D. where research is all important and teaching is ignored.

Oh, no! let's not lower our standards; let's not debase the doctorate, respond the first group. Classifying college and university mathematicians as *either* teachers *or* researchers would be a most unfortunate divisive move. The teacher who is not also a researcher will not have the inspirational qualities, will not have the visceral understanding of what mathematics is all about, these being necessary for the kind of teaching we want. Note the experience of the MONTHLY, which tries to publish exactly the sort of article we were talking about a few moments ago—the thoughtful review of research which gives the nonexpert reader a grasp of where our subject is going. Editors of the MONTHLY have always complained about the dearth of such articles, but we all remember some notable examples: one rich year contained outstanding pieces by Coolidge, Gödel, Kac, and Polya. (Note our debt to Europe.) Well, these are the teachers *cum* researchers who couldn't have done their *teaching* so effectively without their researchers' understanding.

But look, answer our friends on the other side, you're hitting a cripple with his crutch. If our education tried more consciously to develop the analytic and synthetic powers of our young people, perhaps we would find more nonresearch mathematicians who could produce articles as profound as the ones you cited. In any case, the need for talent *of all sorts* appears to be virtually limitless, and the Association has pointed the way toward our involvement in many activities for which we ourselves were not trained; let us help some of our successors to acquire the necessary training.

And so the debate continues—fruitfully, I believe. Five years ago, E. E. Moise was chairman of the Joint Committee on the Doctor of Arts Degree. The decision was *not* to recommend that such a degree be offered; but the problem is still with us, and our discussions may eventually lead to a doctorate with *increased* breadth and *no* loss in depth. We should be mindful of a recent study which disclosed that a high percentage of the ablest American graduate students are bitterly disappointed with their experience—graduate school has not provided the intellectual excitement which they had anticipated. Fortunately for us, the percentage of disappointed students in mathematics (and science) is low, as compared with the situation in the humanities and the social sciences, but we cannot be smug on that account. We have probably made less progress toward a resolution of the teaching-research dichotomy than of any of the others I've listed, and, correspondingly, the mathematical future of the collegiate institution remains in doubt.

This year, when there will be $5\frac{1}{2}$ million students in college, the problem for many of us is not the luxurious one of providing a top-notch teacher-researcher for each of our undergraduate classes, but the desperate one of finding *any* teacher for the job. Not long ago I met a retired naval officer who planned to spend several summers at institutes to prepare himself to teach high-school mathematics. He dropped into the administration building of a college, hoping to get a temporary job as assistant registrar or something similar, until he was ready to undertake high-school teaching. An hour later he emerged as acting chairman of the math department! The teacher shortage is so acute that it won't be solved by other than radical approaches. We must strike out in imaginative ways, utilizing new media (films, TV, programmed materials, computer based instruction, . . .) to obtain a more efficient use of our manpower. We must go to the heart of the matter and encourage greater initiative and independence on the part of the student, so that he won't be so demanding of faculty time. Both our Committee on the Undergraduate Program and our Committee on Educational Media are attacking both aspects of the problem: the immediate one of getting classes "covered" and the deeper, longer range one of giving students some contact with creative, research mathematicians. The CEM began under the leadership of H. M. MacNeille, who is now directing its Calculus Project, aimed at using various media to raise the level of instruction in calculus. Brewster Gere directs a Programed Learning Project; A. N. Feldzamen is Executive Producer of a Film Project which, as successor of the MAA Com-

mittee on Production of Films, will show some of our greatest contemporaries in action. I'll mention the fourth project later.

Many of us believe that a spark can be struck in a brief encounter between an exciting mathematician and a receptive student, and the Program of Visiting Lecturers is designed to strike such sparks. As R. E. Gaskell, the chairman of our MAA Committee on Visiting Lecturers reports, "While the program is supported financially by the NSF, its work would not be possible without the cooperation of the home institutions of the 75 lecturers on our list, who generously grant them time for this work, or without the cooperation of the lecturers . . . , who were given nominal honoraria for the work involved in preparation, travel, and in the visits themselves." As a former member of Gaskell's committee, I can join him in testifying to the hard work and effectiveness of our MAA lecturers. The program has been in operation for over a decade and has been widely imitated in the sciences. During the first several years our lecturers took leave from their universities and went on tour. People like G. B. Price learned of the concert artist's problems with transportation schedules and laundry—with no impresario to smooth the way. More recently the NSF has been willing to support only day-long visits (about 360 of them last year). Our lecturers must make provision for their absence from the campus, and somehow catch up on their return. The NSF and the mathematical community have got their money's worth, and then some, from this program.

There is still a fourth MAA activity aimed at a resolution of the teaching-research dichotomy: the summer institutes and conferences for college teachers. They began in 1953 with an institute directed by B. W. Jones; the lecturers were Artin and Wilder—now there's a pair to try to surpass! This program has given college teachers a chance to get back into the stream of current mathematical work. Under E. A. Cameron's direction, a "multiplier effect" has been incorporated into Summer Seminars: each participant is given released time to conduct a seminar for colleagues at his home institution during the academic year following the seminar.

Let me close my discussion of this topic by referring to a question on which there is no consensus: Just how rare is mathematical talent? Many observers believe that such talent is indeed rare and that we now come close to discovering all of it. According to them we cannot markedly increase the number of Ph.D.'s in mathematics if we wish to maintain traditional standards—the native ability just isn't there. Others (and R. L. Moore might be in this group) believe that many more people could create mathematics if they had the proper instruction. It's pretty hard to devise a controlled experiment to answer this question.

3. Pure vs. Applied. I turn now to the second item on my list, the relationship between pure and applied mathematics. Despite the attention which the Association has increasingly paid to it in recent years, the relationship is uneasy and unsatisfactory. Many mathematicians would agree with the statement of

G. A. Bliss, which appeared in his autobiographical note in the MONTHLY in 1952: "After some careful consideration I concluded that it was really . . . mathematics which had the greatest attraction for me, and I decided to try for a Ph.D. in that subject rather than in astronomy. I have never regretted the decision, though it seems clear to me that one could find great satisfaction in any domain of applied mathematics."

If this were a general sentiment one might expect that there would be no such thing as a dichotomy between pure and applied mathematics, that we would all use problems of the real world to motivate the development of abstract mathematics, that we would then exhibit the "usefulness" of our mathematical results—in short, that we would weave together the pure with the applied to enjoy the fruits of that symbiotic relationship as did our masters Poincaré, Birkhoff, Weyl, and von Neumann. But the actual situation is very different from this utopian prospect. Note a few examples.

Item. One of the world's leading universities has four professorial chairs, two held by pure mathematicians, and two held by applied mathematicians. The four men are socially a congenial group, but professionally they divide into two pairs—in the obvious way—and mathematical interchange between the pairs is nonexistent. Is this not unfortunately all too common? But it's not universal—A. J. Kempner and C. A. Hutchinson of Colorado furnish a counter-example.

Item. The director of a National Laboratory has recently blasted the trend in undergraduate mathematical education, asserting that, in the hands of "purists," the needs of nonmathematicians are being ignored.

Item. A famous physicist is co-author of a new book to teach calculus quickly to freshmen, because he feels that the usual freshman mathematics course does not provide the material which physical scientists need.

It's not that we in the MAA are ignoring the problem. Our Committee on the Undergraduate Program in Mathematics has four panels, of which one is devoted to the Physical Sciences and Engineering, and another to the Biological, Management, and Social Sciences. As R. J. Walker has made clear, the push to introduce linear algebra very early in the undergraduate curriculum has come from the physicists and engineers, not from the "purists." CUPM publications in the offing include sets of computer-oriented problems and engineering problems, to "enrich" our freshman and sophomore mathematics courses. Seminars, conferences, and institutes (some of them year-long) have been held for social scientists and for faculty in business schools. In considerable measure these involve bringing "our" mathematics to nonmathematicians, but there is a substantial feed-back. Almost 50 years ago the pioneering freshman book of F. L. Griffin introduced numerous examples from the social sciences and the life sciences, but this lead was not widely imitated until recently. Now we find in elementary textbooks not only the topics which Griffin's book contained (such as finite probability, and statistics), but also the new tools of the decision maker—linear programming and game theory. The MAA itself has contributed a publication—*Modern Mathematical Methods and Models.*

At a more advanced level our Association collaborated with the Society by publishing as a Slaught paper in 1954, *Special Topics in Applied Mathematics*, the proceedings of an AMS symposium whose purpose F. J. Weyl described in these terms: " . . . the emphasis was on diversity of topics, on highlighting some exciting developments of the day and doing it in a manner to induce broad understanding among the mathematically literate. What we wanted was a collection of significant examples, not brought as yet too often or accessibly to the attention of . . . mathematicians, which would show just how varied and substantial the contributions of the genuinely secular mathematician (as distinct from the monastic one) can be in the interaction of his modes of thought and analysis with the scientific endeavors in other fields." (While I applaud Dr. Weyl's attempt to replace the adjectives "pure" and "applied" with more appropriate ones, I am not persuaded that "monastic" and "secular" are accurately descriptive, when I observe, on the one hand, an academician in Washington resembling a frenetic lobbyist while chasing financial support for his "pure" project, and, on the other hand, an IBM employee sucking reflectively on the stem of his pipe in his "cell" at Yorktown Heights.) To return to the point: the 1953 Symposium on Special Topics in Applied Mathematics is but one of many examples of our Association's attempts to make us old conservatives see the light. I predict that, for the continued health of our subject, this aspect of our education will have to be intensified.

4. Classical vs. Modern. In turning to those features of collegiate mathematics for which I have chosen the catch phrases classical vs. modern, intuition vs. rigor, science vs. humanities, I remind you of where we stood 20 years ago. In a 1946 review in the MONTHLY of Murnaghan's *Analytic Geometry*, C. C. MacDuffee, the first post-war president of the MAA, quoted with approval a remark of the author of the book, "The demands of modern physics and engineering for a knowledge of the elements of calculus are so pressing that it is no longer practicable to postpone the teaching of calculus to the sophomore year." How things have changed, and how the pace of change has quickened!* Just before World War II, Arnold Dresden, one of the most respected, and usually one of the most effective, leaders in both Association and Society affairs, persuaded some colleagues in colleges and universities to meet for a discussion of needed reforms in the undergraduate curriculum. The group met, came to general agreement—and there it ended. The urgency which Dresden felt (and doubtless some of the others felt it, too) was not translated into action.

After the war the atmosphere was different. In fairness to ourselves, it should be noted that our concerns did not stem from Sputnik—these concerns

* In 1940 the General Education Board subsidized a study, directed by F. L. Griffin of Reed College, to suggest an integrated mathematics curriculum for a small college. Now, only 25 years later CUPM has done something similar with its General Mathematics Curriculum. How our subject has changed in this period!

had been growing over a period of years, and their time had come. Of course, the Russian space shots accelerated our activities and facilitated our obtaining the funds necessary to implement our ideas. But we would have gone ahead anyhow. Indeed, support for experimentation, both by the MAA as an organization and by MAA members, came initially from private foundations rather than from the government, and the so-called private sector of our economy continues to give us vitally important help. Not only the large foundations, like Carnegie, Ford, Rockefeller, and Sloan, not only the giant corporations like U. S. Steel, IBM, and Bell Telephone, but also the smaller organizations like Research Corporation have made and continue to make grants which encourage us to keep striving for perfection. And we shouldn't forget the number of man-hours of MAA committee time, and the number of man-hours of individual MAA members' time, that have also been devoted to this striving for perfection, *without* any financial support.

The people who started CUPM did their initial work without subsidy. I needn't say much about these developments, because William Duren will speak about them at our fifth session, on Thursday. He can be relied upon to be a faithful historian in all but one respect—he will undoubtedly underplay the significance of his own work in making CUPM the effective arm of the MAA which it is. We are all much in his debt.

Earlier I mentioned one of the MAA publications which grew out of CUPM work. There are three others which are related to the topics I am now discussing:

Elementary Mathematics of Sets, with Applications;
A Freshman Honors Course in Calculus and Analytic Geometry;
Universal Mathematics.

Without too much forcing, I can use these publications as foci for the points I want to mention.

The language of sets, and the theory of sets (whether elementary or advanced), have become synonymous with "modern" mathematics, in some quarters. It is doubtlessly true that the notion of a set now plays an important rôle in collegiate mathematics; and that, whereas the word occurred infrequently in the MONTHLY in 1945, it is now sprinkled liberally through that journal's pages. Perhaps more startling is the situation with topology: mention of the subject in the MONTHLY was rare in 1945; articles and problems involving topology are now common. In the undergraduate curriculum, number theory and geometries have probably suffered as work in topology is introduced, either in courses on topology per se, or as adjuncts of courses in analysis. The MONTHLY and the MATHEMATICS MAGAZINE may offend some mathematicians by still devoting "too much" space to classical number theory and geometries, but the journals may serve an important balancing function through their conservatisms in this regard. Indeed, our Association seems to push the pendulum into a daring swing when it shows signs of slowing onto dead center, and to pull the pendulum back toward stability when the amplitude gets too large.

The Freshman Honors Course in Calculus and Analytic Geometry is an outline of what Emil Artin offered at Princeton. No course has received more attention than introductory calculus, perhaps in part because of the profit to be made from a successful textbook. Some texts seem to do no more than warm over cold and mediocre mathematics, but there have been many in the past 20 years which contain features of mathematical and pedagogical distinction. One of our dichotomies constitutes a source of apparently endless debate: intuition vs. rigor. We get agreement on the dictum, "The student should not be taught anything that he will have to unlearn later," but here agreement stops. In all likelihood we cannot expect a single approach to be successful with all students, and the constant debate may be useful as a reminder of this fact. Over the years the MONTHLY has published some helpful and provocative general articles on the subject (one by Walter Prenowitz is especially noteworthy for its sensible, judicious tone), and also some papers on specific topics, like A. E. Taylor's on L'Hôpital's rule and E. J. McShane's on Moore-Smith limits, which have made direct contributions to improved textbooks and classroom presentations.

Mathematics may be universal in a nonpersonal sense, but whether it can appeal to all people and be understood by all people is another question. Nevertheless, most of us think that we have to try. Fortunately, some of us have been reasonably successful—our hard-working Secretary, Henry Alder, reported not long ago on a course for nonscience majors. J. Robert Oppenheimer is pessimistic about what can be accomplished in physics—he fears that the concepts are so difficult that life-long study of physics is necessary for real understanding. Very likely he is right, in terms of the kind of understanding which is his ideal, but I'm willing to settle for a much more modest goal—I just want the man in the street to know that balancing my checkbook isn't part of my professional life. So I applaud Henry Alder and the others who are trying to bring mathematics to nonscience undergraduates. *From the point of view of our society there is no more important teaching that we do.*

5. Elementary vs. Advanced. I come now to my last dichotomy—elementary vs. advanced. It's appropriate to consider it last because it can give us a good feeling—we have taken remarkable steps toward a resolution since World War II. Collegiate mathematics stands in the middle, reaching out in one direction to graduate mathematics, the educational goal of some of our majors and the source of our faculty members, and reaching out in the other direction to pre-college mathematics, the professional goal of some of our majors and the source of our students. For many years we reached out much more eagerly in the direction of graduate mathematics than in the direction of pre-college mathematics. The advanced mathematics provided rewards of all sorts, from fun to recognition. The elementary mathematics was boring, grubby, frustrating, dominated by nincompoops, and riddled with politics. Besides, we didn't know anything about it. But we've learned! The willingness of Marshall Stone to under-

take work in mathematical education because of his realization of its importance should be a lesson to us—and it has been. Think of our distinguished Association colleagues who have accepted chairs in mathematics *and education*. Think of our many Association colleagues who have worked on projects from the College Board Advanced Placement Program down to the first grade. Think of the MAA High School Contest, with over 250,000 participants this past year. Think of the Association Program of Secondary School Lectures, which reached over 600 schools this past year; after almost 10 years of Association sponsorship, the high school program in mathematics (like those in the sciences) will be operated by state-level organizations, but our MAA Committee is continuing its concern for useful service. Think of the concern for teacher training, as exemplified by the thoughtful and detailed suggestions of a CUPM panel which began under John Kemeny's leadership and has prospered under the direction of E. E. Moise and G. S. Young, Jr.; these suggestions have been taken up by a CEM project initiated by Bernard Jacobson and now headed by C. B. Allendoerfer; we can expect that useful materials will soon become available.

The number of our members who are involved in this kind of work, their commitment, their willingness to accept criticism from experienced school teachers—all point to a growth toward unity over a spectrum of levels of mathematics which is the most hopeful sign for the future that I can imagine.

6. Other Activities and Problems. What have I left out? Numerous things:

(a) Size. We have over 16,000 members now. By only slight juggling of data we can say that our growth is explosive, i.e., exponential. If we maintain the same rate of growth in the future (and we all know that it would be unworthy of an American organization to settle for a *reduced* rate), then at our sesquicentennial, 100 years hence, the MAA will have 200 million members. In connection with this extrapolation, I am reminded of a story emphasizing rough approximation. An attractive young woman, attired in a striking strapless gown, with a canary perched on her shoulder, entered a bar, drank three martinis, looked over the group of handsome men on nearby bar-stools, and announced, "I'll go out with any man who can guess the weight of this canary." The handsomest man in the place quickly took up the challenge; he stepped forward, made his appraisal, and said, "Forty pounds." The young woman looked at him admiringly, smiled sweetly, and replied, "That's close enough."

As impressive as the size of the MAA is the size of individual departments. Where, 20 years ago, the News and Notices used to report that a certain institution had appointed Dr. X as instructor, we now have universities appointing platoons and regiments of mathematicians, mostly at officer rank—no privates. When we have 200 million members, I suspect that half of them will be in California and the other half in New York. Assuming that we remain as peripatetic as we are now (through leaves, sabbaticals, grants, contracts, and projects), mathematicians on the East and West Coasts will switch positions year by year.

The two MAA Journals—the MONTHLY and the MATHEMATICS MAGAZINE—are growing in size, too. One of the attractive features of these magazines has been their small size—one can hold the journals comfortably and browse through their various departments, almost like the *Saturday Review*. But now so many books are being published that the magazines threaten to get fat just to give adequate notice of new publications. The backlog of articles is comforting to the editors and distressing to the authors. How will we cope with this problem? That's a worry for our Committee on Publications.

(b) Various awards and prizes, including the Award for Distinguished Service, the Chauvenet prize, the L. R. Ford Awards, and the Putnam prize competition (with 1500 undergraduates participating this year).

(c) Various hard-working committees, such as the committee on advisement and personnel, and the finance committee.

(d) Various publications, like the Carus monographs [I mentioned the Slaught papers earlier].

(e) Hedrick Lectures.

In many ways the MAA has grown into a worldly organization. (Can you imagine—we have bellhops to carry our bags to our rooms!) We nonchalantly accept millions of dollars from the NSF to support our major projects. But we're not really in the big leagues, thank goodness—we don't build linear accelerators. Al Feldzamen makes movies for our CEM at a fraction of the cost of those of other scientific organizations. Harry Gehman makes every nickel count and maintains an avuncular relationship to 16,000 nephews and nieces—a sort of Dutch avuncular relationship. Amazingly, we have preserved some of the *personal* qualities of the MAA which characterized its early days. As I said at the start of this talk, I mentioned a few names by way of a *sample* of MAA activity: we enjoy the participation of a tremendous number of our members in our affairs—on the Board of Governors, on Committees, in our Sections, and so forth. This is an impressive source of strength, and I hope that we can continue the tradition. For, mathematics, after all, is made by mathematicians.

CUPM, THE HISTORY OF AN IDEA

W. L. DUREN, JR., Chairman, CUPM, University of Virginia

I. Fifty Years of Undergraduate Mathematics Teaching.

1. *The idea.* It is appropriate for us in this fifty-year anniversary celebration of the Mathematical Association of America, the national organization of American collegiate mathematics, to review the state of undergraduate mathematical education in the past fifty years, look as best we can towards the future, and consider the work of the Association in it. We shall not be concerned only with the history of a certain standing committee, called CUPM, the Committee on the Undergraduate Program in Mathematics, but more generally with the idea of organizing nationally for the improvement of undergraduate mathematics as it occurs in the classroom and in the college.

2. *Before 1915.* Therefore let us begin by sketching very briefly the significant events and forces in American college mathematics in the fifty-year history of the Association. When the MAA was formed in 1915 there was another movement taking shape which was to have far more influence on college mathematics than MAA would have. This was the extension of compulsory general education in the United States from 8 years to 12 years, that is, to age 18 or through high school. Prior to World War I the American high school had been largely academic and college preparatory. Though public and tuition-free, it had a somewhat restricted enrollment. Mathematics was accepted as a standard subject. The student was expected to complete one and a half Carnegie units of high school algebra and one unit of plane geometry as a minimum college entrance requirement, set by tradition. This same mathematics package was also widely adopted as a high school graduation requirement. The mathematics teacher in the high school usually had completed the equivalent of a minimal undergraduate major in mathematics and many held the master's degree from one of the new American graduate schools formed after about 1890.

These new American graduate schools were beginning to produce an oversupply of graduates with the Ph.D. and their graduates had to take positions in undergraduate teaching for which their graduate education involving research capability went beyond anything they needed or could use in their teaching, and tended to make them discontent with their lot. The leading mathematicians in the American Mathematical Society were bent upon the establishment of a true research community in the universities. Undergraduate mathematics was a burden to them, even if it was a source of financial support as a job outlet for their graduates. However, it was to their credit that these mathematicians, in their drive towards research and graduate work, paused long enough to provide for undergraduate mathematics by helping to establish the Association. Indeed all American mathematicians can justifiably take pride in their continued concern for the teaching of our subject in schools and colleges [23].

3. *Twenty-five year depression* (1915–1940). The extension of compulsory general education through high school age soon overwhelmed the high schools. School educators rejected college preparation as the meaning of high school in favor of a long series of "needs" and "life adjustment," student-centered, philosophies which relegated the difficult subject of mathematics to a low priority indeed. Vocational education proliferated. The level of mathematics required for graduation was greatly reduced and the qualifications of mathematics teachers were ignored. In the colleges this hurt mathematics more than any other subject because it is far more dependent upon years of previous school preparation than the others. High school graduates, going to college in greater numbers over low admission requirements which reflected the colleges' need of their tuition money, soon reduced the subjects of college algebra and analytic geometry to remedial high school, and even grade school, mathematics under various euphemistic designations. The job of the well-trained mathematician teaching in an undergraduate college became still more unattractive. Several commissions of national scope, attempting to remedy the situation, failed [24]. This state of affairs continued until after World War II. Indeed, despite a reversal in the more visible sectors of the educational system, it continues today as the dominant environment in very large sectors which are less favored.

It was my privilege to come into the Association about ten years after its founding and in that stage of life when a brash young man says in effect: "Now that I am finally here, will the world kindly step forward and justify its existence." In my limited world I expected MAA to justify its existence. Was it a professional organization, or union, looking after the economic and social well-being of its members? No. Was it an authority maintaining professional or educational standards? No. Was it devoted to the teaching of mathematics? No, not directly. No, I finally decided that MAA existed to give comfort and status to college mathematicians. It was organized to hold meetings, elect officers, select hour speakers, and publish a journal, just like AMS, only at not so high a level. That was not bad. College mathematicians certainly needed comfort and status. This role of the MAA continues today, but no longer as its only role.

4. *Revival of public support* (1941–1957). After World War II American industry perceived its need for more mathematical education for its employees and, before the impact of either the computer or Sputnik, began to advertise this and offer jobs to mathematicians at all levels. So did government. The National Science Foundation entered the picture, expressing the national policy to cultivate science for the public welfare by an early emphasis on graduate study and research.

In 1953, before Sputnik and before the computer had run wild, E. J. McShane, as President of the Association, sensed the opportunity for educational reform which the new public support would give and appointed a special committee on the undergraduate program, where "program" meant the entire system: the curriculum, the teacher, and the student. This *ad hoc* committee made

a quick survey, reported back to the Board of Governors at the Kingston meeting in September 1953 that there was a "widespread dissatisfaction" with the college mathematics program and recommended a national "program of 'doing' to overcome the inertia of the enormously ponderous structure which carries onward the present program with all of its deficiencies [13]." Then the *ad hoc* committee asked to be discharged. The MAA Board approved the report and directed that the committee be reconstituted as an official committee of the Association to get on with the "doing."

We will return later to narrate the story of CUPM in its effort to "get on with the doing." Here let us note the significance of this action of the President and Board of Governors of MAA. By forming CUPM as a standing committee they involved the Association in undergraduate teaching in a role which it had never played before. This policy change was reinforced later by the formation of the committees on institutes, visiting lecturers, and more significantly the Committee on Educational Media. CUPM was established early enough to set up some patterns for national curriculum work in mathematics, notably the use of summer writing groups [2] to produce source books for new texts, early enough so that it could help to establish the School Mathematics Study Group for elementary and secondary mathematics, and in turn that was early enough so that SMSG could ride the crest of the big computer and space science wave and take advantage of it to make "the new math" a reality in public education. The formation of CUPM was early enough for it to furnish resources for the NSF Summer Institutes program in mathematics. Spreading to the sciences, this idea of organized national curriculum work in science education has become a very big thing in the United States and the leadership of the mathematicians in these matters is generally acknowledged. The idea is spreading abroad, as other nations [17] begin to move in similar ways. It has definitely taken hold. However, we are getting ahead of our story. We return to the general history of U.S. college mathematics.

5. *The space age begins* (1957–1965). The new demand for mathematics and the new Federal support quickly transformed life (and salaries) for university mathematicians. The effect accelerated as the computer field developed and went into orbit with Sputnik. It was pleasant for college teachers to read about the new importance of mathematics too, but overall, matters only became worse again for college mathematics teaching. By cruel irony, now that better students were here, especially the much desired majors in unprecedented numbers, and now that undergraduate mathematics could really challenge a teacher with a full graduate education, such teachers were not available to the great majority of colleges. The new demand had taken them away at the same time that it presented the colleges with five times as many advanced students as before [20]. This is the state of collegiate mathematics as of this date in 1965. With the biggest opportunity and the biggest task it has ever had, the U. S. college teaching staff is probably at the lowest average level of qualification in the 50 years of

MAA. By contrast, in the favored institutions, where most of the members of MAA curriculum committees are, the conditions of undergraduate mathematics are markedly improved, especially the students and the honors programs for the good ones designed by interested and capable faculty members. The disparity between the haves and have-nots in college mathematics has greatly increased in the last 10 years.

6. *The next 50 years* (1965–2015). Let us now try to look into the future, something which is always dangerous to do and even more dangerous not to do.

The easiest prediction to make at the present time is that the graduates of "the new math" in the schools will be so much better prepared, so much more advanced, and so much more sophisticated that we must prepare to teach first year students subjects which we formerly taught in graduate school, only in a more modern way. The only difficulty is in predicting exactly what fraction of the students entering college in the future will qualify for anything from one semester's advanced placement in calculus to the high level just described. Certainly the absolute numbers of students entering with advanced capability will increase, and surely they will gravitate to more favored institutions, as they do now. There they will not need CUPM help.

However, we can infallibly and inexorably predict that the most significant factor in American college mathematics in the next 50 years of MAA history will be the massive expansion of college enrollment, initiated by national, state, and local policy to extend general education from 12 years to 16 years, that is through college, and made as near compulsory as possible. The colleges will be overwhelmed by numbers and mediocrity just as the high schools were in the past 50 years as a result of the expansion of general education from 8 years to 12. This fact will far outweigh the influence of the advanced placement students in the majority of the colleges. Unhappily the college mathematics teaching staff of 1965 may be nowhere near as well equipped for the expansion as the high school staff of 1915 was. I am sure that American college teachers of mathematics will carry out their part of this grand educational undertaking with pride and to the best of their ability. We may also be sure that national organization to improve college teaching in mathematics, the CUPM idea, will be called upon to do everything possible, especially to enable teachers to teach more students, to teach subjects which they do not know, and enable both teachers and students to learn more by themselves.

7. *The objectives of a national curriculum committee.* It is clear enough what a national curriculum commission should do when, as in some European countries, it has authority to establish a curriculum and impose rigid standards. In the absence of any desire for or prospect of such power it was not clear in 1953 what the American CUPM should do, and it still is not clear. Several possible objectives have emerged and undoubtedly more will come out of experience. In 1953 the college program was not as good as the existing staff was

capable of teaching, or the students of taking, so the first objective we saw was to rally mathematicians to do better teaching, to halt the pessimistic retreat to remedial mathematics, to modernize and upgrade the curriculum, and to enlist the aid of the best mathematicians in this. This called for radical and bold forward steps, for a certain amount of showmanship. This objective will always be present. Yet now in 12 years we appear to have reached a stage where the curriculum has advanced faster than the teacher and the objective must be shifted to give aid to the teacher who needs help. It is no longer helpful to recommend that he teach more modern material, if he cannot do it.

I know that I am now departing from the role of a historian to express personal opinions but I should like here to make a plea as forceful and urgent as I can to university mathematicians, who work in CUPM, CEM, and institutes for college teachers. Find out what their job really is, what these college teachers are capable of doing, and give them something which will help them. Don't assume that it is like your job in your university. It isn't. Have the courage to keep the mathematics you offer them simple and appropriate to their tasks. It will not always be mathematics that you are proud of, not as general, modern, radical, original, innovative as you would like to represent your contribution. But if you want to share mathematical culture as you know it in your university with the teachers and their students in large and small colleges, you must start where they are and give them something they can use.

II. The History of CUPM.

8. *The original ad hoc committee.* We have related above how E. J. McShane as President of MAA in January 1953 appointed a Committee on the Undergraduate Program to consider what could be done about the state of undergraduate mathematics in the United States. The committee consisted of G. B. Price (Kansas), A. L. Putnam (Chicago), A. W. Tucker (Princeton), R. C. Yates (U. S. Military Academy), and W. L. Duren (Tulane), Chairman. The first internal memorandum called for policy decisions: To avoid criticism of public policies on education, concentrate on the technically capable students of the first college year, maintain liaison with engineers and physicists ("our best customers"), cultivate new relationships with the social scientists, consider a radical, abrupt change or nothing, be wary of overdoing abstract mathematics, reenlist the interest in college teaching of the best young and old mathematicians, and work to unify departments of mathematics and oppose splintering them. The ever-present problems of supported textbook writing in the public domain were considered in the same first memorandum.

The final report of the group was made to the Board of Governors in September 1953 [2] finding the "widespread dissatisfaction" and recommending: 1) That a "working committee" of five be established to "direct the work, prepare syllabi and commission the preparation of experimental teaching materials to implement a widespread trial of new mathematics courses in colleges;

2) That a subsidy of some $75,000 be sought . . . ; 3) That the working committee be allowed to make decisions . . . upon a basis of their mathematical knowledge and judgement, rather than ones which require ratification by any other body than the Board of Governors of MAA; 4) That the working committee be charged with studying especially the needs of the social sciences, the new needs of engineering and physical sciences, of modern statistics, and biological sciences; 5) That in the interest of liberal education a single theoretical core for a freshman course be sought instead of a battery of specialized courses for special programs; 6) That the new program be established on its own merits by experimentation without authority, accreditation or other form of compulsion; 7) That commercial relationships incidental to the preparation of text materials [be so managed] that the Association will not be in the textbook business."

The report was approved by the Board of Governors which directed that the recommended continuing committee be formed. The continuing committee appointed was the same as the *ad hoc* committee except that C. V. Newsom replaced R. C. Yates to effect a liaison with a joint committee of MAA and NCTM on teacher training.

9. *The work of the old CUP.* The CUP immediately began planning for a summer writing session which was to be held at Lawrence, Kansas, under the direction of G. B. Price, supported by the University of Kansas and a small grant from the Social Science Research Council via the Committee on the Mathematical Training of Social Scientists. The 1954 Kansas Summer Writing Group produced a volume of experimental text materials called "Universal Mathematics, Functions and Limits" [27], and some preliminary writing on a second volume called "Elementary Mathematics of Sets" later completed by an enlarged writing group at Tulane University and the University of Virginia [12].

These materials were tried out in classroom use at Tulane, Kansas, and University of Alabama among other places. These departments wrote critiques of their experience with it [14] [15].

A grant from the Ford Foundation supported the printing and distribution of these materials and later CUP enterprises. Even so, the early work of CUP was done largely on a volunteer basis without fees and with meager stipends for summer writing. This was one of its major mistakes, not to have a big grant.

The first major report of CUP in 1955 described these and other activities of CUP and related curriculum groups [2]. The report called for a single first year analysis course with a division thereafter into a classical calculus course and a new course in mathematical analysis for the social sciences. The latter was a new course for which the text material did not exist. The Dartmouth Writing Group of 1957–58 under the direction of J. G. Kemeny undertook to prepare a source book for this course, under the aegis of CUP. They were published in two volumes under the title, *Modern Mathematical Methods and Models* [19].

Other books encouraged by small helping grants from CUP but mostly

moral support in this period included Artin's *Notes for a Princeton Honors Course in Calculus* [1], Polya's *Mathematical Discovery*, vol. I. [26], and two or three *Carus Monographs*. In addition CUP supported preliminary writing or exploratory studies but not the final published version for several other books.

The idea that CUP would conduct open discussions in the mathematical community of important current issues in collegiate mathematics proved to be a fruitful one, especially in MAA meetings but also in conferences and in institutes [21]. Some typical questions were: The place of logic in undergraduate mathematics courses—now pretty well talked out—statistics [28], early calculus, undergraduate honors work or "undergraduate research." In this same pattern the later CUPM sponsored, without commitment of its own, the New Haven conference on the Doctor of Arts Degree in Mathematics.

Although there were some additions to the original committee including E. J. McShane, J. G. Kemeny, and Brockway McMillan, there had been no provision such as we have now for regular rotation. The committee thought that no small group should remain so long in such a pivotal position in mathematical education. Moreover, the limitations of its scope and available energy were not adequate to the rapidly developing national curriculum work, even after establishing a Central Office at the University of Virginia with R. L. Davis as full-time Executive Secretary. Accordingly the old CUP resolved to liquidate itself to clear the way for reorganization of CUPM with new blood and larger organization.

I might comment here upon the forms of organization of national commissions for improving education in a particular subject. The more common form, because of red tape and organization politics in national professional organizations, is to set up an independent commission under its own board usually housed in a privately endowed university. The difficulty with this is that it tends to become a self-perpetuating clique, not representative, and with a small number of individuals too long in control. The MAA idea has the advantage of a permanent national professional organization having a charter as the unique organization concerned with college education in its field, which can keep its CUPM continuously changing by limited term appointments, and can offer not only prestige but also publishing facilities, the communication facilities of its meetings, and liaison services not otherwise obtainable. Thus under the MAA-CUPM plan the national curriculum committee should in theory have assured prestige, viability, better national representation, better communication facilities for long-term operation than the independent commission can achieve.

The original CUP summarized its own work in two reports [3] [4], with some suggestions for new directions in CUP work. Almost as an afterthought these reports included a sketchy outline of a recommended college curriculum in mathematics. The old CUP had avoided making recommendations as being too prescriptive, but came to believe that its successor CUPM should publish recommendations, and it did, as we shall see.

10. *Reorganization of CUPM.* Before proceeding to reorganize CUPM the Association held the 1958 Washington Conference to review the overall program of the Association and formulate a plan of action [16]. This conference re-emphasized the change in character of the MAA to incorporate organized national curriculum work by recommending that the Association recognize five semiautonomous "departments" to carry on its old program of publication and its new programs on the undergraduate program, institutes and visiting lecturers, competitions, manpower and professional guidance.

With CUPM thus endorsed as a distinct arm of the Association, MAA in November 1958 conducted a conference, specifically on CUPM [22] which outlined a structure and a program of work for CUPM, including the preparation of a statement of minimal standards for mathematics teachers "in high schools, junior colleges and colleges," and a statement on preparation for graduate work in mathematics. It emphasized the need of textbooks and urged CUPM to develop a program of subsidized textbook writing. Also it recommended that CUPM delegate specific fields of work to "subcommittees" whose members need not be members of CUPM. These were later called "panels." I might comment that the result of enlarging CUPM itself, forming the panels, and other substructures ultimately brought the number of mathematicians holding appointments in CUPM to about 80, compared with the original 5.

The conference did not neglect to advise CUPM to get a grant from the National Science Foundation.

Early in 1959 incoming President Allendoerfer, upon the recommendation of the outgoing President, G. B. Price, appointed the new committee to consist of G. B. Price, Chairman, E. G. Begle, R. C. Buck, W. T. Guy, R. D. James, J. L. Kelley, J. G. Kemeny, E. E. Moise, J. C. Moore, Frederick Mosteller, H. O. Pollak, Patrick Suppes, Henry Van Engen, R. J. Walker, and A. D. Wallace.

The year 1959 and half of 1960 was devoted to organization and efforts to secure funds, which ended in June 1960 with a grant from the National Science Foundation adequate to support the Committee for full scale work for two years.

11. *CUPM operation under a large grant.* In April 1960 R. C. Buck was appointed chairman of CUPM. It became his responsibility to make good the proposed program of expanded activity. Under his leadership CUPM established a new Central Office at Rochester, Michigan, with Robert J. Wisner as Executive Director. It extended its work by establishing four panels: a Panel on Teacher Training with J. G. Kemeny as Chairman, a Panel on Mathematics for Physical Sciences and Engineering with Robert J. Walker as Chairman, a Panel on Mathematics for Biological, Management and Social Sciences with Patrick Suppes as Chairman, and a Panel on Pregraduate Training of Future Mathematicians with J. C. Moore as Chairman. Each panel had several members

of CUPM on it, other non-member mathematicians, and other scientists from the customer fields who were not professional mathematicians.

From this point this history will not record all of the names of the CUPM family, since they are given in the published annual reports of CUPM [5] and assembled in the appendix to this report.

The newly organized CUPM adopted a basic policy in which it felt that it was diverging from the original group. It involves a critical issue in the idea of organizing nationally for the improvement of mathematical education. As expressed by Buck, it began: "Any approach to college curricula revision must take account of valid differences of opinion about both content and pedagogy; thus there is no attempt to insist upon one specific program." The original CUP, SMSG, and CEM tried to carry out this principle by making no recommendations but, for economy of their own efforts, or for unity in presenting what they had to say, or for reasons of belief that in any one school the basic mathematics courses should be unified as much as possible, they tried to exhibit only one sample of source material for each course. All of them have had difficulty in being understood in this.

The new CUPM chose to carry out this permissive policy of allowable variation in a different way. It made curriculum recommendations but, not only was there diversity by virtue of the independence of the four panels which made them, but, as Buck continued: "In all documents released by CUPM, there have been attempts to indicate alternatives, and indeed to encourage experimentation as an aid to national activity." Accordingly, each of the Panels produced at least one curriculum recommendation for a four-year curriculum for students with special interest in its field and most of these offered alternative structures [6] [7] [8] [9].

Kemeny's Teacher Training Panel, working very hard, came up with its recommendations in a matter of months, then initiated a series of conferences, state by state, to review them with administrators, supervisors, and teachers. Unquestionably these CUPM recommendations have been very influential, especially in the training of elementary teachers, hitherto a difficult group to reach.

The success of this conference plan was due in no small measure to the great energy and imagination of R. J. Wisner as Executive Director. Besides organizing the Central Office, Wisner invented the Consultants Bureau and traveled tirelessly to make the Central Office concept a reality as a means of carrying on the work of CUPM in a magnitude not possible with only the voluntary part-time service of committeemen. Other Executive and Associate Directors of CUPM who have ably carried on the Central Office work while on one or two years leave from their universities are: Bernard Jacobson, A. B. Willcox, B. E. Rhoades, and R. H. McDowell (current Executive Director). For the CUPM idea of organizing nationally for the improvement of mathematical education to succeed, it has turned out that an effective Central Office is an absolute necessity.

When we come to the work of the other panels, in fields where the educational objectives are less specific than in teacher training, it becomes more difficult to evaluate the influence of CUPM recommendations. In reviewing the history of the idea of national curriculum organization in mathematics it would be desirable to do so. The nature of the influence is expected to be different from panel to panel.

The Physical Sciences and Engineering Panel, under the chairmanship of Robert J. Walker, has been remarkable for the patterns of consultation and collaboration that it developed with the Engineers, including at an early stage a conference with some 50 leading engineering educators, continuing through its progressive but realistic curriculum recommendations, its significant recommendations on work in computing [7], its source book of engineering problems [25] produced in collaboration with members of the Commission on Engineering Education, and its continuing work on curricula for space science and mathematical engineering. Its service attitude was expressed by C. W. Jones in his inaugural lecture as Professor of Applied Mathematics at Imperial College [18], comparing CUPM's report with that of a similar European committee: "The difference between these two reports is that whereas the American Committee consists of mathematicians who, having gone to some trouble to find out what is needed, are able to state in detail how it should be provided, the OECD report is the work of engineers and scientists . . . who have a lot to say about the drawbacks of the present state of affairs in Europe"

By contrast, the Panel on Biological, Management and Social Sciences, chaired first by Suppes, then Kemeny, and now chaired by Samuel Goldberg, is working in an underdeveloped country. Its recommendations are a kind of chart for the future, saying what is needed. The future work saying how that should be provided will of necessity demand a partitioning of this too-large collection of fields and will require the preparation of source materials, particularly in the least developed segment, biomathematics.

The work of the important Panel on Pregraduate Training has not yet appeared to be viewed as a whole. Its first document on *Pregraduate Preparation of Research Mathematicians* [8] was aimed too far into the future to be immediately useful to any but young mathematicians in the formative stages of their teaching. The panel's forthcoming recommendations on *Preparation for Graduate Study in Mathematics* is a more contemporary document describing how the goals of the first report can be approached in an existing department with available texts.

Much of the work of the Advisory Group on Communications, chaired by A. W. Tucker, is on publications policy of internal interest in CUPM but affecting the development of the idea of national curriculum work. However, working originally as an *ad hoc* Library Committee under the chairmanship of Arnold Ross, it produced a recommended Basic Library List [10] for a small college library. This may turn out to be quite timely and important in the forthcoming

era of Federal support of new institutions to meet the college enrollment expansion in the U. S.

The published reports of the panels by no means represent the total effort of CUPM. From the beginning the members of CUPM contributed to the programs of the Mathematical Association and its Sections and to the deliberations of many other scientific bodies and Federal commissions on problems of mathematical and scientific education. Creighton Buck, Baley Price, John Kemeny, Henry Pollak, E. J. McShane, A. W. Tucker and others have been very active in this speaking service, which has turned out to be one of the most important activities to stem from CUPM. In addition CUPM designed a number of seminars, conferences, and film projects which were carried out by other organizations with separate financial support.

In March 1963 it was decided to move the Central Office to Berkeley, California. In June 1963 W. L. Duren, Jr. assumed the chairmanship of CUPM for a three-year term. R. C. Buck remained on the Committee. There was no change in the policy to take account of the diversity of opinions and objectives by offering alternative solutions in CUPM reports.

A major effort was then directed to construct the forthcoming *General Curriculum in Mathematics for Colleges* (CMC) which is an economical synthesis of the panel recommendations for colleges whose programs are too limited to mount such a diversity of courses as the panels all together recommend [11]. This report introduces a new pattern for such statements. Instead of a curriculum, the CMC report offers a system of one-semester course modular units from which a large number of curricula can be constructed in a single department.

In another new pattern, the CMC report is being offered, not as a CUPM recommendation, but as a report to the Association with request for evaluation and criticism by the Sections. If the Sections take hold of this assignment it could be the means of extending within the Sections that transformation and enlargement of purpose which the idea of organization for curriculum improvement has already produced in the national Association.

The field of applied mathematics as an undergraduate concentration on its own, has been explored by an *ad hoc* committee of CUPM, chaired by C. R. DePrima. Its report calls for still new patterns of interdisciplinary and interorganization collaboration, extending into graduate work, at least with the Society for Industrial and Applied Mathematics and the Association for Computing Machinery. Other organizations may also be involved.

12. *The problem of the college teacher.* This history has dwelt in some detail with the use of panels and reports for national organization to get at a variety of difficulties in collegiate mathematical education. The panel method is not universally applicable. In fact, in spite of the many ingenious adaptations so far devised in CUPM, the panel and recommendation procedure may be useless in getting directly into the classroom. When the doors of a classroom are shut, it is relatively insulated from outside influences which would seek to improve

what goes on there. To reach into the classroom a national organization must influence either the subject matter being studied (the course of study, the text), or the teaching medium (the teacher, film, programming of self-teaching), or the student himself. Panel recommendations do not enter the classroom. Panels cannot educate teachers, or produce films, or write textbooks, or prepare students, or select them. The critical factor in the improvement of college mathematics has apparently now become the teacher.

Instead of being overtrained for his job, with a surplus of Ph.Ds., such as we had from 1915 to 1940, the teaching staff in strictly undergraduate teaching is now in 1965 so undereducated and overloaded that CUPM recommendations may come to nothing. The college teacher needs help. CUPM did not perceive this clearly enough to adapt its patterns of activity realistically to the post-Sputnik era or to the less favored undergraduate institutions which have fallen so far behind the universities in richness of mathematical environment.

These institutions can do little to help themselves. The leading universities can no longer produce first rate Ph.D. graduates in sufficient number to help much. So it is an opportunity for great service if a national organization for the improvement of mathematical education can bring relief to the bottleneck in the educational medium, either by helping to produce more qualified teachers, or by helping the less qualified teacher to teach better, or by helping the qualified ones to teach more students.

The problem of producing a qualified college teaching staff in mathematics is to be the main concern of CUPM in the next few years. This is a manpower problem in undergraduate colleges, a graduate school problem in education, and so far everybody's and nobody's problem in the Federal government. At present CUPM regards it as a *quality* problem. For unless the qualified teachers, or some reasonable substitute, can be found the many, many young Americans who go to the colleges whose physical capacity is increased by Federal and State dollars will not get a good college education there. Our concern is for the mathematical component of that education in a society in which mathematical skills are more important than they ever were before.

CUPM has some basic preparatory work to use: the reports of its visiting Consultants, a big survey of college environments completed in 1962 for CUPM by the National Opinion Research Center at Chicago, Henry Pollak's studies of what happens to undergraduate mathematics majors when they graduate, some manpower reports on Ph.D. production in mathematics. But whatever CUPM can contribute will depend on finding radically new patterns for an organized national educational improvement group to attack this problem. That is where we are today.

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Appendix. *Membership of CUPM and its Panels after 1959*, (as recorded). See text for membership of earlier committee 1953–58.

I. *The Committee on the Undergraduate Program in Mathematics.*

E. G. BEGLE 1959–63, Ex officio 1964– , R. C. BUCK 1959–65 (Chmn. 1960–63), LEON W. COHEN 1959–61, Ex officio 1964–65, W. T. GUY, JR., 1959–62, RALPH D. JAMES 1959–61, J. L. KELLEY 1959–61, J. G. KEMENY 1959–63, FREDERICK MOSTELLER 1959–62, H. O. POLLAK 1959–65, G. B. PRICE 1959–63 (Chmn. 1959–60), PATRICK SUPPES 1959–62, HENRY VAN ENGEN 1959–61, R. J. WALKER 1959–65, A. D. WALLACE 1959–62, ROBERT J. WISNER, Ex officio 1960–63, ROY DUBISCH 1962–64, DAVID GALE 1962–64, LEON HENKIN 1962–64, E. E. MOISE 1962–64, J. C. MOORE 1962–64, I. M. SINGER 1962–64, R. P. BOAS 1963–66, D. E. CHRISTIE 1963–65, W. L. DUREN, JR. 1962–65 (Chmn. 1963–65), SAMUEL EILENBERG 1963–66, SAMUEL GOLDBERG 1963–65, G. S. YOUNG, JR. 1964–66, R. D. ANDERSON 1965–67 (Chmn. 1965–), C. E. BURGESS 1965–66, M. L. CURTIS 1965–66, C. R. DEPRIMA 1965–67, LEONARD GILLMAN 1965–67, GWENETH HUMPHREYS 1965–66, ROBERT C. JAMES 1965–66, LOWELL J. PAIGE 1965–67, ROBERT M. THRALL 1965–67.

In addition, the Presidents of MAA often actively participated as members of CUPM in their ex officio status in their two-year terms of office. These were in order from 1953: E. J. MCSHANE, W. L. DUREN, JR., G. B. PRICE, C. B. ALLENDOERFER, A. W. TUCKER, R. H. BING, R. L. WILDER 1965–66.

II. *Panel on Mathematics for the Biological, Management, and Social Sciences*, (with dates where recorded).

JOSEPH BERGER 1963–64, ROBERT R. BUSH 1963–66, DOUGLAS CHAPMAN, BERNARD P. COHEN 1964–66, K. K. ESTES, DAVID GALE 1962–65, SAMUEL GOLDBERG 1963–66 (Chmn. 1964–66), R. D. JAMES, LEO KATZ 1964–66, JOHN KEMENY (Chmn. 1963–64), HAROLD W. KUHN 1960–65, H. L. LUCAS 1964–65, FREDERICK MOSTELLER 1959–63, G. BALEY PRICE 1964–65, HOWARD RAIFFA, OTTO SCHMITT 1964–66, MARTIN SHUBIK 1964–66, W. ALLEN SPIVEY 1964–66, THEODOR D. STERLING 1962–65, PATRICK SUPPES (Chmn. 1960–63), GERALD L. THOMPSON 1963–66, ROBERT M. THRALL 1963–65, A. W. TUCKER 1963–65, GEOFFREY WATSON 1960–63.

III. *Panel on Mathematics for the Physical Sciences and Engineering*, (with dates where recorded).

R. CREIGHTON BUCK 1963–65, G. F. CARRIER, BURTON COLVIN 1965–67, E. U. CONDON 1960–64, CHARLES R. DEPRIMA 1962–65, CHARLES A. DESOER, 1959–66, T. P. PALMER, 1960–64, MELBA PHILLIPS 1965–67, HENRY O. POLLAK 1959–65, G. BALEY PRICE 1959–64, MURRAY PROTTER 1960–64, A. H. TAUB 1963–65, R. J. WALKER 1959–65 (Chmn. 1959–65), MILTON WING 1963–65.

IV. *CUPM, Panel on Pregraduate Training*, (with dates where recorded).

R. D. ANDERSON 1962–64, 1965–67, LOUIS AUSLANDER 1965–67, PAUL BATEMAN 1965–67, RALPH P. BOAS (Chmn. 1963–65), D. W. BUSHAW 1965–67, DAN E. CHRISTIE 1963–65, EARL CODDINGTON 1959–64, LEON W. COHEN 1959–66, LESLIE DWIGHT 1964–66, ELDON DYER 1959–64, SAMUEL

EILENBERG 1963–65, LEONARD GILLMAN 1965–67, GEORGE E. HAY 1964–66, LEON HENKIN 1962–65, SAMUEL KARLIN, VICTOR KLEE 1965–67, PETER LAX, HERMAN MEYER 1965–67, KENNETH O. MAY, JOHN MOORE 1959–64 (Chmn. 1959–63), IVAN NIVEN 1962–65, GEORGE SIMMONS, A. D. WALLACE 1959–63, GERALD WASHNITZER, A. B. WILLCOX 1965–67.

V. *Panel on Teacher Training*, (with dates where recorded).

E. G. BEGLE 1959–67, ROY DUBISCH 1962–65, MARY FOLSOM 1963–65, W. T. GUY, JR., CLARENCE HARDGROVE 1963–65, P. S. JONES, JOHN L. KELLEY, JOHN KEMENY (Chmn. 1959–63), BRUCE MESERVE 1962–64, EDWIN E. MOISE 1960–65 (Chmn. 1963–64), HENRY VAN ENGEN, GAIL S. YOUNG, JR., 1963–66 (Chmn. 1964–).

VI. *Advisory Group on Communications.*

JOHN D. BAUM 1963–65, DAN E. CHRISTIE 1963–65, CHARLES R. DEPRIMA 1963–65, MARION K. FORT 1963–64, ROTHWELL STEPHENS 1963–65, ROBERT THRALL 1964–65, A. W. TUCKER 1963–65 (Chmn. 1963–65).

VII. *Executive Officers.*

ROBERT L. DAVIS (Executive Secretary 1956–58), ROBERT J. WISNER (Executive Director 1960–63), BERNARD JACOBSON (Associate Director 1962–63), A. B. WILLCOX (Executive Director 1963–64), B. E. RHOADES (Associate Director 1963–64, Executive Director 1964–65), R. H. MCDOWELL (Associate Director 1964–65, Executive Director 1965–66), W. T. SLABY (Assistant Director 1965–66), L. K. DURST (Associate Director 1966, Executive Director 1966–67).

Central Offices located at: University of Virginia 1956–58, Michigan State, Oakland and Pontiac, Michigan 1960–63, Berkeley, California 1963– .

VIII. *Consultants Bureau.* (No specific terms. All have served at some time.)

R. D. ANDERSON, JOHN D. BAUM, HOWARD E. CAMPBELL, ROY DUBISCH, LINCOLN K. DURST, PHILIP DWINGER, JOHN V. FINCH, ROBERT C. FISCHER, MARION K. FORT, DWIGHT B. GOODNER, W. T. GUY, JR., LEON HENKIN, JAMES HUMMEL, BERNARD JACOBSON, PAUL B. JOHNSON, BURTON W. JONES, PAUL J. KELLY, EUGENE E. KOHLBECKER, ARTHUR LIVINGSTON, ROBERT H. MCDOWELL, ANIL NERODE, ROBERT Z. NORMAN, WILLIAM R. ORTON, JR., B. J. PETTIS, MINA S. REES, CHARLES RICKART, GERALD ROGERS, HARTLEY ROGERS, JR., SEYMOUR SCHUSTER, LELAND SCOTT, WILLIAM R. SCOTT, E. BAYLIS SHANKS, JAMES H. STODDARD, JOHN WAGNER, ELBERT WALKER, FRED WRIGHT, JAMES H. ZANT.

THE HISTORY OF MATHEMATICAL EDUCATION

PHILLIP S. JONES, The University of Michigan

1. The Association and Mathematical Education. The inclusion of a survey of mathematical education as a part of the semi-centennial celebration of the Mathematical Association of America is appropriate for many reasons. For example, the by-laws of the Association state that one of its objectives is "to assist in promoting the interests of mathematics in America, especially in the collegiate field . . . by conducting investigations for the purpose of improving the teaching of mathematics. . . . "

In pursuit of its goals the Association has published the Carus Monographs of which the fifth, *The History of Mathematics in America before 1900*, by David Eugene Smith and Jekuthiel Ginsburg implicitly contains much of significance for the history of mathematical education in the country, especially at the college level. However, the past concern of the Association for mathematical education at the secondary level should neither be ignored nor allowed to lapse. It was particularly significant that the Association was mentioned on page one of the first volume of *The Mathematics Teacher* to be issued under the aegis of the National Council of Teachers of Mathematics. In 1921 the newly founded Council took over this journal from the Association of Teachers of Mathematics in the Middle States and Maryland. In the first issue of that year C. M. Austin, first president of the National Council, explained, in words which have a familiar ring, the needs and reasons for the founding of this new organization as follows: "During this same period (the previous ten years) high school mathematics courses have been assailed on every hand. So-called educational reformers have tinkered with the courses and they, not knowing the subject and its values, in many cases have thrown out mathematics altogether or made it entirely elective. Individual teachers and local organizations have made a fine defense to be sure, but there could be no concerted action. Finally, the Mathematical Association of America came to the rescue and appointed a committee to study the situation and to make recommendations. Already two valuable reports have been issued and others are in preparation. The pity of it is that this work, wholly in the realm of the secondary schools, should have to be done by an organization of college teachers. True, they have generously called in high school teachers to help, but the fact is that it remained for the college people to initiate the work. They could do it because they possessed a live, vigorous organization [1]. (The final report of the committee of the Association to which Austin referred was the famous "1923 report.")

It is, I believe, altogether appropriate that this Association which is still live and vigorous should continue its interest in mathematical education, not only by recognizing it in the program of this, its semi-centennial, but also in the continuing concern showed by the work of its Committee on the Undergraduate Program which has a panel on teacher training.

I was, therefore, greatly pleased when the Committee asked me to speak on

this topic which combines two of my major interests, the history of mathematics and mathematical education. The Committee neither defined these terms nor laid upon me any restrictions as to the time, period, or level of mathematical education which I should consider. I have chosen to consider education to refer to consciously and deliberately planned programs for instruction, and to attempt the task of a broad survey of the major, philosophical-psychological problems of mathematical education from its beginning until today. Such a task, of course, automatically suggests a chronological approach. But I hope that while following a chronological theme I can avoid the impression that I consider the history of either mathematics or mathematical education to be at all well embodied in a sequence of names and dates. I hope, then, that I can point out and trace with you a number of continuing and pervasive themes which I see in the teaching of mathematics through the ages. Many of the problems related to these themes are still with us, and I hope that the function of this talk may be more than a mere interesting and nostalgic review of the past. I hope that a survey of the changing objectives and philosophies of both mathematics and mathematical education can bring into sharper focus the needs and controversies of our time.

2. The Earliest Mathematical Education—Motivation. Although Seidenberg has recently elaborated a conjecture that counting originated in the rituals of prehistoric community life around 10,000 years B.C., which rituals then would no doubt have been taught to successive generations [2], our first knowledge of formal education comes from the Babylonian clay tablets and Egyptian papyri of the early historic period. The Ahmes papyrus of 1650 B. C. was copied from papyri composed as much as two hundred years earlier and gives evidence of having some of the qualities of a copybook or textbook. Not only have excavations in the Near East revealed rooms that appear to have been school classrooms, but also Babylonian cuneiform mathematical texts have, again, schoolbook properties. Scribes, clerks, and priests had to be trained to keep records associated with the calendar, land, taxation, and commerce. Amongst the problems in these ancient texts, however, one often discerns a recreational or theoretical situation such as the recently deciphered Babylonian tablet Plimpton 322 which contains a table of Pythagorean triples. So, from our vantage point in time, we can read both practical motives for formal instruction and evidence of the indulgence of intellectual curiosity in the records of the very earliest days.

However, we turn to Greece, and in particular to Athens of the 6th and 7th centuries B. C., for our first knowledge of formal schooling. In the time of Solon, the state paid tuition for the children of men killed in battle. There was a publically appointed supervisor of education, and parents were required to see that their sons were educated. There were two types of education for two classes of citizens: The liberal arts for the free citizens (here, in fact, is the origin of our word "liberal education"—an education for the free), and education in the so-called practical arts for the lower classes or slaves.

There are a number of interesting anecdotes about early Greek education that are also related to the teaching of Greek mathematics. For instance, there are two versions of the account of the first person to teach mathematics for money. One dating back to 520 B. C., says that a Pythagorean had lost his property and when this misfortune befell him he was allowed by the Brotherhood to make money by teaching geometry. Another story is that Hippocrates, of 460 B. C., who had been a merchant, was captured by pirates. Learning geometry while he was delayed in Athens by a lawsuit in which he attempted to recover his loss, he later taught it to support himself during the remainder of this time. Aristotle gives a slightly different version of this, saying that Hippocrates had been defrauded of his money because, although he was a good geometer, he was stupid and incompetent in the business of ordinary life [3]. This is the earliest citation of evidence in the perennially recurring debate as to the so-called disciplinary value of a mathematical education and its role in teaching people to think well in nonmathematical situations. In more recent times, psychologists have associated this question with the question of "transfer of training."

Pythagoras himself is the hero of the story to the effect that upon his return from Egypt he was eager to transplant to Greece what he had learned and what he had added to Egyptian mathematics. In other words, he was an eager teacher with no students. So he initially paid students the equivalent of six cents a proposition for each theorem learned. He continued this until the student was so interested and enmeshed in the intrigues of mathematics that he was willing to continue even if he had to pay rather than be paid [4]. Here, then, is a fore-taste of another current dispute; namely, does one need to foresee rather immediate profit in order to be well motivated, or can we trust to the intrinsic interest of the subject itself as motivation for the learning of our mathematics students. We have all been questioned by the student who says, "What is the good of all this?" Euclid is supposed to have answered such a question by turning to a slave and saying, "Give the student sixpence since he must need gain by what he learns" [5].

There is another problem that is classical in its antiquity and modern in its occurrence. This is the problem of the student who, impatient of proof and of the slow process of building understandings, wants the formula, the handbook, the quickly achieved result. The story is varyingly told of Alexander the Great speaking to Menaechmus or of Ptolemy speaking to Euclid. It is to the effect that the king wished to learn geometry in a hurry. The geometer's reply was, "For roads over the country there are royal roads and roads for common citizens, but in geometry there is one road for all" [6].

3. The Earliest Mathematical Education—Methods. In any event, all of the three famous Greek philosophers made contributions to the history of mathematical education. Listen to Socrates for a moment or two. This is from Jowlett's translation of Plato's *Dialogues* as quoted by J. W. A. Young who

himself deserves mention in a history of mathematical education presented to the Mathematical Association of America. After writing a dissertation in group theory in 1892, he taught at the University of Chicago where he helped to organize the Chicago section of the American Mathematical Society and became a Professor of the Pedagogy of Mathematics. His book, *The Teaching of Mathematics in the Elementary and Secondary School*, first published in 1906, went through many editions. Young chose to illustrate questioning procedures by this quotation in which Socrates is teaching a demonstration lesson. His class is an illiterate slave and his observer is Meno:

Soc. Tell me, boy, do you know that a figure like this is a square?
Boy. I do.
Soc. And do you know that a square figure has these four lines equal?
Boy. Certainly.
Soc. And these lines which I have drawn through the middle of the square are also equal?
Boy. Yes.
Soc. A square may be of any size?
Boy. Certainly.
Soc. And if one side of the square be of two feet and the other side be of two feet, how much will the whole be? Let me explain: If in one direction the space was of two feet and in the other direction of one foot, the whole would be of two feet taken once?
Boy. Yes.
Soc. But since this side is also of two feet, there are twice two feet?
Boy. There are.
Soc. Then the square is of twice two feet?
Boy. Yes.
Soc. And how many are twice two feet? Count and tell me.
Boy. Four, Socrates.
Soc. And might there not be another square twice as large as this, and having, like this, the lines equal?
Boy. Yes.
Soc. And of how many feet will that be?
Boy. Of eight feet.
Soc. And now try and tell me the length of the line which forms the side of that double square: this is two feet—what will that be?
Boy. Clearly, Socrates, that will be double.
Soc. Do you observe, Meno, that I am not teaching the boy anything but only asking him questions; and now he fancies that he knows how long a line is necessary in order to produce a figure of eight square feet; does he not?

I'll go no further with this, but note that the "Socratic method" of teaching in which students were forced to accept a conclusion, often one which differed from their original intuition, by an ingenious sequence of questions, is not the

same as modern heuristic procedures in which a student is led, sometimes by himself, to *discover* a new relationship or principle for himself. However, there are significant common elements in the methodologies of Socrates and of heuristic teaching.

Though, like Socrates, Plato was no mathematician himself, the famous painting by Raphael of Plato's Academy, a fresco to be found in the Vatican, illustrates a number of mathematical debts to Plato. I have time only to enumerate them rather hurriedly. The first of these is the concept of the academy. The picture suggests that the academy students were not young and immature, but were more nearly faculty members and graduate students. It was an early "institute for advanced study." In it, we see our first examples of group research projects and of specialization into particular subject fields of mathematics.

A second Platonic concept which has been important in the history of mathematics was elaborated in his *Republic* in which he prescribed the training for the various classes which made up the citizens of his "ideal" republic. Mathematics was a major subject in the training of the philosopher-king class in the period from age 20 to 30. The major function of this mathematics was to serve as training for the mind, for mental discipline, a goal claimed for mathematical instruction over many following centuries, and even today.

Mathematics to Plato consisted of arithmetic, geometry, astronomy, and music. These four subjects ultimately became the famous quadrivium of the Middle Ages to which were joined the three subjects termed the trivium—grammar, rhetoric, and logic, to make the curriculum which dominated education for many centuries.

Having thus seen the origins of a liberal education and of a stress on pure or abstract mathematics as the mental discipline which prepares persons to be philosopher-kings and to solve problems in all areas, it is not too surprising that we should find at this time the first major critic of mathematical education in the person of Isocrates. He criticized the concepts of over-specialization and of mathematics as a mental discipline in the following words:

> "I recommend those who are embarking on the study of geometry and astronomy to devote all their energy and intelligence to them. For I declare that, even if these disciplines cannot make better men of them, they have at least the advantage of keeping young people out of mischief; and I am convinced that no more useful or more suitable activity than these could be found for them. But at a later age, after the examinations which qualify one for the rights of an adult, I maintain that these activities are less appropriate. I would, in fact, say that some of those who have carried the study of these disciplines to the point of teaching them in their turn do not know how to put to proper use the sciences they possess and prove in all circumstances of life to be less judicious than their pupils and even—though I hardly dare suggest it—than their own servants" [7].

Aristotle also believed that the liberal education of its free citizens was essential to the building of a cohesive community and a sound state. However,

he felt that small children should gain moral and physical education through play until about the age of 14 when they were were ready for or capable of intellectual pursuits. Here, then, we find an early recognition of the problem of what is called "readiness" in educational jargon which merely asserts that the children who are too young or too ill-prepared cannot understand or profit from the study of abstractions.

4. Mathematical Education in the Middle Ages. The first treatise on educational methods can be attributed to the Roman Quintillian of 35 to 100 A.D. in a book entitled, *Institutes of Oratory*. Briefly, he argued that teachers should take into account the individual differences amongst their students, that students should be given some choice as to the topics studied, in order that their studies might more nearly coincide with the student's particular talent, that student interest was a very important factor in facilitating learning and that interest was aroused by competition and awards, not by punishment (the MAA sponsors contests!). He advocated the study of a foreign language as a device for learning one's own language and stressed mathematics strongly for its teaching of methods of proof. This, you'll note, is a slightly different objective for mathematics instruction than the disciplinary value which I have mentioned previously. Quintillian also stressed memorization as his major teaching technique without seeing anything contradictory between this teaching method and the goal of teaching methods of proof! Quintillian was rediscovered in the Renaissance and some of the blame for the stress on pure memorative learning in the universities of the Middle Ages can be placed upon him.

The monastery schools of the Dark Ages gave way to cathedral schools. As the number of students in these schools increased, it became necessary to appoint a church official to oversee them, and since the school was often in the chancel it was natural to call such a person the *chancellor*. As students organized for mutual protection against a dictatorial chancellor, their groups were called universities. Next faculties were forced to unite to protect themselves against the students, and the earliest forerunner of our modern degrees was the license to teach which also made one a member of the university of professors. The very first formal instruction in pedagogy, however, seems to have been given by the Jesuits of the 16th and 17th century to those members of their order who were going on to teach. In these centuries a double type of education began to develop. One was for middle class students who were often taught by private teachers in the vernacular and with a heavy stress on applications and the vocational uses of mathematics, especially as related to commerce, surveying, and navigation. The other type of education, for upper class students, aimed to create "Christian gentlemen," teaching them little mathematics and with a heavy stress on humanistic studies. This latter was especially true in England and Italy with more emphasis on science and mathematics in France.

However, with the Renaissance we find persons such as Roger Bacon criticizing scholastic education which stressed the reading of Latin and depended

upon memorization. Bacon suggested that more observation and study of science and mathematics should replace the narrow study of Aristotle which he felt led to a concealment of real ignorance by a pretense of knowledge.

Even the Reformation plays some role in an indirect way in the development of our subject. A heavy emphasis on so-called "original sin" very naturally led to a heavy emphasis on corporal punishment as something that was good for both mind and body. This idea of a general treatment, which was "good for mind and body" encouraged the notion of certain types of study being good "mental discipline," and thereby supported both Latin and mathematics for this purpose.

Mathematics has fared better than Latin throughout because it has been varyingly endorsed for its practical utility, for its value as mental discipline, and for its incidental teaching of methods of proof. Because of these claims for mental discipline and training in logic, it has also been stressed as an element of a liberal education. However, different aspects of mathematics were also included in the practical education of the middle class and even opposing philosophies were cited to support mathematical instruction. The philosophical materialism of Bacon and Locke which stressed learning through senses supported mathematics for its practical utility. The rational idealism of Descartes and Leibniz, which stressed that we learn through the reason, also supported an emphasis on the teaching of mathematics.

A slow change in mathematical education is revealed by a glance at some texts of the 17th and 18th centuries. The little book *Mathematicae Totius* by Pater Peter Galtrucius Aurelianensis published at Cambridge in 1683 had sections devoted to arithmetic, geometry, astronomy, chronology (the calendar) "gnomonicae" (the sun-dial), geography, optics, and music. Note that this includes the quadrivium and four additional practical topics. All this was embraced in 305 small pages! However, not to give a wrong impression, we must note that Newton, whose *Principia* appeared in 1687, had studied and taught at Cambridge and that his teacher, Isaac Barrow, had still earlier included preliminary notions leading up to the calculus in his lectures on optics, before resigning his position in favor of his promising young pupil. The ancient works of Apollonius were well known there as were the more modern works of Descartes and Fermat. However, these were not the basic fare of the secondary schools or the undergraduate curriculum.

That the elementary mathematical content of the curriculum on the continent was similar to that just displayed for England is revealed by looking at the table of contents of a *Compendium of Elementary Mathematics Composed for the Use of Young Students* by Christian Wolf of the University of Halle published in 1742. His two volumes included the eight topics of Galtrucius and then added trigonometry, mechanics, hydrostatics, "aerometria," hydraulics, perspective, "pyrotechnia," military architecture, civil architecture, and algebra in 900 small pages! Wolf also represents a growing interest in pedagogical problems which developed, especially in the German universities, at this time.

Wolf, for example, supported a lecture method as against the mere reading and exposition of a standard text which was to be memorized. He also supported the teaching of German versus Latin, allowed some electives in the curriculum, and stressed freedom for the faculty in teaching and for research.

5. The Nineteenth Century—Texts and Journals. The nineteenth century, however, was the period of real development of concern for pedagogy, educational aims and methods. The first book which I have been able to find which dealt explicitly with the teaching of mathematics was published in Paris in "An XIV" (1805). It was *Essais sur l'Enseignement en Général et sur celui des Mathématiques en Particulier* by S. F. Lacroix. In 1810 Gergonne began publication of *Annales de Mathématique Pure et Appliquée* and in the prospectus for this journal, in its first pages, he stated that "these *Annales* will be principally consecrated to pure mathematics and to all the searches which have for their object the perfecting and simplifying of teaching." However, he adds that it will also contain notes on the art of conjecturing, on political economy, military art, chronology, mineralogy, civil architecture, fortification, and the mechanic arts, together with problems and theorems to be demonstrated. These latter were to be included *"to stimulate young geometers."*

As one peruses the pages of this famous journal, he would hardly identify it as an educational journal, but would nevertheless be struck by the frequency with which mediocre and good mathematicians were publishing both short notes and longer articles of a pedagogical nature, especially, of course, dealing with expository and logical problems. For example, in the first issue Gergonne himself published an article on "the identity between the products which resulted from the same factors differently multiplied amongst themselves." This amounts to an attempt to prove commutativity and associativity for the integers without, however, using these words or recognizing these properties as axioms for an abstract system. There is also a discussion of an argument by D'Alembert and others that when an equation has two roots, one positive and one negative, the positive root actually solves the problem in a direct sense and the negative root is only a solution if the statement of the problem is modified in such a manner as to "render additive" that which was subtractive and vice versa.

There is also a proof that all similar triangles are right triangles which involves the use of the cosine law and division by a factor which would have been zero under the conditions of the problem. To this Gergonne has added an editorial note discussing and elaborating the problem of division by zero, the situations in which it arises, and stating that text book writers are most culpable not only if they make such an error but also if they fail to make a point of the difficulty explicitly.

The strength of the French influence upon mathematical education in general and upon American mathematical education in particular is further attested to by a number of additional items. Although the first journal with a

major emphasis upon the pedagogy of mathematics appears to be the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* founded by J. C. V. Hoffman of the Gymnasium at Freiberg and published in Leipzig beginning in 1870, France was not far behind. *L'Enseignement Mathématique* was founded in 1899 by C. A. Laisant and H. Fehr. This was the first journal devoted *solely* to the teaching of mathematics and in few years became the official journal of the International Commission on the Teaching of Mathematics which was appointed at the Fourth International Congress of Mathematicians at Rome in 1908. Laisant was also the author of one of the early books on the teaching of mathematics, *La Mathématique, Philosophie-Enseignement* (1898), but I found a book on methodology published in 1896 by Felix Dauge, *Cours de Méthodologie Mathématique*, to be most interesting. Its introduction stated that it was actually a second edition, with a different name, of a book first published in 1883 and used in a *"l'école normale"* which had since been discontinued and replaced by the inclusion of a course in methodology in the program of the doctorate in science and mathematics. Time does not permit a detailed analysis of any of these books, but since I was led to Dauge by a citation in the book by J. W. A. Young, I think it may be regarded as more influential in this country than the others. Dauge began by stressing the importance of the intimate connections among the different branches of mathematics, especially between algebra and geometry. He moved on to stress the importance of being concerned for the objectives of mathematical instruction which he puts under two categories, for applications to the needs of society and for personal careers, and secondly, for a *"moyen d'exercice d'esprit."* Dauge went on to say that actually there are only a few general precepts that can be given with reference to the teaching of any subject, that the major process of helping people to develop into good teachers is to furnish them with examples of good teaching in their classes. However, he did propose the following general rules: (1) Don't present only rigorous reasoning to youth, (2) resist sacrificing rigor to brevity in order to include more topics in the course, (3) teach fundamental principles with care but don't insist on too much abstraction nor digress too far into discussions of truth, being, and existence which will only confuse, not clarify, the mathematical concepts for students. Dauge stated that (4) at the beginning it is not necessary to prove all propositions, that it is possible to postulate those requiring more obscure reasoning or difficult proofs, that making students learn such proofs by heart does not actually make the proposition more clear. However, he stressed that (5) memory is important, that students should learn by heart some basic theorems, not only for the sake of the theorems, but to accustom the students to mathematical language and to the need to express themselves with precision and elegance. He warned against (6) extending memory, however, to the memorization of *proofs* and suggests that teachers should ask questions frequently in order to be sure that the student sees how the parts of proofs fit together. He advises that (7) when a student finds himself stopped on a proof, the professor should restrain himself from immediately pointing out the se-

quence of ideas; let the student find it for himself, an error corrected may be more profitable than several theorems proved. Other words of advice deal with very mundane affairs such as drawing clear diagrams, but also included gems of wisdom that could still be taken to heart, such as (8) do not lecture too long, the students won't pay attention and will be confused by too many terms. (9) Frame some questions to be sure that the students understand. He goes on to urge that although he had warned against consideration of metaphysical questions with the students, the professor should be informed of and concerned for such. Further, the professor should be an inspiration to the students, should not assign work that's above them, should present applications of the theory that is being developed, should sustain their attention by keeping an eye on them during his explanations, and, without departing from a proper and correct demeanor, should give proof of his good will towards the students, encouraging them with words, even for those who worked and did not succeed and punishing only in grave cases where he is certain it is needed! Above all, the instructor should maintain a spirited and zealous attitude himself.

For those who are close to current new materials in the elementary and secondary school, it is interesting to note that Dauge recommended the study of different systems of numeration in order that students might better understand the numeration system to base 10. However, he had been anticipated in this recommendation by DeMorgan in 1831 and was seconded in it by Laisant. A good share of Dauge's book, as was true of those of Lacroix, Laisant, and DeMorgan, was devoted to detailed consideration of certain typical aspects of subject matter; a favorite topic is the treatment of negative numbers, a topic of which DeMorgan wrote with great firmness. DeMorgan insisted that it was absurd to consider that any number could be less than zero. It is a little difficult, I think, for us to realize that considerable controversy over the nature of negative numbers was carried on through the greater share of the nineteenth century not only with respect to pedagogical problems, but also with respect to philosophical problems as to their very existence.

The book in which Augustus DeMorgan argued against negative numbers was *On the Study and Difficulties of Mathematics* published by the Society for the Diffusion of Useful Knowledge in 1831. This antedates all other methodological books except Lacroix. However, apart from this, the English do not seem to have been as concerned for the pedagogy of mathematics as were French and German mathematicians and educators.

DeMorgan stressed the value of mathematics as mental discipline, and an interesting feature of his essays, though not a surprising one, is his stress on the importance of teaching a unit on logic earlier in the course for secondary school students. He also made a strong plea for abolishing the rote memorization of rules and of proofs. He concluded this plea with the prediction that non-rote memoriter teaching may "excite some to become inquirers who would otherwise have been workers of rules and followers of dogma." He, too, advises teachers against entering into purely metaphysical discussions of the founda-

tions of algebra, but insists that they themselves should be alert to these and devotes some time to the absurdities of such expressions as the negative a, square root of negative a, and zero divided by zero. He ends his introduction with the statement, "The number of mathematics students, increased as it has been of late years, would be much augmented if those who hold the highest rank in science would condescend to give more effective assistance in clearing the elements of difficulties which they present."

DeMorgan does present a certain ambivalence not unknown today when in successive paragraphs he stresses that *understanding* of algebra is a most important objective which is not, however, achieved by reading but by doing, and then goes on to state, "It is a remarkable fact that the first elements of mathematics, a science which demonstrates its results with more certainty than any others, contain difficulties that have been the subject of discussions for centuries." These difficulties, he says, are not such as would suggest themselves to the beginner. The young student sees or fancies he sees the truth of every result which can be stated in a few words, or arrived at by a few simple operations. This arises, he says, from the fact that the young student from his earliest infancy learns no fact from his own observation, deduces no truth by the exercise of his own reason, and thus a habit of examination is not formed, and he is fully prepared to believe in the truth of any rule that is set before him. I hardly need point out that in 1831 DeMorgan was stressing the importance of understanding, of student derivation or "discovery" and thought as opposed to memorization.

6. Mathematical Education in the United States. Having noted some of the highlights of mathematical education in Germany, France, and England, the countries having the greatest influence on early American mathematics, it is time we considered developments in the western hemisphere and the United States. The relative recency of mathematical progress in this country may be brought out by citing a few little-known facts. The first book with mathematical content printed in the Americas was printed in Mexico City in 1556. It was Juan Diez Freyle, *Summario compendioso—de plata y ore*, which had for its major concern the treatment of the value of and conversions between different coinages of different purities of gold and silver ore. How consistent this is with the period of Spanish-American conquest and exploration! Written by a missionary, a member of a religious order, but dealing with the value of gold ore!

It is rather interesting, however, that this first book with mathematical content, aimed at this very practical problem of arithmetic, did also contain some algebra, although there was no apparent very close connection with the rest of the book or its purpose.

I will not elaborate on the other early works, but in the Western Hemisphere prior to 1700 there were 7 Mexican and 4 Peruvian books. These books, though they contained mathematics in their titles and content, dealt with military matters, with navigation, surveying, with the calendar and the determination of fast (feast) days and religious celebrations.

The first book with mathematical content printed in North America was John Hill's *The Young Secretary's Assistant*, which appeared in Boston in 1703. As its title implies, it dealt with how to write a business letter, how to keep books, etc. The secretary to the man who was sending out ships to trade, for instance between the Indies and Europe, had to know a little bit of arithmetic, of weights, measures and coinages and how to translate them into different currencies.

The first mathematics book written by a person in this country was Isaac Greenwood's *Arithmetic, Vulgar and Decimal*, printed in 1729.

The first book with any substantial geometric content was Hawney's *Complete Measurer*, printed in 1801. It dealt largely with simple surveying, weights, measures, cordwood, barrels of wine, etc.

The points are that mathematics was very late in arriving in North America, it was imported, and it stressed practical uses.

Now turning to the early schools, and their programs. The predecessor of our secondary schools was the Latin Grammar School. The Boston Latin Grammar School was established in 1635. It was a private school for a middle class clientele, preparing for the professions, ministry and law, medicine and college.

A little later we have the beginning of the academies. Benjamin Franklin founded one in 1751, but they did not really develop until sometime later, flourishing from 1787 to 1870. They were a sort of a common man's version of the Latin Grammar school. That is, they still were private, but they put a little more stress on preparation for commercial enterprises, business, etc.

About the mathematics curriculum of these grammar schools and academies I would quickly note three things: their purposes, methods of presentation, and content. There were two purposes: *practicality*, the uses of arithmetic and geometry in mensuration, calculating money in different currencies, etc., and "*mental discipline*," the idea was that it's good for your mental muscles to wrestle with Latin and arithmetic.

Their methods of presentation involved the presentation of many rules, straight-forward statements, often in italics, to be memorized, a worked example or so for each, and then problems to be done. This may be illustrated by a quotation from Quackenbos' *Elementary Arithmetic*, copyright in 1821, but used by my father in 1892. Page 78 begins the discussion of fractions as follows: "Halves and halves, thirds and thirds, etc. can be added just as we add pears and pears, dollars and dollars." That was paragraph 130 and there was just a bit more discussion in it of adding fractions with common denominators.

Paragraph 131 on the same small page read, "Halves, thirds, etc. cannot be thus directly added any more than we can add pears and dollars." This was followed by an example of how to add halves and thirds. Following the example came a rule. Following the rule, were "exercises for the slate." And that was it! On two small pages he covered adding fractions with and without common denominators, gave examples and a set of problems.

Although this procedure of stating rules, doing examples and giving exer-

cises was typical, there were, of course, persons trying to do other things. Warren Colburn wrote *Intellectual Arithmetic Upon the Inductive Method of Instruction* in 1821. I took down from my shelf his *Introduction to Algebra Upon the Inductive Method of Instruction*, which in the copy I have is dated 1830. Its preface, in part, read:

> "The first object of the author of the following treatise has been to make the transition from arithmetic too algebra as gradual as possible. The book therefore commences with practical questions in simple equations such as the learner might readily solve without the aid of algebra . . . "
>
> "The most simple combinations are given first, then those which are more difficult. The learner is expected to derive most of his knowledge by solving the examples himself; . . . "
>
> "In fact, explanations rather embarrass than aid the learner because he is apt to trust too much to the man, and neglect to employ his own power; and because the explanation is not made in the way that would naturally suggest itself to him if he were left to examine the subject by himself. . . . This method besides giving the learner confidence, . . . is much more interesting because he seems to be constantly making new discoveries . . . "
>
> "In the ninth article the learner is taught to generalize particular cases and to form rules. . . . The learner should solve every question. When the learner is directed to turn back and to do in a new way, something he has done before, let him not fail to do it, for it will be necessary for his future progress. . . . "

The content of the grammar school curriculum and the nature of beginning college work is further delineated by taking a glance at college entrance requirements. Harvard required arithmetic for admission in 1807 and algebra in 1820. Harvard was followed by Yale in 1847, and by Princeton in 1848. In geometry Yale led all the rest. It required geometry in 1865. Princeton, Michigan, and Cornell had geometry as an entrance requirement in 1868, Harvard added geometry in 1870, but when Harvard did do it they required logarithms as well.

An extract from the code of laws governing the University of Michigan in 1837 shows implicitly both its mathematics curriculum and three important influences upon early mathematics education in this country, that of (1) the French, (2) the United States Military Academy, (3) Charles Davies. The texts in mathematics listed for that year were:

Davies' *Arithmetic*
Davies' *Bourdon's Algebra*
Davies' *Legendre's Geometry*
Davies' *Surveying*
Davies' *Descriptive Geometry*
Bridges' *Conic Sections*
Gregory's *Mathematics for Practical Men*

Davies seems nearly to have made a clean sweep of textbook adoptions at the University of Michigan in 1837—(at this time I believe there were seven students!), and indeed his books were extremely popular and went through many editions. In later editions the names of Bourdon and Legendre were dropped and many people forgot that these were translations from the French, originally for use at the United States Military Academy which also brought over several

French mathematicians as teachers. Charles Davies was chairman of the department there for a number of years. As his textbooks became popular he added to them the first book on the teaching of mathematics published in this country, *The Logic and Utility of Mathematics with the Best Methods of Instruction Explained and Illustrated,* published in New York in 1857. It is interesting that, along with DeMorgan, he stressed the importance of explicit instruction in logic. The introduction to this book read, in part:

> "The following is not a series of speculations. It is but an analysis of that system of mathematical instruction which has been steadily pursued at the Military Academy over a quarter of a century, and which has given to that institution its celebrity as a school of mathematical science."
>
> "It is of the essence of that system that a principle be taught before it is applied to practice, . . . the union of the French and English systems of mathematics. . . . "

7. Foreign Influences on Mathematical Education in the United States. However, I should not overemphasize the French influence upon American mathematical education. In methodology Colburn was influenced by a Swiss educator, Pestalozzi, although the French also played a part in the spread of inductive methods. As recently as 1920 the title page of an arithmetic book of which John Dewey was listed as an author bore the following quotation from Laisant, who was a Belgian:

> "The problem is always the same: to interest the pupil, to induce research, to give him the notion continually, the illusion if you please, that he is discovering for himself, that which is being taught him."

Further, although all American mathematicians know something of Felix Klein, many may not fully appreciate his rather direct effect upon American mathematics. I refer to the effect upon graduate study and research of his presence in this country for the Congress of 1893 and the colloquium at Evanston which followed, and also the publication in this country of his pedagogical volumes, *Elementary Mathematics from an Advanced Standpoint* in a translation by one of the Association's founders, Earle Raymond Hedrick. Finally, in this later period we need to consider the effects in this country of the reform movement in Britain. In particular, E. H. Moore, who spoke out strongly for reform in mathematical education in his American Mathematical Society retiring presidential address in 1902, was greatly influenced by John Perry. For the background of Perry's views I quote from a recent pamphlet from the British Ministry of Education, *Teaching Mathematics in the Secondary Schools,* which states, "Mathematics hardly existed in schools until the time of Queen Victoria, whose reign began in 1837."

In 1861 The Royal Commission on Nine Public Schools said, "The chief honors and prizes in our public schools go for the Classics which are to the great number of the boys, intrinsically more attractive than Mathematics . . . but mathematics, at least, has established a title of respect as an instrument of mental discipline." Another quotation from this same group refers to a situation which has been much more significant in England than in this country, but it

may become increasingly significant in this country. This is the relationship between the secondary schools and the college entrance examinations, or as they are called in this report, "the external examinations." The quotation is, "The introduction of mathematics as an integral part of the school curriculum coincided with the establishment of external examinations. This circumstance tended to set a pattern of mathematics education before much thought had been given to the matter." The curriculum in England at that time included algebra, geometry, surveying, navigation, mensuration and trigonometry.

Following this, toward the end of the 19th century, student failures in mathematics and Latin increased and colleges complained of the poor preparation of their students.

These quotations and facts point out the following "morals": (1) College criticisms have motivated establishing and changing external examinations which in turn have very significant effects on the school curriculum. (2) The changing nature of the school population has also been a factor in the design of school programs. (3) Practical and intellectual objectives have always had a place in our school programs, but have varied in their relative importance.

John Perry spoke in 1901 to the British Association for the Advancement of Science in England, and out of his speech came a committee that reported a year later. Its report did make significant changes in the examinations and syllabus for geometry in England in the direction of allowing more flexibility in answers rather than reproduction of Euclid. I'm not going to go into that, but among Perry's own emphases were: the use of the laboratory method of teaching, stress on the correlation between mathematics and science, acceptance of some theorems without proof, and discussion of practical uses.

I cannot resist the temptation to read to you a quotation from this discussion in 1901. You will be interested that among persons who contributed to this discussion, either written comments or through their own presence, were Mrs. Boole, Miss Scott from Bryn Mawr, Horace Lamb, David Eugene Smith, Lord Kelvin, and Oliver Heaviside. Here is the quotation from Heaviside:

> "Boys are not philosophers and logicians. . . . Now the prevalent idea of mathematical works is that you must understand the reason first before you practice. This is fuss and fiddlesticks. . . . I know mathematical processes that I have used with success for a very long time of which neither I nor anyone else understands the scholastic logic. . . . There is something wanting, no matter how logical people may pretend. . . . Geometry should be entirely observational and experimental at first. . . . "

I hardly need note that this attitude has its modern advocates.

8. More Recent Developments in the United States. J. W. A. Young called it a remarkable coincidence that a marked impetus toward reform was given in two countries, in two successive years, in addresses before major scientific bodies, Perry in 1901, and E. H. Moore in 1902. Moore called for more "fusion" or "correlation" between mathematics and other subjects, especially science, and for more concern for concrete and developmental work in teaching in the secondary schools.

In addition to these events and the "1923 report" [8] noted earlier there were other landmarks in the progress of mathematical education in the United States in the twentieth century. Some of these were: 1894—the Committee on College Entrance Requirements; 1911–1918, the International Commission on the Teaching of Mathematics. The almost tumultuous progress of the past decade was anticipated by the activities of the Commission on Post-War Plans of the National Council of Teachers of Mathematics which operated from 1944–1947. Recent vigorous activity was then touched off in 1952, prior to Sputnik, by the publication of a report *General Education in School and College* by a committee from three eastern colleges and nearby preparatory schools, and the founding of the University of Illinois Committee on School Mathematics. Although the current N.S.F. institutes were anticipated by summer programs for secondary school teachers at Duke University, 1953 marked the beginning of National Science Foundation support with the Summer Conference on Collegiate Mathematics held at the University of Colorado.

However, I have chosen in this talk to slight these developments of the twentieth century and the last decade, because they are familiar to many, and the literature on them is adequate and reasonably available [9]. I have preferred to use my time to present a longer range point of view. But, since this is the semi-centennial celebration of the Mathematical Association of America, and since some of its charter members present here today were active in the second of two projects it sponsored which were of great significance for mathematical education, I believe a bit more should be said of these two reports.

In 1923 the National Committee on Mathematical Requirements of the Mathematical Association of America, which was appointed by E. R. Hedrick in 1916, brought in its famous report after seven years of work. It classified the objectives of mathematical instruction as practical, disciplinary and cultural, and stressed the role of the function concept in the teaching of mathematics. Most texts for years thereafter cited this report in their prefaces and honored the then current interpretation of the "function concept" by stressing the interrelationships between tables, graphs, formulas, and equations. Its stress on the function concept and general mathematics reflected the influences of Felix Klein and E. H. Moore respectively. The representatives of the Association on this Committee were: J. W. Young, A. R. Crathorne, C. N. Moore, E. H. Moore, D. E. Smith, H. W. Tyler. The other seven members of the committee represented secondary school teachers and organizations.

Two groups reported in 1940. One had been appointed by the Progressive Education Association, and the other was the Joint Committee of the Mathematical Association of America and the National Council of Teachers of Mathematics appointed to study the place of mathematics in secondary education.

The first group expounded a long range philosophy. They stressed the importance of teaching problem solving and the nature of mathematical structure and wrote about data, approximation, function, operation, proof, and symbolism.

This report suffered some in its influence for being associated with the Progressive Education Association, I believe. There were many very good and rather modern ideas in the report.

The Joint Committee had the same number of representatives from the Association as from the National Council. The Association representatives were: K. P. Williams, A. A. Bennett, H. E. Buchanan, F. L. Griffin, C. A. Hutchinson, H. F. MacNiesh and U. G. Mitchell.

The Joint Committee stressed individual differences. It recommended a two or more "track" curriculum. This was again a product of the time, the condition of the schools, and the condition of mathematics in the schools. They also wrote about transfer of training. The notion of transfer of training was for a time substantially discredited by psychologists. It is now beginning to be regarded again as an important educational element but with several theories, some new, to describe and explain it and the conditions under which it takes place.

In conclusion, I have omitted much: (1) the names of many writers whose accounts I have read, (2) expositions of the reports of some famous committees which I listed and others which I did not list, (3) an elaboration of the role of the Mathematical Association of America in mathematical education.

I hope that my recurring reference to recurring themes was not too boring and, more important, not misunderstood. I do *not* mean to imply that there is nothing new in mathematical education (although it is a pity how often truths have to be rediscovered before they can be extended and applied). I *do* mean that looking at how fundamental ideas and perceptions have changed may shed light on the future and suggest what are the real invariants in good mathematical education.

For example Euclid's geometry and modern Euclidean geometry both are based on axioms and definitions, *but* the modern view of the nature and source of these and the connections between them and the physical world is significantly different from Euclid's. I think Newton may have had some concept of the relationship between the physical world and a mathematical model, *but* I don't believe that his view was our view. I believe that students may be brought to understand axiomatics and mathematical models better if they are shown something of their growth and the changing nature of our perceptions of them.

Similarly, today mathematical education is faced with many important problems which are not essentially new, but which, likewise, are not merely the same old ones out of which they developed. Some of these are:

(1) What are the *objectives* of mathematical instruction? We need to recognize that they are not the same for all and that our school population includes *all.*

(2) What is the *role of applications* and *physical models* in both clarifying and motivating instruction?

(3) What is the *level of rigor* which is sound and desirable at different stages of the student's development?

(4) How do we *teach the excitement and beauty* of mathematics, as well as its theorems?

(5) Can we teach students to *discover* or to be more creative?

(6) What are *the relative roles of intuition, induction and deduction* and how do we communicate them?

I hope a historical survey of these continuing themes may direct and stimulate your thinking and suggest that there will always be a continuing problem of cooperating to adapt our curricula and teaching procedures to the changing philosophies and definitions of mathematics itself, as well as to the changing needs of our culture—both intellectual and practical, and to our uncertain but growing knowledge of the psychology of learning, applying, and extending knowledge.

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CHALLENGING CONJECTURES

R. H. BING, University of Wisconsin

1. Importance of research to teaching. Mathematics is an alive and growing subject. It is best taught by teachers who are mathematically alive and mathematically growing.

The editorial in the June 18, 1965 issue of *Science* was entitled "What Are Professors For?" It says that the professor's primary activities should be teaching and research. Concerning the teaching portion of his duties the editorial has this to say, "If transfer of information were enough, society could dispense with most professors and education could be almost completely mechanized. The professor's most important role is to provide various forms of guidance for the students. As a result of his university experience, a student should be motivated to pursue continuing scholarship throughout his life."

Motivated to pursue continuing scholarship! If the teacher himself does not pursue continuing scholarship, how effective is he in encouraging his students to do so?

If the teacher pretends to be interested in getting solutions to difficult problems, pretends to be interested in proving challenging theorems, pretends to be seeking knowledge, pretends to have an inquiring mind—how many students are dumb enough to be fooled by the fakery if the teacher is merely pretending but in fact is resting on his background?

The effective teacher is the one who moves ahead. He should not be like the little boy who asked his mother, "Why do I always go around in circles?" and gets the reply "Shut up or I'll nail your other foot to the floor." The teacher should move ahead on two feet—both in teaching and in acquiring mathematical proficiency.

There has been much publicity given recently to the emphasis on research to the extent that teaching is neglected. The layman is disturbed by this. A recent article in the *Reader's Digest* was entitled "Is There a Teacher on the Faculty?" University and college people are less frightened because they know that most researchers do teach and are all the more effective as teachers because of their research experience. We occasionally hear of fellowship bums and those who shun teaching to get contracts. My experience suggests that there is relatively little of this abuse. If it were widespread it would be most unfortunate since those working at the frontier can be and frequently are our most effective teachers. Mathematicians as a whole are an energetic, intelligent, dedicated lot who try for original results themselves but who also see teaching as a way of promoting mathematics.

Let us look at the other side of the coin. It is most unfortunate for the teaching profession when the professor goes to sleep.

The slogan "Publish or Perish" might be interpreted to suggest that the teacher who does not make progress in contributing to knowledge by discovering new mathematical truths or finding new uses of mathematics loses his effective-

ness as a mathematics teacher. Recent institutes for college teachers have turned up examples of teachers who have not kept up and have practically perished mathematically.

One of the important innovations of the last decade has been the massive fellowship and grant programs instituted to stimulate more research and better teaching. Governmental agencies such as the National Science Foundation, Office of Naval Research, Air Force Office of Scientific Research, U. S. Army Research Office, evidencing the nation's concern that the United States obtain and maintain a position of scientific leadership in the world, have developed fellowship and grant programs. Public-spirited private foundations such as the Sloan Foundation, the Ford Foundation, the Research Corporation, the Carnegie Corporation, the Rockefeller Foundation, the Guggenheim Foundation, I.B.M., Bell Telephone, U. S. Steel, and others have made valuable contributions.

Many of these programs were aimed at promoting more and better research but a side effect has been that we are getting better teaching. Of course, even those programs aimed at better teaching will eventually result in more and better research. Thanks in a large measure to these fellowship and grant programs and to the leadership and hard work of many of you, I believe that we are providing students in the United States today with the best mathematical training that any group has ever had in our history.

There is room for improvement and we should continue to move ahead. In particular I hope we find better means of keeping the younger faculty of our smaller colleges active. Experience has shown that it is far easier to keep working people alive than it is to resurrect the dead.

Although doing research and finding new applications of mathematics are perhaps the most interesting ways of working at mathematics, other means of maintaining mathematical vigor include keeping acquainted with new developments in mathematics by reading mathematical articles, attending seminars, participating in the development of new curricula and teaching techniques, solving mathematical puzzles and problems, and coaching mathematics teams. The two publications *Mathematics Magazine* and *The American Mathematical Monthly* put out by the Mathematical Association of America can be of help in pursuing some of these activities.

In the better schools, teaching and research move forward hand in hand. That mathematics is an alive and growing subject makes it interesting from the standpoint of both teaching and research. Research acts as a leaven—as a catalytic agent that improves the whole loaf.

Let us look at some challenging conjectures that have stimulated much mathematical thought. We shall take a brief look at each of the following:

Continuum hypothesis.
Normal Moore Space conjecture.
Fixed point conjecture for cellular plane sets.
Poincaré conjecture.
Annulus conjecture.

2. Continuum hypothesis. Perhaps the oldest of these conjectures and the one to which most thought has been given is the continuum hypothesis—which is that C is the first uncountable cardinal.

I would like to discover whether one could prove on the basis of a reasonable set of assumptions that for each uncountable subset X of the Euclidean line E^1 there is a 1-1 correspondence between the points of X and the points of E^1, or whether there is some uncountable subset of E^1 for which there is no such correspondence.

There have been some very interesting theorems by Paul Cohen and others showing that these questions cannot be answered in certain restricted logical systems. This reminds us a bit of the situation in which Euclid and his associates were unable to prove certain theorems without the Parallel Postulate. However, these logical systems which have been shown to be inadequate to handle the continuum hypothesis have countable models and axiomatic systems based on these logical systems do not purport to describe adequately the working model of the real number system as I and many other mathematicians envision it. When I ask if these things can be proved on the basis of a reasonable set of assumptions, I do not know what these might be but would hope to include only things that would be viewed by many as "self evident truths" and not use assumptions as far fetched as the continuum hypothesis itself.

For example, I would be quite willing to have as one of my reasonable assumptions the assertion that any uncountable subset of E^1 is intrinsically uncountable—it is uncountable, not merely because one does not have a handy effective way of setting up a 1-1 correspondence between it and the integers, but rather, it has so many points that even with additional consistent assumptions, there is no such 1-1 correspondence. Not everyone would agree to this assumption, but enough do to meet my criterion of reasonableness.

Also, I would regard the "integers" as a well defined set on which most mathematicians agree even without axiomatization, rather than things satisfying certain noncategorical systems. Similarly, I regard the "reals" and E^1 as fixed, invariant, agreed upon sets. These viewpoints do not lend themselves well to systems with countable models where any uncountable set is at best conditionally uncountable, rather than intrinsically uncountable. It would be interesting to know if one steps outside these abstract "reals" with countable models and into the real reals, whether or not the continuum hypothesis is true.

3. Normal Moore space conjectures. The normal Moore space conjecture perhaps stands with one foot in Foundations and the other in Topology. It suggests that a certain set of conditions imposed on a topological space is enough to insure that the space is metrizable.

The notation of a distance is important in geometry, analysis, and topology—but its meaning in analysis and topology is slightly more abstract than its meaning in geometry. We note that to each pair of points x, y of a Euclidean space there is associated a number $D(x, y)$ called the distance between x and y, satisfying the following conditions:

1. $D(x, y) \geqq 0$, the equality holding if and only if $x = y$.
2. $D(x, y) = D(y, x)$ (symmetric condition).
3. $D(x, y) + D(y, z) \geqq D(x, z)$ (triangle inequality).
4. D preserves limit points. The last condition means that p is a point of the closure of a set X if and only if $D(p, X) = 0$.

These conditions alone do not furnish us with all the properties of a Euclidean distance, any more than a set of axioms for the reals based on a countable model of set theory gives a complete picture of the reals as mentioned in the discussion of the continuum hypothesis. However, the properties are powerful enough to be useful in proving many theorems and in getting some counterexamples.

A topological space with a distance function has a sequence of open coverings such that the meshes of the open coverings approach zero. Such sequences of open coverings are very useful in studying topological spaces since they provide us with means of proving theorems, building arcs, approximating the space with polyhedra, and computing certain algebraic properties of the space.

In his book, *Foundations of Point Set Theory*, R. L. Moore takes as axiomatic the condition that a space has a decreasing sequence $G_1, G_2, \cdots$ of open coverings. He does not assume a distance function so cannot discuss the meshes of the coverings, but uses a substitute. He does this by supposing that if N is a neighborhood of a point p, there is a covering G_i of the sequence of coverings such that each open set in G_i which contains p lies in N. A regular Hausdorff space with such a decreasing sequence of open coverings is called a *Moore space.*

A Moore space has many of the properties of a metric space and many theorems about metric spaces can be proved equally well in such a space. There are examples known of Moore spaces which do not have a distance function since they are not normal—that is, they contain two mutually exclusive closed sets which do not lie in mutually exclusive open sets. Such an example is given by the following flow space.

Suppose water is flowing across a square at a varying rate—say varying

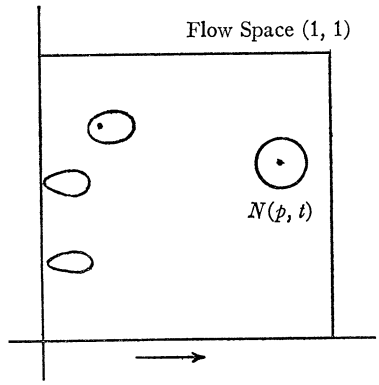


FIG. 1. (rate $= 1 - x$)

linearly from one foot per second on the left to zero on the right. Points of the flow space are points of the square and for each point p of the square and each $t>0$, $N(p, t)$ is the set of all points of the square to which one could swim from p in time less than t if one swims at a rate of one foot per second. The $N(p, t)$ neighborhoods on the right of the square are nearly round while those on the left are oblong as shown in Figure 1.

No two points of the left side of the square belong to the same $N(p, t)$. It may be shown that the square whose basis of neighborhoods are the $N(p, t)$'s is a Moore space. It is not normal because if one takes as A and B, the rational and irrational points on the left side of the square, these are mutually exclusive closed sets which do not lie in mutually exclusive open sets.

Many research hours have been spent in trying to decide if each normal Moore space is metrizable. Partial solutions have been given. Just as the continuum hypothesis has been translated into equivalent problems, the normal Moore space conjecture has been translated. In the August 1965 issue of the *Proc. Amer. Math. Soc.*, I show relations between the conjecture and the existence of certain functions from abstract sets into the positive integers. Because of such considerations, the normal Moore space conjecture lies partially in Foundations and partially in Topology.

4. Fixed point conjecture for cellular plane sets. A straight line segment has the fixed point property in the sense that if f is a map of I^1 into I^1, then some point of I^1 goes onto itself under f. To see why this is true, one can consider I^1 as horizontal and let R be the set of points of I^1 that move to the right under f and L be those which move to the left. If there is no fixed point, the left point of I^1 is in R and the right point in L. It may be shown that each of L and R is open in I^1. Since I^1 is connected, it is not the sum of two non-null mutually exclusive open sets. Hence there is a point of I^1 in neither R nor L. This point is left fixed under f.

It may also be shown that a square has the fixed point property. If one were to pick up a piece of paper, wad it up, and drop it back on the spot where it lay, there would be some point directly above where it once was.

Cells of all dimensions are known to have the fixed point property.

A set is cell-like if it is the intersection of a decreasing sequence of topological cells. There are many properties that are inherited in the sense that a set has the property if it is the intersection of a decreasing sequence of sets with the property. Being closed and being compact are such properties. People once wondered if the fixed point property was such an inherited property. They now know that it is not.

However, it is still unknown whether or not a set X in the plane has the fixed point property if it is the interesection of a decreasing sequence of topological disks. I have received papers to referee in which erroneous proofs were given. I saw a preprint of another paper giving an erroneous counterexample. Once I attended a lecture in which the speaker was giving a proof and as a result of a question from the audience during his talk, he stopped his talk because the

question uncovered a flaw. It is a tantalizing question because it seems that an answer should be available.

In 1948 Borsuk discovered an example of a set which did not have the fixed point property even though it was the intersection of a decreasing sequence of topological cubes. Here is another example.

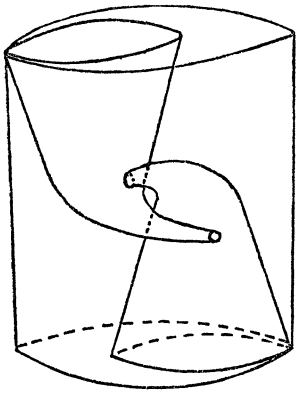


FIG. 2

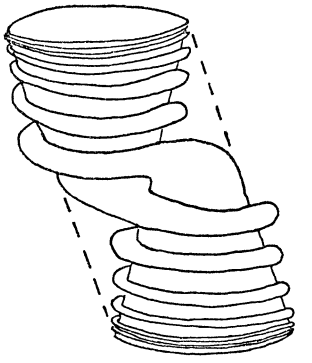


FIG. 3

Start with a solid cylinder as shown in Figure 2. Bore a hole from the top, a hole from the bottom, and whittle away at the sides. Finally, have left a set like that shown in Figure 3. The horizontal cross sections of the ends are circles and other horizontal cross sections are the unions of tangent circles. The set of points of tangency spirals about the bottom circles and about the top circle. It is what one would obtain by taking two cones, one right side up and the other upside down and sewing them together so that the vertex of one spirals around the base of the other as shown in Figure 4. To get a map that leaves no point fixed,

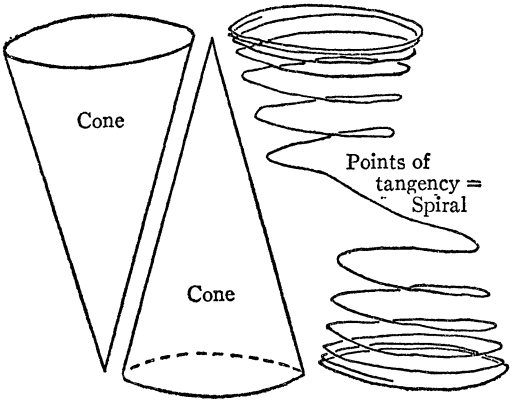


FIG. 4

one merely needs to insist that the top circle goes onto itself under a rotation, the bottom circle goes onto itself under a rotation, and each other horizontal cross section goes onto a slightly higher cross section.

5. The Poincaré conjecture. In high school geometry we work with a set of axioms that characterize the plane. In this spirit it is of interest to consider sets of conditions such that Euclidean 3-space satisfies these conditions and any topological space which satisfies the conditions is topologically equivalent to Euclidean 3-space. The Poincaré conjecture gives a set of conditions that appear to some to be enough to insure that any topological space satisfying them would be topologically like closed 3-space—that is Euclidean 3-space with a point added at infinity to make it a 3-sphere.

The Poincaré conjecture perhaps arose as a result of a mistake. In 1900 Poincaré announced that any two n-manifolds were topologically alike if they had the same homological properties. He perhaps indicated a proof, but it was in error since he showed in a 1904 paper that there is a 3-manifold with the same homological properties as a 3-sphere but which is topologically different.

Homotopy properties are stronger than homological properties but in 1919 Alexander showed that there are lens spaces with the same homotopy properties that are combinatorially different. As a result of some more recent work of Moise we now know that these lens spaces are topologically different.

There have been many false solutions to the Poincaré conjecture—most of them never reached print. One by Whitehead appeared in 1934, but he made a retraction in 1935. I gave a partial solution in 1958 when I showed that a stronger condition than that stipulated in the Poincaré conjecture was enough to insure that the space actually was topologically a 3-sphere. For a while it looked like I might withdraw my paper since it was rumored in 1957 that both Moise and Papakyriakopoulos had complete solutions. Naturally, I did not wish to publish a partial solution if others got the whole thing. Both of their proofs sprung leaks.

In 1957 Poenaru wrote a long paper trying to give a solution but an error was discovered in his proof by the referee of a journal and it did not reach print. Koseki published a long proof in 1958 but the proof is not clear at one crucial spot.

Last summer, it appeared that each of Haken and Poenaru might have solutions. Although each tried novel methods and got interesting results, neither actually got a solution to the Poincaré conjecture.

The precise statement of the Poincaré conjecture is that a compact connected 3-manifold is topologically a 3-sphere if it is simply connected. We shall not discuss it further, but those interested in its history and various partial solutions of it might find of interest the chapter entitled "Some aspects of the topology of 3-manifolds related to the Poincaré conjecture" in volume II of *Lectures on Modern Mathematics* (Wiley, New York, 1964). Any set homeomorphic to an annulus is called an annulus.

6. The Annulus Conjecture. The closure of the part of the plane between two concentric circles is called an annulus. Generalizing this notion we say that the closure of the part of E^n between two concentric $(n-1)$-spheres is an n-dimensional annulus.

Suppose S is a smooth $(n-1)$-sphere in E^n and h is a homeomorphism of E^n onto itself such that $h(S)$ lies in the bounded component of E^n-S. Although $h(S)$ may not be smoothly embedded, it has a Cartesian product neighborhood since h is a space homeomorphism. Is the closure of the part of E^n between S and $h(S)$ an annulus? The answer is yes if $n=1$, 2, or 3. These are precisely the dimensions in which we can triangulate manifolds. If we knew how to prove the annulus conjecture for $n\geqq 4$, this would be useful in studying higher dimensional manifolds.

The annulus conjecture has been shown to be equivalent to other hard conjectures. False solutions to it have been proposed and submitted for publication, but so far as I know, no solutions (false or otherwise) have appeared in print.

7. Mathematics moves forward. The conjectures we have discussed are difficult and it is only infrequently that problems as difficult as these are solved. In fact, perhaps as many will be settled by counterexamples as by proofs. I have a feeling that most topology theorems that are true are true because of some simple underlying geometric principle. If there is such a principle we may write a clumsy complicated proof at first but once the underlying principle is understood, the proof clears up. If there is no such easy underlying geometric principle, the chances are that the conjecture is false.

Although the above conjectures may be hard to solve, I believe it is fruitful to ponder them. Good results are often found when one is looking for something else. With more space I could point out many interesting results that were obtained in unsuccessful attempts to answer the preceding conjectures.

Today, new results are being proved at an increasing rate but these proven theorems reveal many interesting things yet to try. As evidence of the amount of research being done I mention that in 1964 over 1200 abstracts were presented at various meetings of the American Mathematical Society.

Of course, only a fractional part of the mathematical research being done in the world is done in the United States. *Mathematical Reviews* which summarizes papers from all over the world reviewed 7000 papers during the first six months of 1965. Estimating the average length of a research paper to be 10 pages (some are much longer and some shorter) this comes to 140,000 pages of mathematics a year—more pages than many students read in an entire college career. I would not suggest that you read much of it. It contains few expository articles or textbooks. Most of the review articles are concerned with careful proofs, new methods of doing things, new applications of mathematics, reasons why certain things are true.

One nice thing about working at mathematics is that you do not need to

know everything about everything. You should have a solid background, and if you are to work successfully at the frontier you need to know almost everything about something.

For those students who follow mathematics as a career, I hope we can get more of them to remain mathematically active. To train them properly we need teachers with mathematical vigor who will remain learners.

I have heard that when learning ceases, rigor mortis sets in. Whether or not this is true, I feel that our schools today are turning out too many finished people—people whose learning is essentially finished when they graduate.

In the United States today there is a need for scholarly people: those who will remain students, scholars, learners, thinkers after college—teachers who can devise and teach a different and better course than they were ever taught.

The progress of our country depends in a large measure on active people. Our most effective teachers are the sort of people who subscribe to professional journals, participate in mathematical activities, read, think, ponder, plan, forge ahead. They work at mathematics whether it be at solving problems in the *Monthly* or doing profound mathematical research.

This is a good day to be a mathematician and teacher. Things are moving. There is excitement and controversy. Changes are being made. The rewards are better than ever before.

But on the other hand, more is expected of us. The high school teacher is supposed to teach not only the facts of mathematics but also an understanding of why these facts are true. It is hoped that the college teacher will impart not merely facts and understanding but also a glimpse of the charm and challenge of mathematics.

I wish you success in leading your better students to see the beauty of mathematics, to give them a taste of the type of mathematics that grips a mathematician's interest, tugs at his imagination, and makes him willing and eager to work long and hard to get a result that challenges him.

We realize that most students will not be research mathematicians, most will not reach the frontier—but they appreciate mathematics more if they know that the frontier is there, know that it is not inaccessible, get an occasional glimpse of it, and realize that it is being pushed back—sometimes steadily but often spasmodically.

Mathematics is not merely a subject of the past, but it is an alive and growing subject. It is not just a subject for the exceptional genius but it is a subject with a challenge for all of us.

TRENDS IN ANALYSIS

E. J. McSHANE, University of Virginia

1. Introduction. Before I begin on the subject matter of this talk, I wish to explain the sense in which it will live up to its title. First, I shall not try to discuss recent important advances in analysis. The "Mathematical Reviews" is being published at a rate that threatens the shelf-room of everything else in my mathematical library, and nearly half of it consists of reviews of papers on analysis. If I tried to describe the thousandth part of these in one hour, I could not hope to do so intelligibly. Nor would this be appropriate for a meeting of the Mathematical Association of America, which is concerned with the teaching of mathematics rather than with the latest mathematical research. Of the recent mathematical discoveries, any one student can be taught only a small part. But whatever we teach, we teach from some point of view and with some mental attitude, and what the student learns from us about mathematical thought-patterns will long outlast the individual mathematical facts that he picks up. I intend to speak about the changes in the prevailing points of view in analysis that have taken place since this Association was founded.

The most conspicuous trend in analysis, as well as in most other parts of mathematics, during the twentieth century, may be said to have started twenty-two centuries ago, with Euclid. It was he who made a coherent logical structure of the mathematics of his time, analyzing the assumptions needed to establish the theorems of geometry. But in the development of mathematics in the eighteenth and nineteenth centuries such an analysis of structure was largely ignored. There was then much more interest in acquiring new facts, in discovering interesting properties of special functions and inventing new methods of solving rather specific problems. But today the tendency is to consider large classes of mathematical objects that exhibit some common properties and to investigate to find the common structural elements that produce the common properties. Often we find that among the mathematical objects sharing the common properties there is some old acquaintance from familiar geometry or algebra. Then we usually choose a terminology suggested by that familiar example. This helps to remove the boundaries between the different fields of mathematics, and thus to unify the subject. More than this, by emphasizing the similarities between less familiar objects and those whose properties are well known to us, it helps us in reasoning about the new objects. As an example of what I mean, among the most conspicuous properties of vectors in finite-dimensional spaces are the linear properties; the vectors can be added, subtracted and multiplied by numbers in a way familiar to all of us, but many other collections of objects can be added and subtracted and multiplied by numbers just as those vectors can. When we wish to study the additional properties of such collections that are closely related to the linearity properties, we call the collection a "linear space," and the analogy with elementary geometry is a guide to our reasoning.

I do not at all intend to suggest that all good mathematical papers, or even the majority of them, are devoted to this sort of analysis of structure. Perhaps a majority of today's published papers in analysis could be read by Cauchy, if he could return from the grave and if the literary style of contemporary writing did not drive him back into it. It is well that this is so. It was about thirty years ago that Hermann Weyl warned that we had been living for a long time on abstracting and generalizing the special examples that our grandfathers had found, and that the continued life of mathematics required finding new interesting examples. There is not much profit in finding a common generalization of several mathematical objects whose chief common property is dullness. But when several somewhat similar special cases have been developed, or when one strikingly new instance appears, we may count on it that several of us will soon be busy trying to find just what it is that is essential in the situation.

One strong incentive toward the development of the concept of linear spaces was the great interest in integral equations about the turn of the century. It was evident that the classes of functions being studied had the linearity properties of vectors in finite-dimensional spaces, and that their transformations by means of integral kernels strongly resembled the transformations of vectors by means of matrices. Ehrhard Schmidt, for example, published his researches in a geometric terminology practically the same as that most favored today. At about the same time Fréchet, F. Riesz and Hausdorff were developing point-set topology and introducing the concepts of metric and norm and completeness and many others that we now expect any beginning graduate student to have at his finger-tips.

2. Lebesgue Integrals. In 1904 a new element of huge importance made its appearance. This was the Lebesgue integral. If by R_2 we denote the set of functions f on $[0, 1]$ such that f and $|f|^2$ are Riemann integrable, and define

$$\|f\| = \left[\int_0^1 |f(x)|^2 dx\right]^{1/2},$$

this becomes a normed linear space (provided we identify functions f, g for which $\|f-g\|=0$). But it is not complete; Cauchy sequences in R_2 exist which have no limit in R_2. Nor is the situation improved if we allow "improper" Riemann integrals, permitting, say, finitely many infinite discontinuities of the integrand. With the Lebesgue integral it is different; the corresponding space L_2 is a complete normed linear space. This alone, without the other advantages of the Lebesgue integral, would have made it valuable.

The completeness of the space L_2 is a mathematical fact, and is proved by logical processes. But the reason for the great success of the geometrization of function spaces is that mathematicians use logic only in the final stages of their research. Before a mathematician tries to prove something, he must conjecture that something; before he succeeds in proving it he must have decided on some pattern of proof, ordinarily pieced together from familiar devices. Less formally,

he needs good hunches before he knows what he wants to prove, and he needs good tricks to make a proof. The big help offered by thinking of function-aggregates as linear spaces is that it immediately offers analogies out of geometry, and provides mental pictures that help to suggest both the theorems and their proofs. To give a familiar example of what I mean, let us consider Fourier series of functions in L_2. In L_2 (I now allow complex-valued integrands) we can introduce an inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt$$

which strongly resembles the inner product of vectors in Euclidean spaces. The functions

$$e_n = (e^{2\pi nit}: 0 \leqq t \leqq 1) \qquad (n = 0, \pm 1, \pm 2, \cdots)$$

have norm 1:

$$\|e_n\| = \langle e_n, e_n\rangle^{1/2} = 1,$$

and any two of them have inner product 0, so they are analogous to orthogonal unit vectors in Euclidean spaces. The Fourier coefficients

$$a_n = \int_0^1 f(t)e^{-2\pi nit}dt = \langle f, e_n\rangle$$

are the components of f along these axis-vectors. Whether we use the Riemann integral or the Lebesgue, it is rather easy to prove that if f and $|f|^2$ are integrable, a generalization of the theorem of Pythagoras holds:

$$\left[\sum_{n=-\infty}^{\infty} |a_n|^2\right]^{1/2} = \|f\|.$$

Analogy with finite dimensional spaces would suggest that if $\sum|a_n|^2$ is finite, then $\sum a_n e_n$ should represent a function with finite norm whose Fourier coefficients are the a_n. But since this is an infinite series there is a limit process involved, namely taking partial sums

$$\sum_{n=-N}^{N} a_n e_n$$

and letting N tend to ∞. These partial sums form a Cauchy sequence in L_2, and because L_2 is complete they have a limit f in L_2, and this is readily shown to be the function we were seeking. The completeness of L_2 was crucial in the analogy between L_2 and the Euclidean spaces. No such close analogy with R_2 exists; in fact, in the study of Fourier series in the nineteenth century no such geometric picturization would have been at all likely to occur.

3. Abstract Linear Spaces. The popularity of Banach's famous book helped to make practically a commonplace of the ideas of abstract linear spaces and linear operations on them, with their concrete realizations in terms of aggregates of functions and operations defined chiefly in terms of summation or integration. It would be difficult to overestimate the impact of these ideas on contemporary analysis. When studying an unfamiliar class of functions it is now natural to establish its possession of at least some of the properties that have proved important in previous research. This done, it is to some extent classified, and one is thereby practically commanded to show that it has other properties that its congeners are known to possess.

It has been suspected that this sort of highly abstract mathematics is ill suited for use in applications. Whether a mathematical theory is or is not applied depends less on its content than on the mathematicians concerned. Now it may be true that there is a positive correlation between an interest in abstraction and a disregard for applications; this would be hard to prove or disprove. However, the correlation is certainly not 1.00. Consider for example the situation in quantum theory forty years ago. New experimental facts could be fitted into the Bohr theory only by wild contortions. Then Schrödinger and Heisenberg almost simultaneously developed new theories that were much more satisfactory, but unfortunately appeared very different from each other. Schrödinger used partial differential equations, Heisenberg used matrices (infinite matrices, but handled as though finite). A reconciliation or a choice had to be made for the sake of advancing. In 1929 John von Neumann settled the question by introducing abstract Hilbert spaces. These are complex linear spaces on which there is defined an inner product $\langle x, y\rangle$ with certain simple properties; the norm is $\|x\| = \langle x, x\rangle^{1/2}$, and the space is complete in this norm. A linear operator L defined on a dense subset D of the space is self-adjoint if for each y in the space, $\langle Lx, y\rangle$ is bounded for all unit vectors x in D if and only if y is in D, and then $\langle Lx, y\rangle = \langle x, Ly\rangle$. For such operators he developed a spectral theory. Now both the sequences used by Heisenberg and the functions used by Schrödinger were elements of Hilbert spaces, and Heisenberg's transformations by matrices and Schrödinger's transformations by differential operators were corresponding self-adjoint linear operators. So the two theories were two concrete realizations of one abstract theory. This did not merely resolve the apparent conflict. It also gave a sound foundation for further advances in quantum theory. On the other hand, it also opened a field for pure mathematical research that has continued to prove fertile ever since.

Although this application of abstract Hilbert space theory in quantum mechanics is of great importance, for the purposes of this talk it has a serious defect. Either a listener will know quantum theory, in which case I should merely have used a few words to remind him of what he knows; or else he does not know quantum theory, in which case I am asking him to take quite a bit on faith. So it is worth while to look at another example, in which the abstract

structural study and the concrete uses are both easily visible, and in which the basic ideas can be explained in a few minutes. This is the Mikusiński operational calculus.

4. Operational calculus. In the 1890's Oliver Heaviside was busily solving differential equations by replacing the operation of differentiation by the letter D and then optimistically acting as though D behaved in all respects like a real number. His objects were to solve the equations and to infuriate the mathematicians, and he did both. Much later, it was found that most of the differential equations solvable by Heaviside's methods could also be solved by use of the Laplace transform. However, this presupposed a knowledge of the theory of the Laplace transform, and restricted the operational calculus to those functions possessing transforms.

Important in the Laplace transform theory is the convolution. For brevity we shall understand that all functions mentioned are defined and real-valued on $(-\infty, \infty)$ and vanish on $(-\infty, 0)$, and are Lebesgue integrable over every interval $[a, b]$. The convolution $f*g$ of two such functions is that function whose value at x is

$$\int_{-\infty}^{\infty} f(x-t)g(t)\,dt.$$

This operation is commutative and associative. In 1924 Titchmarsh proved an easily stated but deep theorem about convolutions. If $f*g$ is almost everywhere 0, then either f is almost everywhere 0 or else g is.

Now let us remember how we can construct the rational numbers from the integers, by defining them to be equivalence-classes of ordered pairs of integers. We can duplicate this faithfully using convolution in place of multiplication. We define two ordered pairs (a, b), (f, g) of functions to be equivalent if $a*g=b*f$. This relation is reflexive and symmetric; by Titchmarsh's theorem, it is also transitive. It therefore classifies all ordered pairs of functions into equivalence classes, which we call operators. If we define multiplication and addition as though the pairs were fractions, we obtain the usual laws. In particular, to each function f corresponds an operator $f\dagger$ consisting of the pairs $(a*f, a)$ for all non-zero a; and we verify at once

$$(f*g)\dagger = f\dagger \cdot g\dagger,\ (f+g)\dagger = f\dagger + g\dagger,\ (cf)\dagger = cf\dagger$$

(c real). But there are operators other than these. For the class of pairs (a, a) is the identity for multiplication, and there is no function δ for which $f\dagger \cdot \delta\dagger = f\dagger$ for all f. Engineers and physicists have for years been making profitable use of this non-existent function; it is commonly called the "Dirac δ-function," and it is described neatly and nonsensically as the function which vanishes everywhere except at $x=0$, and is there so very infinite that its integral is 1. Ignoring the fact that this is impossible, we find that $f(x-t)\delta(t) = f(x)\delta(t)$ for all t, since it is

obviously true if $t=0$, and both have the factor $\delta(t)=0$ otherwise. Hence

$$[f*\delta](x) = \int_{-\infty}^{\infty} f(x-t)\delta(t)\,dt$$

$$= \int_{-\infty}^{\infty} f(x)\delta(t)\,dt$$

$$= f(x).$$

Hence $f\dagger\cdot\delta\dagger=f\dagger$, and $\delta\dagger$ is the identity in operator space. Now there is no function δ on $(-\infty, \infty)$ with the properties wished for; but in operator space there is an operator, the identity, which will truly and rigorously possess the properties that we would find in the operator corresponding to the wished-for "δ-function." Thus the success of the engineers in using the δ-function is explained. If we map all our functions into operator space and carry out our computations in that space, ending up with operators that are images of functions, it is quite possible that in the intermediate steps we may have used operators which are not images of functions. If we now re-write our work as though it were in function space, in place of these operators we have to insert symbols such as δ, δ', etc., that are not really names of functions. Nevertheless, the final result is sound because we have merely written a notational caricature of a logically sound operational calculation.

If h is the Heaviside function, that is $h(x)$ is 0 or 1 according as $x<0$ or $x\geqq 0$, and if for a function f of the type under consideration we define

$$F(x) = \int_0^x f(t)\,dt,$$

then $h*f=F$. Hence $h\dagger\cdot f\dagger=F\dagger$; the operation of indefinite integration among functions corresponds to multiplication by $h\dagger$ among operators. But every operator (a, b) with a and b not almost everywhere 0 has a reciprocal (b, a). In particular, $h\dagger$ has a reciprocal; let us call it s. (It is *not* the $\dagger$-image of a function.) Now we know that this operator s behaves among operators the way Heaviside presumed that D behaved wherever he wanted it to, and the long-used operational treatment of linear differential equations with constant coefficients is on a firm basis.

The theory we have just sketched exhibits several of the properties of abstract structural analysis at its best. It eliminates analytic machinery that had been inescapable in former treatments; it does not merely postpone the tedious part, leaving it to be done each time that we wish to apply the general theory to some specific situation. It does not attain a spurious generality by omitting the "nonessential" or "uninteresting" parts of the more special theories (these are by definition the parts that the generalization fails to cover); in fact, it covers much more territory than the Laplace transform did, and no one knows what

Heaviside may have thought he covered. It does not lose contact with the uses for which the original theory was devised; in fact, Mikusiński's book is full of applications to such things as electric circuits.

It is evident that a common generalization of several mathematical statements cannot contain every detail of each instance—if it did, we could not distinguish it from the instances. A good generalization has a satisfying extensiveness while retaining the good flavor of the interesting instances—and this is a highly subjective criterion. Clearly present-day taste runs much more to a liking for wide coverage and much less to a liking for individual charm than did that of earlier times. Sometimes it has been realized that the process of generalization discarded properties that were too valuable to ignore, and then a different generalization was developed to recover what had been bypassed. Thus in treating aggregates of functions as linear spaces we ignored the fact that functions can be combined otherwise than by linear combination, for instance by multiplication or convolution. So along with Banach spaces we now study Banach algebras. Again, in extending Fourier theory to abstract harmonic analysis one ignored the possibility of extending Fourier series and transforms from the real axis to the complex plane. But now the Fourier series in the complex domain are back as subjects of active study. However, at the present rate of proliferation of mathematics the only alternative to inundation is a process of coalescence, by which the mass of accumulated information is molded into coherent general schemes. Probably in the process we lose many details that ought to have been kept, whatever that means. But as gain we have a structure whose beauty often repays the loss of the decorative individual peculiarities—and survival is a gain too.

5. Theory of Distributions. The theory of distributions, or generalized functions, which is pervading more and more of analysis, is one which, even if it had been invented, could hardly have attained wide acceptance except in a mathematical community already imbued with the concepts of linear topological spaces. We have already seen how the "Dirac δ-function" and its derivatives can be made intelligible and safely usable by first mapping functions into the space of Mikusiński operators and then noticing that other well-defined operators behave the way that the correspondents of δ, δ', etc. should. The method of Sobolev and of Schwartz involves a different imaging of the ordinary functions. To make it easy, we consider real functions f of one real variable that are Lebesgue integrable over every interval. Let $\mathfrak{D}$ denote the class of all infinitely differentiable functions that vanish outside some interval. For each function ϕ in $\mathfrak{D}$ there is a number

$$f\dagger(\phi) = \int_{-\infty}^{\infty} f(t)\phi(t)\,dt,$$

so $f\dagger$ is a function on $\mathfrak{D}$. It is clearly linear. If we define a very stringent limit concept in $\mathfrak{D}$ by saying that $\phi_n \to \phi$ if and only if there is an interval outside of

which all the ϕ_n vanish, and $\phi_n(x) \to \phi(x)$ and $d^m\phi_n(x)/dx^m \to d^m\phi(x)/dx^m$ for all x and all positive integers m, then $f\dagger$ is continuous; if $\phi_n \to \phi$, then $f\dagger(\phi_n) \to f\dagger(\phi)$. Moreover, two functions f_1, f_2 that differ on a set of positive measure will have $f_1(\phi) \neq f_2(\phi)$ for some ϕ, so that f is uniquely determined by $f\dagger$, up to sets of measure 0. But there are far more continuous linear functions on $\mathfrak{D}$ than those of the type $f\dagger$ just discussed. These, all together, are the "distributions," or "generalized functions." For instance, if there were such a function as the δ-function, we would have

$$\delta\dagger(\phi) = \int_{-\infty}^{\infty} \delta(t)\phi(t)\, dt = \phi(0)$$

for all ϕ in $\mathfrak{D}$. Now $(\phi(0) : \phi \in \mathfrak{D})$ is a perfectly good linear continuous functional on $\mathfrak{D}$; it is a distribution, and can perfectly well be called the δ-distribution. The preceding equation is, properly speaking, nonsense, since it involves the "δ-function" $\delta(t)$. Nevertheless, either in recognition of the convenience to physicists of pretending that such a function exists, or else in the spirit of "if you can't lick 'em, jine 'em," mathematicians are now quite willing to speak of δ-functions and to write them in integrands whenever convenient. After all, neither in mathematics nor out of it is there any harm in using intrinsically nonsensical expressions, provided that the listener knows what is nonsensical about them and can be trusted to replace them by what we really mean to say.

If f is absolutely continuous, the distribution corresponding to f' has value at ϕ

$$f'\dagger(\phi) = \int f'(t)\phi(t)\, dt = \int f(t)[-\phi'(t)]\, dt = f\dagger(-\phi').$$

Accordingly, for any distribution F we define the distributional derivative DF by $DF(\phi) = F(-\phi')$. Now all distributions are differentiable, hence infinitely differentiable. However, we must remember that in general $Df\dagger$ is not the distribution corresponding to f'. For example, for the Heaviside function h we have $h'(t) = 0$ except at $t = 0$, hence $(h')\dagger = 0$; but

$$h\dagger(\phi) = \int_{-\infty}^{\infty} h(t)\phi(t)\, dt = \int_{0}^{\infty} \phi(t)\, dt$$

so

$$Dh\dagger(\phi) = \int_{0}^{\infty} [-\phi'(t)]\, dt = \phi(0),$$

and thus $Dh\dagger$ is the δ-distribution. More generally, if f is of bounded variation on every interval, the value of the distributional derivative $Df\dagger$ at the test-function ϕ is

$$\int_{-\infty}^{\infty} \phi(t)\, df(t),$$

which is not in general the same as the integral of $\phi f'$.

The usefulness of distributions was their birthright, since their invention was motivated by the insistence of physicists and engineers that δ-functions were too useful to abandon merely because they were meaningless. Now more and more mathematicians are becoming comfortable in their presence, and are using δ-functions and other distributions in their everyday thinking and in their published papers, and they are finding these objects as convenient as did the engineers. There is a whole theory of Fourier series and Fourier transforms of distributions, more or less paralleling the classical theory, going far beyond it in some directions, but necessarily paying the price of not giving all the wealth of known detail that abounds in the older theory. (A theory in which everything is differentiable can hardly draw subtle distinctions based on continuity and differentiability properties.)

6. Differential Equations. Another field in which distribution theory plays a large and increasing rôle is differential equations. It would probably be wrong to say that one of the trends in analysis is toward the study of differential equations. I do not know of any time since the beginnings of analysis at which differential equations were not among the objects of major attention of analysts. If there is a trend to note in this connection, it might be a trend to assimilate all relevant advances in mathematics and use them to increase our knowledge of differential equations. This has of course been done with distributions. Even if these gave no new results, they would be helpful in clarifying our ways of thinking about differential equations, making the essential and crucial ideas stand out more clearly from the analytical machinery. When we say, for example, that the Green's function $g(x, t)$ of a differential operator L (in one variable) is the solution of $Lg = \delta(x-t)$, we have not only given a compact definition, we have almost given directions what to do with it.

Once again, let me say that this incorporation of new constructions is by no means the only activity in differential equations. One of the most novel results of recent years is contained in a short paper by Hans Lewy. If $F(x_1, x_2, y_1)$ is any reasonably well behaved function of three variables and c_1, c_2, c_3 are complex numbers not all 0, it is known that there exists a complex valued function $u(x_1, x_2, y_1)$ that satisfies

$$c_1 \partial u/\partial x_1 + c_2 \partial u/\partial x_2 + c_3 \partial u/\partial y_1 = F(x_1, x_2, y_1).$$

It was reasonable to conjecture that if the c_j were replaced by well-behaved functions of x_1, x_2 and y_1, the solution should still exist. Lewy proved the unexpected theorem that with $c_1 = 1$, $c_2 = i$, $c_3 = 2x_2 - 2ix_1$ there is an infinitely differentiable function F for which the equation has no solution with Hölder-con-

tinuous derivatives, not even on any neighborhood of any point. This has led to considerable further study. Hörmander has found conditions on the coefficient functions that determine whether or not the equation is of the solvable kind or not (and also got rid of the Hölder continuity); and we now know a good bit more than before about solvability of partial differential equations. But it remains true that this growth has its roots in one highly instructive and concrete example.

Before going on to the next topic, I want to talk for a moment like a physicist. Suppose that $f(x, t)$ $(-\infty < x < \infty,\ 0 \leqq t < \infty)$ represents the temperature at place x and time t in an infinitely long uniform rod. By the elementary physics of heat conduction, if f is a sufficiently well-behaved function it should satisfy the differential equation

$$\frac{\partial f}{\partial t} = k\frac{\partial^2 f}{\partial x^2}, \tag{1}$$

where k is a constant whose value depends on the structure of the rod. Subject to the condition that the initial temperature distribution $f(x, 0)$ is a twice-differentiable function $g(x)$ that does not increase faster than exponentially, we can solve this easily and find

$$f(x, t) = \frac{1}{2k\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left[-(\xi - x)^2/4k^2t\right] g(\xi)\, d\xi. \tag{2}$$

But what the physicist really wants is not to solve the differential equation, for this is just a mathematical device to help in analyzing the physical situation. What he is looking for is the temperature distribution $f(x, t)$ that results t seconds after an initial distribution $g(x)$. The right side of the preceding equation defines a transformation of g into $f(x, t)$; we can call this T_t, and condense the equation to $f(x, t) = [T_t g](x)$.

From the physics of the problem it is clear that if we start with distribution g and wait t seconds, and then start again with the resulting distribution $T_t g$ and wait s more seconds, the resulting distribution $T_s[T_t g]$ is the same as the result $T_{s+t}g$ of starting with distribution g and waiting $s+t$ seconds. At this point it is tempting to use an argument of a type of which physicists are occasionally guilty: because our solution is the mathematical model of a physical process in which we evidently have $T_s T_t = T_{s+t}$, our expression for T_t (deduced from the differential equations) must have this property. As cautious mathematicians, all we can say is that either the transformation defined by our integral formula satisfies $T_s T_t = T_{s+t}$ for all non-negative s and t, or else our mathematical model fails to match the physics. The equation $T_s T_t = T_{s+t}$ must be verified from the mathematics, not from a physical analogy. Fortunately the verification is not difficult; we readily show that if g is continuous and does not increase faster than exponentially, the same is true of $T_t g$, and $T_s T_t g = T_{s+t} g$ for $s, t \geqq 0$. That is, the

$T_t (t \geqq 0)$ constitute a *one-parameter semi-group* of linear transformations on a function-space.

If we would permit ourselves the luxury of the most unexamined use of formal procedures, from the identity

$$h^{-1}(T_{t+h} - T_t) = [(T_h - T_0)/h]T_t$$

we would deduce $T_t' = T_0' T_t$, and by iteration $T_t^{(n)} = [T_0']^n T_t$. Recalling that T_0 is the identity, the value of $T_t^{(n)}$ at $t=0$ is $(T_0')^n$, and if we expand in a Maclaurin series

$$T_t = \sum_0^\infty [(T_0')^n/n!]t^n = \exp(tT_0').$$

Thus T_0' determines T_t for all t; it is the "infinitesimal generator" of the semi-group. This is pure formalism. Nevertheless, given the semi-group defined by (2), we find that if g has a continuous second derivative that increases at most exponentially with $|x|$, then

$$\frac{d}{dt} T_t g \Big|_{t=0} = kg''. \tag{3}$$

[The calculation uses the fact that the second partial derivatives of

$$\exp[-(\xi - x)^2/4k^2t]$$

with respect to x and with respect to ξ are equal, and involves two integrations by parts.] So the infinitesimal generator of our particular example of semi-group is the operation $k\partial^2/\partial x^2$, and (3) is the same as (1) at $t=0$. This indicates that there is an intimate mutual relationship between one-parameter semi-groups and differential equations of the form

$$\frac{\partial f}{\partial t} = Af,$$

where A is a differential operator, namely the infinitesimal generator of the semi-group. There has been a good deal of significant research on the properties of semi-groups and their infinitesimal generators. More relevant to the purpose of this talk is the observation that the concept of semi-group has become very widespread. The defining property of a semi-group is so simple, and the kind of heuristic argument by which we were led to it can be applied to so many situations both in pure mathematics and in applications, that it is not surprising that examples are abundant. Now, when one meets such an example, he is ready to observe the fact, and thereby to increase his grasp of the mathematical object.

7. Probability Theory. When I was an undergraduate, the standard freshman course in mathematics included a small amount of probability theory, confined

to the simplest problems concerning random variables that could take on only finitely many values. But I believe that most mathematicians then regarded probability theory as at best dubiously respectable. Quite unrelated to this, the concept of Lebesgue measure had been widened to apply in spaces other than finite-dimensional; this extension was one of the evidences of the power of the ideas used, but seemed of rather esoteric interest only. In 1906, Einstein published his famous study of the Brownian motion. If a fluid contains a large number of similar microscopic particles, it can be regarded as a mixture of two fluids, the second consisting of huge "molecules," i.e. the particles. If the statistical mechanical theory of diffusion is correct, these particles should have random motions such that, if $x(t)$ is the x-coordinate of a particle at time t, the displacement $x(t_2)-x(t_1)$ during a time interval $[t_1, t_2]$ should be independent of the history of the particle before time t_1, and should be a normally distributed random variable with variance proportional to t_2-t_1; that is, the probability of being in a set A at time t_2 is

$$\frac{1}{k\sqrt{2\pi(t_2 - t_1)}} \int_A \exp\left[-(y - x(t_1))^2/2k^2(t_2 - t_1)\right] dy, \tag{4}$$

where the proportionality constant k involves the diffusion coefficient of the fluid. Einstein thus found striking evidence for the validity of statistical mechanics. Moreover, given times $0<t_1<t_2< \cdots <t_n$ and intervals $[a_1, b_1]$, $\cdots$, $[a_n, b_n]$, it is easy to calculate the probability that a particle starting at $x(0)=0$ will have $x(t_j)$ in $[a_j, b_j]$ for $j=1, \cdots, n$. That is, we have assigned a numerical probability to the event that a Brownian-motion path belongs to the set of paths satisfying

$$a_j \leqq x(t_j) \leqq b_j \qquad (j = 1, \cdots, n). \tag{5}$$

In 1923 Norbert Wiener combined these assorted ideas into a powerful unit. The set of all real-valued functions on $[0, \infty)$ is a linear space. The sets (5) and their finite unions form a small but important family of subsets of this space, and on them we have a finitely-additive measure defined, namely their probability measure. Wiener showed that this measure can be extended to be defined and countably additive on a σ-algebra of subsets of the function space. Moreover, this can be done in such a way that the set of all discontinuous functions (although of a higher order of infinity than the set of continuous functions) has measure zero. This immediately made all the powerful machinery of the Lebesgue integration theory available for studying probability theory in this function space. Moreover, it served as a point of entry into the theory of random functions.

To give a quick glance at the relationship of this theory with other parts of analysis, we shall ignore all the inescapable and sometimes difficult questions of measurability, integrability, etc., and consider a set of probability measures $p(t, x, A)$, where $t \geqq 0$, x is in Euclidean n-space, and A is a Borel set in n-space.

We suppose that for fixed t and x, $p(t, x, A)$ is a probability measure defined for Borel sets; it is to be interpreted as the probability that a particle starting at place x at an arbitrary time t_0 shall be located in set A at time t_0+t. This is supposed to be independent both of the time t_0 and of what happened to the particle at times before t_0. Then, speaking somewhat sloppily, the probability that the particle, starting at time t_0 at place x, will at time t_0+t be in a tiny cube dy containing a point y, and at time t_0+t+s will be in A, is the product of the probability $p(t, x, dy)$ that it gets into dy at time t_0+t and the probability $p(s, y, A)$ that s seconds later it gets from there to A. If it is in A at time t_0+t+s, it had to be at some place y at time t_0+t, so by integrating with respect to y we find

$$p(t+s, x, A) = \int p(t, x, dy)p(s, y, A). \tag{6}$$

This is the Chapman-Kolmogorov equation. Now for all functions g in some class of functions on n-space (including surely all bounded Baire functions) the integral

$$\int g(y)p(t, x, dy) \tag{7}$$

is defined for each x in the space, and is therefore a function on the space, which we call $T_t g$. It is evident that T_t is a linear operation, and by the use of the Chapman-Kolmogorov equation we can easily show that $T_s T_t = T_{s+t}$, so the T_t form a one-parameter semi-group of transformations on a function space. But we have already seen that the semi-group is closely related to the equation

$$\frac{\partial}{\partial t} f(x, t) = Af(x, t),$$

where A is the infinitesimal generator of the semi-group and is often expressible as a differential operator involving the partial derivatives of f with respect to $x_1, \cdots, x_n$. To give a specific example, let the transition probabilities be those assumed by Einstein in his study of the Brownian motion (see (4)). If we substitute this in the definition (7) of $T_t g$, we obtain exactly the right member of equation (2). So the transformation semi-group T_t ($t \geqq 0$) of the Brownian motion is the same as the semi-group associated with the heat equation (1), which in the present setting should be renamed the diffusion equation. Since the Brownian motion is merely one of many Markov processes, equation (1) is merely one of many differential equations related to stochastic processes. If a mathematician is interested in any one of the three theories (differential equations, semi-groups, stochastic processes), the more he knows of the other two the better off he is.

To show the connection of stochastic processes with other types of differential equations, let us consider the Dirichlet problem; given a bounded open set B (say in the plane) and a reasonably well-behaved function g on its boundary ∂B, we wish to find a function harmonic on B, continuous on $B \cup \partial B$, and agreeing

with g on ∂B. Let x be in B and let A be a Borel subset of ∂B. (The letters x, y, z will stand for vectors in the plane, $x=(x_1, x_2)$, etc.) If a particle begins Brownian motion at x, with probability 1 it will follow a continuous path that at some time leaves B. We denote by $p(x, A)$ the probability that the first point on its path that is out of B (which surely is in ∂B) will be a point of A. This is a probability measure on the Borel subsets of ∂B, and by integration we can define a function

$$f(x) = \int_{\partial B} g(y)p(x, dy). \tag{8}$$

It is not hard to give an indication of proof that as $x \to y$ ($x \in \partial B$, $y \in B$) we have $f(x) \to g(y)$, at least where ∂B is not too complicated. To indicate that f is harmonic we by-pass all proofs of measurability, although they are both essential and difficult, and argue thus. Let S be a circle centered at x and contained, with its interior, in B. We denote by $z(\vartheta)$ the point of S such that the vector from x to $z(\vartheta)$ makes angle ϑ with the positive x_1-axis. With probability 1 the moving particle leaves the interior of S; and by the rotational symmetry of the quantities involved, the probability that the first point at which it meets S will be in an arc A_0 subtending an angle $\triangle\vartheta$ is proportional to $\triangle\vartheta$. Since this must be 1 if A_0 is S, the probability is $\triangle\vartheta/2\pi$. Now if A is a Borel subset of ∂B and A_0 is a very short arc containing a point $z(\vartheta)$, it is clearly a reasonable conjecture that the probability that the particle will first hit S in A_0 and then go on to hit A before hitting any other point of ∂B is very nearly $[\triangle\vartheta/2\pi]\cdot p(z(\vartheta), A)$. But to hit A it must go through some point of S, so if we integrate this expression over all of S we get the probability that it will first hit ∂B in a point of A:

$$p(x, A) = \frac{1}{2\pi}\int_S p(z(\vartheta), A)\, d\vartheta.$$

Substituting this in (8) yields

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_S \int_{\partial B} g(y)p(z(\vartheta), dy)\, d\vartheta \\ &= \frac{1}{2\pi}\int_S f(z(\vartheta))\, d\vartheta. \end{aligned}$$

That is, the value of f at x is the mean of its values over the circle S with center x, and this characterizes f as a harmonic function.

Anyone speaking on the subject of stochastic processes here in Ithaca would be sadly remiss if he did not mention that outstandingly important contributions to this theory have been made by mathematicians associated either now or previously with Cornell University.

It is clear that activity in probability theory is waxing. This may quite possibly have an important side-effect. The geometrization of analysis influenced the direction of research by directing mathematicians' minds towards theorems

in analysis that are analogues of geometric statements that spring readily to mind, because of our life-long familiarity with geometry. Probability theory provides an additional source of images and analogies. One can hardly expect that the heuristic substitution of a "chance happening" model for an analytic statement will ever be quite as self-suggesting as the substitution of a "geometric figure" model, since we are unlikely ever to feel quite as informally at ease with chance events as with geometric diagrams. Nevertheless, to those who acquire some degree of relaxedness with probability, it may be an important auxiliary in conjecturing theorems and proofs.

8. Prophecy. I will finish off by allowing myself the luxury of two prophetical remarks, with full confidence that if my predictions are wrong no one will remember them anyhow. First, my old love, the calculus of variations, although rather thoroughly neglected for some years, is beginning to show signs of starting a new kind of development. Several recent research papers have been devoted to studying the minima of functionals involving random elements or integrals over function spaces. These are at present isolated results, not part of a coherent theory. But in view of the present interest in analyzing the abstract structures of mathematical theories, it would be quite natural to have these concrete examples embedded in a new and interesting kind of calculus of variations.

Second, the rapid growth of computing may well have a long-range effect on the direction of mathematical thinking. I do not mean that hordes of mathematicians will become computers or rush to put problems on the machines. However, in the process of developing those completely novel uses of computers that surely lie in our future, new methods of analyzing complicated situations will be called for, and when developed will be new parts of mathematics. When they become familiar, they will provide a new collection of familiar situations from which new abstract structures will be developed. But it is just as hopeless to predict the nature of these new structures by an extrapolation from today's numerical techniques as it would have been to predict the contents of Banach's "Théorie des opérations linéaires" by an extrapolation from plane geometry.

THE CLASSICAL GROUPS AS A SOURCE OF ALGEBRAIC PROBLEMS

C. W. CURTIS, University of Oregon

Introduction. Rather than attempting a general survey of what has been accomplished in algebra, I shall try to describe the influence on the development of algebra of one of the great sources of algebraic problems and ideas—the classical linear groups. I intend to show that these groups have provided an experimental, empirical basis for large parts of algebra. They are a family of examples which serve as raw material from which problems, conjectures, abstract ideas and general theorems have emerged, and which have provided direction and continuity to the research based on them. Surveys of recent developments and problems of current interest have been given by Brauer [5], [6], Carter [11], Dieudonné [24], M. Hall [26], Kaplansky [35], and MacLane [36].

It is impossible in this lecture to give an account of all the major results which involve the classical groups. In making a selection, I have simply included the material that has been most helpful to me.

1. Definitions and background. The story begins with Galois, who proved that a polynomial equation was solvable by radicals if and only if a certain finite group of automorphisms of a field was solvable. Thus he raised the problem of investigating groups in terms of their structure, and observed, as an example, that the set of all 2 by 2 invertible matrices with coefficients in a finite field of more than 3 elements, was a finite group that was not solvable.

Let us first give an up-to-date version of Galois' example. Let K be an arbitrary field, and V a finite dimensional vector space over K. The set of all linear transformations T of V which possess multiplicative inverses (and which we shall call invertible transformations) forms a group with respect to multiplication, called the general linear group $GL(V)$ of V. Hermann Weyl invented the term *the classical groups* to stand for $GL(V)$ and certain subgroups of $GL(V)$, which include the following ones.

(a) $SL(V)$, the unimodular group, consisting of all invertible transformations of determinant one.

(b) $O(V, f)$, the orthogonal group of a non-degenerate symmetric bilinear form f on V, consisting of all invertible transformations T that leave f invariant:

$$f(Tx, Ty) = f(x, y),\ x, y \in V.$$

For example, if V is n-dimensional euclidean space and $f(x, y)$ the usual inner product, $O(V, f)$ is the familiar group of linear transformations T that preserve lengths: $\|Tx\| = \|x\|$, where $\|x\| = f(x, x)^{1/2}$.

(c) $Sp(V, F)$, the symplectic group, consisting of all invertible transformations leaving invariant a non-degenerate skew-symmetric ($F(x, y) = -F(y, x)$) bilinear form F on V.

In order to understand why the list of classical groups does not continue, we have to return to Galois' observation about the structure of the classical groups.

Suppose we have, more generally, a family of algebraic systems $\{G_\alpha\}$ for which the fundamental theorem of homomorphisms is valid, such as groups, rings, algebras, etc. Whenever we have a homomorphism

$$\phi: G \xrightarrow[\text{onto}]{} H$$

with kernel L, G is called an extension of H with kernel L. G is called *simple* if whenever G is an extension of H with kernel L, then either G is isomorphic to H, or G is isomorphic to L. Given a family $\{G_\alpha\}$ of simple systems, the *extension problem* asks for the construction of all systems G for which there is a sequence of homomorphisms

$$\phi_1: G \xrightarrow[\text{onto}]{} G_{\alpha_1} \in \{G_\alpha\}$$

with kernel G_1,

$$\phi_2: G_1 \xrightarrow[\text{onto}]{} G_{\alpha_2} \in \{G_\alpha\}$$

with kernel G_2, etc., and such that for some k, the kernel G_k belongs to the family $\{G_\alpha\}$. The systems G determined in this way are built up from the simple systems $\{G_\alpha\}$ by solving the extension problem.

Solvable groups are the result of applying this process to the family $\{G_\alpha\}$ of cyclic groups of prime order. Galois' observation that the classical groups are not solvable shows that the classical groups are built up from non-abelian simple groups by solutions of the extension problem.

The problem of which simple groups are derived from the classical groups was investigated first by Jordan and Dickson ([34], [21]) with Dickson's book in 1901 containing fairly complete results for the classical groups defined over finite fields. Dickson's proofs, while satisfactory at the time, involved exhausting calculations with matrices which are a tribute to his persistence, but often left obscure the essential ideas. The subject was taken up again by Dieudonné and Artin ([22], [23], [3]) who identified the simple groups derived from the classical groups over arbitrary fields by elegant methods using the geometric properties of linear transformations and subspaces of the underlying vector space.

The result of their investigations, which extended and clarified Dickson's work, can be summarized as follows. First of all consider all homomorphisms f_α of a group G onto abelian groups A_α with kernels G_α. It turns out that there is a unique minimal such kernel G' called the *commutator group* of G, which is generated by all commutators $(x, y) = xyx^{-1}y^{-1}, x, y, \in G$. Thus an arbitrary group is an extension of an abelian group with kernel equal to the commutator group. The first step in investigating the structure of classical groups was to find the commutator groups. In the case of $GL(V)$ the multiplication theorem for determinants gives a homomorphism of $GL(V)$ onto the abelian multiplicative group

of the field. With one exception (when V is the two dimensional space over the field of two elements), the commutator group turns out simply to be the kernel of the determinant homomorphism, $SL(V)$. The symplectic group is its own commutator group. The commutator group of the orthogonal group is more complicated. When isotropic vectors $v \neq 0$ such that $f(v, v) = 0$ are present, the commutator group of the orthogonal group is the kernel of a homomorphism called the spinor norm, whose construction depends on a certain associative algebra called the Clifford algebra of the form f, (see [23], [3]).

The main structure theorem on the classical groups, proved in special cases by Dickson, and in full generality by Dieudonné, is that, with a manageable number of exceptions, the commutator group G' of a classical group G is an extension of a non-abelian simple group with kernel Center(G'). In other words G'/Center(G') is a simple group, in most cases.

Dickson accumulated many other facts about the classical groups that proved to be useful for experimental purposes: he found the orders of the finite simple groups he had obtained, and partial information on their conjugacy classes and subgroups. All these results provided evidence that there should be general methods for studying these groups, without considering them one family at a time, as is the case in the approach both of Dickson and Dieudonné-Artin.

2. First attempts at a classification; Lie algebras. With the examples of simple groups furnished by the classical groups, and the alternating and Mathieu groups in the finite case, the problem arises to classify the non-abelian simple groups in some reasonable way, and in particular, to ask whether there are others. This problem, for arbitrary groups, still seems well beyond our reach, but is basic because the simple groups are the building blocks of arbitrary groups.

The first successful classification of simple algebraic systems which are relevant to our problem was contained in E. Cartan's thesis [9] in 1894, in which the simple Lie algebras over the field of complex numbers were classified. Since then Lie algebras have developed a life of their own (see [38], [32], [35]). The link between Cartan's work and group theory, however, can only be understood by plunging into the deep waters of Lie group theory, invented by Cartan's predecessor, Sophus Lie. For an up-to-date account of Lie group theory see Chevalley [13], Helgason [27], or Hochschild [29]. Thus the first successful classification of some kind of simple groups required heavy use of nonalgebraic methods, and shows that it may be shortsighted to expect a problem stated in algebraic terms to have a purely algebraic solution.

Without giving details, a complex Lie group G is a group which is at the same time a complex analytic manifold, such that the group operations are given locally by analytic functions. The classical matrix groups, with the operation of matrix multiplication, are the prototypes of Lie groups. From the manifold structure, a Lie group G has a tangent space $\mathfrak{g}$ at the identity, which is a finite dimensional vector space over the complex numbers. It is possible to define on $\mathfrak{g}$

a bilinear multiplication $[X, Y]$, satisfying the axioms of a Lie algebra:

$$[XX] = 0, \qquad X \in \mathfrak{g}$$
$$[[XY]Z] + [[YZ]X] + [[ZX]Y] = 0, \; X, Y, Z \in \mathfrak{g}.$$

It can then be shown, by an interesting combination of algebraic and analytical reasoning, that the group multiplication in a certain neighborhood of the identity is completely determined by the multiplication in the Lie algebra. Simple Lie groups have simple Lie algebras (where the concept of a simple Lie algebra is defined by the general remarks on the extension problem in section 1). Then the purely algebraic problem of classifying the simple Lie algebras implies a local classification of simple Lie groups.

The classical groups over the field of complex numbers are Lie groups, and to the simple groups derived from the classical groups correspond simple Lie algebras. Cartan's great achievement was to prove, completing a program already begun by Killing, that besides the simple Lie algebras corresponding to the classical groups, there are only 5 additional simple ones, called the exceptional Lie algebras, of dimensions 14, 52, 78, 133, and 248. There is a simple Lie group belonging to each of the exceptional simple Lie algebras, and the actual construction of the exceptional algebras and groups, has been one of the most striking applications of the theory of alternative and Jordan algebras. (See e.g. [30], [31], [32], [16], [47].)

The importance of Cartan's classification is that questions about properties of simple Lie algebras and Lie groups can be answered by checking against each type of algebra (or group) in the classification. Many interesting general problems and conjectures have been made in this experimental way, and later general solutions, independent of the classification, have been found for those that survived the case-by-case examination.

One difficulty in studying Lie groups via Lie algebras is that different Lie groups can have the same Lie algebra, and although homomorphisms of Lie groups always generate homomorphisms of the Lie algebra, the reverse statement is not necessarily true. But the most fundamental criticism from the point of view of algebra is that the classical groups are defined for arbitrary fields, while the Lie group machinery works smoothly only for groups which are real or complex analytic manifolds.

3. Representation theory. The next milestone after E. Cartan's work was Hermann Weyl's series of papers in 1925–1926 [50] on the representation theory of simple Lie algebras and Lie groups.

There were two problems, partly suggested by physical applications, that motivated the work on the representations.

One arises from the observation that a classical group G on a vector space V admits a homomorphism into the general linear group of the space of tensors

$$V^{(m)} = V \otimes \cdots \otimes V, \; (m \text{ factors})$$

given by

$$A \to A \underbrace{\otimes \cdots \otimes}_{m} A = A^{(m)}.$$

The problem was to decompose the tensor space $V^{(m)}$ into minimal invariant subspaces relative to the set of transformations $A^{(m)}$, $A \in GL(V)$.

A second problem, also suggested by physics, was to split up the space of continuous functions on the 2-sphere into minimal invariant subspaces relative to the action of the rotation group in 3-dimensional space.

These problems are both special cases of the following general problem. A group G is said to have a representation on a vector space M over a field K if there is a homomorphism $T: G \to GL(M)$. A subspace N is called a G-subspace if $T(g)N \subset N$ for all $g \in G$. The representation on M is called *irreducible* if the only G-subspaces are the trivial subspace $\{0\}$ and the whole space M, and *completely reducible* if M is a direct sum of irreducible G-subspaces.

At the time of Weyl's papers, there was already an extensive theory of representations of finite groups, due mainly to Frobenius, Schur and Burnside. In particular it was known that every representation of a finite group in a field of characteristic zero was completely reducible. (See [20], p. 41. The bibliography of [20] contains references to the original papers on representations of finite groups.)

Weyl, completing work begun by Cartan, classified explicitly the irreducible representations of the simple Lie algebras over the complex field, and proved also the theorem of complete reducibility for representations of simple Lie algebras, by using integration over a compact group associated with the Lie algebra. The Lie group—Lie algebra correspondence then solves the corresponding problems for representations of Lie groups. The failure of homomorphisms of the Lie algebra to generate homomorphisms of the group is strikingly illustrated by the spin representation of the rotation group. In this case, a representation of the orthogonal Lie algebra gives rise to a representation of a two-sheeted covering group of the rotation group, which cannot be viewed as a representation of the rotation group itself ([7], [13], Chapter 2, Section XI).

Weyl's work led first to a purely algebraic proof of complete reducibility of representations of simple Lie algebras by van der Waerden and the physicist Casimir [12]. Then J. H. C. Whitehead gave a new and profoundly original proof of the theorem on complete reducibility ([52], [53]) that can be viewed as the beginning of the cohomology theory of Lie algebras, and contains the motivation for at least a chapter of homological algebra ([38], [10]).

In 1939, Weyl returned to the subject, and in his book on the classical groups [51], he gave a new determination of the irreducible tensor representations of the classical groups over fields of characteristic zero, using the Wedderburn structure theory of associative algebras in an ingenious manner due to Schur, Brauer and himself. This method avoided the Lie algebras and their accompanying

analysis, and gave one of the first solutions since Dickson's book of a deep problem involving the classical groups by purely algebraic methods.

Weyl's paper ([50] p. 358–359) also contained an explicit formula for the characters of the irreducible representations of the simple Lie groups, which has inspired other combinatorial investigations on the representations of Lie groups and Lie algebras by Freudenthal, Kostant, Steinberg and others (see [32], Chapter VIII, for a complete account). This work includes a generalization of the Clebsch-Gordan formula for splitting up the tensor product of two irreducible representations of the unimodular group on a two dimensional space, into irreducible components (see [49], pp. 127–131).

4. Algebraic groups. The key to the discovery of a uniform approach to the classical groups was contained in Weyl's 1925–26 papers, where he proved (see [50] pp. 338–342) that a simple Lie algebra over the complex numbers was determined up to isomorphism by a certain finite group of permutations of one-dimensional subspaces of the Lie algebra. Weyl proved that these groups were all isomorphic to finite groups generated by reflections in euclidean space E^n.

In 1934–35, Coxeter proved ([17], [18]) that a finite group was isomorphic to a group generated by reflections in E^n if and only if as an abstract group, it had generators $S_1, \cdots, S_n$ which satisfied the defining relations

$$S_i^2 = 1,\ 1 \leqq i \leqq n, \text{ and } (S_iS_j)^{n_{ij}} = 1,\ i \neq j.$$

He determined all such groups, and proved that the ones corresponding to simple Lie algebras were precisely those groups generated by reflections satisfying a certain crystallographic condition on the angles between the reflecting hyperplanes.

In 1941, Stiefel proved that the Weyl-Coxeter group of a Lie algebra could be constructed within a compact Lie group belonging to the Lie algebra [41]. Specifically, the Weyl-Coxeter group of G is isomorphic to $N(T)/T$ where T is a maximal torus in G, and $N(T)$ its normalizer. Using Lie theory, it followed that the structure of G was determined, at least locally, by the finite group $N(T)/T$.

The stage was now set for attempting to classify simple groups of Lie type over arbitrary fields, now that an internal key to their structure had been found. To replace the analytical machinery of Lie groups, algebraic geometry was used, since by that time the foundations of algebraic geometry over arbitrary ground fields had been laid by Zariski and Andre Weil.

Hermann Weyl had already observed ([51], p. 147) that the classical groups could be viewed as the intersection of an algebraic variety in the space $M_n(K)$ of all n by n matrices over K with the general linear matrix group $GL(n, K)$. Such a group is called an *algebraic group*, and again the prototypes are the classical groups. For example the unimodular group is defined by the polynomial relation

$$\det(X) - 1 = \sum \pm x_{i_1 1} x_{i_2 2} \cdots x_{i_n n} - 1 = 0.$$

In 1956–58, Chevalley succeeded in classifying all simple algebraic groups over algebraically closed fields of arbitrary characteristic [15]. He proved that such a group was determined up to isomorphism by a Weyl-Coxeter group exactly as in the case of compact Lie groups, and that the Weyl-Coxeter groups which appeared satisfied the crystallographic restriction. Since the groups with these Weyl-Coxeter groups were known, it followed that the only simple algebraic groups were the classical groups and the five types of exceptional groups. The proof of Chevalley's result makes heavy use of the techniques of algebraic geometry. It depends on some basic work of A. Borel on solvable algebraic groups.

5. Finite groups of Lie type. Until 1955 no simple finite groups were discovered that were not already known to Dickson. In his famous Tôhoku Journal paper [14] of 1955, Chevalley constructed a simple group associated with every simple Lie algebra over the complex field and every field K, with some exceptions when K is the field of 2 or 3 elements. His construction yielded finite simple groups corresponding to the exceptional Lie algebras, and gave a uniform method for investigating the groups, which is based on new structural properties of the groups.

Chevalley proved first that a simple Lie algebra $\mathfrak{g}$ over the complex numbers has a basis $X_1, \cdots, X_n$ such that

$$[X_iX_j] = \sum c_{ijk}X_k,$$

where the $\{c_{ijk}\}$ are integers. This multiplication table serves to define a Lie algebra $\mathfrak{g}_K$ over an arbitrary field K, since the structure constants $\{c_{ijk}\}$ can be viewed as elements of K. The Chevalley group G associated with the Lie algebra $\mathfrak{g}$ and the field K is defined to be a certain subgroup of the automorphism group of $\mathfrak{g}_K$. Chevalley then proved by a long argument, but one that broke with tradition by treating all the groups at once, that with a few exceptions, the groups G were all simple.

For a Lie algebra $\mathfrak{g}$ associated with one of the classical groups, the Chevalley group of $\mathfrak{g}$ and K coincides with the simple group derived from the corresponding classical group [37]. In case K is the complex field, the Chevalley groups over K yield a complete set of simple Lie groups with complex parameters. If K is algebraically closed, of arbitrary characteristic, then the Chevalley groups give a complete set of examples of simple algebraic groups defined over K, by the classification theorem for algebraic groups. If K is a finite field, then the Chevalley groups include the finite simple groups investigated by Dickson, Dieudonné and Artin, and those associated with the exceptional Lie algebras give infinite families of finite simple groups not isomorphic to those in Dickson's original list.

The proof of this last fact requires formulas for the orders of the Chevalley groups defined over finite fields. Chevalley [14] derived these formulas using some topological properties of the Lie group of the same type, and recently L. Solomon [40] has given another derivation of the formulas using some topo-

logical properties of the Weyl-Coxeter group of the Lie algebra. No purely algebraic derivation of the formulas is known. A knowledge of the formulas is important because of a number theoretical study by Artin ([1], [2]) of the orders of the known simple groups which, among other things, gives a method of showing that different Chevalley groups in general cannot be isomorphic.

6. Further results on finite groups. The excitement over Chevalley's paper had hardly died down before new infinite families of finite simple groups were discovered, by Hertzig, Ree, Steinberg, Suzuki, and Tits (see [11]). Some of these were defined by analogy with the classification of simple algebraic or Lie groups, where it was known that among the groups parametrized by fields which were not algebraically closed, certain twisted versions of the Chevalley groups could occur. All are obtained from a Chevalley group by the following sort of construction. Certain Chevalley groups G, defined for certain fields K, admit automorphisms of period 2 or 3 such that the set of elements in G left fixed under the automorphism contains a new simple group. Of course we do not know whether we have seen the last of these constructions, especially in view of a new simple group defined by Janko [33], of order $11(11+1)(11^3-1)$, which is a subgroup of a Chevalley group defined over the field of 11 elements, but is not obtained by the preceding general method.

Nevertheless the situation has crystallized enough for us at least to hope for uniform description of all the known finite simple non-abelian groups. For this we are indebted above all to J. Tits, whose work is still in the process of publication. From a close inspection of the construction of the Chevalley groups, Tits developed the following very simple but still rather mysterious set of axioms which are satisfied by all the known finite simple groups except possibly the Janko group, and are suggested by the structural properties of the groups derived in Chevalley's Tôhoku Journal paper. (See [44], [45], [46].)

A group G (not necessarily finite) is said to have a BN-pair if G is generated by subgroups B and N such that

$$B \cap N \text{ is a normal subgroup of } N; \text{ let } H = B \cap N.$$

$$N/H = \langle w_1, \cdots, w_n \rangle, \; w_i^2 = 1,$$

and for all cosets $w \in N/H$, and generators w_i of N/H,

$$w_i B w \subset BwB \cup Bw_iwB$$

and for each i

$$w_i B w_i \neq B.$$

The group $W = N/H$ is called the Weyl group of G, for the reason that the group N/H of an arbitrary finite group with a BN-pair is isomorphic to a group generated by reflections in euclidean space. Abstract groups with BN-pairs can be expressed as a union of (B, B) double cosets, $G = \bigcup_{n \in N} BnB$, with the double

cosets in 1–1 correspondence with the elements of the Weyl group, a decomposition discovered first for the classical groups by Bruhat [8]. All the Chevalley groups, as well as the twisted types, have BN-pairs. Among other things Tits has found an elegant proof of simplicity for certain types of groups with BN-pairs [46], that provides in almost all cases a much shorter and easier proof of simplicity of the Chevalley groups and the twisted types than the one given in Chevalley's paper.

It has also been possible, using the methods of Chevalley and Tits in combination with Lie algebra methods similar to those in Weyl's 1925–26 papers, to classify all the irreducible representations of the finite Chevalley groups and the twisted types, in an algebraically closed field of the same characteristic as the field by which the groups are parametrized, completing the work begun by Weyl for the classical groups over fields of characteristic zero ([42], [19]).

The complex representations and characters of the finite Chevalley groups have not yet been classified, partly because of our lack of knowledge about the conjugacy classes.

In this connection, the Jordan normal form theorem tells us that two elements X and Y of $GL(V)$ are conjugate, i.e., $X = ZYZ^{-1}$, if and only if they have the same Jordan normal form. The Bruhat decomposition $G = \cup BnB$ fails to give a normal form up to conjugacy for elements of a general Chevalley group. R. Steinberg [43] has recently given a normal form for certain conjugacy classes of elements in "simply connected" versions of all the Chevalley groups over algebraically closed fields, that includes the Jordan normal form as a special case. This powerful result has already had applications to the classification of algebraic groups over non-algebraically closed fields and may suggest a method of determining the conjugacy classes in finite Chevalley groups.

Behind all the fascinating special results and problems suggested by these developments, lies the austere and immensely difficult problem of classifying the finite simple groups. The classical groups on 2- and 3-dimensional spaces were determined in terms of the structure of their subgroups by Zassenhaus, Suzuki, and Brauer and his students. See Section 92 of [20] for a survey of this work up to 1962. Then came the great achievement for which the Cole prize in algebra was awarded at the Denver meeting of the American Mathematical Society last winter—Walter Feit and John Thompson's proof, published in 1963, of Burnside's conjecture that a non-abelian simple group must have even order [25]. Since then the pace in finite groups has picked up, with Thompson's proof that the only finite simple groups, all of whose proper subgroups are solvable, are the known ones, and related works of Suzuki and Gorenstein and Walter. This work brings together all that is known about finite groups because the inductive method of proof used in these questions presents one with a group whose structure is to be determined, given fairly complete information on the structure of its subgroups which may be solvable groups, p-groups, etc.

If we continue to believe, as of course we must, in the possibility of communicating mathematical proofs as well as statements of results, then we are faced with the difficult and important task of simplifying some of these Hercu-

lean arguments, which often run to two and three hundred pages, to the point where they can meaningfully be presented in lectures and text books.

A bright ray of light in this direction is Tits' classification of finite groups with BN-pairs. It was known for some time that in a BN-pair G whose Weyl group was the symmetric group S_3, a coset geometry could be defined that satisfied the axioms for a projective plane. D. G. Higman and McLaughlin [28] proved that if the group G was finite, the plane was Desarguesian, and that G was an extension of the simple classical group $PSL(3, K)$ for some finite field K. Tits has proved that there is a geometry associated with every finite group with a BN-pair, and has succeeded in classifying the possible geometries in most cases [45]. The difficulty in applying his work to arbitrary finite groups is that it seems to be very difficult to tell whether a group has a BN-pair or not.

7. Conclusion. There are many other important directions I have not been able to take up. One basic experimental fact we have observed is the strong dependence of the methods used for studying the simple groups and their representations on the field over which the groups are defined. To make precise and formalize this kind of dependence is one of the tasks of homological algebra. In this direction we have, for example, applications of Galois cohomology [39] to the study of algebraic groups defined over nonalgebraically closed fields, and important new constructions in the category of modules over a ring—the algebraic K-theory of Bass [4]—which are at least partly suggested by the study of linear groups over the ring.

I have been able, nevertheless, to tell enough of the story to show that the classical groups have successfully resisted the best efforts of three generations of mathematicians to subdue them, and continue to suggest interesting problems for further research. The methods that have been used represent strong improvements and refinements over those available to Jordan and Dickson and have found their way into our elementary teaching. Their continual interaction with other branches of mathematics, and physics, would have heartened Hermann Weyl, who expressed in the preface of his book [51] his concern over the dangers of a too thorough specialization of mathematical research. The permanent feature of all this work has been the source of the problems—the groups themselves.

I hope that what I have said makes it clear that in our capacity as teachers we should introduce our students to the empirical sources, the physical facts of our science, as well as to the general ideas and methods we use in our work.

Other sources of algebraic ideas that have had a similar influence are the theory of algebraic numbers, representations of finite groups, and algebraic geometry; but these I leave to another time and other speakers.

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THE FUTURE ROLE OF THE FEDERAL GOVERNMENT IN MATHEMATICS

SAUNDERS MacLANE, University of Chicago

1. Preamble. Mr. Chairman, members of the Mathematical Association of America, and guests. At this meeting our Association comes to full maturity and wisdom, and this at a time when that wisdom is much needed to understand the rapid changes buffeting the mathematical world. First, a very minor example of these changes. A dozen years ago most of you wouldn't be here, and our chairman would be standing at the lectern reading my speech and trying not to let on where he disagreed with it. Today the government liberally supports travel to meetings—bringing you here to listen to my disembodied voice. Many examples, all more important than this one, show how the government support of mathematics has enormous effects on our science. However, there is no general agreement as to whether these effects are good, bad, or just indifferently expensive. Today I will try to describe some of the future effects, good or bad, as I see them. I expect that each of you will disagree with some of my conclusions and that some of you may disagree with all of them.

2. Present Situation. First, consider the present situation. Mathematics in the United States today is lively. Fundamental advances are being made. Famous hard problems have been solved. There is wide activity in many different mathematical specialties; applied mathematics is more vigorously developed than before. It is especially pleasant to contemplate this lively state of mathematics in our country, because this has not always been the case.

In the United States there is large scale government support of mathematics. This support on this scale is so new that many of our judgments are not adjusted to the situation. Let me give a personal example. A dozen years ago I was a department chairman, deciding about applications for government support; with my colleagues I tried to plan this support so that *nothing* vital would be lost if all government support suddenly disappeared—at that time such counsels of caution were common. Now they would be wholly unrealistic. Today there is no prospect that the government will *withdraw* its support (just because, if it did, many scientific activities would simply collapse). The problem is rather: What happens when the government *shifts* the direction of its support? The real problem before us is not the existence of government support of mathematics, but the wise *direction* of this support.

Here is a first controversial question. Is or is not the present lively state of mathematics in the United States the direct result of government support? The answer is not obvious. Indeed, I incline both to a *yes* and a *no*. "Yes," in the sense that the amount and volume of our mathematical activity would be impossible without government support. "No" in the sense that other important factors have been more vital stimuli to the quality of mathematics here. Ob-

serve that *all* the physical and *all* the biological sciences have partaken of government support. We must account somehow for the fact that mathematics is at present one of the healthiest of all these sciences. I hold that the health comes from certain internal sources of strength in mathematics. One source is the recent development of the habit of collaborative research. Twenty-five years ago a joint research paper was a rarity and, in most departments, seminars hardly existed. Today there is a lively flow of joint papers in nearly all fields of mathematics; these joint papers almost always represent an active sharing of ideas, quite different from the process present in some other sciences where the director likes to sign all papers which come from his laboratory. At the same time, every Department of Mathematics worth the name has a long weekly directory of research seminars; in each seminar a few people get together to discuss in detail what is going on in their specialty. I submit that this new habit of collaborative work has brought many American mathematicians to a much higher level of activity and development than would have been otherwise possible. I hold that the beneficial effects of these habits have been just as important as government support in developing the present lively state of mathematics.

3. The Future. This was a question of the past effects of government support. What about the future? Money directed toward mathematics may or may not be useful for the development of the science and may or may not be useful for other proper government purposes. It is very important to bear in mind that more money need not mean better science. I remind you of the Eilenberg proposal to ruin American mathematics in five years: Just give a five year leave of absence to all those research mathematicians able to direct Ph.D. theses. In the spirit of collaboration I offer the MacLane method of thwarting all proper government purposes: Offer to triple the salary of all those and only those mathematicians who work in *Categorical Algebra.* These fanciful examples I cite to emphasize the vital point: Money for mathematicians is not the main thing; it's how you spend it!

4. The mechanism of decision. What I now propose to do is to examine the mechanism of decision as to how the money is spent.

Decisions about the government support of science are fundamentally political decisions and not just scientific ones. Federal funds come from the taxing power of the government. The disbursement of these funds is, in the last analysis, made by decisions of the Congress, acting for what it judges to be the general welfare. Therefore, questions about government money for mathematics are political questions. In case you incline not to believe this, consider, please, the situation in physics. Recently a number of high energy physicists decided that a new and bigger accelerator was needed—perhaps up to 1000 billion electron volts. For this purpose, they prepared a pamphlet called "The Nature of Matter—Purposes of High Energy Physics," edited by Luke C. L. Yuan. This pamphlet, with articles by many able physicists, was not a scientific document but a propaganda document with scientific overtones. It set forth the political

reasons for spending the public monies not just on applied physics, but on physical studies aimed to discover basic facts about the structure of elementary particles. I believe this pamphlet has had the desired effect.

Now mathematics is surely not yet that expensive and consequently not yet that political, but on a smaller scale there is an exact parallel. In the long run, we must face the prospect that there will be documents entitled "The Nature of Mathematics—Purposes of High Abstraction," edited by Nicholas Bourbaki, and that these documents will be propaganda documents with scientific overtones. These documents should set forth the political reasons for spending the public monies not just on applications of mathematics, but on mathematical studies aimed to discover basic facts about the structure of abstract mathematical entities.

5. The establishment. Since decisions about federal funds for science are political decisions, it is most fortunate that this country has developed the practice of making these decisions with the help and the collaboration of scientists. The fact that so many scientists do take part in this decision process makes the process much more effective than it otherwise would be. In the course of this development there has grown up in Washington what I would like to call a "scientific establishment." Here I use the term "establishment" in the precise British sense. I might almost say that an establishment is a private club, complete with rules, memberships, and blackballs for outsiders, functioning in the public domain. In the United States this scientific establishment consists of many devoted scientists who work hard and travel far, all to no personal advantage, save that of possible influence on policies. The character of this establishment is much influenced by the wartime conditions of secrecy which conditioned its early years. By now the establishment has its own way of life, with frequent committee meetings and wordy and unanimous joint reports. The fact that these reports habitually express no minority opinions is in my view an anachronism arising from the early impression that such reports are science, where there can be agreement, and not politics, where there should be minority opinion.

It is a striking fact that mathematics is simply not a part of this scientific establishment. Let me expand this flat statement. If I define a mathematician to be one who has contributed to our science at least one idea, process, or theorem that others have used, then I know of exactly two mathematicians who are part of the establishment. Put differently, the scientific establishment shows the ratio of two mathematicians to sixty or more physicists and to fifty or sixty chemists.

This raises the question: Why is this so? This is a question full of fun, fascination, and controversy. There are many different explanations as to why mathematics does not belong to the establishment. One extreme view holds that other scientists deliberately keep the mathematicians out. A milder and more realistic version holds that the physicists and others just don't know many mathematicians and so don't happen to think of them when setting up committees. Still

another version (and I have seen real evidence for this version) is this: The scientific establishment was first forged during the Second World War under the fervor of work on important classified applied problems. When the groups to carry out this work were set up, about 1941, those who were responsible simply didn't know who the mathematicians were and even had sadly mistaken notions as to who the good mathematicians were. Still other reasons are offered. There is an extreme view that mathematicians almost always refuse to serve on committees; some part of this may be true; some mathematicians I have known very well are likely to be more critical than most scientists of the waste of time involved in committee meetings. A more modest version of this explanation holds that mathematicians have a sort of ingrown dislike for committees and administration. This may indeed have some basis in the nature of our science. In experimental science the facts found in the laboratory are controlling; in mathematics, it is the discipline found in the head. The personal nature of this discipline has meant that mathematicians have had to develop a special sort of devotion to the significance of their own work and of the importance of sticking right to it, no matter the distraction.

Another version of the explanation reads as follows: Each tightly knit group of scientists has its own group ethos. I am told that in physics an older man is often expected to give up active research work in favor of committee assignments, and he is honored when he does this. In mathematics it is clear that an older man who gives up research for such other activity really loses caste with his fellow mathematicians.

There are plenty of other purported explanations as to why mathematics is not part of the establishment. I don't intend to try to decide here between these explanations, first, because I'd rather leave the debate to my audience, and second, because I think the relevant question is: Should mathematics in the future join the establishment?

There is an unselfish reason why mathematics should be part of the establishment. Science is an interplay between theory and experiment; the American empirical temper tends to give priority to experiment. As science matures—and with massive financial support it will do so rapidly—the theoretical aspect will become more important and with it mathematics as the clearest and most general form of this aspect. For this reason, if for no other, mathematics belongs in the establishment. It might even be that the President's Science Advisory Committee, which now and again likes to pontificate about mathematics, should consider again having a mathematician as member.

There is also a selfish reason why mathematics should be part of the establishment. Unless mathematics does become part of the establishment, fundamental decisions about its support will be made by people of enormous good will who profoundly misunderstand the nature of mathematics.

Sometime *after* I wrote that apparently grandiloquent phrase about people of enormous good will who profoundly misunderstand the nature of mathema-

tics, I came across an apt illustration. On August 6th last occurred two acts of folly: Saunders MacLane tried to demolish an automobile windshield and Alvin Weinberg tried to turn the teaching of calculus back 40 years. MacLane's act showed an abysmal ignorance of the laws of physics. I hope Alvin will pardon me if I say that his article in *Science* ("But is the Teacher also a Citizen," *Science*, August 6, 1965) displayed great political sagacity, enormous good will, and abysmal ignorance of mathematics. He observed—and I would say correctly—that good teaching of calculus is of interest to other sciences. He noted that a physicist friend of his had written a text on calculus which (in Weinberg's words) "presents the traditional parts of the subject that have been bypassed by the professionals" (such as skill in handling simple integration). Now as long as it is clear that calculus is not a branch of physics we can certainly welcome physicists—or chemists or economists or logicians—who take the interest to write texts in mathematics. In this case Weinberg doesn't make clear whether his physicist friend wanted to train students to reproduce B. O. Pierce's "Tables of Integrals" or to train students to do many problems in volumes, work, moments of inertia, water pressure, and the like. Now any good calculus course must have many good problems; these particular ones are of the duller sort, of less interest to most users of calculus and even of less interest to those physicists who believe in the eight-fold way. In any case, Dr. Weinberg should have realized that his vague statements could only be interpreted as a call to return to the style of teaching best described to this audience as Granville's *Calculus* (now Granville, Smith, and Longley). Since that was the style 39 years ago when I first started studying calculus at the school where Granville, Smith, and Longley had all taught, I have some evidence for the observation that Weinberg inclines to turn the clock back 40 years.

Pardon me. Weinberg's article recommends (though it may not exemplify) careful scholarship. If I draw my conclusion carefully, I must argue that it is proposed to turn the clock back not 40 years, but just 39 years.

6. The apportionment of federal funds. This should be enough to demonstrate that the country and mathematics both suffer when mathematics is not represented in the establishment. This will matter more in the future because there will be hard decisions to make. Perhaps the toughest decision will be this one: Given substantial federal support of science as a whole, how does one divide this support between the various sciences or between the different branches of one science? Once the government is giving large scale support to science, it becomes essential to ask this question. I wish this were not so, not just because *I* don't know how to answer the question, but because, as far as I can see, *no one* now really knows how to answer this question. Neither the scientists nor the politicians have been accustomed to this sort of decision about science; in the past, this allocation decision has usually been avoided. (One might almost say: The decision is "locally avoidable" but not "avoidable in the large.") At the beginning, when the government first supported science, one could say: Support

all competent scientists. This will work for a few years. It simply doesn't work indefinitely, partly because it is not too easy to define what one means by a "competent" scientist, and partly because the distribution of government funds presently influences the subjects which the competent people take up.

Let me offer some examples to show how hard it is to apportion money between the sciences.

First example: What sciences do you include? Do you include social sciences? At present, it is my understanding that the National Science Foundation includes those social sciences which choose to bear the label "Behavioral Sciences" and which are in addition quantitative. Personally, I don't think that a label "Behavioral," even if invented by the Ford Foundation, is an answer to our problem. I also wonder if the requirement "quantitative" might not in a few years lead to some fanciful quantitative decorations of important research which ought not to be quantitative at all.

Second example: How do you divide money between "pure" and "applied" mathematics? I have no answer to propose. I simply observe that all the present noisy controversy about applications does not provide an answer either. In mathematics, it is right and proper that each of us be enthusiastic about his own work. Well placed enthusiasm for pure mathematics does not decide against applied mathematics. Well placed enthusiasm for applied mathematics does not decide against pure mathematics, even in those cases when the public enthusiasm for the application is coupled with considerable ignorance of what goes on in pure mathematics. At present, I see no statesmen with a good view of the whole picture.

My *third example* is a question to each of you personally: How many specialists should there be in your own field of mathematics? You say "more." How many more? Think a bit and you may find some of the hidden booby traps. If, say, there were ten times as many mathematicians supported in your field, there would be ten times as many papers to read and ten times the chance that your own results would be scooped.

Let us continue. Suppose that someone has finally answered the question about distribution of government funds between fields of science. But now, just within one field, the government supports many types of activities: research, education, teaching, facilities, conferences, and the like. For mathematics, most of the support funnels through the National Science Foundation, more exactly, through a wide variety of "Programs" in that Foundation. For many years I have been puzzled as to how and where someone with a real understanding of mathematics makes a wise decision on the relative levels of these activities. Gradually I have come to the regretful conclusion that *no one* makes a wise decision here. It just happens.

An example. In 1966 the International Congress of Mathematicians will be

held in Moscow. Now international congresses are big and unwieldy and have all sorts of troubles, but the fact remains that mathematics (with astronomy) is the most international of sciences and that our ablest young research mathematicians get vital stimulus and encouragement for future projects from meeting their colleagues from other countries. At previous international congresses, I have myself seen many explicit examples of such profitable contacts between young mathematicians of different countries. Let me add that in my judgement the United States today has the best mathematics in the world. I can say this because it is not my generation of American mathematicians which brought this about. Ten years ago we were not best; ten years from now we may not be best but in recent years our younger mathematicians have done wonderful things; at the last International Congress one of the two Fields medals for outstanding achievement by younger mathematicians was awarded to an American. (Added in proof: At the 1966 Congress, two of the four Fields medals were awarded to Americans.)

What conclusion? At the next Congress there should be a strong representation of the ablest young American mathematicians. How will they get there? The chief source is a possible NSF grant to the American Mathematical Society. I am informed that this grant will be $20,000. Think a minute. Twenty thousand dollars. Enough to send twenty young men, possibly twenty-five. How does this compare? How much does the NSF spend on travel expenses for people coming to *this* meeting in Ithaca? Much more, I suspect. How much does the NSF spend on summer institutes for high school teachers of mathematics? Much more. How much does it spend on summer salaries? More. How much does it spend in a four-year period to support the committee on the Undergraduate Program in Mathematics? About one hundred times as much. How much does it spend on the Committee on Educational Media? More. How much did another government agency spend this summer on a conference on categorical algebra (my subject)? More.

It is *not* my intent to question *any* of these expenditures. I admire the vigor of the program director for each of these activities and the devotion and self-sacrifice of the mathematicians who administer each.

It *is* my intention to draw a comparative judgment about these activities. Do you share my conclusion that a grant of $20,000 for travel expenses to the International Congress of Mathematics is just ridiculously small? If you share my conclusion, you are at liberty to write to John Wilson of the National Science Foundation. Feel free to differ with my own numerical estimate, which is that there should be about $100,000 for such travel expenses.

This was just an example. The intended point is this: Given one field of science, say mathematics, there are hard decisions as to the relative levels of Federal support of different sorts of activity in that science. I am personally not convinced that the National Science Foundation (or for that matter, anybody else) knows how to bring *wise* and *active* knowledge of the science itself to bear on these important decisions.

7. Centers of excellence. Here is a third type of hard decision about science. Where do you locate scientific facilities? At present, the Science Foundation has a program to develop what it chooses to call "centers of excellence." This is a program which is understandably popular: 90 percent of the universities of the country at first blush might hope to be one of the new centers. This popularity doesn't necessarily mean that the program is a good one. Some wag, noting that excellence is a rare quality and that "centers of excellence" was just a slogan, observed that the program might really turn out to be one developing "centers of mediocrity." Be that as it may, the process of deciding which of ten competing institutions in a given area is the one to become a center of government support is a pretty tough one. I have carefully examined the texts of all the congressional hearings where the Science Foundation defended this program. From this examination I was unable to find one scintilla of evidence that the Science Foundation had given any careful and considered thought to the decision process which will be required. To put my conclusion bluntly, I hold that the administration of the Science Foundation and the NSF Board have been irresponsible in starting this program without careful advanced consideration of the decision process involved.

Since I am sure this conclusion will shock many of you, let me ask you to examine quickly the situation in mathematics. Recent developments have clearly demonstrated that the major mathematical centers are no longer individual universities but congeries of first class universities. I give you Princeton, New Jersey; Cambridge, Massachusetts; Paris, France; and Moscow, USSR as examples. In other words, the development of our science appears to flourish best when mathematicians come together in larger groups. At the same time the development of our politics suggests that each of 50 states would like a center of mathematics in that state. This paradox is the source of a real problem.

Let me put the matter differently. Well into the 20th century American mathematics, though vigorous, was essentially provincial. Our great heroes were often typical provincial heroes. One such example is my own revered teacher, E. H. Moore, to whom I certainly owe a great deal. Today, as a result of long efforts, American mathematics has come of age, thanks to a massive injection of European talent, much of it from Hitler Germany. As late as 1950, a count at the then three leading universities for mathematics in this country, showed that approximately 50 percent of the full professors were not trained in this country. Today American Mathematics does stand on its own feet, and this thanks to a new generation. However, the doctrine of regional development and regional centers of excellence is often advanced hand in hand with the doctrine that provincial heroes of American mathematics should not only be admired—and they should indeed be admired—but that they should be imitated. I hold that Government support of such reversion to the provincial would be fatal.

8. Summary. To summarize, I have cited three basic questions of government policy about mathematics: How do you distribute money between branches of mathematics? How do you distribute money between different

activities in mathematics? How do you make geographical decisions? I have not hesitated to indicate here and there my own opinions. My chief opinion is this. At present we do not have either the mechanism of decision or the wisdom necessary to make such decisions properly. Also, I think I can explain why these decisions are so hard. Political decisions are made by majority vote, and scientific decisions cannot be made by majority vote. The decisions which face us are both political and scientific—what do we do?

In closing I would like to make three explicit recommendations.

First: In selecting members of the scientific establishment, eagerness to serve has in the past played far too great a role. Actual and significant scientific accomplishment—and I mean scientific accomplishment and not administrative accomplishment—should in the future play a much bigger role if we are to be able to decide well the problems before us.

Second: Minority opinions as of science policy should be vigorously encouraged. Committee reports should not try to be unanimous.

Third: There should be a recognized avenue by which working scientists can express their opinions on science policy. I suggest that such an avenue might be a journal of science policy open to contributions from any qualified scientist. At present, what we have are effective means of disseminating *to* the scientific community the opinions of the Washington establishment. What we need is a feedback.

Members of the Association: Now it is your turn to feedback.

WHAT CAN MATHEMATICIANS DO FOR THE FEDERAL GOVERNMENT?

JOHN W. TUKEY, Princeton University and Bell Telephone Laboratories

1. Introduction. All of my title but "ian" was assigned to me; I assume that we all understand the title to mean "for the Federal Government and thus for the Nation, and for the World." If I wanted to give a useful impression by making untrue statements, I could answer: "almost everything except mathematics" and "by working on uncomfortable problems." These statements are untrue because, as we shall see, there is much of what every mathematician judges to be his natural work, pleasant to do, to be done "for the Federal Government." On the other hand, these statements are useful and help to give a correct impression, because many mathematicians seem to feel that *they* themselves should do only mathematics. Undoubtedly some are right.

Each of you owes it to yourself, to your college, university or other organization, and to your country to work out the places and ways where you can do the most.

2. Kinds of "us." A decade or so ago someone made amusingly distorted maps labelled "A New Yorker's idea of the United States," "A Bostonian's idea of the United States" and so on. They reflected the attitudes back of the story of the maiden lady from Boston who made a safari by automobile to California and back, and who, when asked about the route, said: "I forget how we went, but we came back by way of Dedham."

What is the parallel caricature of a mathematician's idea of society? I suggest a simply-ordered structure of seven classes:

(1) Mathematicians in the narrow sense.

(2) Mathematicians in the broader sense, including mathematical physicists, mathematical statisticians, and mathematical logicians.

(3) Legitimate users of mathematics, including physicists, statisticians, some engineers, some chemists, etc.

(4) Other technicians and scientists.

(5) Other intellectuals.

(6) People at large.

(7) Administrators.

I am glad that this is so largely a caricature, for the good of each of us, our profession and our country. But it would be unrealistic not to recognize that it has shreds of truth.

One who speaks on my subject should make it clear where he stands among these classes. Once upon a time, I was briefly a mathematician in the narrow sense, say from 1938 to 1941. Since then I have been a mathematician in the broad sense on a part-time basis, with a large share of my time spent as a legitimate user of mathematics, mainly in the fields of statistics and data analysis. Please discount my words accordingly.

3. Education, including research. The first of the three heads under which it is natural to outline the ways that mathematicians do things "for the Federal Government" and thus for the nation and mankind, is education, including research carried out in educational institutions.

For the last 20 years, the major supporter of mathematical research, of all kinds, has been the Federal Government. By doing better and more effective research many mathematicians serve the nation and the world through the Federal Government.

The Federal share in the support of graduate training is not quite as large as it is in research, but good, efficient, and diversified graduate training is more and more important to our country and more and more supported through Federal actions.

The improvement of mathematics education in the schools, which I am happy to see becoming more and more a continuing activity entered into by all kinds of mathematicians, has been largely supported by Federal funds.

These are certainly among the things that all mathematicians feel that mathematicians should do. For all of this agreement, such areas are not free of strains, either now or in the future. Year by year, a larger fraction of our young people complete high school, enter college, complete college, enter graduate school, get master's degrees, get doctor's degrees, do post-doctoral work. For the next decade or two, at least, not only fractions, but sizes of age groups will increase. We face real problems of teaching both those who will leave the academic world and those who are so badly needed to become teachers themselves. There have always been readjustments in the details of how we train students; these readjustments will have to come more rapidly and in larger steps. In particular, we are going to have to provide greater diversity, both in types of training and in fields. The rise of the computer, to take one example, will call for more and more people with a strong master's background in most fields of mathematics combined with an increasing logico-linguistic understanding of formal modes of communication and translation.

Those who speak out against the "new mathematics" in the schools will have to be encouraged to bring together the necessary insights and skills and develop "new curricula" of their own. There is no substitute for fair and open competition; I hope to see a strong enough group willing to work hard on *new* approaches in yet another direction to deserve and receive Federal support.

Throughout the educational system, the responsibility of individual mathematicians must be to support diversity and competition wherever these have strength enough to survive and be effective. In many cases, support need only mean not standing in the way, not supporting some single monolithic position, not impressing on students that only one way is right. In others, support means setting out to cooperate or lead in new ventures. All can help.

4. Technical contributions. The second main head is the doing of mathematics outside departments of mathematics.

At one extreme stands direct government service. Once the people who worked directly for the Federal Government and appeared on our membership lists were few and exceptional. Today the number is vastly increased, and so is the diversity of the branches of government that employ them. Between membership lists, articles in journals, and conversation with friends, most of you have some idea of what kinds of people are involved and what they do. I wish I could help you understand the situation better, but my own contacts are about as spotty as your own. It would be a major effort to come to a clear understanding of where we stand today, to say nothing of where we will be tomorrow.

I ought, however, to point out, rather than describe, the major contributions of mathematical thinking, by mathematicians, behind the classification barriers of the National Security Agency. It is probably here that mathematical thinking has contributed the largest share to the activities of any Federal agency.

Throughout the Federal government, with few exceptions, the mathematician contributes as part of a mixed team. Almost all the exceptions correspond, at least roughly, to a "department of mathematics" in a substantial research establishment. In my view, this is as it should be. The best place to get research for research's sake done is still the university full of graduate students, best for the research, and best for the by-products. This has not kept Federal agencies, however, from employing many mathematicians in interesting and rewarding careers.

There are also many mathematicians involved in work under government contracts, in industry proper, in the "porch laboratories" that are really neither in nor out of the government's "house," and in universities and colleges outside of mathematics departments. Again what is done is very diverse, and hard to picture as a whole. Times are changing, perhaps slowly; the fraction of such work that is directly connected with defense is dropping steadily; the types of work being done are steadily diversifying. Take a few "porch laboratories" as examples. How different from each other are the things done at RAND, at the National Center for Atmospheric Research, at Los Alamos or at Brookhaven and Argonne. Both the kinds of problems, and the kinds of mathematics are diverse and becoming more so.

Finally, consider technical consultation on government work, which I should like to separate clearly and carefully from advisory consultation, which will be the next main topic. There is more of this, I suspect, than most of us realize. Mathematicians can do mathematics unobtrusively: for government contractors, for university projects in other fields, or directly for government agencies. As a consequence, it is very hard to get any clear idea of how much of this is done, on what sort of problems, involving what sort of topics.

I have no feeling that too much is being done, rather I would suspect not enough. It is not necessarily easy to be effective in such work; someone must build a bridge from the problems to the mathematics. Usually the mathematician must at least help with this task, which is often rather unfamiliar to him. I do not know what can be done to make such cooperation easier and more

effective. Perhaps there is a need for local or regional unofficial employment centers or mediators. Opportunities for such work are much more diversified than most of us realize. In particular, some of my friends who study the human brain from a physiological point of view are concerned about how much more mathematical help some of their Russian competitors seem to have. This is but one example.

5. Advice and administration. The third main head under which mathematicians can serve the Federal Government, the one which presumably accounts for my selection to appear here today, is advice and administration. Let us survey the field briefly, and then return to the areas where my own experience may be illuminating.

One of the major paths of Government consultation is provided by the National Academy of Sciences—National Research Council. The individual members of the Academy, elected by the general membership at large on the basis of demonstrated scientific ability, accept a responsibility to advise the Federal Government as part of their acceptance of membership. The divisions of the closely-affiliated National Research Council are made up of both representatives of scientific organizations and members at large, together with liaison representatives from appropriate government agencies. Activity of mathematicians in advising the government through the NAS-NRC, both through the Division of Mathematics and through committees more broadly responsible to the Academy-Research Council as a whole, has increased substantially in recent years. Some of our colleagues, including my friend Sam Wilks, now gone from among us, look to a very considerable expansion of NAS-NRC activity by mathematicians, especially in connection with problems not primarily mathematical. For my own part, this seems right, natural, and inevitable.

One major activity of the Academy-Research Council's Office of Scientific Personnel is the annual screening and ranking of applicants for all kinds of fellowships offered by the National Science Foundation. Here the mathematicians have always played their part, thoroughly and effectively.

Advice to granting agencies—to NSF, ARO, OAR, ONR, NIH, and so on, or, if you are so unmathematical as to prefer words to symbols, to the National Science Foundation, the Army Research Office, the Office of Aerospace Research, the Office of Naval Research, the National Institutes of Health, etc.—is given in varying amounts and patterns, both on the relative quality of individual grant proposals and on broader questions of policy. Much of this advice is discipline-by-discipline. Mathematicians have long participated effectively at a variety of levels. Not all such advice on policy concerns grants for research in mathematics, since the NSF Office of Science Information Services has a statutory Science Information Council upon which mathematicians have served.

Then there are the working agencies of the Federal Government, by which I mean agencies concerned with doing things "in-house." (Clearly an agency can be both a granting agency and a working agency.) There are many advisory

committees here, whose scope varies from quite narrow to very broad, from specific aspects of the work of a bureau to large parts of the work of a department. Here mathematicians are also to be found, perhaps not in as large numbers as there ought to be, but probably in larger numbers than many here in the audience would expect.

Finally there is the Executive Office of the President. Here too mathematicians play a part: on committees advisory to the Office of Statistical Standards, on panels and subpanels working in the Office of Science and Technology, even at times on the President's Science Advisory Committee.

Between advice and administration stands the National Science Board, with statutory responsibility for many of the activities of the National Science Foundation. The Board's role is still evolving, but however this evolution may turn, its importance for science in this country will remain great. Here "Jimmie" (E. J.) McShane upholds the laurels of mathematics with wisdom and humor and Mina Rees makes it clear that mathematics is something important to learn, something to be more learned than taught.

Administration, which is fortunately almost always a matter of individuals rather than committees, even in the Federal Government, is not barred to mathematicians. In my student days, for example, Luther P. Eisenhart and R. G. D. Richardson were deans of important graduate schools. Today mathematicians help to administer the Federal Government. Of the five or six Ph.D.'s who bothered to come to commencement in person for their Princeton diplomas in 1939, three were mathematicians. I am here today, Ralph Fox is in Japan recovering from an emergency appendicitis operation, and "Joe" (F. J.) Weyl is shouldering a large share of responsibility for the affairs of the Office of Naval Research. Today Brockway MacMillan is the Undersecretary of the Department of the Air Force. The administrative doors are no more closed to mathematicians in government than they are in industry or the universities.

6. The advisor. Of all these roles for a mathematician, being an advisor is the one with which most of you will have had the least experience. Research and teaching have long been before our eyes; examples of most kinds of technical contributions are easy to find, university and college administrators are always nearby, even if often misunderstood. The advisor is something else, and for good reason. One requirement for effectiveness is often unnoticeability.

What can we say of his duties, his motives, his actions, his rewards?

First, and perhaps foremost, he is not there as a representative. If there is a mathematician on an advisory group, he ought not to be there as a representative either of some abstract impalpable entity called "mathematics" or of an informal trade union of mathematicians. If he acts as such a representative, he will do less than justice to his opportunities, less than justice to himself, to "mathematics" and to the government. If he were put on to "represent" mathematics, those who selected him failed to do justice to their own opportunities.

There are many good reasons why he may have been chosen: sometimes

because his background will tend to ensure a certain point of view being properly considered; occasionally, perhaps, because his technical knowledge will be helpful (this has happened to me very infrequently and, even then, usually in statistics rather than in mathematics proper); even more likely, perhaps, because he has a fresh and able mind willing to take a look at a new problem. Those who pick mathematicians for such reasons, and those mathematicians who serve as if they had been picked for such reasons, do take proper advantage of their opportunities.

All of us, mathematicians or not, have individual backgrounds and interests. How much should we let these influence us? When are we being unduly biased? These questions come up, consciously or subconsciously, when any reasonable man serves as an advisor. There is no simple answer to these questions. The middle way is needed, and may not be easy to find. The man who tries to gain as much as he can for "his own" interests usually fails to get those things which he might have had, for the rest of the group will soon recognize his bias and discount his views overwhelmingly. The man who leans over backward fails to serve anyone. The proper choice, the middle way, is to think as hard as you can about the general welfare, as seen from your own background and experience, and then urge those things that seem to you best fitted to meet its overall needs. Your background will then be appropriately represented, you will be doing what you can for those who appointed you, and you can be happy with your conscience. You owe the government the best insight and understanding you can offer, not just your initial prejudices, not just what you think your friends would like you to say.

I think it is fair to say that there are two separable qualities of importance in an advisor, besides the ability to work with others in reaching a common position on both policy recommendations and action recommendations. These are acuteness and wisdom. As you serve on advisory groups you cannot help learning that these are not the same. A good advisory group needs both. Members who provide either are well worth their salt; those who provide both are doubly valuable. If you become an advisor, you will have to learn what balance of acuteness and wisdom you ought to bring to each particular group.

There is often even a place in an advisory group for a mild enough form of an *enfant terrible*, but only to the extent that such behavior comes naturally, and with some concern for others. I have heard respected advisory colleagues speak with affection of a former member, indicating that it was too bad that so-and-so is no longer here; he was always wrong, but his remarks often helped to clarify matters.

Being a good advisor, then, requires a willingness to work with others, and to argue with them when necessary, an honest attempt to assess the general good in broad and honest terms and to be acute, or wise, or both, as the needs for the particular group and your own abilities combine to indicate.

7. A consultant defined. Both an advisor in the sense just described, and a technical advisor in the usual sense, are instances of consultants. There is a

simple definition which can teach us much: *A consultant is a man who thinks with other people's brains.* He rarely has the time to become an individual expert on all the things involved in the issue at hand. He must—and often this is easier to do as part of a group—come to use the various experts' brains as essential tools in understanding the problems before him. Too often he will need to use their brains, and not just their words. Doing this is important in many different circumstances, from the most technical to the highly administrative.

The ability to do this, much of which can be learned, is not a substitute for wisdom, but is often essential in providing wisdom what it needs to feed upon.

8. Apprenticeship. As mathematicians we are familiar with growth, experience, and testing. Any institution that chooses department heads, research professors, or even long-appointment assistant professors unwisely, will suffer. Any agency that picks its advisors unwisely will suffer too. As a result there has to be some sort of an apprenticeship system for advisors, as well as for professors, a working up from less dangerous committees to more dangerous ones.

An advisor usually has to begin by advising on rather unimportant problems at first. To make matters worse, working on these problems is likely to be almost as uncomfortable as working on the larger ones.

9. Privacy and publicity. The problem of privacy is always important, and sometimes crucial. Both those who are being advised and those who advise have rights and responsibilities. At one extreme stands the President, who, with his personal advisors, is entitled to complete privacy, even as against the inquiries of the Congress. His advisors have no recourse, even when they are publicly criticized for not giving specific advice they may, or may not, have already given privately and urgently. At the other extreme stand groups publicly announced as having been asked to study some problem and to make a report with recommendations.

There is no single principle of universal application, but there is surely no excuse for the advisor in private who cannot bear to have his recommendations not accepted, and who goes directly or indirectly to the newspapers. There is a real difference between being asked to volunteer advice and agreeing to work as an advisor. The latter implies a sense of responsibility and an acceptance of team play that is often absent from the former.

So far as I have seen, those concerned with giving advice related to matters scientific and technical, whatever their own discipline, have behaved responsibly about publicity. I have emphasized their responsibility here, because I hope, and expect, that this will continue to be true in the future.

The relation of the advisor to politics may also be crucial. There is a place for both the advisor who has a public position on political issues, and the advisor who is prepared to offer the best advice he can, no matter who is in power. It is right that advice about matters scientific and technical should affect politics quite explicitly; it is important, on the other hand, that administrators should have competent advice whatever their prejudices or policies (although successive groups of advisors may find it necessary to resign).

The President's Science Advisory Committee has established a short but important tradition of responsible advice to the President, whoever he may be. As a consequence, new Presidents, though they had every right to do so, have felt no need to interrupt the orderly progression of terms of service on this Committee.

I received one letter, I presume from a member of our Association, who felt that Viet Nam made the topic of my talk wholly uninteresting, since he wanted nothing at all to do with the Federal Government. This is a view any mathematician is clearly privileged to take, but there are other forms of protest better suited to most of us. If this nation is to survive, and its people are to live decently, there will be a real need for the best advice any of us can give about scientific and technical matters, whatever the broad policies of the government, foreign or domestic, may be. Revolution may or may not be a constitutional privilege, but short of revolution some of us must support the orderly workings of our government with whatever wisdom or acuteness we can find.

10. Advice to legislatures. Much has been learned by experience about the practice of advising the executive branch of the Federal Government. Much less has been learned about advising the Congress, where there is still much to be learned. Recent developments suggest that both sides are indeed learning how this is to be done effectively. It is to be earnestly hoped that whatever we learn will find further use, not only in providing helpful advice to a greater variety of congressional committees, but also in the advising of state legislatures.

11. Some problems of the "day." It is easy to look ahead and see major problems before us, before all the people of this country, in which scientific and technical matters play an important part. In a moment I shall mention a few of the more obvious and difficult. No one of them can be decided on purely scientific or technical grounds. To be spared the responsibility of working on any of them would make anyone's life simpler and more pleasant. But many, both scientists and others, will have to do what they can to clarify the issues and offer good advice.

Is science being subverted? This question is raised too frequently these days to be avoided in the long run, though it seems to mean many different things to different people. There are those who feel that the name of science is holy, and should be separated from everything that is only a reasonable choice, rather than an established fact. There are those who are concerned that science is becoming "little more than an instrument for the continuation of politics by other means." These are questions on which no category of people is likely to find unanimity. Yet there will be a great need for a gradual consensus.

How is the Federal Government's support for science and technology to be allocated? Notice that I said "allocated," NOT "divided," NOT "shared." We come ever closer to the point where "either-or" decisions will involve widely different fields.

How will we learn to live with our environment? Here there are many subsidiary questions, each of major proportions:

What is a just balance between those who would "preserve everything" and those who would "exploit freely"?

How can we pay a fraction of the attention that we have so long paid to how we take substances from our environment to the equally important problem of how we return substances to it?

What are the real technical issues in connection with population growth? Are rates of growth or total numbers more crucial?

And so on. As mathematicians we know that in mathematics on the average, every solution uncovers several new problems. Matters are not different in the affairs of our country.

12. Mathematicians and the problems of the day. Mathematicians can help with all of these problems if they truly wish, but not by doing mathematics, not by working on things that are enjoyable to work on. Instead they will have to give up the protection of unquestioned hypotheses and contribute their best acuteness and wisdom under uncomfortable circumstances. Some will do this; I trust more than in the past. They will not begin at the top by working on the sort of problem just illustrated. Rather they will serve apprenticeships on small bothersome problems, and work their way up.

For others this will not be the better path of service. They too can do much for the Federal Government and the nation by serving in more "mathematical" ways.

There is one thing that all of us could do; I believe almost all of us can learn to do it. That is to come to believe, and to impress upon our students, that all of these varied kinds of service are good, that some are called to one and some to another, that each should seek a way in which he or she will come as close to excellence as possible.

It is not easy to give such an impression, for it is not enough to say such things. If you do not feel them, if you do not live them, your students will behave as if you never said them.

This Association was founded because mathematicians had to be concerned with teaching as well as research. Its members need now to recognize even broader responsibilities.

THE AMERICAN MATHEMATICAL MONTHLY, Vol. 74, No. 1, Part II, January 1967, pp. 1-110



GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 74 NUMBER 2

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FEBRUARY 1967

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

R. A. ROSENBAUM, *Editor*

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $6.00. For nonmembers the subscription price is $10.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July,
August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

THE ROLE OF THE AXIOMATIC METHOD

R. L. WILDER, University of Michigan

Ever since its importance to the development of mathematical theory has been realized, much has been written on the axiomatic method. However, except for the work of a few pioneers of the early part of the present century, such as Skolem and Tarski, most of this writing has been done during the past 20 years, roughly speaking. It took the mathematical world a long time to realize the potentialities of the method—and, we should add, to recognize its limitations —and it is undoubtedly true that even today some eminent mathematicians deplore its widespread use. The point of view taken by both Poincaré [**10**] and H. Weyl still survives. Although recognizing the accomplishments of the method, Weyl felt that its fertility was "approaching exhaustion" [**15**]. He made a firm plea for a return to problems of "mathematical substance" which, he maintained, form the basis for the concepts subsequently cast in axiomatic form. Both Poincaré and Weyl were intuitionists of sorts; they believed that mathematics can not be reduced to axiomatic form without sacrificing something of its true nature. They did not deny that axiomatics is a legitimate tool, but only as a device for giving precision to already created mathematical entities—a kind of classifying or cataloging function, much as the present-day intuitionist considers it to be.

It therefore seems timely to reappraise the role of the method, particularly in view of recent developments in mathematical logic; and to gain proper perspective for the task, it appears that we should begin with a short review of its evolution. In studying the evolution of mathematical concepts, I have found it convenient to distinguish between those cultural stresses which influence the development of concepts, according to their "hereditary" and their environmental aspects. For example, the inception of most early mathematics, such as arithmetic, and geometry in its primitive forms, was due to environmental forces; but the axiomatic method occupies a unique position in that it appears to have owed its inception chiefly or even entirely to hereditary stresses. In other words, the factors which forced its creation were based on internal rather than external needs. Of course, because of the dearth of knowledge concerning early Greek developments, one cannot be absolutely positive about this. However, most historians seem to agree that crises, attendant upon the attempts to cope with paradoxes such as those of Zeno, compelled the formulation of a basic set of principles upon which to erect the geometrical edifice. The introduction of the method carried with it a tacit acceptance of certain types of logical reasoning; thus logical deduction was applied to mathematics for the first time.

Accordingly, it seems reasonable to regard as the chief role played by the axiomatic method in Greek mathematics, that of providing a consistent foundation. In considering this role, we should note two aspects of the method: (1) The manner in which logic was brought into mathematics; indeed, both the *motive* (namely, to achieve a *consistent* system) and the *methodological aspect* (logical

deduction), tended to make mathematics subordinate to logic; (2) there was presumably only one interpretation of the geometry described in the *Elements* —in modern terms, only one model. Primitive geometry, like arithmetic, grew out of the need for coping with the physical environment; and by the Greeks, geometry, even after it was formulated as an axiomatic system, was apparently still considered a science of physical space. Consequently, what we would call the primitive terms were not variables, but constants referring to the single spatial model.

For the next major change we must turn to the 19th century. Space does not permit going into the factors that produced the change; with the advantage of hindsight we can perceive signs of it even earlier. It has been customary to attribute the change to the introduction of non-euclidean geometry. But as Freudenthal recently recalled [4], there was little evidence of excitement or even interest in the mathematical community regarding the work of Bolyai and Lobachevski, for at least 30 years after its publication. And meanwhile other writers such as Peacock in algebra, Whewell in mechanics, and Boole in logic, clearly brought out the concept of attributing to the primitive terms of an axiomatic system the undefined character that we give them today. The later impact of the admissibility of non-euclidean geometry into mathematics only promoted an evolution already under way.

Thus, in his *Mathematical Analysis of Logic*, 1847, Boole stated, relative to his symbolic algebra, "the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon the laws of their combination. Every system of interpretations which does not affect the truth of the relations supposed, is equally admissible" Nevertheless, after it was recognized that the non-euclidean geometries are just as consistent as the euclidean, there seems to have been a general feeling that both the euclidean and the non-euclidean systems were still only candidates as potential sciences of *space*. Witness the attempts to determine, both on an earthly and on a cosmic scale, whether the sum of the angles of a triangle is really 180°.

However, geometrical thought gradually fell into line with that in other fields; the publication of Pasch's *Vorlesungen über neuere Geometrie* in 1882 helped especially to establish the variable character of the primitive terms. In Nagel's words, "No work thereafter held the attention of students of the subject which did not begin with a careful enumeration of undefined or primitive terms and unproved or primitive statements; and which did not satisfy the condition that all further terms be defined, and all further statements proved, solely by means of this primitive base" [7].

However, considering the primitive terms as undefined is not the same as conceiving of them as variables, and evidently the transition in point of view was, like most fundamental mathematical conceptions, a slow process. Certainly the works of Grassman, Pasch, Peano, Padoa, Hilbert, and Russell, to name but a few of those involved, all contributed to the evolution; and by the early part of the present century the notion was certainly recognized if not com-

pletely assimilated. This is evident in such works as those of Fréchet on abstract spaces, and of E. H. Moore in general analysis; in the axiomatic formulation of the theory of groups; and in Hausdorff's axioms for a general topological space. In lectures given during the summer of 1909 at the University of Illinois, subsequently published in his *Fundamental Concepts of Algebra and Geometry* in 1911, J. W. Young stated at the end of his concluding lecture: "We may regard the undefined terms in an abstract science (by which he meant an axiomatic system) as *symbols* (ital. mine) representing *any* entities for which the fundamental assumptions appear to be satisfied." Certainly here the notion of the primitive terms as variables appears to be recognized.

I am trying to bring out by these remarks another historically important role played by the axiomatic method. I pointed out that the role of the method in Greek mathematics seems to have been to provide a consistent foundation for mathematics—a twofold objective, namely the provision of a *foundation* which at the same time met the then current charges of *inconsistency* (the second objective may well have been the motivating factor). During the period covering roughly the latter third of the 19th century and the first third of the 20th century, we find the axiomatic method playing a new role, namely that of introducing increasing abstraction into mathematics. E. T. Bell [1; p. 239] dates the period from 1879 to 1920, placing the emphasis upon the theory of groups which during this period received a thorough axiomatic treatment. But it is clear from Bell's own subsequent remarks that the credit really should go to the axiomatic method itself; some twenty-eight pages later he states explicitly, "The postulational method is the source of both the modern critical movement and abstractness."

(Incidentally, I have no intention of commenting upon whether the abstract character of modern mathematics is a good thing. I wish only to point out the role of the axiomatic method in the development.)

Before passing on to the current situation, I must mention an instance in which the axiomatic method was again called upon to serve the function, originally exemplified in Greek mathematics, of trying to establish a consistent foundation for a theory. After the Cantorian theory of sets was found to harbor contradictions, Zermelo published [14] his famous set of axioms using "set" and "element of" as primitive terms. The analogy with the Greek situation, as the latter appears to us through the haze of centuries, is striking. In both situations, crises had developed which threatened the security of mathematics; and in both cases resort was taken to explicit axiomatic statement of the foundations upon which one hoped to build without fear of further charges of inconsistency. Moreover, although Zermelo's system could avoid the somewhat inconsequential, but nevertheless serious, defects of Euclid's *Elements*, it is probably correct to assume that, like the Greek system, Zermelo's had only one model in mind, namely, a portion of set theory sufficient for ordinary mathematical purposes, yet carefully limited to avoid the known contradictions.

One should also recall, in this connection, the work of the 19th and early 20th

century pioneers in what is now usually called "Mathematical Logic." Here again the role of the axiomatic method as a mode of foundation was called into play. Even though Russell called his axioms "universally true," or tautological, they were clearly postulational and had later to be augmented by "multiplicative axioms" (axioms of choice) and "axioms of infinity" which could hardly be regarded as other than strictly axiomatic in nature. It is interesting to note that Russell, like Euclid and Zermelo, had only one model in mind, and he aimed at making the conditions surrounding his system so stringent that there could be only one model. In the first edition of his *Introduction to Mathematical Philosophy* he stated, "Logic is concerned with the real world just as truly as zoology, though with its more abstract and general features." The trouble with this (as perhaps Russell himself later concluded, since the statement is omitted in the later edition) is that while the model which zoologists study is already existent, that which Russell was studying was only an abstraction in process of evolution.

We note, too, that Russell and Whitehead were largely motivated by the desire to meet the crisis in the foundations caused by the set-theoretic contradictions. This was in contrast to logicians such as Peano and Frege who preceded them and who furnished a basis upon which they could proceed to build mathematics. Also, we should note the contrast between the axiomatic systems of Zermelo and of Russell and Whitehead in regard to the language used. Zermelo's was in the tradition of Euclid, as modernized by the 19th century mathematicians; primitive terms "set" and a relation between sets, "is an element of," were employed along with the natural language. Russell and Whitehead held a much grander conception, namely a foundation of the whole of mathematics, starting with what they termed primitive logical truths.

As it turned out, both systems exhibited deficiencies which had to be remedied. In Zermelo's case, the notion of "property" crept in under the guise of the natural language used, and since "property" and "set" were soon recognized to be virtually synonymous, it was necessary to substitute a more precise notion while at the same time eliminating the vicious circle—a task assumed by Fraenkel [3] and Skolem [12]. In the case of Russell and Whitehead's *Principia Mathematica*, aside from criticisms of the Reducibility Axiom, Axioms of Choice and the like (which elicited the question of just how far one can stretch the concept of "primitive logical truth"), the syntactical rules were deficient. This deficiency was remedied by those successors of Russell and Whitehead who adopted the formalist philosophy and, following the initial impulse given by Hilbert, made a clear separation between the primitive terms and axioms and the metamathematical rules for derivations of formulas (i.e., proving of theorems) from the axioms. Using our prerogative of hindsight, we can see that the two streams (that continued by Zermelo in the tradition of Euclid, and that started by Russell in the tradition of the symbolic logic of Boole and Frege) were heading for confluence.

An indication of this can be found in the historically important paper of Skolem given at the 5th Scandinavian Congress in 1922 [12]. Skolem, a logician

well acquainted with the logical system of Schröder, pointed out that the Zermelo axioms could be formulated as elementary statements within the framework of the Schröder logic, and that as a consequence the indefinite notion of "property" used by Zermelo could be adequately replaced by finite expressions formed from the "ϵ" and "$=$" relations. More important yet, he pointed out that a theorem proved by Löwenheim in 1915, concerning the Schröder algebra, could be simplified and adapted to show that if the Zermelo system is consistent, then its formulas must have *countable* models. Since formulas asserting the existence of powers higher than aleph-null can be proved in the Zermelo system, Skolem rightly concluded that "The axiomatic foundation of the theory of sets leads to a relativity of the set notion, and this is inseparably connected with every axiomatization." In his conclusion, he made clear that he did not believe the axiomatic method the proper tool to employ for a *foundation* of set theory—indeed, he expressed astonishment that so many mathematicians had so considered it.

Fortunately such a pessimistic attitude did not deter later investigators from continuing to apply axiomatics to set theory, even though the correctness of Skolem's conclusions was recognized. Von Neumann, for example, whose classical papers on axiomatic set theory were published soon after, remarked that the methods of Löwenheim and Skolem were applicable to his own system and that they consequently placed upon his, and every other axiomatic theory of sets, "the stamp of unreality." And after remarking that "it appears—that no categorical axiomatization of the theory of sets exists," he went on to state: "Since there exists no axiom system for mathematics, geometry, etc., which does not assume the theory of sets, so will there certainly be given no categorical axiomatic infinite systems. This circumstance" he continued, "appears to me to be an argument for intuitionism" [8]. In his general conclusions, then, von Neumann agreed with Skolem.

Undaunted, however, the "axiomatikers" continued where von Neumann left off, and in a series of papers beginning in 1937, Paul Bernays presented an effective and usable simplification of the von Neumann system. Although couched in the natural language, the system is clearly formalizable in the first-order predicate calculus; and about 1940 Gödel used it, with modifications, to establish the consistency of the axiom of choice and Cantor's generalized continuum hypothesis with the other axioms of set theory [5].

Now, aside from its strictly mathematical significance, this result was of prime importance from the standpoint of the evolution of the axiomatic method. In order to maintain this assertion, we must fill in one or two points that we have omitted in our historical survey.

I remarked earlier on the role of the axiomatic method in the increasing abstraction of mathematics. Parallel with this, during the latter part of the 19th century, the researches of Dedekind, Weierstrass, Cantor and others on the structure of the real number continuum were forcing set-theoretical ideas on the mathematical community—an example of what I earlier termed "hereditary"

stress. And although Cantor was astute enough to discover the so-called "continuum hypothesis," he never noticed that even in his first paper, in which he had to prove that every infinite set contains a set of cardinality aleph-null, he was using the axiom of choice. Peano noticed the choice principle (and objected to it) as early as 1890 [9], but the proof of the well-ordering theorem (published in 1904 by Zermelo [13]), was required to startle the mathematical community with the power of the axiom of choice.

This was a most important event from the standpoint of the evolution of mathematics, as well as a prime example of discovery and explicit formulation of a basic, hitherto unnoticed feature of a current concept. The continuum hypothesis, by way of contrast, was a natural derivation from the continuum *problem*; once the real numbers were shown to form a totality of power higher than aleph-null, one was immediately forced to ask whether it is the *next* higher power. Cantor mentions it in his 1883 paper in *Mathematische Annalen*, (vol. 21), and his first reaction was to state that he hoped soon to be able to prove that the cardinality c is the same as that of the second number-class. However one went on, naively, using the choice principle without ever recognizing its existence. Hilbert stated the continuum problem as the first of the famous set of problems which he gave at the 1900 Congress [6], and did not notice that he was assuming that every infinite set has a subset of cardinality aleph-null—which is, as we have already remarked, dependent on the choice principle. One had to wait for the most striking use of the principle, in the proof of the well-ordering theorem by Zermelo, to force general perception of its existence. It makes us wonder just how many other fundamental assumptions we are habitually making in our conceptual thinking, without either discovering them or giving them precise formulation.

Once the power and fundamental character of the principle was evident, Zermelo included it in his set of axioms for set theory [14]. On the other hand, he did not include the continuum hypothesis as an axiom, and thus was left open the question whether it could be proved from the other axioms for set theory.

For our purposes, the importance of Gödel's 1940 result, namely the consistency of the choice principle and continuum hypothesis with the parts of set theory that express generally recognized set-theoretic principles, is that it also showed the power of the axiomatic method as a mode for investigating and establishing relationships between various aspects of a theory. No matter whether one feels, as Skolem did, that set theory should not be axiomatized, one is impressed with Gödel's accomplishment. Many (including Gödel) were of the opinion that as a principle of general set theory, the continuum hypothesis is false; and that this might ultimately be deduced from generally accepted set-theoretic principles. Now one knew that this was a vain hope; and this was an accomplishment of the axiomatic method. One can still maintain that axiomatization is not the proper way to *formulate*, i.e., *found* set theory, while admitting that, as a tool for investigating the relationships between the various aspects of

that theory, and perhaps discovering new aspects, it has a legitimate role. Of course, the recent results of Paul Cohen [2], showing that *denials* of either choice axiom or continuum hypothesis are also consistent with the usual principles of set theory, have strengthened this thesis.

I have emphasized the type of role played by the axiomatic method in the evolution of set theory, because I think it is destined to receive more emphasis in the future. For centuries, mathematics got along with very imprecise conceptual tools. Except for the level of their achievements due to the centuries' accumulation of mathematical materials, the mathematicians of the pre-modern era were not on much surer ground than the ancient Babylonian number scientists. Can one imagine, however, an "outsider" such as Bishop Berkeley making valid criticisms of the basic concepts of mathematics today? Also, let us reflect for a moment that it is less than 100 years since mathematicians devised what has since been considered a precise formulation of the structure of the real number system—the real number continuum. At the time, this was considered an achievement of the highest order; the vaguely conceived notion of the "continuous" had at last been examined in its totality and seemingly described with precision. In Russell's words, "Whether or not the kind of series which Cantor defines as continuous is thought to be the most similar to what has hitherto been vaguely denoted by the word, the definition itself, and the steps leading to it, must be acknowledged to be a triumph of analysis and generalization" [11, p. 303]. Unfortunately, another vaguely conceived notion had been used in the process, that of "set." And this notion was so little suspect that even so competent and philosophically minded a mathematician as Poincaré could comment (albeit not ignoring the illusory nature of such beliefs among his forebears), "We believe that we no longer appeal to intuition in our reasoning." And, in analysis, he remarked "absolute rigor has been attained." (See Bell [1; 295].) Today, thanks to the axiomatic method, we know that Cantor, Dedekind, and all the rest of those 19th century pioneers only moved a step forward—an important step, to be sure. What they did was at least a further approximation. But so far as a precise formulation of the structure of the real number continuum is concerned, we know today that we still depend upon our intuition. Shall we accept the continuum hypothesis? Shall we declare all sets of real numbers measurable? In the present state of our mathematical intuition, we cannot answer these questions except by *edict*. But mathematical questions are not decided by edict. They are ultimately answered by hereditary stress and by other forces active in the evolution of mathematics. I surmise that in the future this will be one of the chief roles played by the axiomatic method: to help us analyze more deeply the consequences of what our intuitions suggest. Just as the demands of function theory forced a more precise analysis of the nature of the real number continuum, during the latter part of the last century, so probably will the demands of an increasingly abstract mathematics force decisions regarding such matters as the axiom of choice and the continuum hypothesis. As of now, I would not venture an opinion upon what the answer will be. Regarding the axiom of choice,

upon which the existence of nonmeasurable sets appears to depend, general acceptance seems highly probable. Too many intuitively necessary relations seem to depend upon it—for example, that if a set A is mapped upon a set B, then the number of elements in A is at least as great as the number of elements in B.

Let me repeat: I do not say that the axiomatic method is the proper way of founding either set theory or logic. Frankly, I am of the opinion that it is not. I do not believe that we should base mathematics upon variable systems of numbers, set theory, or logic—which is all that we can do with the axiomatic method at present. To this extent I am an intuitionist, and so far as can be told from their *behaviour*, so are most mathematicians. As working mathematicians, we use only one set theory, only one logic, and only one type of number system; and these are intuitively based, whether we admit it or not. Moreover, I believe that in contrast to the assumption that they are fully matured concepts—sort of "old-hat" ideas, as we might say—they are still in the process of evolution. I also believe that the role of the axiomatic method in this process will be fundamental. Up to now the axiomatic method has certainly shown itself to be the most important tool for the codification of what our intuitions tell us.

I do not wish to terminate these observations without mentioning what to many mathematicians are even more important roles of the axiomatic method. I should probably question the use of the definite article "the" in the expression "the axiomatic method"; because there seem to be several "axiomatic methods" in use today, as well as several ways of classifying them. There are roughly three types if we arrange them according to degree of formalization:

1. In what we might call the "Euclidean type," because of its similarity to Euclid's *Elements*, the primitive terms are not treated as undefined and only one model is described. The model is usually a concept suggested by our perceptions of the physical or social world, and the purpose is to set forth the fundamental properties and relations of the model. It is not the more refined type that we mathematicians commonly use, and is to be found mostly in studies of the physical and social science areas, where it seems to serve a useful function. Indeed, one can question whether in these areas (with the possible exception of certain physical theories) the types of axiomatics used in mathematics would yield significant results.

Interesting problems can be found in the application of this type of axiomatics, especially in regard to choice of models. The 17th and 18th centuries offer numerous examples of both the use and the misuse of the method; the best known are, I suppose, Spinoza's *Ethics* and Newton's *Principia*. The misuses of the method during this period may have had something to do with the virtual lapse in its application to the social sciences until recent times.

I am confident, nevertheless, that close study of that period would show relations to the emergence of symbolic logic. For example, Descartes and Hobbes both proposed what they called a "geometrization"—meaning an axiomatization

in the Euclidean manner—of all knowledge. In Descartes' case this included both medicine and ethics, and in Hobbes' case the proposal was to include psychology, ethics and politics. In both cases, a "universal calculus of reasoning" was plainly conceived. Much later, when Leibniz came upon the scene, the same kind of project was clearly in the air. Leibniz even attempted to apply the axiomatic method to politics. For instance, in 1669, when he was 23 years old and had studied both philosophy and law, he was in the service of the elector of Mainz and at this time the crown of Poland became vacant. It therefore fell to his lot to support the claims of his patron. For this purpose he wrote a little book, *Specimen demonstrationum politicarum pro rege Polonarum eligendo*, giving, in the "geometrical" manner, some sixty propositions (along with corollaries), demonstrating that three of the candidates merited exclusion while the fourth should be elected. (N.B. The fourth was not elected.) Two years later, using another "geometrical" system, he tried to convince Louis XIV that the French king should send troops against the Turks.

One can conjecture that examples such as these, along with Leibniz' companion ideas regarding a "calculus of reasoning," are among the ancestors of what later evolved into Mathematical Logic. There were many such examples, written by lesser lights (Leibniz' own teacher, Erhard Wiegel, a mathematician of repute at the University of Jena, wrote both on *Ethica Euclidea* and an *Idea Matheseos Universae*). Their lack of success in accomplishing their immediate objectives (due partially, no doubt, to the primitive conception of the axiomatic method then prevailing), should not obscure their significance in the evolution of human thought.

Probably most of us have seen attempts, that seemed only pretentious and ineffectual, at applying mathematical methods in areas that were previously purely descriptive or classificatory. However, I am confident that the axiomatic method of the "Euclidean type" does have a function in many such cases, especially of clarification and revelation of unnoticed properties—a function which the more refined type of axiomatics served, within mathematics itself, during the first part of the present century.

I think that the modern lack of success of the axiomatic method in the social sciences is probably due to two causes: (1) The notorious lack of theory in these sciences, and (2) the lack of adequately structured models (a deficiency probably caused by failure to perceive or sense certain important details of social patterns). The situation is somewhat analogous to that in set theory. In the area of physical reality, we perceived collections long before they had any mathematical significance; this is attested to by the fact that the words "class," "collection," and terms for special collections such as "flock," "herd," and "covey" were in the natural language long before mathematicians noticed that this aspect of reality is of vital importance. After Cantor's work established the importance of the notion of set, our intuitions of set theory developed to the point where a consistent model seemed to have been achieved; but today, nearly a century

later, we still have no unique or precisely clear model for set theory. Returning to the role of the axiomatic method in the social sciences, we notice an analogous situation today in both sociology and anthropology, where the concept of *culture* has had a history similar to that of set theory, except that it is still in the strictly intuitive state; it has not yet had the benefit of an analysis such as set theory has undergone.

The reason why the Greeks—and this applies also to the mathematicians of the 16th, 17th, 18th and 19th centuries—did not run astray in their reasoning is that they had developed a mature intuitive model for what they fancied to be physical reality. I do not wish to get too deeply involved in these matters, but what Euclid and later mathematicians were describing was not physical reality; it was a mental model which their perception of physical reality had suggested. And when we wonder at their not running into logical difficulties, we should recall that both the model and the logic they used were derived from this reality. It is interesting to observe that the first event of importance in axiomatic theory after the discovery of the non-euclidean geometries was the finding of models for the new theories. Thus, after venturing into what might have seemed domains of fantasy, one sought solace again in "reality"—at least within a structure, namely orthodox euclidean geometry, whose foundation was a conceptual model of certain aspects of physical reality. Moreover, in proposing the use of models as a test for logical consistency, the mathematician was only following the same path. Today, the theory of models in mathematical logic is a mature doctrine; but originally the use of models was only an appeal to reality, and the belief that reality is *consistent* was matched by the belief that only consistent theories are real—both Poincaré and Hilbert agreed upon this.

So far as mathematical theory is concerned, the "euclidean type" of axiomatics may now be dead, after having played an important role in the evolution of mathematical thought. But it will probably continue to play an important role in those areas of intellectual effort which do not have the venerable antiquity of older sciences such as mathematics.

2. The second type of axiomatics, according to my classification, is the one with which we are all familiar both in our research and in our teaching; it is the type used by the so-called "working mathematician"—a phrase I would rather not use in this connection, since I certainly consider my colleagues in mathematical logic to be working mathematicians—without quotation marks. We are careful to list our primitive terms, but we list neither the logical nor the set-theoretic rules by which we shall abide; hence the term "naive axiomatics." Tôward these latter, i.e. the logical and set-theoretic rules, we take what is operationally like an "absolute," or "Platonic" point of view. We know this method as a principal tool for modern research in all branches of mathematics, especially algebra, topology and analysis, and it is unnecessary to recite here the details of its important role; as I remarked earlier, it is responsible to a great extent for the increasing abstraction of modern mathematics.

The method also plays an increasingly important didactic role. I think a fairly straight line could be traced from the activities of the E. H. Moore school of mathematics at the University of Chicago during the first two decades of the century, down to what some call "the new school mathematics." One of the "products" of the E. H. Moore school was R. L. Moore, and most of us have heard of the "Moore method" of teaching—for those who haven't, an opportunity to hear it described by its chief exponent will be furnished by one of the films now being produced by the Committee on Educational Media under auspices of the Mathematical Association of America. Axiomatics is at the heart of this method. Incidentally, if E. H. Moore were alive today, he would certainly be active in the so-called "reforms" of elementary and secondary teaching, since he was much interested in teaching methods on these levels. Although I have never participated in writing the texts that are being produced for use in the elementary schools today, I get the impression that the axiomatic method, in the form of one of the first two types, is being employed to some extent. I have encountered not only young students who can quote to me the commutative and associative laws, but also some who talk set-theoretic language!

However, I have mixed feelings about the use of the axiomatic method for setting up the natural and real number systems. When I was a young instructor, I did not care for the method then prevailing in the theory of real functions, of starting with the Peano axioms; but I think my reaction was based largely on intuitive grounds. As I look back, I think that I was probably a semi-intuitionist, in that I felt the students had spent years using the arithmetic of natural and rational numbers, and that it was a waste of time giving it an axiomatic basis. Conversely, for the real number system, I felt that the students came to us in function theory with only the same understanding and familiarity with real numbers that a mature mathematician of a century ago had; and that here it was necessary to acquaint them with the structural theories of Dedekind and Cantor. Subsequent developments have provided, I think, valid arguments in support of this attitude. Moreover, I have the same feeling in regard to set theory. Sets, like natural numbers, have a natural background in our social, scientific and physical environment, and I believe that we can save considerable time in our curriculum if we proceed with the so-called "naive" approach. The *intuitive* comprehension of sets and their elementary properties should be built up first. Of course, as in the case of the natural numbers, once the student has developed an intuitive understanding of set theory, he can profit from seeing how a set theory and a theory of natural numbers based thereon can be set up using an axiomatic approach. But he should be aware that here the axiomatic method plays only a clarifying and investigative role; that so far as a *foundation* for the natural numbers, the real numbers, or set theory is concerned, only a relative characterization is achieved. Let me emphasize that I am referring here to the *pedagogical* role of the axiomatic method. I hope I have already made it clear that I approve heartily of its role in the *study of the foundations* of set

theory, where it serves to pinpoint aspects of our intuition that we would not otherwise recognize. The intuitive conceptions of the structure of the real number system prevalent 200 years ago may have been adequate for the mathematics of the time, but during the succeeding century its inadequacy became more and more apparent and provided the motivation for the more precise formulations by the axiomatic method; and with these formulations we have managed until the present day. Whether this will continue to be the case remains to be seen. At any rate, there can be no question regarding the importance of the role which the axiomatic method has played here. Nevertheless, we should distinguish between the uses to which the method is to be put. I do not question that the precise formulations of the laws of arithmetic may form a useful and clarifying function for the elementary schools, answering for the student some of the questions which the intuitive approach must raise; but they should follow, not precede the latter.

3. The third type of axiomatics, in which the logical apparatus not only enters the discussion, but is explicitly formalized, has proved one of the major tools in modern foundational research. Using an axiomatized functional calculus in which to imbed the technical structure of the theory, Tarski, Henkin, and others cleared up such matters as the relationship of consistency and models and the relation of completeness with models. We know now why a system such as the Peano axioms can be categorical when the set theory involved is not axiomatized, and why it is not categorical when it is axiomatized; and we shall have to rely upon future researches in mathematical logic to answer unsettled questions on set theory and on the real number continuum.

For example, Gödel's proof of the compatibility of the generalized continuum hypothesis with the other axioms of set theory was based on the notion of "constructible" sets; if we use the axiom requiring all sets to be constructible, the generalized continuum hypothesis holds. But we know now, from Paul Cohen's work, that the generalized continuum hypothesis is a weaker axiom itself than the axiom of constructibility. Of course, one may get all that is desired by simply adding the generalized continuum hypothesis to the axioms of set theory, but there are two objections to this: (1) it lacks the intuitive character associated with the other axioms of set theory, and (2) it lacks motivation in that its mathematical fruitfulness, up to now, is too meager.

In summary, as I consider the three types of axiomatics that I have arbitrarily marked off, I sense that modern mathematics, despite its abstractness, looks more and more like an applied science. When the applied scientist uses the Euclidean type of axiomatics to analyze and describe a conceptual model in a physical or social situation, he is deriving his axioms from a perceptually induced concept. When the so-called "working mathematician" uses the "naive" method, he is deriving his axioms from an intuitively induced concept—from what his *mathematical* intuition has sensed to be an important structure or pattern. The purest form of the axiomatic method, type 3, has enabled the logician

to analyze our logical methods and our set theory—matters taken for granted in "naive" axiomatics. Also, the creative role which axiomatics has played on the second level is paralleled by its more fundamental creative role in the use of type 3, in which our most basic intuitions are strengthened and clarified by the discovery of new axioms as, e.g., the axiom of constructibility of Gödel.

Despite its limitations, and despite its failure to provide the type of foundation originally conceived by Peano, Hilbert and other pioneers, the axiomatic method is not yet at the end of its hey-day—perhaps its greatest glory is even yet to come. Its expanding role in pedagogy, as a research tool for the analysis of abstract mathematical structures and in the investigation of the methods and structures offered by our intuitions, are perhaps only a harbinger of more fundamental roles to come. Although the axiomatic method seemingly cannot fill the role of providing an intuition-free foundation for mathematics, I think it will continue to be one of our most effective tools for the creation of new mathematics and the teaching of mathematics.

This is a slightly revised version of an address delivered before sectional meetings of the Association during the spring of 1965.

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VECTORIAL INVERSIVE AND NON-EUCLIDEAN GEOMETRY

H. W. ALEXANDER, Earlham College (Now at University College, Nairobi, Kenya)

The purpose of this paper is to show how vectorial methods, familiar from linear algebra, may be used to provide an analytic treatment of inversive and non-Euclidean geometry. As an application of the central ideas of linear algebra, this type of analytic geometry is quite different in flavor from the familiar vectorial treatments of affine and Euclidean geometry. As against the usual inner product of vectors, this geometry employs a type of "inner product" of a pair of vectors which arises as naturally in the context of the geometry of circles as does the familiar inner product in the context of Euclidean geometry. The tools are developed for an extensive preliminary exploration of the very rich geometry of circles and spheres, which tends to be neglected in the modern curriculum.

Let (x_1, x_2) be a point in a Cartesian plane. Consider the equation

$$a_0(x_1^2 + x_2^2) - 2a_1x_1 - 2a_2x_2 - 2a_3 = 0, \quad a_i \text{ real.} \tag{1}$$

If $a_0=0$, $a_1^2+a_2^2\neq 0$, then (1) corresponds to a straight line. If $a_0\neq 0$, then (1) corresponds to a circle with center $(a_1/a_0, a_2/a_0)$ and radius

$$r_A = \frac{1}{|a_0|}(a_1^2 + a_2^2 + 2a_0a_3)^{1/2}. \tag{2}$$

Thus the circle will be real, null or imaginary according as $a_1^2+a_2^2+2a_0a_3$ is positive, zero or negative.

The key idea we wish to introduce is that of associating a 4-dimensional vector $A=(a_0, a_1, a_2, a_3)$ with the equation (1), and with the corresponding locus. Let $A=(a_0, a_1, a_2, a_3)$, $B=(b_0, b_1, b_2, b_3)$ be two vectors. We define a type of inner product, which we shall call the "number product" of A and B, denoted by $A\#B$, as follows:

$$A \# B = a_1b_1 + a_2b_2 + a_0b_3 + a_3b_0. \tag{3}$$

In particular,

$$A \# A = a_1^2 + a_2^2 + 2a_0a_3, \tag{4}$$

so that if A corresponds to a circle, its radius is, from (2),

$$r_A = \frac{1}{|a_0|}(A \# A)^{1/2}. \tag{5}$$

It is readily verified that the number product has the following properties:

$$A \# (B + C) = A \# B + A \# C; \quad (aA) \# B = a(A \# B); \quad A \# B = B \# A. \tag{6}$$

Let us now agree to use the term "stircle" to mean either a straight line or a circle, and let us speak of "the stircle A" to mean the stircle corresponding to the 4-vector A and to any nonzero multiple of A. We exclude from all consideration the vector (0, 0, 0, 0), which corresponds to no locus.

It is convenient to say that two vectors are *equivalent* if one is a nonzero scalar multiple of the other. With this understanding, there corresponds to a given stircle an *equivalence class* of vectors, which may be denoted by $\langle A \rangle$, where A is any member of the class. If the vectors A and B both correspond to the same stircle, then we may write either $A = kB$, or better $\langle A \rangle = \langle B \rangle$. The equation $A = B$ means, as it usually does, that the two 4-vectors are equal; that is, $a_i = b_i$, $i = 0, 1, 2, 3$. Every such equation occurring in this paper may, however, also be interpreted as stating that the stircle A is identical with the stircle B, and thus that $\langle A \rangle = \langle B \rangle$.

Suppose that A and B are both circles, so that $a_0 \neq 0$, $b_0 \neq 0$. Let d be the distance between the centers. Then it is readily shown that

$$d^2 = r_A^2 + r_B^2 - 2(A \# B)/a_0 b_0. \tag{7}$$

The condition of orthogonality of circles A and B is $d^2 = r_A^2 + r_B^2$, which in view of (7) reduces to

$$A \# B = 0. \tag{8}$$

It is readily verified that (8) is still the condition of orthogonality when either A, or B, or both, are straight lines. Thus (8) is the condition of orthogonality of any pair of stircles.

Suppose P is a null circle (or point circle). Then $P \# P = 0$. For convenience, since $p_0 \neq 0$, put $p_0 = 1$. Then from $P \# P = 0$ we deduce that $p_3 = -\frac{1}{2}(p_1^2 + p_2^2)$, so that a point may always be written in the form

$$P = (1, p_1, p_2, -\tfrac{1}{2}(p_1^2 + p_2^2)). \tag{9}$$

The orthogonality condition $A \# P = 0$ now becomes $-\frac{1}{2}a_0(p_1^2 + p_2^2) + a_1 p_1 + a_2 p_2 + a_3 = 0$; and this, from (1), is the condition that (p_1, p_2) shall lie on the stircle A. In fact,

$$A \# X = 0 \tag{10}$$

is seen to be a compact way of writing (1), provided $X \# X = 0$.

Suppose A and B are real circles. Let us make the definition

$$\Delta(A, B) = (A \# A)(B \# B) - (A \# B)^2. \tag{11}$$

Let us use (5) and (7) to eliminate $A \# A$, $B \# B$ and $A \# B$ from the right of (11). With a little algebra we find

$$\Delta(A, B) = \frac{a_0 b_0}{4}(r_A + r_B + d)(r_A + r_B - d)(r_A + d - r_B)(r_B + d - r_A). \tag{12}$$

If A, B intersect in two points, then their centers together with either point of intersection form a triangle with sides d, r_A, r_B, and (12) then shows that $\Delta(A, B)$ will be positive. If A, B are tangent, then (12) shows that $\Delta(A, B) = 0$; while if A, B have no common points, then $\Delta(A, B)$ will be negative.

Suppose A, B intersect, and θ is the angle between the radii from the two centers to either point of intersection. Then from the law of cosines it is easily established that

$$\cos\theta = (A \# B)\left[(A \# A)(B \# B)\right]^{-1/2}. \tag{13}$$

The angle θ may also be regarded as the angle between A and B. The formula continues to hold good when either, or both, of the stircles A, B become straight lines.

Let A and B be distinct stircles; this means that the vectors A and B are linearly independent. We shall use the notation $\langle A, B\rangle$ for the set of all stircles linearly dependent on A and B. That is,

$$\langle A, B\rangle = \left\{C \mid C = aA + bB,\ a, b \text{ real},\ a^2 + b^2 \neq 0\right\}. \tag{14}$$

The system of stircles so defined is called a *pencil* of stircles. In the older literature such a set is called a *coaxal system* of circles, and the straight line belonging to the pencil is called its *radical axis*. More explicitly, we shall call $\langle A, B\rangle$ the pencil based on A, B. We now distinguish three cases according to whether A, B intersect, are tangent, or have no common points.

(a) $\Delta(A, B) > 0$. Let P be one of the points of intersection of A and B. Then from (1), $P\#A = 0$, $P\#B = 0$. Hence $P\#(aA+bB) = 0$, and we conclude that every stircle of the pencil goes through both points of intersection of A and B. A pencil of this type is called *elliptic*.

(b) $\Delta(A, B) = 0$. Then a little algebra shows $\Delta(A, aA+bB) = 0$. The consequence is that all the stircles of the pencil are mutually tangent at a single point. This type of pencil is called *parabolic*.

(c) $\Delta(A, B) < 0$. Let us ask whether the pencil contains one or more null circles. We find $(aA+bB)\#(aA+bB) = a^2A\#A + 2abA\#B + b^2B\#B$. Under what circumstances will there exist a ratio a/b which will make this expression vanish? If $\Delta(A, B) < 0$ there are two such ratios. Thus the pencil contains two points. Such a pencil is called *hyperbolic*. Note that in the case of a parabolic pencil there is just one point. An elliptic pencil contains no points.

It is a rather simple exercise to show that the centers of all the circles in a pencil lie on a straight line, and that every pencil contains just one straight line, unless both A and B are straight lines, in which case we have a pencil of straight lines.

Suppose we have two pencils $\langle A, B\rangle$ and $\langle C, D\rangle$ such that $A\#C = 0$, $A\#D = 0$, $B\#C = 0$, $B\#D = 0$. It is then readily verified that every member of $\langle A, B\rangle$ will be orthogonal to every member of $\langle C, D\rangle$. Under such circumstances we say that the pencils $\langle A, B\rangle$ and $\langle C, D\rangle$ are mutually orthogonal. Let P, Q be points, and R, S two stircles passing through P, Q. Then $\langle P, Q\rangle$ is a hyperbolic pencil, and $\langle R, S\rangle$ is an elliptic pencil orthogonal to $\langle P, Q\rangle$.

Suppose that A, B, C are linearly independent vectors, so that the corresponding stircles do not belong to the same pencil. The *bundle* of stircles based on A, B, C, denoted by $\langle A, B, C\rangle$, is defined to be the set of all stircles linearly

dependent on A, B, C:

$$\langle A, B, C\rangle = \{X \mid X = aA + bB + cC;\ a, b, c \text{ real},\ a^2 + b^2 + c^2 \neq 0\}. \tag{15}$$

Elementary linear algebra tells us that there is a vector D, unique to within a multiplied constant, which is such that $A\#D=0$, $B\#D=0$, $C\#D=0$. It then follows that $X\#D=0$, where $X=aA+bB+cC$, so that D is orthogonal to every stircle of the bundle. Thus $\langle A, B, C\rangle$ may be described as the set of all stircles orthogonal to D. If D is real, we have a *hyperbolic* bundle; if D is a point, we have a *parabolic* bundle; if D is imaginary, we have an *elliptic* bundle. Note that a parabolic bundle consists of all stircles passing through a point. It may also be shown that an elliptic bundle consists of all stircles which cut a fixed circle E at the extremities of a diameter of E. The circle E is known as the *equator* of the bundle.

Suppose A is a real stircle, and P is a point not on A, so that $P\#A\neq 0$. The pencil $\langle A, P\rangle$ is hyperbolic, since P and A are not incident. One of its null circles is P. The other, which we shall denote by P_A, is called the *inverse of P relative to A*. To find it explicitly, write $X=aA+pP$ for the general member of $\langle A, P\rangle$. We then find $X\#X=a^2A\#A+2apA\#P$; we have used the fact that $P\#P=0$. In order that X may be a point, we must either have $a=0$, in which case X is identical with P; or $a=-2p(A\#P)/(A\#A)$. In the latter case,

$$X = p\left[P - 2\frac{A\#P}{A\#A}A\right].$$

It is convenient to put $p=1$, and define P_A by means of

$$P_A = P - 2\left(\frac{A\#P}{A\#A}\right)A. \tag{16}$$

This is our fundamental formula for inversion. We shall consider it as defining the inverse of P relative to A, even when A is an imaginary circle. Note that (16) is meaningless if A is a point. It is readily shown that P is also the inverse of P_A relative to A. Since the centers of all the circles belonging to a pencil are collinear, it follows that if A is a circle, then P, P_A and the center of A are collinear.

Consider two stircles A and B. The inverse of B relative to A, denoted by B_A, is defined to be the set of points which are inverses, relative to A, of points lying on B. We shall show that B_A is given by essentially the same formula (16) as was obtained for the inverse of a point P relative to A:

$$B_A = B - 2\left(\frac{A\#B}{A\#A}\right)A. \tag{17}$$

Preparatory to showing this, let C be a second stircle, and define

$$C_A = C - 2\left(\frac{A\#C}{A\#A}\right)A.$$

Now form, and evaluate, the number product $B_A \# C_A$; it is readily found that

$$B_A \# C_A = B \# C. \tag{18}$$

Now let P_A be the inverse of P relative to A, defined by (16), and let B_A be defined by (17). Then, using (18), $P_A \# B_A = P \# B$. Thus P_A is incident with B_A if and only if P is incident with B. We conclude that B_A is actually the set of inverses, relative to A, of all points lying on B.

Let $C=(1, c_1, c_2, c_3)$ be a circle which is not null, so that $c_3 \neq -\frac{1}{2}(c_1^2+c_2^2)$. Its center is $P=(1, c_1, c_2, c_3^*)$, $c_3^* = -\frac{1}{2}(c_1^2+c_2^2)$. We readily find $C \# C = 2C \# P$. If we now use (16) to find the inverse of P relative to C, we obtain $P_C = P - C = (0, 0, 0, c_3^* - c_3)$. Thus the "inverse" vector is of the form $(0, 0, 0, a)$, $a \neq 0$, which corresponds to no point, straight line or circle, real or imaginary. In order to provide every point with an inverse relative to every stircle we associate $(0, 0, 0, 1)$ with an "ideal" stircle, which may be called the "stircle at infinity" or the "point of infinity." The set of points of the Euclidean plane, with $(0, 0, 0, 1)$ adjoined, is called the *inversive plane*. In this plane, every stircle possesses an inverse relative to every other stircle.

Inversive geometry may be defined to consist of the collection of definitions and theorems expressing properties which are preserved under inversion. For example, consider the following theorem of inversive geometry: let A, B, A' be such that A and A' are mutually inverse relative to B. Let C be another stircle, and let A_C, B_C, A'_C be the inverses of A, B, A' relative to C. Then A_C and A'_C are mutually inverse relative to B_C. Using (13) and (18) we may show that the angle between two stircles is preserved under inversion. We thus say that inversion is a *conformal* transformation.

Let us briefly indicate how these ideas may be extended to three dimensions. The extension to n dimensions is then rather straightforward. An attractive feature of this extension is that formulas (16) and (17) are unchanged. Let us begin by defining, for any pair of 5-vectors $A=(a_0, a_1, a_2, a_3, a_4)$ and $B=(b_0, b_1, b_2, b_3, b_4)$, the number product

$$A \# B = a_1b_1 + a_2b_2 + a_3b_3 + a_0b_4 + a_4b_0. \tag{19}$$

Thus we have

$$A \# A = a_1^2 + a_2^2 + a_3^2 + 2a_0a_4. \tag{20}$$

If (x_1, x_2, x_3) represents a point in Cartesian 3-space, then the equation

$$a_0(x_1^2 + x_2^2 + x_3^2) - 2a_1x_1 - 2a_2x_2 - 2a_3x_3 - a_4 = 0 \tag{21}$$

represents a sphere if $a_0 \neq 0$, with center $(a_1/a_0, a_2/a_0, a_3/a_0)$ and radius given by (5). If $a_0=0$, but $a_1^2+a_2^2+a_3^2 \neq 0$, then (21) represents a plane. We shall say, in general, that (21) corresponds to a *splane*, including both sphere and plane in this concocted term, and we shall associate with the splane (21) the vector $A=(a_0, a_1, a_2, a_3, a_4)$. The special case $(0, 0, 0, 0, a_4)$, $a_4 \neq 0$, we shall call the

point at infinity. When this special point is adjoined to Euclidean 3-space we obtain *inversive space.*

As in the 2-dimensional case, (8) is the condition of orthogonality of two splanes, and (10) is the condition of incidence of the point X and the splane A; (10) may be regarded as a compact version of (21). If A and B are intersecting splanes, their angle is given by (13). Pencils and bundles of splanes are defined by (14) and (15). If A, B, C, D are linearly independent (i.e., do not belong to the same bundle), then $\langle A, B, C, D\rangle$ will denote the *hyperbundle* defined by

$$(22)\quad \langle A, B, C, D\rangle = \{X \mid X = aA+bB+cC+dD;\ a, b, c, d \text{ real},\ a^2+\cdots+d^2\neq 0\}.$$

Suppose we have a point P and a real splane A. Then $\langle A, P\rangle$ is an elliptic pencil whose second null sphere (i.e., point) is P_A given by (16). We again call P_A the inverse of P relative to A. As in the earlier case, we may show that P, P_A and the center of A are collinear, provided A is a sphere. The more general formula (17) will again give the inverse of any splane B relative to any non-null splane A. It will again turn out that the angle between splanes is preserved under the inversion.

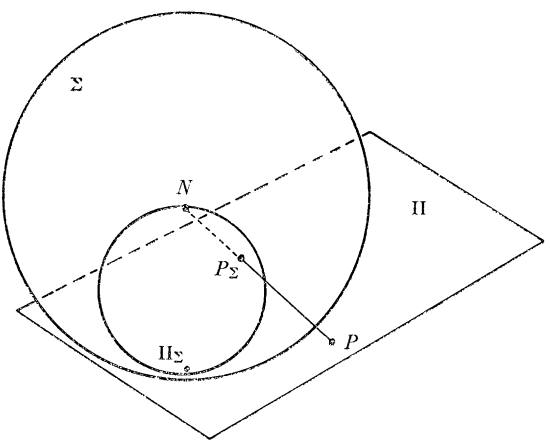


FIG. 1

We now consider an important specialization of inversion known as stereographic projection. In Figure 1, let Π be the plane $x_3=0$, which as a splane has the vector $\Pi=(0, 0, 0, 1, 0)$. Let Σ be the sphere which is tangent to Π at the origin, with center at N:(0, 0, 2), whose 5-vector is (1, 0, 0, 2, 0). According to (17), the inverse of Π relative to Σ is $\Pi_\Sigma=(1, 0, 0, 1, 0)$, which is a sphere tangent to Π at the origin, of unit radius. It is seen that Π_Σ passes through the center of Σ; in fact, if the origin is regarded as the "south pole" of Π_Σ then the center of Σ, namely N, is the "north pole" of Π_Σ. We know that if P is any point of Π, then its image P_Σ under the inversion will lie on Π_Σ; and furthermore, N, P and P_Σ will be collinear. Thus we have (as part of the inversion relative to Σ) a mapping

from Π onto Π_Σ in which corresponding points are collinear with N. Such a mapping is known as a stereographic projection from Π to Π_Σ. We shall use I as the symbol for the inversion relative to Σ. Thus P_Σ can be written $I(P)$, and we may also write $P=I(P_\Sigma)$.

We may now deduce easily that stereographic projection carries circles on Π_Σ into stircles on Π. For suppose that C is a circle on Π_Σ, and that Π^* is the plane in which C lies. Then we may write $C=\Pi^*\cap\Pi_\Sigma$. Its stereographic image is $I(\Pi^*)\cap I(\Pi_\Sigma)$. Hence $I(C)$ is the intersection of a splane $I(\Pi^*)$ and the plane $\Pi=I(\Pi_\Sigma)$; it is thus a stircle of Π.

Consider an elliptic pencil $\langle A, B\rangle$ on Π, whose stircles go through the distinct points P, Q. Its image under the inversion I is a set of circles on Π_Σ, which pass through the image points P_Σ, Q_Σ of P and Q. We can readily see that this set of circles is the intersection of the sphere Π_Σ and a pencil of planes which pass through the line joining P_Σ, Q_Σ. We call such a set an elliptic pencil of circles on Π_Σ. Similarly, a hyperbolic pencil on Π will be mapped into a hyperbolic pencil on Π_Σ, and the corresponding pencil of planes will either consist of mutually parallel planes, or of planes which pass through a line which does not cut Π_Σ.

Finally, a parabolic pencil in Π will map into a parabolic pencil on Π_Σ; and the line common to the corresponding pencil of planes will be tangent to Π_Σ.

Just as we have pencils of three types on the sphere Π_Σ, so we have bundles of three types on Π_Σ, each being the image of a bundle of the same type on Π. An elliptic bundle on Π_Σ possesses the important property that the set of planes of the circles of the bundle all pass through a common point P *which is interior to* Π_Σ. Thus if A and B are any two circles of this elliptic bundle their planes must intersect in a line which cuts Π_Σ in two points. Thus every pair of circles of an elliptic bundle must intersect. The same must be true of pairs of stircles belonging to an elliptic bundle of stircles in the plane Π. Hence, if B, C belong to an elliptic bundle, $\langle B, C\rangle$ is an elliptic pencil.

Example 1. Let E be the equator of Π_Σ, whose plane is parallel to Π and goes through the center C: (0, 0, 1) of Π_Σ. The image E_Σ of E will be a circle of radius 2 about the origin; its 4-vector is $E_\Sigma=(1, 0, 0, 2)$. Now consider the bundle of planes through C. A suitable basis for this bundle consists of the three planes (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, -1). The inversion I carries these into the splanes (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (1, 0, 0, 0, 2), which cut Π in the stircles $L=(0, 1, 0, 0)$, $M=(0, 0, 1, 0)$, $E_\Sigma=(1, 0, 0, 2)$. These, then, form a basis for the bundle of stircles in the plane Π into which the great circles on Π_Σ are projected. We note two facts about this bundle. First, since the three stircles above listed are orthogonal to $A=(1, 0, 0, -2)$, the bundle must be orthogonal to A. This is, as it should be, an imaginary circle. Second, we note that the pencil $\langle L, M\rangle$ consists of all diameters of E_Σ. Thus the bundle $\langle L, M, E_\Sigma\rangle$, which we are considering, consists of all stircles in the plane Π which cut E_Σ in a pair of diametral points.

We shall now use the tools we have developed in order to investigate various forms of plane geometry. A "geometry" is here understood as a system involving

two sets $\mathcal{P}$ and $\mathcal{L}$ and an incidence relation. The set $\mathcal{P}$ consists of "points," the set $\mathcal{L}$ of "lines." A "point" P and a "line" L are said to be incident if P lies on L, or L goes through P.

On the sphere we have a natural geometry in which $\mathcal{P}$ consists of the points of the sphere, and $\mathcal{L}$ consists of its great circles which, as we have seen, form an elliptic bundle. This suggests that an interesting geometry on π may arise when we take for $\mathcal{L}$ the stircles of an elliptic bundle.

In Euclidean geometry the "lines" are the straight lines of the plane. Let us project these lines from Π to Π_Σ. They are projected into a parabolic bundle on Π_Σ, whose associated bundle of planes consists of all planes through the north pole N. This suggests that a parabolic bundle, on the plane Π or on the sphere Π_Σ, ought to give rise to an "interesting" geometry.

Motivated by these considerations, let us ask what kind of geometries arise when we take a bundle of stircles in the plane as the system of "lines," $\mathcal{L}$, of the geometry we are building. We know that all stircles of a bundle are orthogonal to a stircle, which is a real stircle, a point, or an imaginary circle according as the bundle is hyperbolic, parabolic or elliptic. This stircle we call the absolute, A, of the geometry; and we call the geometry hyperbolic, parabolic, or elliptic according as the bundle is hyperbolic, parabolic, or elliptic.

Hyperbolic geometry is, in fact, the type of non-Euclidean geometry which was investigated in the first quarter of the nineteenth century by Gauss, Bolyai and Lobachevski; it is often known as Lobachevskian geometry. Elliptic geometry was so named by Felix Klein, following suggestions made by Bernhard Riemann in his famous inaugural dissertation of 1854. The use of the terms elliptic, parabolic and hyperbolic as names for these geometries is due to Klein.

We have defined the set $\mathcal{L}$ consisting of the "lines" of the system of geometry. As a preliminary to our definition of the set $\mathcal{P}$, we note that two "lines" may have two common points. If the "lines" B and C, both orthogonal to the absolute A, have in common the points P and P', then every stircle of the elliptic pencil $\langle B, C\rangle$ will go through P and P', and it follows that P, P' and A belong to a hyperbolic pencil orthogonal to $\langle B, C\rangle$. If A itself does not reduce to a point (as it does in parabolic geometry) this means that P and P' are mutually inverse relative to A. Such a pair of points, $\{P, P'\}$, we shall call a "point" in our geometry, and $\mathcal{P}$ will consist of all possible point-pairs, each pair mutually inverse relative to A. We have now defined $\mathcal{P}$ in the elliptic and hyperbolic cases. In the parabolic case, we define $\mathcal{P}$ to consist of all point-pairs of the form $\{A, P\}$, where A is the absolute, and P is a point of the inversive plane. In all three cases, we define a "point" to be incident with a "line" if both points of the point-pair are on the stircle which represents the "line." We have now ensured that two "lines" will intersect in no more than one "point." In the case of elliptic geometry two "lines" will always have precisely one common "point." Thus elliptic geometry is a type of projective geometry.

It is often convenient to choose, from each point-pair, a single representative, and regard these representatives as the "points" of the geometry. In hyper-

bolic geometry A is real. If A is a circle we can take the Euclidean points interior to A as representatives; we then have the so-called Poincaré model of hyperbolic geometry. If A is a straight line, we may take the Euclidean points on one side of A as representatives; this is the half-plane model of hyperbolic geometry. If A is imaginary, and the geometry elliptic, we return to the viewpoint of Example 1, in which the great circles on the sphere were stereographically projected into the elliptic bundle orthogonal to $A=(1, 0, 0, -2)$. The southern hemisphere of Π_Σ is projected into the interior of the circle $E_\Sigma=(1, 0, 0, 2)$. It is convenient to take as a set of representatives of $\mathcal{P}$ the interior of E_Σ together with one-half of the points of E_Σ itself; for example, those on any open semi-circle of E_Σ, with one or other of the endpoints of the semi-circle.

In the case of parabolic geometry, the representative points will be the points of the inversive plane, except the point A itself.

We must now introduce a set of mappings which will be the analog of the rigid motions (isometries) of Euclidean geometry. It is reasonable to require that these mappings carry "lines" into "lines," and preserve "angle" and "length." Our strategy is to select a set of mappings which carry "lines" into "lines," and then define "angle" and "length" so as to remain invariant under these mappings. Let A be the absolute; let B be the bundle of stircles orthogonal to A which constitutes the set of "lines," and let M be any inversion. Under M, A and B are carried into $M(A)$ and $M(B)$, and the latter is a bundle orthogonal to the stircle $M(A)$. But is it the *same* bundle? Not unless $M(A)=A$. In view of this, we restrict ourselves to inversions which leave A fixed, and call these "reflections." Each "reflection" is an inversion relative to a "line" of the geometry. We may now define an "isometry" to be the composite of a finite sequence of "reflections."

Next, consider "angle" and "length." The first is easy to dispose of, since an inversion preserves the Euclidean angle between stircles. We accordingly define "angle" between "lines" to be the Euclidean angle between the corresponding stircles.

We can derive, without difficulty, a concept of "length" in the case of elliptic geometry, using as our starting point the arc length of a circle on the sphere, or, what amounts to the same thing, the angle between radii joining the center to two points on the sphere. In Example 1, a sphere Π_Σ was projected stereographically onto the plane Π from the north pole N. Let C:(0, 0, 1) be the center of Π_Σ. The stereographic projection is identical with the inversion I relative to the sphere $\Sigma=(1, 0, 0, 2, 0)$. Given two points P, Q of Π, we find their image points P_Σ, Q_Σ on Π_Σ, and we define the "distance" $d(P, Q)$ to be one-half of the smaller arc length along the great circle joining P_Σ, Q_Σ; if P_Σ, Q_Σ are antipodal, $d(P, Q)$ will be one quarter of the circumference of a great circle. In general, then, $d(P, Q)=\frac{1}{2}\angle P_\Sigma C Q_\Sigma$.

Let P, considered as a 5-vector, be

$$P = (1, p_1, p_2, 0, p_4), \quad p_4 = -\tfrac{1}{2}(p_1^2 + p_2^2).$$

Then we have $\Sigma\#\Sigma=4$, $\Sigma\#P=p_4$, so that

$$P_\Sigma = P - \tfrac{1}{2}p_4\Sigma = (1 - \tfrac{1}{2}p_4, p_1, p_2, -p_4, p_4).$$

This is a point on the sphere Π_Σ. As a point in 3-space its coordinates are $(\alpha p_1, \alpha p_2, -\alpha p_4)$, where $\alpha=(1-\frac{1}{2}p_4)^{-1}$. The space-vector $\overrightarrow{CP_\Sigma}$ is $(\alpha p_1, \alpha p_2, -\alpha p_4-1)$. Similarly the point $Q=(1, q_1, q_2, 0, q_4)$, where $q_4=-\frac{1}{2}(q_1^2+q_2^2)$ corresponds to the space-vector $\overrightarrow{CQ_\Sigma}=(\beta q_1, \beta q_2, -\beta q_4-1)$, where $\beta=(1-\frac{1}{2}q_4)^{-1}$. The vectors $\overrightarrow{CP_\Sigma}$ and $\overrightarrow{CQ_\Sigma}$ are unit vectors, since Π_Σ is of unit radius. Hence

$$\cos\angle P_\Sigma C Q_\Sigma = \overrightarrow{CP_\Sigma}\cdot\overrightarrow{CQ_\Sigma} = 1 + \alpha\beta(p_1q_1 + p_2q_2 + p_4 + q_4) = 1 + \alpha\beta(P\,\#\,Q).$$

Considered as stircles in the plane Π, $P=(1, p_1, p_2, p_4)$, $Q=(1, q_1, q_2, q_4)$, and $A=(1, 0, 0, -2)$. Hence we have $A\#P=p_4-2=-2\alpha^{-1}$, $A\#Q=-2\beta^{-1}$, and

$$\sin^2(d(P, Q)) = \sin^2\tfrac{1}{2}\angle P_\Sigma C Q_\Sigma = -\tfrac{1}{2}\alpha\beta(P\,\#\,Q) = \frac{-2(P\,\#\,Q)}{(A\,\#\,P)(A\,\#\,Q)}.$$

This is nonhomogeneous with respect to A; that is, if the vector A is multiplied by a nonzero scalar k, then $\sin^2(d(P, Q))$ will be multiplied by k^{-2}. To overcome this flaw, we introduce $A\#A=-4$ in the numerator, obtaining

$$d(P, Q) = \sin^{-1}\left[\frac{(A\,\#\,A)(P\,\#\,Q)}{2(A\,\#\,P)(A\,\#\,Q)}\right]^{1/2}. \tag{22}$$

This is our basic distance function in the elliptic case.

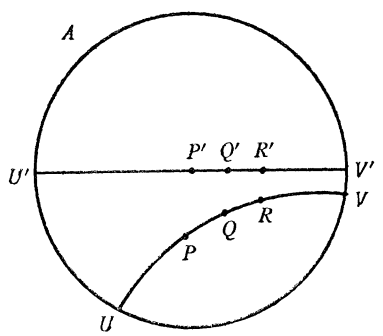


FIG. 2

In order to determine what form $d(P, Q)$ shall take in the hyperbolic case, let L be a "line" which meets the absolute A in the points U and V (see Figure 2). Let P, Q, R be three points of L, where Q is *between* P and R in the sense that if we traverse L from U to V we encounter first P, then Q, then R. We wish to show that the appropriate distance function, in this case, is

$$d(P, Q) = \sinh^{-1}\left[\frac{-(A \# A)(P \# Q)}{2(A \# P)(A \# Q)}\right]^{1/2}. \tag{23}$$

We may show, without difficulty, that a rigid motion (in fact, a sequence of no more than two reflections) exists which will carry P to the center of A, which we call P', and Q and R to the points Q' and R' on the positive x_1-axis, with Q' between P' and R'. Let

$$A=(1, 0, 0, a^2/2),\quad P'=(1, 0, 0, 0),\quad Q'=(1, q, 0, -q^2/2),\quad R'=(1, r, 0, -r^2/2),$$

where $0<q<r<a$.

It is reasonable to require that the distance function $d(P, Q)$, as in the elliptic case, be expressible in terms of the quantity $[(A\#A)(P\#Q)]/[2(A\#P)(A\#Q)]$. It turns out to be convenient to consider, not this quantity itself, but a function of it:

$$K(P, Q) = \sqrt{\frac{-(A \# A)(P \# Q)}{2(A \# P)(A \# Q)}} + \sqrt{1 - \frac{(A \# A)(P \# Q)}{2(A \# P)(A \# Q)}}. \tag{24}$$

Let us evaluate $K(P', Q')$, $K(P', R')$, $K(Q', R')$. We find

$$K(P', Q') = \sqrt{\frac{a+q}{a-q}},\quad K(P', R') = \sqrt{\frac{a+r}{a-r}},\quad K(Q', R') = \sqrt{\frac{(a+r)(a-q)}{(a-r)(a+q)}}.$$

From these expressions it is evident that

$$K(P', R') = K(P', Q')\cdot K(Q', R').$$

We thus have a multiplicative relationship between $K(P', R')$, $K(P', Q')$ and $K(Q', R')$; whereas we clearly want an *additive* relationship $d(P', R')=d(P', Q')+d(Q', R')$ to prevail between adjacent arcs on the same "line." To bring this about, we define $d(P, Q)=\log K(P, Q)$, and appeal to the formula $\sinh^{-1}x=\log(x+\sqrt{(1+x^2)})$. Thus we find, from (24), the desired distance function (23).

We are now in a position to derive the various properties of hyperbolic and elliptic geometry. Of these, we shall concern ourselves only with *parallelism* and the *angle sum of a triangle*. Given a line L and a point P not on L:

(25a) in *Euclidean geometry*, there is a unique line L' through P which does not meet L;

(25b) in *elliptic geometry*, there is no line through P which does not meet L;

(25c) in *hyperbolic geometry*, there are infinitely many lines through P which do not meet L.

(25a) is, of course, Euclid's fifth postulate in the Playfair form. We can show that this statement holds true, also, for any parabolic geometry. Given the "point" P and the "line" L, consider the parabolic pencil $\langle A, L\rangle$, where A is the absolute. There is a unique member, say L', of this pencil which passes through P. It has no "point" in common with L.

Proposition (25b) follows from the fact, mentioned earlier, that in an

elliptic bundle *every pair of stircles intersects.* To see that (25c) is true, consider the Poincaré model in which the absolute A is a real circle. The proof for the halfplane model would be quite similar. The "line" L will meet A in two points Q and R. Consider the two parabolic pencils $\langle A, Q\rangle$ and $\langle A, R\rangle$. In the first of these there is one member, say L', which passes through P; and in the second there is likewise one member, say L'', which passes through P. The two stircles L', L'' must be distinct. If they were not, the stircle $L'=L''$ would be orthogonal to A, and pass through Q and R. But there is just one stircle orthogonal to a given stircle and passing through two points on it. Thus $L'=L''$ implies $L'=L''=L$, and thus P lies on L, which has been assumed not to be the case. The stircles ("lines") L' and L'' are members of the elliptic pencil passing through P and P_A, the inverse of P relative to A; furthermore, L' and L'' are tangent to L. These two stircles divide the elliptic pencil into two sets of stircles, such that all the stircles of one set cut L, while all the stircles of the other set do not cut L. Since this second set contains infinitely many members, (25c) is justified.

We now prove that in hyperbolic geometry the angle sum in any triangle is less than π. To do this, we deal first with a right-angled triangle (see Figure 3):

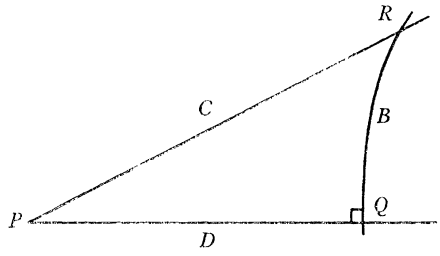


FIG. 3

THEOREM. *If PQR is a right triangle in the hyperbolic plane, with Q the right angle, then*

$$\cos R = \sin P \cosh 2r, \tag{26}$$

where r is the length of the side opposite R.

Proof. In the Poincaré model, let $A=(1, 0, 0, \frac{1}{2})$, and let P be at the center of the circle A. Let PQ coincide with the positive x_1-axis and let PR lie in the first quadrant. Then

$$P = (1, 0, 0, 0), \qquad Q = (1, a, 0, -a^2/2),$$

where a is some positive number less than 1. Using the distance formula (23) we find that $r=d(P, Q)$ is given by $\sinh r = a(1-a^2)^{-1/2}$, or $\cosh 2r = (1+a^2)/(1-a^2)$. Let us denote the 4-vectors corresponding to the sides opposite P, Q, R by B, C, D. Then, using P for the angle between C and D we have

$$C = (0, \sin P, -\cos P, 0), \qquad D = (0, 0, 1, 0).$$

In order to determine the 4-vector B, we have the conditions $B\#A=0$, $B\#D=0$, $B\#Q=0$. These, together with the arbitrary choice of $b_0=1$ for the first component of B, lead to $B=(1, (a^2+1)/2a, 0, -\frac{1}{2})$, from which

$$\cos R = \frac{B\#C}{\sqrt{(B\#B)(C\#C)}} = \sin P \frac{1+a^2}{1-a^2} = \sin P \cosh 2r.$$

Since $\cosh 2r$ is greater than 1, (26) implies that $\cos R > \sin P = \cos(\frac{1}{2}\pi - P)$. Hence, $P+R<\frac{1}{2}\pi$. Thus the angle sum of a right triangle is less than π. Given a general triangle, we divide it into two right triangles by a perpendicular from a vertex to an opposite side, and readily conclude that its angle sum is less than π.

Formula (26) is one of a system of results in hyperbolic trigonometry, analogous to the formulas of spherical trigonometry, all of which can be derived on the same basis as (26).

In references 1 and 2 will be found details on the interpretation of the geometry of this article in 3-dimensional projective space. Reference 3 includes a summary of more advanced results in inversive and conformal geometry. Reference 4 is the standard treatment of bundles of circles and their properties.

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A PARALLELOGRAM LAW FOR CERTAIN L_p SPACES

D. C. KAY, University of Oklahoma

A slight variation of the parallelogram law was recently found to play a role in extending Haantjes' proof of a theorem on curvature [3] to normed vector spaces. This condition replaces the ptolemaic inequality used in that proof, but applies to a wider class of linear spaces than those which are ptolemaic. In [4], Schoenberg has shown that the only linear spaces which are ptolemaic are Euclidean. The property we wish to study here however holds for certain spaces which are not Euclidean:

DEFINITION. *A vector space V with norm $\| \ \|$ is said to obey a weak parallelogram law provided there exists a positive constant $\gamma<1$ such that for all x and y belonging to V,*

$$\|x+y\|^2 + \gamma^2\|x-y\|^2 \leqq 2\|x\|^2 + 2\|y\|^2. \tag{1}$$

The constant γ is restricted in this manner because the inequality cannot hold without qualification if $\gamma > 1$ as is clear by setting $y = -x$. Also, the inequality for $\gamma = 1$ implies (as is shown by Schoenberg's result) that V is an inner product space with the usual parallelogram law $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2$, and finally, (1) imposes no condition at all if $\gamma \leqq 0$.

Since the parallelogram law implies the weak parallelogram law in a normed vector space (with $\gamma = 1/2$, say), but presumably not conversely, our terminology is apparently justified. The immediate question to answer is whether there exist vector spaces which satisfy the weak but not the full parallelogram law. It is the purpose of this paper to answer this question affirmatively for certain finite-dimensional spaces and in particular to prove that all finite-dimensional L_p spaces $(1 < p \leqq 2)$ obey a weak parallelogram law. We do not deal with the curvature theorem of Haantjes since that appears elsewhere (see [6]).

1. A sufficient condition for the weak parallelogram law. The following argument will yield a property of the unit sphere which guarantees the weak parallelogram law. Let x and y be two linearly independent points in a normed vector space V, the truth of (1) being self-evident otherwise. With $z = x+y$, define

$$x' = \frac{x}{\|x\|}, \qquad y' = \frac{y}{\|y\|}, \qquad z' = \frac{z}{\|z\|}.$$

By the linear independence of x and y, x', y', and z' are three distinct points lying on the unit sphere U. Let o be the zero vector of V. If U is strictly convex, o, x', z', and y' are the vertices of a convex quadrilateral which lies in the two-flat F determined by o, x', and y', and therefore, there is a unique ellipse E in F with center o passing through x', y', and z'. Let $\| \ \|_E$ denote the norm in F with unit sphere E. That is, for $p \neq o$ in F and with p' either point of intersection of E and the line determined by o and p, define

$$\|p\|_E = \frac{e(o, p)}{e(o, p')},$$

with $e(u, v)$ a Euclidean metric for F. From now on, $e(u, v)$ will be abbreviated by the simpler notation, uv. The norm $\| \ \|_E$ is "Euclidean" since the affine transformation ϕ mapping E onto the unit circle with center o yields

$$\|p\|_E = \frac{op}{op'} = \frac{o\phi(p)}{o\phi(p')} = o\phi(p).$$

Thus, the ordinary parallelogram law holds for this norm. Setting $w = x-y$ we have

$$\|z\|_E^2 + \|w\|_E^2 = 2\|x\|_E^2 + 2\|y\|_E^2.$$

Moreover, $\|z\|_E = \|z\| = \|x+y\|$, $\|x\|_E = \|x\|$, and $\|y\|_E = \|y\|$. If we set $w'' = w/\|w\|_E$,

then $\|w\|_E = \|w\|/\|w''\| = \|x-y\|/\|w''\|$. Making substitutions we obtain

$$\|x+y\|^2 + \|w''\|^{-2}\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

This gives us (1) provided $\|w''\|$ is bounded. Since w'' is a point on E, we need only ask for the boundedness of the diameters of all ellipses such as E. But E was merely an ellipse with center o and having six points of contact, $\pm x'$, $\pm y'$, and $\pm z'$, with U. Thus we may state:

LEMMA 1. *The weak parallelogram law is valid in a normed linear space with strictly convex unit sphere U and zero vector o provided the diameters of all ellipses with center o having six distinct points of contact with U are bounded, as measured by the norm.*

This lemma is of primary interest in the case of two-dimensional spaces where the condition given can be studied in terms of the curvature of the unit sphere U. Intuitively, a unit sphere having positive curvature at every point cannot have a sequence of ellipses, such as those defined above, whose diameters are unbounded. Thus, the norm defined by U will obey a weak parallelogram law. We shall prove a more general theorem which will apply to a two-dimensional L_p space, $p \geqq 1$, where we cannot assume that the curvature of U exists at every point. We shall subsequently find the following lemmas useful. Since the first is an easy exercise in elementary calculus, while the second is a standard result in the theory of convex bodies, we omit the proofs.

LEMMA 2. *If $h(x)$ is a real-valued function of the real variable x on the interval $a \leqq x \leqq b$ where $a < b$, and the derivative $h'(x)$ exists except possibly at $x = c$ for $a < c < b$ but $h(x)$ is strictly increasing (decreasing) in a neighborhood of c, then Rolle's Theorem holds for $h(x)$ on that interval. That is, if $a \leqq a_1 < b_1 \leqq b$ and $h(a_1) = h(b_1)$ then there exists a c_1 such that $h'(x)$ exists at c_1, $h'(c_1) = 0$, and $a_1 < c_1 < b_1$.*

LEMMA 3. *Let θ represent the Euclidean angle $\angle(o, p, q)$ where p and q are points of a bounded two-dimensional convex curve U with o any fixed interior point of U, with uv denoting a Euclidean metric, as before. Then if $a = \inf_{x \in U} ox$ and $pq < 2a$, θ is bounded away from 0 and π.* (For later reference we put $b = \sup_{x \in U} ox$.)

We shall also need the following rather trivial result:

LEMMA 4. *With the notation of Lemma 3, suppose that U is symmetric about o and that E_n is a sequence of ellipses centered at o meeting U in six distinct points whose Euclidean diameters d_n tend to infinity. Then the width w_n of E_n (defined by $w_n = 2 \inf_{x \in E_n} ox$) is bounded away from zero.*

Proof. Suppose w_n is not bounded away from zero. We may assume without loss of generality that $\lim_n w_n = 0$. We make the observation that there is therefore some subsequence $\{k\}$ of $\{n\}$ such that the six points in common between E_k and U converge to either u or $-u$ for some point u on U. Since E_k meets U

in six points, we can find two of those points in the same quadrant formed by the axes of E_k. Let us choose the notation so that p_k and q_k are those two points and $op_k > oq_k$. Thus, p_k and q_k converge to the same point on U and consequently, $p_kq_k < 2a$ for all k sufficiently large. From the elementary properties of ellipses, if ϕ_k is the angle between the tangent of E_k at p_k and the line through o and p_k, while $\theta_k = \angle(o, p_k, q_k)$, then $|\tan \theta_k| < |\tan \phi_k|$ for each k. But setting $r_k = op_k$ we can easily compute $|\tan \phi_k|$ in terms of r_k, d_k, and w_k:

$$|\tan \phi_k| = \frac{w_k}{\sqrt{(1 - 4r_k^2/d_k^2)(4r_k^2 - w_k^2)}}.$$

It then follows that $\lim_k \tan \theta_k = 0$, which contradicts Lemma 3.

2. Two-dimensional Minkowski spaces. In the following theorem we shall assume that U, as a plane convex curve, is differentiable at every point and twice differentiable at all but finitely many points. That is, if U is represented locally by $y = f(x)$ in some rectangular coordinate system (x, y), then $f(x)$ has first derivative $f'(x)$ for all x in the interval of definition of $f(x)$, while the second derivative $f''(x)$ exists at all but a finite number of points in that interval.

THEOREM 1. *If U is the unit sphere of a two-dimensional Minkowski geometry having the differentiability properties mentioned above, the weak parallelogram law holds if and only if the curvature of U wherever it exists is bounded away from zero.*

Proof. Sufficiency. Suppose the curvature of U is bounded away from zero. In view of Lemma 1, it suffices to show that no sequence of ellipses E_n having six points of contact with U and centered at o, the center of U, and whose diameters d_n tend to infinity, exists. To that end, suppose that such a sequence does exist, and let the six points of contact for E_n and U be $\pm p_{ni}$ $(i = 1, 2, 3)$. Since these sequences of points have convergent subsequences, we may assume without loss of generality that $\lim_n p_{ni} = p_i$ for each i. The hypothesis implies that U is strictly convex, so it is clear that three of the points $\pm p_i$ $(i = 1, 2, 3)$ coincide with p_1 while the others coincide with $-p_1$. Let the notation be such that $\lim_n p_{ni} = p_1$ for each i. Choose a rectangular coordinate system (x, y) with origin o and the positive y-axis passing through p_1. Then E_n has an equation of the form

$$a_nx^2 + 2b_nxy + c_ny^2 = 1, \tag{2}$$

where $a_n > 0$, $c_n > 0$ and $b_n^2 - a_nc_n < 0$. Now the diameter of E_n is given by

$$d_n = 2\sqrt{2}[a_n + c_n - \sqrt{(a_n + c_n)^2 + 4(b_n^2 - a_nc_n)}]^{-1/2} \tag{3}$$

while the width (as defined in Lemma 4) is

$$w_n = 2\sqrt{2}[a_n + c_n + \sqrt{(a_n + c_n)^2 + 4(b_n^2 - a_nc_n)}]^{-1/2}. \tag{4}$$

By Lemma 4, w_n is bounded away from zero so that the quantity inside the bracket in (4) is bounded. Thus, a_n and c_n, being positive, must both be bounded,

and therefore so is b_n. It then follows that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ have convergent subsequences, say $\{a_k\}$, $\{b_k\}$, and $\{c_k\}$, and hence, $\lim_k a_k = a$, $\lim_k b_k = b$, and $\lim_k c_k = c$. Since $d_n^{-1} \to 0$, (3) implies that $\lim_k b_k^2 - a_k c_k = 0$ and therefore $b^2 = ac$. If the coordinates of p_{ki} are (x_{ki}, y_{ki}) for each k and for $i = 1, 2, 3$, which must satisfy equation (2) for $n = k$, then, since $x_{ki} \to 0$, we have $c > 0$ and obtain $(0, 1/\sqrt{c})$ as the coordinates of p_1. In some neighborhood of p_1 U has the representation $y = f(x)$, and by hypothesis, we may assume the neighborhood is such that $f'(x)$ exists for all x and $f''(x)$ exists for all x except, possibly, $x = 0$. For all sufficiently large k and each i, x_{ki} will fall in the interval of definition of $f(x)$. For each such k define the function

$$g_k(x) = a_k x^2 + 2b_k xy + c_k y^2 - 1, \quad \text{where} \quad y = f(x).$$

Since $g_k(x_{ki}) = g_k(x_{k2}) = g_k(x_{k3}) = 0$ (we shall assume $x_{k1} < x_{k2} < x_{k3}$), we may apply Rolle's Theorem and find numbers x_{k4} and x_{k5} such that $g_k'(x_{k4}) = g_k'(x_{k5}) = 0$, with $x_{k1} < x_{k4} < x_{k2} < x_{k5} < x_{k3}$. Then

$$(5) \quad (1/2)g_k'(x) = a_k x + b_k y + y'(b_k x + c_k y) = 0, \qquad \text{for } x = x_{k4},\ x = x_{k5}.$$

As $k \to \infty$ we have $x = x_{k4} \to 0$, and therefore $y = f(x_{k4}) \to 1/\sqrt{c}$, so we find that since $y' < 0$, $y' = f'(x_{k4}) \to -\sqrt{a/c}$. The convexity of U implies that $f'(x)$ is continuous where it exists, so we have $f'(0) = -\sqrt{a/c}$. Next, compute $g_k''(x)$, $x \neq 0$:

$$(6) \qquad (1/2)g_k''(x) = a_k + 2b_k y' + c_k y'^2 + y''(b_k x + c_k y).$$

For all x sufficiently close to 0 the quantity $b_k x + c_k y$ is positive for all k. Hence, for all such x, $y''(b_k x + c_k y) < -\epsilon(b_k x + c_k y)$, where $\epsilon > 0$ is the number guaranteed by hypothesis (and our choice of coordinate system) such that $y'' < -\epsilon$ for all x. That is,

$$(7) \qquad (1/2)g_k''(x) < a_k + 2b_k y' + c_k y'^2 - \epsilon(b_k x + c_k y).$$

Now, the limit of the right-hand member of (7) as x tends to zero is

$$a_k - 2b_k\sqrt{a/c} + c_k(a/c) - \epsilon c_k/\sqrt{c}$$

and the limit of this as k tends to infinity is $-\epsilon\sqrt{c}$. Therefore, for all sufficiently large k there exists a neighborhood N_k about $x = 0$ for which

$$g_k''(x) < -\epsilon\sqrt{c}/4 < 0$$

holds for all x. That is, $g_k'(x)$ is decreasing for x in N_k. By Lemma 3 Rolle's Theorem applies to $g_k'(x)$ and we can find an x_{k6} such that $g_k''(x_{k6}) = 0$ and $x_{k4} < x_{k6} < x_{k5}$. Then (6) becomes

$$a_k + 2b_k y' + c_k y'^2 + y''(b_k x + c_k y) = 0, \qquad x = x_{k6}.$$

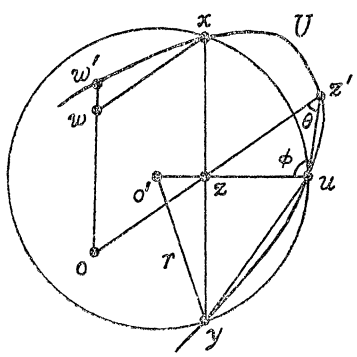
As $k \to \infty$, $x_{k6} \to 0$ and the above equation implies $f''(x_{k6}) \to 0$, a contradiction. It follows that no sequence E_n exists, and therefore, the weak parallelogram law is valid.

Necessity. Suppose that the weak parallelogram law holds, and that the curvature K of U exists at some point p. By hypothesis, the curve U is twice differentiable in some neighborhood of p. Let $\{x_k\}$ and $\{y_k\}$ be two sequences of points on U converging to p, $x_k \neq y_k$ for each k. Dropping the subscripts for the moment, define $z=(x+y)/2$, $z'=z/\|z\|$, $w=(x-y)/2$, $w'=w/\|w\|$, and let u be that point on U closest to x such that $xu=uy$. Further, let o' be the center of the circle passing through x, u, and y, and r its radius, and set $\theta=\angle(u, z', z)$, $\phi=\angle(z, u, z')$. Now since $\|x\|=1$ and $\|y\|=1$, we have

$$\frac{2\|x\|^2+2\|y\|^2-\|x+y\|^2}{\|x-y\|^2}=\frac{1-\|z\|^2}{\|w\|^2}=\frac{1-(oz/oz')^2}{(ow/ow')^2}.$$

Simplifying and using the law of sines for triangle (u, z, z') the latter becomes

$$\frac{zu}{zy^2}\cdot\frac{\sin\phi}{\sin\theta}\cdot\frac{ow'^2}{oz'^2}(oz'+oz)=\frac{1}{2r-zu}\cdot\frac{\sin\phi}{\sin\theta}\cdot\frac{ow'^2}{oz'^2}(oz'+oz).$$



By the weak parallelogram law the above remains no smaller than the constant $\gamma>0$. Lemma 3 shows that $\sin\theta\geqq\delta>0$ for some constant δ, and if a and b are the numbers defined, we have $ow'\leqq b$, $oz'+oz\leqq 2b$, and $oz'\geqq a$. Thus,

$$0<\gamma\leqq\frac{1}{r_k-(z_ku_k)/2}\cdot\frac{b^3}{\delta a^2},$$

replacing the subscripts. Since the osculating circle exists at p we have $\lim_k 1/r_k=K$. Both, z_k and u_k converge to p so we have $\lim_k (r_k-(z_ku_k)/2)^{-1}=K$. Hence there exists k for which $(r_k-(z_ku_k)/2)^{-1}<2K$. We have therefore proved that wherever K exists it must satisfy the inequality

$$0<\gamma<\frac{2Kb^3}{\delta a^2}.$$

Thus K is bounded away from zero, and this completes the proof of the theorem.

Theorem 1 gives us full information regarding two-dimensional L_p spaces, for then U is the set of points (x, y) satisfying $|x|^p+|y|^p=1$, $p\geqq 1$. One can readily observe the curvature behavior of U in the first quadrant, $0\leqq x<1$. For $0<x<1$ and $0<y<1$ we have $y=(1-x^p)^{1/p}$, so the usual formula for curvature K applies. Thus,

$$K=\frac{(p-1)(xy)^{p-2}}{(x^{2p-2}+y^{2p-2})^{3/2}}. \tag{8}$$

Now, if $p=1$ we find $K=0$ for all x, $0<x<1$. If $1<p<2$ the curvature is positive for all x, becoming infinite as x tends to zero or unity, so K is bounded away from zero, where it exists, in $0\leqq x\leqq 1$. For $p=2$, $K=1$ for all x (of course, the full parallelogram law holds in this case), and for $p>2$, K tends to zero as x approaches zero. By the symmetry of U with respect to the coordinate axes, the curvature behavior for the first quadrant carries over to the other three. Hence, we have proved:

THEOREM 2. *For two-dimensional L_p spaces $(p\geqq 1)$, the weak parallelogram law holds iff $1<p\leqq 2$.*

3. Finite-dimensional L_p spaces. For $n\geqq 1$, an n-dimensional L_p space is the set of all n-tuples $x=(x_1, x_2, \cdots, x_n)$, x_k real, with the norm defined by $\|x\|=(\sum_{k=1}^n |x_k|^p)^{1/p}$. The next theorem establishes the fact that the restriction to dimension two in Theorem 2 is inessential.

THEOREM 3. *In an L_p space of dimension $n>1$ $(p\geqq 1)$, the weak parallelogram law holds iff $1<p\leqq 2$.*

Proof. The necessity of the condition is obviously a direct consequence of Theorem 2. Hence, suppose V is an L_p space with $1<p\leqq 2$, and let $x=(x_1, x_2, \cdots, x_n)$ and $y=(y_1, y_2, \cdots, y_n)$ be any two points of V. We must then prove for some positive $\gamma_n<1$ independent of x and y, that

$$\Big(\sum |x_k+y_k|^p\Big)^{2/p}+\gamma_n^2\Big(\sum |x_k-y_k|^p\Big)^{2/p} \leqq 2\Big(\sum |x_k|^p\Big)^{2/p}+2\Big(\sum |y_k|^p\Big)^{2/p}, \tag{9}$$

where for convenience $\sum$ denotes summation on k from 1 to n. Since (9) is trivial for $x=y$ we may assume that for at least one k, $x_k\neq y_k$. Therefore, but for a reordering of the subscripts, $|x_n-y_n|$ is maximal among $|x_k-y_k|$, $k=1, 2, \cdots, n$, and is positive. The validity of the weak parallelogram law for two dimensions enables us to produce a positive $\gamma<1$ such that for any four given real numbers z_1, z_2, w_1, w_2,

$$(|z_1+w_1|^p+|z_2+w_2|^p)^{2/p}+\gamma^2(|z_1-w_1|^p+|z_2-w_2|^p)^{2/p} \leqq 2(|z_1|^p+|z_2|^p)^{2/p}+2(|w_1|^p+|w_2|^p)^{2/p}. \tag{10}$$

The triangle inequality for dimension $n-1$ applied to the points $(x_1, x_2, \cdots, x_{n-1})$ and $(y_1, y_2, \cdots, y_{n-1})$ yields

$$(11) \qquad (\sum' \mid x_k + y_k \mid^p)^{1/p} \leqq (\sum' \mid x_k \mid^p)^{1/p} + (\sum' \mid y_k \mid^p)^{1/p}$$

where $\sum'$ indicates summation on k from 1 to $n-1$. Therefore,

$$\begin{aligned} 2(\sum \mid x_k \mid^p)^{2/p} + 2(\sum \mid y_k \mid^p)^{2/p} - (\sum \mid x_k + y_k \mid^p)^{2/p} \\ (12) \qquad \geqq 2(\sum \mid x_k \mid^p)^{2/p} + 2(\sum \mid y_k \mid^p)^{2/p} \\ - (\mid (\sum' \mid x_k \mid^p)^{1/p} + (\sum' \mid y_k \mid^p)^{1/p} \mid^p + \mid x_n + y_n \mid^p)^{2/p}. \end{aligned}$$

Now set $z_1 = (\sum' \mid x_k \mid^p)^{1/p}$, $z_2 = x_n$, $w_1 = (\sum' \mid y_k \mid^p)^{1/p}$, $w_2 = y_n$, and apply (10). The right-hand member of (12) is not less than

$$\gamma^2(\mid (\sum' \mid x_k \mid^p)^{1/p} - (\sum' \mid y_k \mid^p)^{1/p} \mid^p + \mid x_n - y_n \mid^p)^{2/p} \geqq \gamma^2(\mid x_n - y_n \mid^p)^{2/p}.$$

But from our choice of $\mid x_n - y_n \mid$ we have $n \mid x_n - y_n \mid^p \geqq \sum \mid x_k - y_k \mid^p$. Thus,

$$\begin{aligned} 2(\sum \mid x_k \mid^p)^{2/p} + 2(\sum \mid y_k \mid^p)^{2/p} - (\sum \mid x_k + y_k \mid^p)^{2/p} \\ \geqq \frac{\gamma^2}{n^{2/p}} (\sum \mid x_k - y_k \mid^p)^{2/p} > \frac{\gamma^2}{n^2} (\sum \mid x_k - y_k \mid^p)^{2/p} \end{aligned}$$

and hence, (9) holds with $\gamma_n = \gamma/n > 0$, which completes the proof.

The fact that $\gamma_n \to 0$ as $n \to \infty$ means that we cannot extend our proof to the infinite-dimensional case. It is not known whether an infinite-dimensional space, besides Hilbert space, possesses a weak parallelogram law. Such a space, if it exists, must of necessity be uniformly convex, an immediate consequence of (1) restricted to points on the unit sphere. (See [1] for a discussion of uniformly convex spaces and certain inequalities resembling the parallelogram law, valid for L_p spaces of arbitrary dimension.) The question remains whether, in particular, an infinite-dimensional L_p space possesses a weak parallelogram law.

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FINITE ADDITIVITY, INVERTIBILITY, AND SIMPLE EXTENSIONS OF TOPOLOGIES

DAVID RYEBURN, Kenyon College

1. Introduction. Doyle and Hocking [2] defined a topological space X to be *invertible* if, for every nonempty open subset O of X, there is a homeomorphism $f:X\to X$ such that $f(X-O)\subset O$. Then f is said to be an *inverting homeomorphism* for O. Doyle and Hocking [2] proved that if an invertible space X possesses a nonempty open subspace which is a T_i-space ($i=0$, 1, or 2) or whose closure is compact, then X has the property in question. Levine [6] proved that if an invertible space X possesses a nonempty open subspace which is separable, satisfies the first or second axiom of countability, is regular, or is normal, then X has the property in question. Gray [3] proved that if an invertible space X possesses a nonempty open subspace which is metrizable, then X is metrizable. In this paper it will be noted that certain topological properties are possessed by spaces having finite open or closed coverings the members of which possess the properties, and some of the results of Doyle and Hocking [2], Levine [6], and Gray [3] will follow from these facts as will similar results for certain other topological properties. In addition, invertibility can be defined analogously for uniform spaces, and again the whole space will possess certain properties holding for subspaces.

In another paper, Levine [5] defined one topology, $\mathfrak{T}^*$, to be a *simple extension* of another topology, $\mathfrak{T}$, on the same set X, in case $\mathfrak{T}^*=\{U\cup(V\cap A): U, V\in\mathfrak{T}\}$ for some $A\notin\mathfrak{T}$. He investigated the question of whether $(X, \mathfrak{T}^*)$ has certain properties possessed by $(X, \mathfrak{T})$; the properties included regularity, complete regularity, the Tychonoff property, and normality. Some of the results, and others like them, may be obtained by using methods similar to those used to investigate invertibility.

2. Summary of finite additivity results. A topological property P is said to be *finitely o-additive* in case whenever $X=A\cup B$ with A, B open subspaces of X possessing property P, X must also possess property P. Similar definitions may be given for *finitely c-additive*, where the sets A and B are closed, and for *finitely s-additive*, where the sets A and B are complementary clopen sets (constituting a separation of X).

It is immediate that the T_0 property, the T_1 property, metacompactness (every open covering has a point finite open refinement), pseudocompactness (every real-valued continuous function is bounded), local compactness, σ-compactness, the first countability property, the second countability property, separability, the Lindelöf property, local connectedness, and the countable chain property (every disjoint class of open sets is countable) are finitely o-additive. It is likewise immediate that regularity, the T_3 property, the property of being an $\aleph_0$-space (a regular space with a countable class $\mathcal{S}$ of subsets such that if C is a compact subset and U is an open subset with $C\subset U$, then $C\subset S\subset U$ for some $S\in\mathcal{S}$) [7], compactness, countable compactness, sequential compact-

ness, pseudocompactness, separability, the Lindelöf property, and the countable chain property are finitely c-additive. Mrówka [9] proved that normality is finitely c-additive, and thus the T_4 property, hereditary normality, and the T_5 property are finitely c-additive. Hence it is immediate that perfect normality and the T_6 property are finitely c-additive. Michael [8] proved that paracompactness is finitely c-additive, and thus hereditary paracompactness is also finitely c-additive. De Groot [1] and Nagata [10] proved that metrizability is finitely c-additive. It is immediate that complete regularity, the Tychonoff property, collectionwise normality (for every discrete family $\{F_i: i\in I\}$ of closed subsets there is a disjoint family $\{U_i: i\in I\}$ of open subsets with $F_i\subset U_i$ for each $i\in I$), and strong paracompactness (every open covering has a star-finite open refinement) are finitely s-additive.

3. Invertibility and topological properties.

THEOREM 1. *Let X be a topological space invertible in one of its nonempty open subsets, O, and suppose that O enjoys a finitely o-additive topological property P. Then X enjoys P.*

Proof. Let f be an inverting homeomorphism for O. Then $f(O)$ is open and $X=O\cup f(O)$. Since O and $f(O)$ enjoy P, and P is finitely o-additive, X enjoys P.

THEOREM 2. *Let X be a topological space invertible in one of its nonempty open subsets, O, and suppose that $Cl(O)$ enjoys a finitely c-additive topological property P. Then X enjoys P.*

Proof. Let f be an inverting homeomorphism for O. Then $f(\mathrm{Cl}(O))$ is closed and $X=\mathrm{Cl}(O)\cup f(\mathrm{Cl}(O))$. Since $\mathrm{Cl}(O)$ and $f(\mathrm{Cl}(O))$ enjoy P and P is finitely c-additive, X enjoys P.

THEOREM 3. *Let X be a topological space and O a nonempty open subset enjoying a c-hereditary, finitely c-additive property P. If X is invertible and if P implies normality then X enjoys property P.*

Proof. Let f be an inverting homeomorphism for O. Then $X=O\cup f(O)$, and both O and $f(O)$ are open in X. X is invertible and O is normal, hence X is normal [6]. Thus there are closed sets F_1 and F_2 with $F_1\subset O$, $F_2\subset f(O)$, and $X=F_1\cup F_2$. Since F_1 and F_2 enjoy P and P is finitely c-additive, X must enjoy P.

Theorems 1, 2, and 3 and the additivity results of Section 2 immediately yield the results referred to in Section 1.

4. Invertibility in uniform spaces. If X is a uniform space and Y is a nonempty subset of X, then X is said to be uniformly invertible in Y in case there is a uniform isomorphism $f: X\to X$ for which $f(X-Y)\subset Y$.

THEOREM 4. *Let $(X, \mathfrak{U})$ be a uniform space having a nonempty subset Y for which $(Y, \mathfrak{U}\cap(Y\times Y))$ has a countable uniformity base and in which there is an inverting uniform isomorphism f such that $(Y\times Y)\cup(f(Y)\times f(Y))\in\mathfrak{U}$. Then $(X, \mathfrak{U})$ has a countable uniformity base.*

Proof. Let $\mathfrak{B}$ be a countable uniformity base for $(Y, \mathfrak{U}\cap(Y\times Y))$. Then for each $B\in\mathfrak{B}$ we have $B=W\cap(Y\times Y)$ for suitable $W\in\mathfrak{U}$. Since f is a uniform isomorphism, $(f\times f)(W)\in\mathfrak{U}$. By hypothesis $(Y\times Y)\cup(f(Y)\times f(Y))\in\mathfrak{U}$. Thus

$$W^{\#} = [W \cap (f\times f)(W)] \cap [(Y\times Y)\cup(f(Y)\times f(Y))] \in \mathfrak{U}.$$

But $W^{\#}\subset[W\cap(Y\times Y)]\cup[(f\times f)(W)\cap(f(Y)\times f(Y))]=B\cup(f\times f)(B)$. Thus for each $B\in\mathfrak{B}$ we have $B\cup(f\times f)(B)\in\mathfrak{U}$. Now let $U\in\mathfrak{U}$. Then $(f^{-1}\times f^{-1})(U)\in\mathfrak{U}$ and thus for some $B\in\mathfrak{B}$ we have $B\subset U\cap(f^{-1}\times f^{-1})(U)\cap(Y\times Y)$. Hence $B\subset U$, $(f\times f)(B)\subset U$, and $B\cup(f\times f)(B)\subset U$. Thus $\{B\cup(f\times f)(B): B\in\mathfrak{B}\}$ is a countable uniformity base for X.

LEMMA. *Let $(X, \mathfrak{U})$ be a uniform space uniformly invertible in every nonempty open subset, and let O be a nonempty open subset such that $(O, \mathfrak{U}\cap(O\times O))$ is a Hausdorff space. Then $(X, \mathfrak{U})$ is a Hausdorff space.*

Proof. Let $x\neq y$, $x\in X$, $y\in X$. First suppose that $x\in O$ and $y\in O$. Then for some $V\in\mathfrak{U}\cap(O\times O)$ we have $(x, y)\notin V$. If $V=W\cap(O\times O)$, $W\in\mathfrak{U}$, then $(x, y)\notin W$. Now suppose that $y\notin O$. Then $O-\{x\}$ is a Hausdorff space and is open in X, and by inverting in $O-\{x\}$ we return to the case already considered.

Using Theorem 4 and the lemma we obtain

THEOREM 5. *Let $(X, \mathfrak{U})$ be a uniform space uniformly invertible in every nonempty open subset, and let O be a nonempty open subset for which $(O, \mathfrak{U}\cap(O\times O))$ is metrizable and in which there is an inverting uniform isomorphism f such that $(O\times O)\cup(f(O)\times(f(O))\in\mathfrak{U}$. Then $(X, \mathfrak{U})$ is metrizable.*

The following example shows that the condition $(O\times O)\cup(f(O)\times f(O))\in\mathfrak{U}$ cannot be omitted. Let I be an uncountable set, S a metric space, $X=S\times I\times\{1\}$, $Y=S\times I\times\{2\}$, $Z=X\cup Y$. For each cofinite subset J of I and each natural number n let

$$A(n) = \bigcup_{i\in I}\{((s, i, 1), (t, i, 1)): s, t\in S \text{ and } d(s, t) < 1/n\},$$

$$B(n) = \bigcup_{i\in I}\{((s, i, 2), (t, i, 2)): s, t\in S \text{ and } d(s, t) < 1/n\},$$

$$C(J, n) = \bigcup_{j\in J}\{((s, j, 1), (t, j, 2)): s, t\in S \text{ and } d(s, t) < 1/n\},$$

$$D(J, n) = \bigcup_{j\in J}\{((s, j, 2), (t, j, 1)): s, t\in S \text{ and } d(s, t) < 1/n\},$$

and let $E(J, n)=A(n)\cup B(n)\cup C(J, n)\cup D(J, n)$. Then the class $\mathfrak{E}$ of all sets of the form $E(J, n)$ is a base for some uniformity $\mathfrak{U}$. Now suppose that $\mathfrak{B}=\{B_k: k=1, 2, \cdots\}$ is a countable base for $\mathfrak{U}$. For each k there is a cofinite set $J(k)$ and there is a natural number $n(k)$ such that $E(J(k), n(k))\subset B_k$. Thus the sets $E(J(k), n(k))$ form a countable base also. Note that $\bigcap_{k=1}^{\infty} J(k)$ is cocountable, while I is uncountable. If L is a cofinite subset of I with $L\supset\bigcap_{k=1}^{\infty} J(k)$ then for some k we should have $E(J(k), n(k))\subset E(L, 1)$, but this contradicts the

choice of L. Thus no such countable base exists. Yet $\{A(n): n=1, 2, \cdots\}$ and $\{B(n): n=1, 2, \cdots\}$ are countable bases for the subspaces X and Y respectively. There is a natural inversion interchanging X and Y, and thus by Theorem 5 the set $(X\times X)\cup(Y\times Y)$ must not be a member of $\mathfrak{U}$. Note that this example shows that the condition $(O\times O)\cup(f(O)\times f(O))\in\mathfrak{U}$ cannot be replaced by the condition that O be a clopen set. Note further that although the uniform space $(Z, \mathfrak{U})$ is not metrizable, the associated topological space is metrizable, since metrizability for topological spaces is finitely c-additive. The difficulty lies in the fact that X and Y come near each other for the uniformity, although they are complementary clopen sets.

THEOREM 6. *Let $(X, \mathfrak{U})$ be a uniform space uniformly invertible in a nonempty subset Y such that $(Y, \mathfrak{U}\cap(Y\times Y))$ is totally bounded. Then $(X, \mathfrak{U})$ is totally bounded.*

Proof. Let f be an inverting uniform isomorphism for Y. If $U\in\mathfrak{U}$ then $(f^{-1}\times f^{-1})(U)\in\mathfrak{U}$ and thus $U\cap(f^{-1}\times f^{-1})(U)\cap(Y\times Y)\in\mathfrak{U}\cap(Y\times Y)$. Hence, since $(Y, \mathfrak{U}\cap(Y\times Y))$ is totally bounded, there is a finite subset F of Y with

$$[U\cap(f^{-1}\times f^{-1})(U)\cap(Y\times Y)](F) = Y.$$

Thus $Y\subset U(F)$, $f(Y)\subset U(f(F))$, and $U(F\cup f(F))=X$. Hence $(X, \mathfrak{U})$ is totally bounded.

THEOREM 7. *Let $(X, \mathfrak{U})$ be a uniform space uniformly invertible in a nonempty subset Y such that $(Y, \mathfrak{U}\cap(Y\times Y))$ is complete. Then $(X, \mathfrak{U})$ is complete.*

Proof. Let $S: D\to X$ be a Cauchy net in X. Then either (i) S is frequently in Y, or (ii) S is eventually in $X-Y$. In case (i), let $D^*=S^{-1}(Y)$ and let $i: D^*\to D$ be the inclusion mapping. Then $S\circ i$ is a subnet of S and hence a Cauchy net in X. $S\circ i$ is still a Cauchy net when regarded as a net in Y, and hence converges (to a point $y\in Y$) by completeness of Y. Thus S has y as a cluster point, and since S is a Cauchy net S must converge to y. In case (ii), let $D^\#=S^{-1}(X-Y)$ and let $j: D^\#\to D$ be the inclusion mapping. If f is an inverting uniform isomorphism for Y, then $f\circ S\circ j$ converges to a point $y'\in Y$, and S converges to $f^{-1}(y')$.

5. Additivity and simple extensions. Throughout the remainder of this paper $(X, \mathfrak{T})$ will be a topological space, A will be a subset of X not a member of $\mathfrak{T}$, and $\mathfrak{T}^*=\{U\cup(V\cap A): U, V\in\mathfrak{T}\}$ will be a simple extension of $\mathfrak{T}$. Levine [5] observed that $(A, \mathfrak{T}^*\cap A)=(A, \mathfrak{T}\cap A)$ and $(X-A, \mathfrak{T}^*\cap(X-A))=(X-A, \mathfrak{T}\cap(X-A))$.

THEOREM 8. *If P is a finitely s-additive, c-hereditary topological property, $(X, \mathfrak{T})$ has P, and A is a $\mathfrak{T}$-closed subset of X, then $(X, \mathfrak{T}^*)$ has P iff $(X-A, \mathfrak{T}\cap(X-A))$ has P.*

Proof. Since $(X, \mathfrak{T})$ has P, A is $\mathfrak{T}$-closed, and P is c-hereditary, $(A, \mathfrak{T}\cap A)$ has P. But $(A, \mathfrak{T}^*\cap A)=(A, \mathfrak{T}\cap A)$, and A is $\mathfrak{T}^*$-clopen. If $(X-A, \mathfrak{T}^*\cap(X-A))=(X-A, \mathfrak{T}\cap(X-A))$ has P, then since $X-A$ is also $\mathfrak{T}^*$-clopen and P is

finitely s-additive, $(X, \mathfrak{T}^*)$ has P. Conversely, if $(X, \mathfrak{T}^*)$ has P then $X-A$ is $\mathfrak{T}^*$-closed and thus $(X-A, \mathfrak{T}^* \cap (X-A)) = (X-A, \mathfrak{T} \cap (X-A))$ has the c-hereditary property P.

Theorem 8 may be applied if $(X, \mathfrak{T})$ is normal, a T_4-space, paracompact, strongly paracompact, collectionwise normal, metacompact, compact, countably compact, sequentially compact, locally compact, σ-compact, or a Lindelöf space.

THEOREM 9. *If P is a finitely s-additive, hereditary topological property, $(X, \mathfrak{T})$ has P, and A is a $\mathfrak{T}$-closed subset of X, then $(X, \mathfrak{T}^*)$ has P.*

Proof. Since P is hereditary, $(X-A, \mathfrak{T} \cap (X-A))$ has P. By Theorem 8, $(X, \mathfrak{T}^*)$ has P.

Theorem 9 may be applied if $(X, \mathfrak{T})$ is regular, a T_3-space, completely regular, a Tychonoff space, hereditarily normal, a T_5-space, perfectly normal, a T_6-space, hereditarily paracompact, an $\aleph_0$-space, metrizable, first countable, or second countable.

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GROUPS, GRAPHS, AND FERMAT'S LAST THEOREM

STEVEN BRYANT, San Diego State College

If we assign a point $P(x)$ in space to every element x of a group and join $P(x)$ with $P(x^2)$ by a directed line, we obtain a graph here called the graph of the group. We will show that any Abelian group with fewer than $(1093)^2$ elements is determined up to isomorphism by its graph, i.e., two such groups with isomorphic graphs are isomorphic. A stronger statement can be made: If the square of every element is known then the group is known up to isomorphism. This says that if the diagonal of the group table is given, then the isomorphism type of the group can be computed.

The prime 1093 is the first prime which satisfies $2^{p-1} \equiv 1 \pmod{p^2}$. Such primes are related to Fermat's Theorem, i.e., if no one of x, y, z are divisible by p and $2^{p-1} \not\equiv 1 \bmod p^2$ then $x^p + y^p \neq z^p$, [1].

For convenience, call p a *bad prime* if $2^{p-1} \equiv 1 \bmod p^2$. Now we can state the theorem.

THEOREM. *If G and H are Abelian groups of order n, and n is not divisible by the square of a bad prime, and there is a square preserving function on G onto H, then G and H are isomorphic. Furthermore, the isomorphism type of G can be computed if the square of every element of G is given.* (In the sense described above, i.e., if the main diagonal of the multiplication table of the group is known.)

Proof. The main part of the argument rests on the fact that under the hypotheses, the graph of the group G consists essentially of cycles and the number of elements in various size cycles determines the number of elements in the group G of each prime power order and this in turn determines G up to isomorphism.

CASE 1. n odd and n not only not divisible by the square of a bad prime but also not divisible by a bad prime, i.e., $n \not\equiv 0 \bmod p$ for any bad prime p. Then $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ with p_i odd and p_i not a bad prime for $i = 1, \cdots, m$.

So G is a direct product of groups $G_1, \cdots, G_m$, where $G_i = \{x: x \in G$ and the order of x is a power of $p_i\}$. Then, if x is an element of G, x is a unique product $x_1 \cdots x_m$ with x_i in G_i for $i = 1, \cdots, m$.

Now let k_i be the order of 2 mod p_i, i.e., $2^{k_i} \equiv 1 \bmod p_i$ and k_i is the smallest positive integer for which this holds. We need the following:

(1) *If p is not a bad prime and k is the order of* 2 mod p *then the order of* 2 mod p^α *is* $kp^{\alpha-1}$.

Proof. If $2^t \equiv 1 \bmod p^\alpha$ then $2^t \equiv 1 \bmod p^{\alpha-1}$. By induction $kp^{\alpha-2}$ divides t so $t = akp^{\alpha-2}$. If $a < p$ then $(a, p) = 1$ and $(2^k)^{ap^{\alpha-2}} \equiv 1 \bmod p^\alpha$. But $2^k = 1 + bp$ with $(b, p) = 1$ so $(1+bp)^{ap^{\alpha-2}} = 1 + cp^\alpha$. Expand the left side and cancel the ones, then divide by $p^{\alpha-1}$ this yields $ab \equiv 0 \bmod p$, a contradiction so $t \geqq kp^{\alpha-1}$. Finally $2^{kp^{\alpha-2}} \equiv 1 \bmod p^{\alpha-1}$ implies $2^{kp^{\alpha-1}} \equiv 1 \bmod p^\alpha$ by an easy argument, so $kp^{\alpha-1}$ is the order of 2 mod p^α.

(2) *Call the set $x, x^2, x^4, x^8, x^{16}, \cdots$ etc., the cycle to which x belongs and call the number of elements in this set the cycle order of x. If the (group) order of x is a power of a prime p then the (group) order of x is p^{α_0} if and only if the cycle order of x is kp^{α_0-1}.*

Proof. Let p^{α_0} be the order of x then $x^{p^{\alpha_0}} = 1$ with α_0 minimal. Let t be the cycle order of x. Then $x^{2^t} = x$ with t minimal so $2^t \equiv 1 \bmod p^{\alpha_0}$ and t is the order of 2 mod p^{α_0} so $t = kp^{\alpha_0-1}$. (This follows from (1).) Conversely, if the cycle order of x is $kp^{\alpha-1}$ and the order of x is p^{α_0}, let $t = kp^{\alpha-1}$; then $x^{2^t} = x$ with t minimal so $2^t \equiv 1 \bmod p^{\alpha_0}$ with t minimal hence $t = kp^{\alpha_0-1}$ so $\alpha = \alpha_0$.

Now let k_i be the order of 2 mod p_i. From (1) and (2) we have:

(3) *If x_i is an element of G_i then x_i has order p_i^α if and only if x_i has cycle order $k_i p_i^{\alpha-1}$.*

We also need to observe,

(4) *If the order of x is $p_i^{\beta_1} \cdots p_m^{\beta_m}$ then the cycle order of x is*

$$[k_1 p_1^{\beta_1 - 1}, \cdots, k_m p_m^{\beta_m - 1}]$$

i.e., the least common multiple of $k_1 p_1^{\beta_1 - 1}, \cdots, k_m p_m^{\beta_m - 1}$.

If the square of every element of G is given then the cycles of G are given and for each cycle order the number of elements which have that cycle order is given. What we have to see is that this determines the number of elements of G of prime power order for each prime divisor of n, the order of G. This will determine the primary subgroups of G up to isomorphism and hence determine G up to isomorphism.

Let $N_G(k_i p_i^\alpha)$ be the number of elements of G which have cycle order a divisor of $k_i p_i^\alpha$. The numbers $N_G(k_i p_i^\alpha)$ are known for $i = 1, \cdots, m$ and every α. We want to see that the isomorphism type is known, i.e., can be computed, if these numbers are known. The proof is by induction on m.

If $m = 1$ then $G = G_1$ and $N_G(k_1) - 1$ is the number of elements of order p_1 in G, $N_G(k_1 p_1) - N_G(k_1)$ is the number of elements of order p_1^2 in G_1 etc. So in this case G is known since $N_G(k_1 p_1^\alpha)$ is known for every α.

Now let the primes be labeled so that $p_1 > p_i$ for $i \neq 1$. Then $p_1 > p_i > k_i$ for $i \neq 1$ because k_i is the order of 2 mod p_i and hence $k_i \leqq p_i - 1$. Let p_1^γ be the maximal order of elements of G_1, so $y^\sigma = 1$ (where $\sigma = p_1^\gamma$) for every y in G_1 and γ is the smallest integer for which this holds. We continue by induction on γ, and for this we need:

(5) *If $x = x_1 \cdots x_m$ and the cycle order of x is $k_i p_i^\alpha$ with $i \neq 1$ then the cycle order of x_1 is either 1 or k_1.*

Proof. If the cycle order of x_1 is $k_1 p_1^\delta$ with $\delta \geqq 1$, then using (4), we see that p_1 divides k_i but $p_1 > k_i$.

(6) *If $\beta \geqq 1$ and $x = x_1 \cdots x_m$ and the cycle order of x is $k_1 p_1^\beta$ then the cycle order of x_1 is $k_1 p_1^\beta$ for if the cycle order of x_1 is $k_1 p_1^\delta$ with $\delta < \beta$ then using (4) we see that p_1 divides k_j for some $j \neq 1$ but this is impossible since $p_1 > k_j$ for every j. If $\delta > \beta$ then p_1 divides k_i which is also impossible since $k_1 \leqq p_1 - 1$.*

Let $N_G^*(p_1^\beta)$ be the number of elements in G_1 of order a divisor of p_1^β and let $\delta(i, \alpha) = N_G^*(p_1)$ or 1 depending on whether k_1 divides $k_i p_i^\alpha$. Then using (5) and (6) we have:

(7) *For $i \neq 1$ $N_G(k_i p_i^\alpha) = \delta(i, \alpha) N_{G/G_1}(k_i p_i^\alpha)$ for every $\alpha \geqq 0$.*

(8) *$N_G(k_1 p_1^{\alpha-1}) = N_{G_1}^*(p_1^\alpha) N_{G/G_1}(k_1 p_1^{\alpha-1})$ for every $\alpha \geqq 1$.*

Now if $\gamma = 1$, (8) consists of one equation: $N_G(k_1) = N_{G_1}^*(p_1) N_{G/G_1}(k_1)$ and since n the order of G is known, $N_{G_1}^*(p_1)$ is known. (In fact, $N_{G_1}^*(p_1) = p_1^{\alpha_1}$ where $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$). Then for $i \neq 1$, $N_G(k_i p_i^\alpha)/\delta(i, \alpha)$ is known for every $\alpha \geqq 0$, hence

G/G_1 is known by inductive assumption. But G_1 is also known, i.e., G_1 is the product of α_1 cyclic groups of order p_1. Finally then G is known. (This means that we could write down equations containing $N^*_{G_j}(p_j^\alpha)$ as unknowns and solve these equations and that the solution is unique.)

To conclude the induction on γ, let $G^{p_1} = \{y: y \in G \text{ and } y = x^{p_1} \text{ for some } x \text{ in } G\}$. Then for $\alpha < \gamma$,

$$N_G(k_1 p_1^{\alpha-1}) = N_{G^{p_1}}(k_1 p_1^{\alpha-1}).$$

To see this observe that $G^{p_1} = G_1^{p_1} \times G_2 \times \cdots \times G_m$ and for $\alpha < \gamma$ any element in G_1 of order a divisor of p_1^α is in $G_1^{p_1}$ and hence any element in G_1 of cycle order a divisor of $k_1 p_1^{\alpha-1}$ is in $G_1^{p_1}$.

We also have for $i \neq 1$ and every $\alpha \geqq 0$,

$$N_G(k_i p_i^\alpha) = N_{G^{p_1}}(k_i p_i^\alpha).$$

To see this let x be an element of G of cycle order $k_i p_i^\alpha$, $i \neq 1$, then $x = x_1 \cdots x_m$ with x_1 in $G_1^{p_1}$ (this holds because of (5)), and x_j in G_j for $j \neq 1$. From this we have using (7) and (8).

(9) *For* $i \neq 1$, $N_G(k_i p_i^\alpha) = N_{G^{p_1}}(k_i p_i^\alpha) = \delta(i, \alpha) N_{G/G_1}(k_i p_i^\alpha)$ *and* $\alpha \geqq 0$.
(10) *For* $1 \leqq \alpha < \gamma$, $N_G(k_1 p_1^{\alpha-1}) = N_{G^{p_1}}(k_1 p_1^{\alpha-1}) = N^*_{G_1}(p_1^\alpha) N_{G/G_1}(k_1 p_1^{\alpha-1})$.
(11) $N_G(k_1 p_1^{\gamma-1}) = N^*_{G_1}(p_1^\gamma) N_{G/G_1}(k_1 p_1^{\gamma-1})$.

Using (9), (10), and (11) we have the following argument: The maximal order of elements in $G_1^{p_1}$ is $p_1^{\gamma-1}$ and $N_{G^{p_1}}(k_i p_i^\alpha)$ is known for $i \neq 1$ and $\alpha \geqq 0$. Also $N_{G^{p_1}}(k_1 p_1^{\alpha-1})$ is known for $\alpha < \gamma$. Hence $G_1^{p_1}$ is known by inductive assumption. But $G^{p_1} = G_1^{p_1} \times G_2 \times \cdots G_m$ so G/G_1 is known, therefore $N^*_G(p_1^\gamma)$ is known (because $N_G(k_1 p_1^{\gamma-1})$ is known). But then $N^*_{G_1}(p_1^\alpha)$ is known for every $\alpha \geqq 0$ so G_1 is known. Finally then G is known since G_1 and G/G_1 are known.

CASE 2. If n is odd and divisible by one or more bad primes but still not by the square of a bad prime, the analysis is essentially the same. For any bad prime p, in this case, the primary subgroup G_p is cyclic. The entire argument goes through because every element in G_p has order p (except for the identity) and the order of an element in G_p is p if and only if its cycle order is the order of 2 mod p.

CASE 3. If n is even let G_2 be the primary subgroup consisting of all elements whose order is a power of 2. The graph of G will contain trees as well as cycles but the trees in the graph determine the primary subgroup G_2 and in fact the isomorphism type of G_2 can be computed from these trees. The cycles which do not have trees sticking on them are the graph of G/G_2 and G/G_2 satisfies the hypothesis of Case 2, so then G/G_2 is known as well as G_2, hence G is known.

Remarks. The cyclic group of order $(1093)^2$ and the product of two cyclic groups of order 1093 have the same graph. To see this let p be any bad prime so $2^{p-1} \equiv 1 \bmod p^2$ and let k be the order of 2 mod p and let k' be the order of 2 mod p^2. Then $2^k \equiv 1 \bmod p$ and $2^{k'} \equiv 1 \bmod p^2$. From this we have $2^{kp} \equiv 1 \bmod p^2$

and $2^{k'} \equiv 1 \bmod p$, hence k' divides kp and k divides k' so $k' = k$ or $k' = kp$ but k' divides $p-1$ so $k' = kp$ is impossible and we have $k' = k'$.

In particular since 1093 is a bad prime, every element in the cyclic group of order $(1093)^2$ has the same cycle order. In the product of two cyclic groups of order 1093 every element has the same order, so these two groups have isomorphic graphs.

It is not known whether there are infinitely many bad primes or, for that matter, whether there are infinitely many primes which are not bad.

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BING'S HOUSE WITH TWO ROOMS FROM A 1-1 CONTINUOUS MAP ONTO E^3

L. C. GLASER, Rice University

In [5], Proizvolov claimed to prove that if f is a 1-1 continuous map taking X onto E^n, where X is connected, locally compact and paracompact, and E^n is Euclidean n-space, then f is a homeomorphism. If all his results were correct one would have as an immediate corollary a difficult and well-known problem in point set topology. That is, if f is a monotone map of E^n onto E^n then is f a compact map? (A map γ taking X onto Y is monotone if counter images of points in Y are continua. A map f is compact if for each compact set $A \subset Y$, $f^{-1}(A)$ is compact.)

In [7], Kenneth Whyburn gave a particularly nice counterexample to Proizvolov's result by demonstrating a simple 1-1 continuous map taking a connected 3-manifold A with nonempty boundary onto E^3. The result obtained in [7] was first announced in [6]. An extensive generalization of [6, 7] was announced before either of the above in [2]. Earlier yet, in [1], numerous different examples were given showing a result similar to that of [7] (i.e., the domain spaces here were also 3-manifolds with boundary and the 1-1 continuous maps piecewise linear). In fact, the first counterexample to [5] was shown to the author by John Cobb in a class each of us had at the University of Wisconsin in 1963. His domain space was not a 3-manifold with boundary, in fact, it was not simply connected. Cobb's example also appears in [1].

It also was shown in [1] that the 3-ball, I^3, and the 3-sphere S^3, are the images of 1-1 continuous maps on noncompact 3-manifolds with nonempty boundary. These gave counterexamples to two other theorems of [5]. In [4] it is shown that under certain general conditions if f is any 1-1 continuous map taking M^n, an n-manifold with nonempty boundary onto E^n, then the 1-point compactification of $f(\text{Bd } M^n)$ is a contractible noncollapsible $(n-1)$-subcomplex of

S^n. In particular the 1-point compactification of $f(\text{Bd } M)$ in [4] is merely the Dunce Hat discussed in [8]. In [3], it is shown that E^n is in fact the 1-1 continuous image of E^n_+, the closed half space of E^n given by $\{(x_1, \cdots, x_n) \in E^n \mid x_n \geqq 0\}$.

The purpose of this note is to show, using only elementary techniques, that in addition to giving another example of such a map it is possible to construct the map so as to get another well-known contractible noncollapsible 2-complex as the end result.

THEOREM. *There exists a 3-manifold M with nonempty boundary and a 1-1 continuous map f taking M onto E^3 such that the 1-point compactification of $f(\text{Bd } M)$ is the contractible noncollapsible 2-complex R, known as Bing's house with two rooms.*

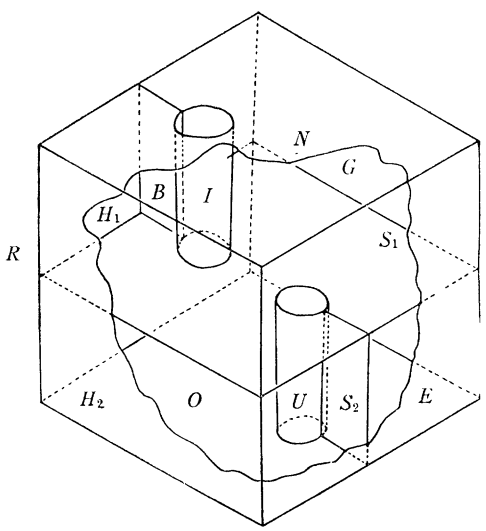
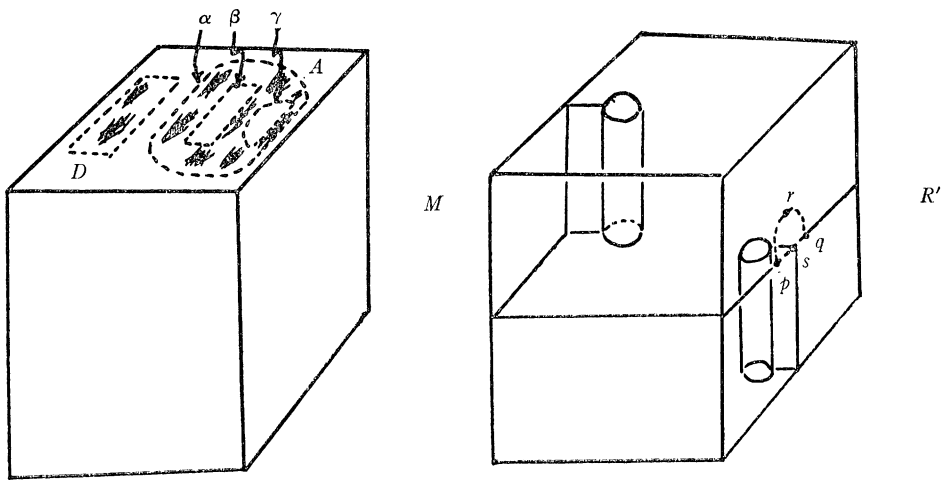


FIG. 1

The contractible noncollapsible 2-complex R, due to R. H. Bing, is most easily described by showing a picture of this 2-polyhedron. This is done in Figure 1. We see in Figure 1, that N is a square disk with an open circular disk removed, S_1 is a square disk with two open circular disks removed, O is a square disk with an open circular disk removed, H_1, H_2, E and G are square disks, I and U are right circular cylinders, and B and S_2 are rectangular disks.

The 3-manifold M with nonempty boundary that we map 1-1 continuously onto an open subset V of E^3 homeomorphic to E^3 will be made up of the interior of the unit 3-cell plus an open disk D and an open two-holed annulus A on the top face of this 3-cell. The 1-1 continuous map f can be most easily described by indicating how we could deform M topologically in E^3 so as to obtain a set V which is topologically E^3.



(i) FIG. 2 (ii)

M is indicated in Figure 2 (i). By first pushing the annulus A down slightly and then pulling up a neighborhood of part of A and expanding, and also pulling up and deforming a neighborhood of D we can deform M into M' as indicated in Figure 3 (the illustration of Figure 3 was suggested to the author by the referee; although it is slightly more complicated than the original version, it does suggest the final deformation more clearly).

We now push the two rectangular disks $abcd$ and $a'b'c'd'$ together so as to form a circular tunnel in the right top part of M'. Then the right top part of M' is pushed down so that the rectangle $efgh$ is matched up with the outer boundary α of the open annulus A (α is also indicated by $e'f'g'h'$). This leaves the bottom section hollow except for a solid circular tube joining the top and bottom faces of this hollow part. We will use the circular tunnel of the right top section to enter this hollow part with the neighborhood T of D. The outer left portion of M' is then pushed against the part of A bounding the right top section of M' so that this section is completely covered with $p'r'q'$ (indicated by the heavy lines) matched up with prq and so that T can be inserted into the circular tunnel and the hollow part formed above. Finally T is expanded so as to fill this part. During this expansion D is pulled around the solid tube so that $u's'v'w'$ matches $usvw$.

We now have the set V which is just a deformed solid open ball which is homeomorphic to E^3. The boundary of $M=D\cup A$ now lies in V as is shown in Figure 2 (ii). This is exactly the 2-polyhedron R except for a closed semi-circular disk $psqr$ removed. The 1-point compactification of $f(\text{Bd } M)$ then closes this part up and we get precisely R.

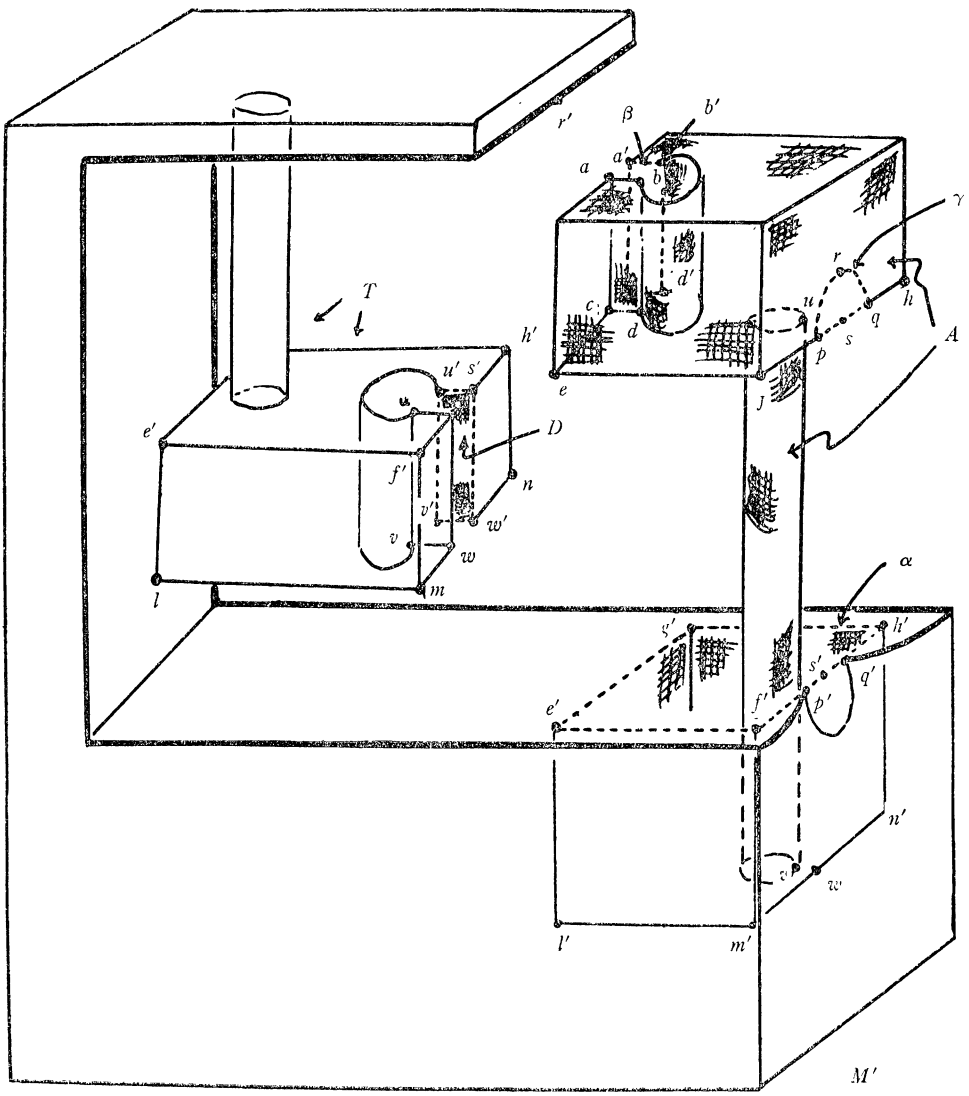


FIG. 3

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MATHEMATICAL NOTES

THE DIRECT PRODUCT OF RIGHT ZERO SEMIGROUPS AND CERTAIN GROUPOIDS

R. J. WARNE, West Virginia University

1. Introduction. An M-groupoid S is a groupoid [4] which satisfies the following conditions:

P.1.1. S contains a left identity.

P.1.2. If y or z is a left identity of S, then $(xy)z=x(yz)$ for all x in S.

P.1.3. For any x in S, there is a unique left identity e (which may depend on x) such that $xe=x$.

Such systems were investigated by Tamura, Merkel and Latimer in [2]. According to their results, an M-groupoid S is the direct product of a right zero semi-group, i.e. a semigroup such that $xy=y$ for x, y and a groupoid with identity and conversely.

We will investigate first the structure of semigroups which are direct products of right zero semigroups and cancellative semigroups with identity. We consider the relationship of these semigroups to right groups (the direct products of groups and right zero semigroups). Finally, we consider groupoids which are direct products of right singular semigroups and unipotent (one-idempotent) groupoids with identity.

We will follow the notation and terminology of Clifford and Preston [1] for all concepts not defined in this paper.

2. Left cancellative semigroups and right groups. A decomposition of a groupoid S is a partition of S, $S=U(S_a: a \text{ in } A)$, $S_a \cap S_b=\square$, $a\neq b$ in which for every a, b in A, there is a c in A such that $S_aS_b\subseteq S_c$ [2, p. 118]. If for every such a, b, $c=b$, and each S_a is a groupoid of type T, we say S is a right zero band of groupoids of type T. Semigroups which are "bands of semigroups of type T" have been studied extensively [1]. A right group [1, p. 37] is a semigroup S such that if a, b in S there exists a unique x in S such that $ax=b$. We refer the reader to [1] for references on right groups. An M-groupoid which is a semigroup is called an M-semigroup.

We now give a different formulation of a lemma stated in [2].

LEMMA 1 [2, p. 118]. *A groupoid (semigroup) S is isomorphic to the direct product of two groupoids (semigroups) A and B if and only if there exists homomorphisms U and V of S onto A and B respectively such that for a in A and b in B, $aU^{-1}\cap bV^{-1}=(x_{ab})$, a set consisting of exactly one element. In this case U and V are said to be orthogonal.*

Proof. If x in $aU^{-1}\cap bV^{-1}$, let $xW=(a, b)$. It is easily seen that W is the required isomorphism. The converse is clear.

For the purpose of illustrating this important principle of "orthogonality of decompositions" [2], we will give a proof of the following result established in [2]. Our proof using Lemma 1, is slightly different from that given in [2].

THEOREM 1 [2, p. 119]. *An M-groupoid (semigroup) S is the direct product of a right zero semigroup and a groupoid (semigroup) with identity and conversely. The groupoid (semigroup) with identity is obtained as Se, where e is a left identity.*

Proof. We first show that if e is a left identity of S, Se is a groupoid with identity e. If se, te in Se,

$$(se)(te) = ((se)t)e \quad \text{in } Se \quad \text{and} \quad (se)e = s(ee) = se.$$

Let R denote the subset of left identities of S. Clearly, R is a right zero semigroup. If x in S, let e_x denote the unique left identity of P.1.3. Let $xU=xe$ and $xV=e_x$ for x in S. Since $(xe)(ye)=x(e(ye))=x(ye)=(xy)e$, U is a homomorphism of S onto Se. Since $(xy)e_{xy}=xy=x(ye_y)=(xy)e_y$,

$$e_{xy} = e_y \quad \text{and} \quad e_{xy} = e_x e_y.$$

If e in R, $e=e_e$. Thus, V is a homomorphism of S onto R. Let xe and g be arbitrary elements of Se and R respectively. If $zU=xe$ and $zV=g$, i.e. $ze=xe$ and $zg=z$, then

$$z = zg = z(eg) = (ze)g = (xe)g = x(eg) = xg.$$

However, $(xg)e=x(ge)=xe$ and $(xg)g=xg$. Thus, U and V are orthogonal and $S=RxSe$ by Lemma 1. The converse follows by routine calculation.

THEOREM 2. *The following conditions on a semigroup S are equivalent.*

(1) *S is left cancellative; $ba=ca$ implies $bx=cx$ for all x in S: a in aS for every a in S.*
(2) *$S=E\times C$ where E is a right zero semigroup and C is a cancellative semigroup with identity.*
(3) *S is a right zero band of cancellative semigroups with identity.*

Proof. (1)$\rightarrow$(2): We first show that S is an M-semigroup. Let E denote the set of idempotents of S. If a in S, there exists e in S such that $a=ae$. Hence $ae=ae^2$, $e^2=e$, and e in E, i.e. $E\neq\square$. If e in E and a in S,

$$ea = e^2a = e(ea), \quad \text{and} \quad ea = a.$$

Thus, every idempotent is a left identity. Next, let g be a fixed element of E, and let us consider Sg. Clearly, Sg is left cancellative. Suppose that $(ug)(sg) = (vg)(sg)$ with s, u, v in S. Thus, $ug = (ug)g = (vg)g = vg$ and Sg is cancellative. Apply Theorem 1.

(2)$\rightarrow$(3) For g in E, let $S_g = ((g, c) : c \text{ in } C)$. Clearly, S_g is a cancellative semigroup with identity $(g, 1)$ where 1 is the identity of C and S is a right zero band of the S_g.

(3)$\rightarrow$(1). Let S be a right zero band of the semigroups $(S_t : t \text{ in } T)$ where S_t is a cancellative semigroup with identity e_t. If x in S_t, y in S_w, and a in S_u where t, u, w in T and $ax = ay$, then $w = t$ and $a(e_t x) = a(e_t y)$. Thus, $(ae_t)x = (ae_t)y$ and $x = y$ since x, y, and ae_t in S_t. Hence, S is left cancellative. If $xa = ya$, $x(e_u a) = y(e_u a)$, $(xe_u)a = (ye_u)a$, and $xe_u = ye_u$. If z in S_v (v in T),

$$e_u z = (e_u e_u)z = e_u(e_u z) \quad \text{and} \quad z = e_u z.$$

Thus, $xz = x(e_u z) = (xe_u)z = (ye_u)z = y(e_u z) = yz$.

COROLLARY 1. *The following assertions concerning a semigroup are equivalent.*

(1) *S is a right group.*

(2) *$S = G \times E$ where G is a group and E is a right zero semigroup.*

(3) *S is a right zero band of groups.*

Proof. (1)$\rightarrow$(2). If $ba = ca$ (a, b, c in S) and x in S choose y in S such that $ay = x$. Thus, $bay = cay$ and $bx = cx$. Similarly, a in aS. Thus, by Theorem 2, $S = C \times E$ where C is a cancellative semigroup with identity 1 and E is a right zero semigroup. Thus, if (a, h), $(1, h)$ in $C \times E$, there exists (b, t) in $C \times E$ such that $(a, h)(b, t) = (1, h)$, i.e. $ab = 1$.

(2)$\rightarrow$(3). The proof is similar to the proof "(2)$\rightarrow$(3)" in Theorem 2.

(3)$\rightarrow$(1). Suppose S is the right zero band of groups $(S_a : a \text{ in } A)$. If x in S_a, y in S_b(a, b in A), xy and y in S_b and hence there exists z in S_b such that $x(yz) = (xy)z = y$. S is left cancellative by Theorem 2.

The equivalence of (1) and (2) is given by [1, Theorem 1.27, p. 38].

3. M-groupoids. A groupoid is called unipotent (one-idempotent) if it contains precisely one idempotent [1]. An idempotent e of a groupoid S is said to be primitive if f is an idempotent of S such that $ef = fe = f$, then $e = f$. Primitive idempotents have played an important role in algebra (see [1], for example).

In [3], a groupoid S is said to satisfy VIII$_3'$ (notation of [3]) if it verifies the following conditions:

P.3.1. Existence of a left identity.

P.3.2. $(xy)z = x(yz)$ if y or z is a left identity.

P.3.3. If e, f are idempotents, $ef = f$.

P.3.4. There is a decomposition $(S_a : a \text{ in } A)$ of S such that S_a is a groupoid with two-sided identity e_a.

We will say that a groupoid S satisfies VIII$_2'$ (notation of [3]) if P.3.1, P.3.2, P.3.4 hold and if S verifies

P.(3.3)'. If a, b in A, $e_a e_b = e_b$.

THEOREM 3. VIII$_2'$ *characterizes an M-groupoid* [3].

Proof. For the proof of sufficiency, we may show that VIII$_2'$, implies P.1.3. If e is a left identity of S, e in S_a for some a in A and hence $e=e_a$. Next, we show that each e_a is a left identity of S. Let e_b be a left identity of S (the existence of such is guaranteed by P.3.1). If x is an arbitrary element of S,

$$e_a x = e_a(e_b x) = (e_a e_b)x = e_b x = x.$$

If x in S_a, say, $xe_a=x$. If $xe_c=x$ with $c\neq a$, $S_aS_c\subseteq S_a$ and $e_c=e_ae_c$ in S_a, a contradiction. Thus, $e_c=e_a$ and P.1.3 is valid.

To prove necessity, we note that S is the direct product of a right zero semigroup E and a groupoid G with identity 1 by Theorem 1. If for e in E, we let $S_e=((e, g): g \text{ in } G)$, (S_e) is a decomposition of S such that each S_e is a groupoid with identity $(e, 1)$. The collection of left identities is $((e, 1): e \text{ in } E)$. It is now a routine calculation to establish P.(3.3)$'$.

THEOREM 4. *The following conditions on a groupoid are equivalent.*

(1) *S is an M-groupoid such that each left identity is a primitive idempotent.*
(2) *$S=E\times G$ where E is a right zero semigroup and G is an unipotent groupoid with identity.*
(3) *S satisfies* P.1.1 *and* P.1.2 *and S is a right zero band of unipotent groupoids with identity.*
(4) *S satisfies* VIII$_3'$.

Proof. (1)$\rightarrow$(2). Let e be a left identity of S and suppose that g is an idempotent of Se. Then, $g=xe$ for some x in S, $ge=(xe)e=x(ee)=xe=g$, and $e=g$. Apply Theorem 1.

(2)$\rightarrow$(3). It is easy to see that $((e, 1): e \text{ in } E)$, where 1 is the identity of G, is both the collection of idempotents and the collection of left identities of $E\times G$. P.1.2 is verified by routine calculation. If $S_e=((e, g): g \text{ in } G)$, S is a right zero band of the (S_e) and each S_e is clearly a unipotent groupoid with identity.

(3)$\rightarrow$(4). Clearly, P.3.1, P.3.2, and P.3.4 are valid. Let S be the right zero band $(S_a: a \text{ in } A)$ of unipotent groupoids with identity e_a.

Clearly $(e_a: a \text{ in } A)$ is the collection of idempotents of S. Let e_a be a left identity of S and suppose e_b is an arbitrary idempotent of S.

If $e_be_a=z$ in S_a, $e_bz=e_b(e_be_a)=e_be_a=z$ and $z^2=(e_be_a)z=e_b(e_az)=e_bz=z$, i.e. $e_be_a=e_a$. If c in A, $e_be_c=e_b(e_ae_c)=(e_be_a)e_c=e_ae_c=e_c$ and P.3.3 is verified.

(4)$\rightarrow$(1). Clearly, VIII$_3'$ implies VIII$_2'$. Thus, S is an M-groupoid by Theorem 3. If e and f are idempotents and $ef=fe=f$, then $e=f$ by P.3.3.

If the semigroup S is the right zero band $(S_a: a \text{ in } A)$ of unipotent semigroups with identity e_a,

$$(e_ae_b)(e_ae_b) = e_a(e_b(e_ae_b)) = e_a(e_ae_b) = e_ae_b \quad \text{and} \quad e_ae_b = e_b(a, b \text{ in } A).$$

If x in S_c, say, $e_ax=e_a(e_cx)=(e_ae_c)x=e_cx=x$ and e_a is a left identity. Thus, the modification of Theorem 4 to semigroups is clear.

We next give the analogue of Theorem 2 for groupoids.

THEOREM 5. *The following assertions on a groupoid are equivalent.*

(1) *S satisfies* P.1.1 *and* P.1.2. *S is left cancellative; $ba=ca$ implies $bx=cx$ for all x in S; if a in S there exists a left identity e such that $a=ae$.*

(2) *$S=E\times C$ where E is a right zero semigroup and C is a cancellative groupoid with identity.*

(3) *S satisfies* P.1.1 *and* P.1.2 *and S is a right zero band of cancellative groupoids with identity.*

Proof. (1)→(2). Clearly, S is an M-groupoid. If g is a left identity of S, we show as in the proof of "(1)→(2)" in Theorem 2, that Sg is cancellative. Apply Theorem 1.

(2)→(3). Since C is unipotent, we proceed as in the proof of "(2)→(3)" in Theorem 4.

(3)→(1). Let S be the right zero band of cancellative groupoids with identity $(S_a: a \text{ in } A)$ where e_a is the identity of S_a. By Theorem 4, $e_ae_b=e_b(a, b \text{ in } A)$ and P.1.1, P.1.2, and P.1.3 are valid. As in the proof of Theorem 3, we show each e_a is a left identity of S. Thus, we may proceed as in the proof of "(3)→(1)" in Theorem 2.

For a fairly complete bibliography on the theory of groupoids, see [4].

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ON THE STRUCTURE OF J-RINGS

JIANG LUH, Indiana State University

1. Introduction. A ring R is called a J-ring if there exists an integer $n>1$ such that $x^n=x$ for every $x\in R$. It is well known that every J-ring is commutative (cf. [2] p. 217).

Raymond and Christine Ayoub [1] have recently proved this theorem for a certain class of exponents n without recourse to transfinite methods. In their paper, it also has been proved that every J-ring is a direct sum of rings with prime characteristic. The purpose of this note is to investigate further properties of these direct summands and to obtain the structure of J-rings.

In the sense of McCoy and Montgomery [4], a ring R is called a p^k-ring if there exist a prime p and a positive integer k such that $x^{p^k}=x$ and $px=0$ for every x in R. A p-ring is simply a p^1-ring.

In the present note, we shall prove the following

MAIN THEOREM. *A ring R is a J-ring if and only if R is a direct sum of finitely many p^k-rings.*

2. Preliminaries. The following important theorem which is proved in [3] will be used in the following proofs.

THEOREM 1 (DIRICHLET). *If $a>0$ and b are integers and if $(a, b)=1$, then there are infinitely many primes of the form $an+b$, where n is a positive integer.*

THEOREM 2. *If $m_1, m_2, \cdots, m_r$ are positive integers $\geqq 2$, then there exists a prime p such that*

$$m_i - 1 \mid p - 1, \quad i = 1, 2, \cdots, r.$$

Proof. Let $a=(m_1-1)(m_2-1)\cdots(m_r-1)$ and let $b=1$. By Theorem 1, there exists a prime

$$p = (m_1 - 1)(m_2 - 1)\cdots(m_r - 1)n + 1$$

for some positive integer n. Thus, $m_i-1 \mid p-1$, $i=1, 2, \cdots, r$.

THEOREM 3. *If n is an arbitrary integer >1 and if q is prime, then there exist integers $k\geqq 1$ and $t\geqq 0$ such that $q^t(q^k-1)\equiv 0 \pmod{n-1}$.*

Proof. Write $n-1=mq^t$, where $q \nmid m$. By the Fermat-Euler Theorem, $q^k\equiv 1 \pmod{m}$, or $m \mid q^k-1$, where $k=\phi(m)$, so that $n-1 \mid q^t(q^k-1)$.

THEOREM 4. *If x is an element of a ring R such that $x^n=x$ for some integer $n>1$ and if h and k are positive integers, $h\equiv k \pmod{n-1}$, then $x^h=x^k$.*

This result can be easily obtained by induction (cf. [1]).

We can now give a characterization of J-rings.

THEOREM 5. *A ring R is a J-ring if and only if there exists a prime p such that $x^p=x$ for every x in R.*

Proof. Sufficiency: Obvious.

Necessity. Suppose that $x^n=x$ for every $x\in R$. By Theorem 2, a prime p exists such that $n-1 \mid p-1$, or $p\equiv 1 \pmod{n-1}$. By applying Theorem 4, we have $x^p=x$ for every x in R.

3. Proof of the Main Theorem. First, let us assume that R is a direct sum of finitely many $p(i)^{k(i)}$-rings R_i, say, $R=R_1\oplus R_2\oplus \cdots \oplus R_r$. Hence, by Theorems 2 and 4, there exists a prime p such that for any $x=(x_1, x_2, \cdots, x_r)\in R$ with $x_i\in R_i$, $x^p=(x_1^p, x_2^p, \cdots, x_r^p)=(x_1, x_2, \cdots, x_r)=x$. Therefore, R is a J-ring.

Conversely, suppose that R is a J-ring and that $x^n=x$ for every $x\in R$. We shall show that R is a direct sum of finitely many $p(i)^{k(i)}$-rings.

Since $(ax)^n=ax$, or $(a^n-a)x=0$ for every $x\in R$ and every integer a, there exists a least positive integer m such that $mx=0$ for every x in R. Clearly,

$m \mid a^n - a$, for every integer a. We claim now that m is a product of distinct primes. To see this, it will be sufficient to show that if $m = p_1 m_1$, where p_1 is a prime factor of m, then $(p_1, m_1) = 1$. Indeed, suppose contrarily that $m_1 = p_1 m_2$ for some integer m_2. Then, for every $x \in R$,

$$m_1 x = (m_1 x)^n = m_1^n x = p_1^n m_2^n x = p_1^{n-2} m_2^{n-1} mx = 0,$$

which contradicts the choice of m. Thus, $m = p(1) \cdot p(2) \cdot \cdots \cdot p(r)$, where $p(i)$ are distinct primes. Set

$$R_i = \{x \in R: p(i)x = 0\}, \quad i = 1, 2, \cdots, r.$$

Evidently, R_i is a subring of R with characteristic $p(i)$ and $R = R_1 \oplus \cdots \oplus R_r$.

Now we are in a position to finish the proof of the main theorem by showing that each R_i is a $p(i)^{k(i)}$-ring. In fact, by Theorem 3, there exist integers $t(i) \geqq 0$ and $k(i) \geqq 1$ such that $n-1 \mid p(i)^{t(i)}(p(i)^{k(i)} - 1)$. Hence, for every $x \in R$, we have

$$(x - x^{p(i)^{k(i)}})^{p(i)^{t(i)}} = x^{p(i)^{t(i)}} - x^{p(i)^{k(i)+t(i)}} = 0,$$

the last equality being a consequence of Theorem 4. But R has no nilpotent element other than 0. Therefore, we obtain that

$$x - x^{p(i)^{k(i)}} = 0, \quad \text{or} \quad x^{p(i)^{k(i)}} = x.$$

Thus, R_i is a $p(i)^{k(i)}$-ring.

REMARK. In view of the main theorem, we can reduce the proof of commutativity of J-rings to consider only p^k-rings.

The author is now at The Wright State University, Dayton, Ohio.

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ON STRONGLY CONTINUOUS FUNCTIONS

S. A. NAIMPALLY, Iowa State University (Now at University of Alberta, Edmonton, Canada)

Let X and Y be topological spaces. Levine [2] defined a function $f: X \to Y$ to be *strongly continuous* iff for each subset K of X, $f(\overline{K}) \subset f(K)$. When Y is a uniform space the function space of continuous functions on X to Y is closed in Y^X in the topology of uniform convergence. It is natural to enquire whether a similar result is true for the function space of strongly continuous functions. This investigation resulted in a surprise: namely, that with some restrictions on X and Y the function space of strongly continuous functions is even closed in the pointwise convergence topology! It is well known that a similar result is not true for the function space of continuous functions.

In what follows S will denote the set of all strongly continuous functions on X to Y and will be assigned the pointwise convergence (or p.c.) topology. (See Kelley [1] page 217.)

The following results which are proved by Levine in [2] are recorded here to make the note self-contained.

(a) If $f\in S$ and K is a nonempty connected subset of X, then $f(K)$ is a single point.

(b) If X is locally connected and f is a function on X to Y such that for each nonempty connected subset K of X $f(K)$ is a single point, then $f\in S$.

1. THEOREM. *Let Y be Hausdorff and $f\in\overline{S}$ (the bar denotes closure in the p.c. topology). Then for each nonempty connected subset K of X, $f(K)$ is a single point.*

Proof. Suppose that there exists a nonempty connected subset K of X such that $f(K)$ contains at least two distinct points $f(p)$ and $f(q)$. Since Y is Hausdorff $f(p)$ and $f(q)$ have disjoint open neighborhoods U and V respectively. Since $f\in\overline{S}$, there exists a net $\{f_n \mid n\in D\}\subset S$ such that for each $x\in X$, $f_n(x)\to f(x)$. So there exists an $m\in D$ such that for all $n\geqq m$, $f_n(p)\in U$ and $f_n(q)\in V$. This contradicts (a) and the theorem is proved.

The following example shows that the hypothesis of Theorem 1 cannot be weakened.

2. *Example.* Let X consist of two points p, q with the topology consisting of $\varnothing$, $\{p\}$, X. Let Y be the set of integers with cofinite topology. It is well known that Y is T_1 but not Hausdorff. Since X is connected all the elements of S are constant functions. If $f\in Y^X$ and $f\notin S$ then $f(p)=m$, $f(q)=n$, $m\neq n$. A neighborhood of f in the p.c. topology is of the form $\{g\in Y^X \mid g(p)\in U \text{ and } g(q)\in V\}$, where U and V are open sets in Y containing m and n respectively. But $U\cap V\neq\varnothing$ for each pair of open sets U, V of Y. Let $r\in U\cap V$. Let $h(p)=h(q)=r$. Then h belongs to the above neighborhood of f and so $f\in\overline{S}$, but f does not take the connected set X into a single point.

Theorem 1 combined with (b) lead to the following result.

3. THEOREM. *If X is locally connected and Y Hausdorff then S is closed in Y^X in the p.c. topology.*

The above Example 2 shows that if X is locally connected and Y is not Hausdorff then the above Theorem 3 is false. The following example shows that X must be locally connected.

4. *Example.* Let $X=\{0,\ 1/n\}$, $n=1,\ 2,\ 3,\ \cdots$ with the usual subspace topology of the reals. Let Y be the reals with the usual topology. Let $f_n\in Y^X$ be defined as

$$f_n(0)=1, \qquad f_n\left(\frac{1}{m}\right)=\begin{cases}\dfrac{1}{m} & \text{for } m<n\\ 1 & \text{for } m\geqq n.\end{cases}$$

Then each $f_n \in S$ and $f_n \to f$, where

$$f(0) = 1, \qquad f\left(\frac{1}{m}\right) = \frac{1}{m} \quad m = 1, 2, 3, \cdots.$$

Clearly f is not even continuous, although f is a pointwise limit of $\{f_n\}$ a sequence of strongly continuous functions.

It is well known that if Y is compact then the space of all continuous functions on X to Y need not be compact even in the topology of uniform convergence. In contrast to this we have the following result for S.

5. THEOREM. *If X is locally connected and Y compact Hausdorff, then S is compact in the p.c. topology.*

Proof. By Theorem 3, S is pointwise closed in Y^X. Also since Y is compact, for each $x \in X$, $S(x) = \{f(x) \mid f \in S\}$ has compact closure in Y. These are the two properties of the hypotheses of Theorem 1 page 218 of Kelley [1], from which it follows that S is compact in the p.c. topology.

6. THEOREM. *If X is locally connected and Y a uniform space with a uniformity $\mathcal{U}$ then S is equicontinuous (i.e. for each $x \in X$ and $V \in \mathcal{U}$ there exists a neighborhood U of x such that $f(U) \subset V[f(x)]$ for all $f \in S$).*

Proof. For each $x \in X$ there exists an open connected neighborhood U of x and so for all $f \in S$, $f(U) = f(x)$. So obviously $f(U) \subset V[f(x)]$ for all $V \in \mathcal{U}$ and $f \in S$. Thus S is equicontinuous.

For the function space of all continuous functions on X to Y, the p.c. topology is not satisfactory and so we use the topology of uniform convergence on compacta (when Y is a uniform space). Theorem 3 has already shown that the p.c. topology is quite adequate for S when X is locally connected and Y Hausdorff. The following theorem which follows easily from Theorem 6 above and Theorem 15 of Kelley [1] page 232, shows that the two topologies are in fact equivalent for S.

7. THEOREM. *If X is locally connected and Y a uniform space then the p.c. topology for S coincides with the topology of uniform convergence on compacta. Consequently, S is evenly continuous (i.e. for each $x \in X$, each $y \in Y$ and each neighborhood U of y there exists a neighborhood V of x and a neighborhood W of y such that $f(V) \subset U$ whenever $f(x) \in W$).*

The author would like to thank Prof. D. E. Sanderson and the referee for their comments and suggestions.

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MAXIMAL NON-T_i SPACES, $i=0,1,2,3$

J. P. THOMAS, University of North Carolina at Charlotte

It is the purpose of this paper to investigate the existence and characteristics of maximal non-Kolmogoroff, (non-T_0), maximal nonaccessible, (non-T_1), maximal nonseparated (non-Hausdorff or non-T_2) and maximal nonregular (non-T_3) topological spaces. (T_i is here to be understood in the sense of Kelley [2].) In general, terminology and notation will be that of N. Bourbaki [1]. The notation $(X, \mathfrak{I})$ will denote a set X with topology $\mathfrak{I}$. A topology $\mathfrak{I}$ will be said to be finer than $\mathfrak{I}'$ if $\mathfrak{I}' \subset \mathfrak{I}$. If A is a set of subsets of X we shall denote by $\Phi(A)$ the topology generated by A.

DEFINITION I. *A topological space $(X, \mathfrak{I})$ is said to be maximal non-Kolmogoroff (resp. maximal nonaccessible, maximal nonseparated, maximal nonregular) if it is not a Kolmogoroff (resp. accessible, separated, regular) space and if any strictly finer topology on X is Kolmogoroff (resp. accessible, separated, regular).*

If a set X has at least two elements, say a, b, then

$$\mathfrak{I}_1 = \Phi(\{\{a, b\}\} \cup \{\{x\} \mid x \in X, x \neq a, b\})$$

is maximal non-Kolmogoroff and $\mathfrak{I}_2 = \Phi(\{\{a, b\}\} \cup \{\{x\} \mid x \in X, x \neq a\}$ is maximal nonaccessible, maximal nonseparated and maximal nonregular. It is easy to see that any maximal non-Kolmogoroff (resp. maximal nonaccessible) topology is of the form of $\mathfrak{I}_1$ (resp. $\mathfrak{I}_2$) and that any non-Kolmogoroff (resp. nonaccessible) topology is coarser than a maximal non-Kolmogoroff (resp. nonaccessible) topology. The problem is more difficult in the case of nonseparated and nonregular topologies.

THEOREM I. *Let X be any set and let $\mathfrak{I}_1$ be a nonseparated topology on X. Then there exists a maximal nonseparated and maximal nonregular topology $\mathfrak{I}_2$ on X which is finer than $\mathfrak{I}_1$.*

Proof. If $\mathfrak{I}_1$ is nonaccessible it has already been remarked that there exists a topology of the form $\Phi(\{\{a, b\}\} \cup \{\{x\} \mid x \in X, x \neq a\})$ where $a \neq b$ which is finer than $\mathfrak{I}_1$. Such a topology is maximal nonseparated and maximal nonregular. Hence, suppose that $\mathfrak{I}_1$ is accessible. Since $\mathfrak{I}_1$ is nonseparated there exists an ultrafilter $\mathfrak{U}$ on $(X, \mathfrak{I}_1)$ with two distinct limit points, say a and b. Let $\mathfrak{I}_2 = \Phi(\mathfrak{U} \cup \{\{x\} \mid x \in X, x \neq a, b\}$. Clearly $\mathfrak{I}_2$ is finer than $\mathfrak{I}_1$ and is non-separated (and hence non-regular) since neighborhoods of a and b always intersect. $\mathfrak{I}_2$ is maximal nonseparated. For, if $\mathfrak{I}_3$ is strictly finer than $\mathfrak{I}_2$ let $A \in \mathfrak{I}_3 \cap C\mathfrak{I}_2$. Then $a \in A$ or $b \in A$ for otherwise $A \in \mathfrak{I}_2$. Suppose that $a \in A$ but $b \notin A$. Since $A \notin \mathfrak{U}$ we have $CA \in \mathfrak{U} \subset \mathfrak{I}_2$. Thus A and CA are disjoint $\mathfrak{I}_3$ neighborhoods of a and b respectively. If $c \neq a$, b there is a neighborhood V of a which does not contain c since $\mathfrak{I}_1$ is accessible. Then V and $\{c\}$ are disjoint neighborhoods of a and c respectively. Similarly we can find disjoint neighborhoods of b and c. Thus in this case $\mathfrak{I}_3$ is separated. In case $a, b \in A$, since $A \notin \mathfrak{U}$ we have $CA \in \mathfrak{U} \subset \mathfrak{I}_3$. Then

$CA \cup \{a\} \in \mathfrak{U} \subset \mathfrak{I}_3$ and $\{a\} = A \cap (CA \cup \{a\}) \in \mathfrak{I}_3$. Similarly $\{b\} \in \mathfrak{I}_3$ and thus $\mathfrak{I}_3$ is discrete and therefore separated.

Finally we find that $\mathfrak{I}_3$ is regular. For if $c \in X$, $c \neq a$, b then c has a fundamental system of closed neighborhoods since c is isolated and $\mathfrak{I}_3$ is separated. Now let V be any neighborhood of b. Let W_1 and W_2 be disjoint neighborhoods of a and b respectively. Then $W_2 \cap V$ is a neighborhood of b and is contained in V and is closed since

$$C(W_2 \cap V) = W_1 \bigcup_{x \in C(W_1 \cup (W_2 \cap V))} \{x\}$$

is open

THEOREM 2. *Let $(X, \mathfrak{I}_1)$ be a separated nonregular topological space. Then there exists a topology $\mathfrak{I}_2$ on X which is separated, nonregular and strictly finer than $\mathfrak{I}_1$.*

Proof. There exist $x \in X$, $V \in \mathfrak{I}_1$ such that $x \in V$ and such that if Y is a neighborhoog of x then $\overline{Y} \cap CV \neq \phi$. Suppose that $x \in W \subset V$, $W \in \mathfrak{I}_1$. We shall show that $\overline{W} \cap CV$ is infinite. For suppose that $\overline{W} \cap CV = \{v_1, v_2, \cdots, v_n\}$ with n finite. Then there exist neighborhoods $V_1, V_2, \cdots, V_n$ of $v_1, v_2, \cdots, v_n$ respectively and neighborhoods $W_1, W_2, \cdots, W_n$ of x, such that $W_i \cap V_i = \phi$, $i = 1, 2, \cdots, n$. Then $W' = W \cap W_1 \cap \cdots \cap W_n \subset W$ is a neighborhood of x. We shall show that $\overline{W}' \subset V$ which will be a contradiction. For let $v \in \overline{W}' \cap CV$. Then $v = v_i$ for some $v_i \in \overline{W} \cap CV$ since $W' \subset W$. But V_i is a neighborhood of v which does not intersect W'. Now let $z \in \overline{V} \cap CV$ and note that z is not isolated for $\mathfrak{I}_1$. Define $\mathfrak{I}_2 = \Phi(\mathfrak{I}_1 \cup \{\{z\}\})$. Then $\mathfrak{I}_2$ is strictly finer than $\mathfrak{I}_1$ and is separated since $\mathfrak{I}_1$ is separated. And $\mathfrak{I}_2$ is nonregular, for if U is an open neighborhood of x for $\mathfrak{I}_2$ and $U \subset V$, then $\overline{U} \cap CV$ is infinite for $\mathfrak{I}_1$ by the above argument and is therefore infinite for $\mathfrak{I}_2$, since all points except z have the same neighborhoods as for $\mathfrak{I}_1$.

COROLLARY I. *There exist no separated maximal nonregular topologies.*

Proof-Trivial.

COROLLARY 2. *Let X be any set. Then the maximal nonseparated topologies are the maximal nonregular topologies on X.*

Proof. Let $\mathfrak{I}_1$ be a maximal nonseparated topology on X. By Theorem I there exists a maximal nonseparated, maximal nonregular topology $\mathfrak{I}_2$ on X which is finer than $\mathfrak{I}_1$. But this implies $\mathfrak{I}_1 = \mathfrak{I}_2$.

Let $\mathfrak{I}_3$ be a maximal nonregular topology on X. By Corollary I, $\mathfrak{I}_3$ is nonseparated. Hence by Theorem I there exists a maximal nonseparated, maximal nonregular topology $\mathfrak{I}_4$ on X such that $\mathfrak{I}_4$ is finer than $\mathfrak{I}_3$. But since $\mathfrak{I}_3$ is maximal nonregular this implies that $\mathfrak{I}_3 = \mathfrak{I}_4$.

THEOREM 3. *Let X_1 and X_2 be topological spaces, each having more than one element. Then $X_1 \times X_2$ with the product topology is neither maximal non-Kolmogoroff, maximal nonaccessible, nor maximal nonseparated.*

Proof. Let $\mathfrak{I}$ be the product topology on $X_1 \times X_2$. If both X_1 and X_2 are Kolmogoroff, $(X_1 \times X_2, \mathfrak{I})$ is Kolmogoroff, so assume X_1 is non-Kolmogoroff. There exist $a, b \in X$, $a \neq b$ such that $a \in \overline{\{b\}}$, $b \in \overline{\{a\}}$. Let $c, d \in X_2$. Then $(b, c) \in \overline{\{(a, c)\}}$, $(a, c) \in \overline{\{(b, c)\}}$, $(b, d) \in \overline{\{(a, d)\}}$ and $(a, d) \in \overline{\{(b, d)\}}$. Thus $(X_1 \times X_2, \mathfrak{I})$ is not maximal non-Kolmogoroff. The proof for nonaccessible and nonseparated spaces is similar.

The author is grateful to Dr. E. E. Enochs for supervision in the preparation of this paper.

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REMARKS ON DETECTION PROBLEMS

PETER FRANK AND ROBERT SILVERMAN, Syracuse University

Given a finite set $S = \{a_1, a_2, \cdots, a_n\}$, denote by 2^S the class of all subsets of S, and by $|X|$ the number of elements in the set X. For X, Y in 2^S, define $d(X, Y) = |X \triangle Y|$, where $X \triangle Y = (X \cup Y) - (X \cap Y)$. Thus the "distance" between the sets X and Y is the number of elements in their symmetric difference. One readily verifies that d is a metric on 2^S. The structure of the metric space of 2^S is intimately connected with many of the basic problems in combinatorics [3].

The sets $R_1, \cdots, R_k$ are termed a *metric basis* of 2^S provided that $|X \triangle R_i| = |Y \triangle R_i|$, $i = 1, \cdots, k$ implies that $X = Y$. A metric basis is *minimal* provided no other metric basis has fewer elements. Denote by $F(n)$ the number of elements in a minimal basis.

One may also look for sets $R_1, \cdots, R_k$ such that $|R_i \cap X| = |R_i \cap Y|$, $i = 1, \cdots, k$ implies that $X = Y$. We call such a family of sets an *intersection detecting* family. Again one may seek the least value of k for which there exists an intersection detecting family with k sets. Denote this value by $f(n)$.

Söderberg and Shapiro [4] considered $f(n)$, and Erdös and Rényi [1] studied both $f(n)$ and $F(n)$. Lindström [2] finally showed that $\lim_{n\to\infty} [f(n) \log n]/n = \log 4$, and by an additional special and somewhat lengthy argument established that also $\lim_{n\to\infty} [F(n) \log n]/n = \log 4$. We show here that either result can be derived from the other, and indeed that $|F(n) - f(n)| \leqq 1$. This follows immediately from the relations

$$(1) \qquad |R \triangle X| + 2|R \cap X| = |R| + |X|,$$

$$(2) \qquad |X| = |Y| \Leftrightarrow |S \cap X| = |S \cap Y| \Leftrightarrow |S \triangle X| = |S \triangle Y|.$$

From (1) we obtain

(3) if $|X| = |Y|$, then $|R \triangle X| = |R \triangle Y| \Leftrightarrow |R \cap X| = |R \cap Y|$.

Combining (2) and (3) we see that

$$|S\cap X| = |S\cap Y| \text{ and } |R_i\cap X| = |R_i\cap Y| \Rightarrow |R_i\triangle X| = |R_i\triangle Y|.$$

Hence if $R_1, \cdots, R_k$ is a metric basis, then $R_1, \cdots, R_k, S$ is an intersection detecting family. Similarly, if $R_1, \cdots, R_k$ is intersection detecting, then $R_1, \cdots, R_k, S$ is a metric basis.

The two problems above may be given matric formulations and indeed, this is the form in which they are treated in the references. Form the "incidence" matrix $A=[a_{ij}]$, where $a_{ij}=1$ if a_j is in R_i, and 0 otherwise. Then $R_1, \cdots, R_k$ is an intersection detecting family if and only if $Ax=Ay\Rightarrow x=y$ for all 0, 1 column vectors x, y. Now obtain the matrix B from A by replacing all 0's in A by -1's. Then $R_1, \cdots, R_k$ is a metric basis if and only if $Bx=By\Rightarrow x=y$ for all 0, 1 column vectors x, y. The latter result is evident if we observe that using (1) together with $|X|-2|R\cap X| = |X-R| - |X\cap R|$, we obtain

$$(4)\quad |R\Delta X| = |R\Delta Y| \Leftrightarrow |X\cap R| - |X-R| = |Y\cap R| - |Y-R|.$$

But $|X\cap R_i| - |X-R_i|$ is precisely the inner product of the ith row of B with the 0, 1 vector corresponding to X.

We now consider more general detection problems. Let $f(R, X)$ denote a Boolean set operation (expressible in terms of intersection and complementation, and thus union). If $f_1, \cdots, f_k$ are k such set operations (distinct or not) we say that $R_1, \cdots, R_k$ is $f_1, \cdots, f_k$ detecting for S provided that $|f_i(R_i, X)| = |f_i(R_i, Y)|$, $i=1, \cdots, k$ implies that $X=Y$. If $f_1, \cdots, f_k$ are all equal to f, we say that $R_1, \cdots, R_k$ is f-detecting for S. We see that $R_1, \cdots, R_k$ is $\triangle$-detecting for S if and only if $R_1, \cdots, R_k$ are a metric basis of 2^S.

There are 16 Boolean set operations. Of these, 6 are of no use for detection purposes: $f(R, X)=\phi$, S, X, X', R, or R', where " $'$ " denotes complementation. This leaves 10, which we now list:

$$\begin{aligned}
T_1(R, X) &= R\cap X & T_6(R, X) &= (R\cap X)'\\
T_2(R, X) &= R'\cap X & T_7(R, X) &= (R'\cap X)'\\
T_3(R, X) &= R\cap X' & T_8(R, X) &= (R\cap X')'\\
T_4(R, X) &= R'\cap X' & T_9(R, X) &= (R'\cap X')' = (R\cup X)\\
T_5(R, X) &= R\triangle X & T_{10}(R, X) &= (R\triangle X)'.
\end{aligned}$$

The following lemma generalizes (3).

LEMMA. *If* $|X| = |Y|$, *then for all* $i, j=1, \cdots, 10$

$$|T_i(R, X)| = |T_i(R, Y)| \Leftrightarrow |T_j(R, X)| = |T_j(R, Y)|.$$

The proof follows from (3) and the following identities:

$$\begin{aligned}
|T_1| + |T_6| &= |T_2| + |T_7| = |T_3| + |T_8| = |T_4| + |T_9|\\
&= |T_5| + |T_{10}| = n
\end{aligned}$$

$$|T_1(R, X)| + |T_2(R, X)| = |X|$$
$$|T_1(R, X)| + |T_3(R, X)| = |R|$$
$$|T_3(R, X)| + |T_4(R, X)| = |X'|.$$

The lemma can be restated as a general theorem about detection, which shows that the most general detection scheme gives an advantage of at most one over the intersection detection scheme.

THEOREM 1. *Let* $f_1, \cdots, f_k, f_1^*, \cdots, f_k^*$ *and* f *be in* $\{T_1, \cdots, T_{10}\}$. *If* $A_1, \cdots, A_k$ *is* $f_1, \cdots, f_k$ *detecting for* S, *then* $A_1, \cdots, A_k, B$ *is* $f_1^*, \cdots, f_k^*, f$ *detecting for* S. *Here* $B = S$ *when* f *is* T_1, T_3, T_5, T_6, T_8 *or* T_{10}, *and* $B = \phi$ *when* f *is* T_2, T_4, T_7 *or* T_9.

As a special case, $f_1^*, \cdots, f_k^*$ can all be taken equal to f.

We conclude by stating a duality theorem.

THEOREM 2. $R_1, \cdots, R_k$ *is* $\cap$*-detecting for* S *if and only if* $R_1', \cdots, R_k'$ *is* $\cup$*-detecting for* S.

The proof follows immediately from the identity $|R' \cup X'| + |R \cap X| = n$.

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A NOTE ON NEAR-RINGS OVER VECTOR SPACES

R. E. WILLIAMS, Westminister College

The near-rings considered in this note satisfy the ring axioms with the possible exception of the right distributive law. These are a special class of left near-rings.

The problem of determining the near-rings definable on a finite cyclic group has been considered in this MONTHLY [1] by J. R. Clay with the following result: A necessary and sufficient condition that $(Z_n, +, *)$ be a left near-ring is that there exist a mapping π: $p \to \pi(p)$ of Z_n into Z_n with the properties that:

1. $\pi(p) = p * 1$ for each p,
2. $\pi(p)\pi(q) = \pi(p\pi(q))$ for each p, q.

The addition in the near-ring is the same as that in Z_n and the products $\pi(p)\pi(q)$, $p\pi(q)$ are the usual products in Z_n. A partition of this class of near-rings into isomorphic classes was also obtained.

In this note similar results are established for arbitrary (right) vector spaces. Briefly, (1) is replaced by the condition that the function of p be a linear transformation subject to (2). This gives the following theorem, where V is a right vector space over a division ring D and $\mathfrak{L}$ denotes the complete ring of linear transformations of V.

THEOREM 1. *A necessary and sufficient condition that* $N=(V, +, *)$ *be a near-ring with the usual addition in* V *and such that* $x*(y\alpha)=(x*y)\alpha$, x, y *in* V, α *in* D, *is that there exist a mapping* $L: x\to L(x)$ *of* V *into* $\mathfrak{L}$ *with the properties*:

a. $x*y=yL(x)$, b. $L(y)L(x)=L(yL(x))$,

where $L(x)$ *is the image of* x *in* $\mathfrak{L}$.

Proof. For the necessity, let N be a near-ring such that $(N, +)$ is a right vector space over D. For x in N, denote the mapping $x'\to x*x'$ by $L(x)$. $L(x)$ is linear as

$$(y+z)L(x) = x*(y+z) = x*y + x*z = yL(x) + zL(x),$$
$$(y\alpha)L(x) = x*(y\alpha) = (x*y)\alpha = (yL(x))\alpha,$$

for each y, z in N, α in D. Thus $L(x)$ is in $\mathfrak{L}$ for each x, showing the existence of a mapping $L: x\to L(x)$ of V into $\mathfrak{L}$.

The condition $L(y)L(x)=L(yL(x))$ on L follows from

$$x*(y*z) = (y*z)L(x) = zL(y)L(x) = zL(x*y) = zL(yL(x)).$$

To establish sufficiency, choose $L: x\to L(x)$ such that $L(y)L(x)=L(yL(x))$ and define $x*y=yL(x)$. Clearly

$$x*(y+z) = (y+z)L(x) = yL(x) + zL(x) = x*y + x*z$$

so the product is left distributive. Also, from

$$x*(y*z) = (y*z)L(x) = zL(y)L(x),$$
$$(x*y)*z = zL(x*y) = zL(yL(x)),$$

it follows that the product is associative. Finally, as V is a vector space over D the condition $x*(y\alpha)=(x*y)\alpha$ is valid since $L(x)$ is in $\mathfrak{L}$.

THEOREM 2. *Let* $N_i=(V, +, *_i)$, $i=1, 2$ *be near-rings determined as in Theorem* 1, *by* L^1, L^2. *A necessary and sufficient condition that* N_1 *be isomorphic with* N_2 *with* $(N_1, +)$ *and* $(N_2, +)$ D*-isomorphic is that there exist a nonsingular element* T *in* $\mathfrak{L}$ *such that* $L^1(x)=TL^2(xT)T^{-1}$ *for each* x *in* N.

Proof. Let $L^1(x)=TL^2(xT)T^{-1}$. Clearly $x\to xT$ is a D-isomorphism of $(N_1, +)$ onto $(N_2, +)$. Then

$$(x*_1 y)T = yL^1(x)T = yTL^2(xT)T^{-1}T = (xT)*_2(yT),$$

so the condition is sufficient.

Now let $N_1\approx N_2$ such that $(N_1, +)$ is D-isomorphic with $(N_2, +)$. Considera-

tion of the additive groups shows that the isomorphism is achieved by a non-singular linear transformation $T: N_1 \to N_2$. Thus

$$(x *_1 y)T = yL^1(x)T = (xT) *_2 (yT) = yTL^2(xT),$$

so $yL^1(x) = yTL^2(xT)T^{-1}$ for each x, y in V. Hence the condition is necessary.

In considering the sums $L(x)+L(y)$, it may be verified that $L(x+y)=L(x)+L(y)$ is a sufficient condition for N to be a ring. In any case, it is clear that there is a smallest subring of $\mathfrak{L}$ which contains all of the elements $L(x)$, x in N and hence contains the finite sums, regardless of any condition on L.

DEFINITION. *Let A be the subring of $\mathfrak{L}$ generated by the $L(x)$, x in V. A is called the associated ring of N. The set $[L(x)] = [L(x) \mid x \in V]$ is called the set of generators of A.*

THEOREM 3. *Let $N_i = (V, +, *_i)$, $i=1, 2$ be isomorphic near-rings with D-isomorphic additive groups and let A_1 and A_2 be the associated rings. Then the inner automorphism $a \to a^T = T^{-1}aT$ of $\mathfrak{L}$ induces the isomorphism $A_1 \to A_1^T = A_2$.*

Proof. Let $[L^i(x)]$, denote the generators of A_i, $i=1, 2$.

$$[L^1(x)]^T = [T^{-1}L^1(x)T] = [L^2(xT)],$$

so $[L^1(x)]^T = [L^2(x)]$ for x and hence xT range over all of V.

For arbitrary a_i in $\mathfrak{L}$,

$$(\sum a_i)^T = \sum a_i^T, \qquad (-a_i)^T = -a_i^T, \qquad (a_i a_j)^T = a_i^T a_j^T$$

so A_1 is mapped isomorphically onto A_2 by the given inner automorphism of $\mathfrak{L}$.

Example. The converse of Theorem 3 is not valid, as shown by the following example. As before, V denotes a right vector space. Let the identity and zero linear transformations of V be denoted by I and 0 respectively.

Let $L^1(x) = I$ for each x in V. A check of conditions (a) and (b) show that L^1 defines a near-ring N_1. Now select an x in V and let $L^2(x) = I$, $L^2(y) = 0$ for $x \neq y$. L^2 defines a near-ring N_2 and clearly N_1 is not isomorphic to N_2. However, the associated rings A_1 and A_2 are identical.

The encouragement given by Dr. J. L. Zemmer in preparing this note has been most helpful.

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ENUMERATION OF CANONICAL SETS BY RANK

J. V. BRAWLEY, JR., Clemson University

1. Introduction. Let M be the set of square $n \times n$ matrices over the finite field $GF(q)$. By a canonical set for M we shall mean a subset C of M with the property that each $A \in M$ is similar to one and only one matrix in C. It is clear that every canonical set for M has the same number $N(n, q)$ of elements. This number has

ON A CONJECTURE OF ERDÖS, HAJNAL AND MOON

B. BOLLOBÁS, L. Eötvös University, Budapest

In the following we shall prove a conjecture of Erdös, Hajnal and Moon, published in [1]. For the sake of completeness, we repeat some of the definitions of that article.

A *graph* consists of a finite set of vertices, some pairs of which are joined by an edge. We do not permit loops or multiple edges. An n by m bipartite graph has n "green" and m "blue" vertices with all its edges joining vertices of different color. We call a k by l bipartite graph complete if all green and blue vertices are connected. We say that an n by m (bipartite) graph has property (k, l) if any new edge increases the number of complete k by l bipartite graphs in this graph (naturally we assume $k \leqq n$, $l \leqq m$). Erdös, Hajnal and Moon conjectured that every n by m bipartite graph with property (k, l) has at least $(k-1)m+(l-1)n-(k-1)(l-1)$ edges, and now we verify this conjecture.

THEOREM. *Every n by m bipartite graph with property (k, l) has at least $(k-1)m+(l-1)n-(k-1)(l-1)$ edges.*

Note that this is the same as saying that the graph cannot lack more than $(n-k+1)(m-l+1)$ edges. We use induction on k and l. For $k=l=1$ the statement is trivial. Because of the symmetry of k and l it is sufficient to prove that if it holds for property (k, l) it also holds for $(k+1, l)$.

Let G be an n by m graph with property $(k+1, l)$ and let $\overline{G}$ be its complement (i.e., $\overline{G}$ has the same vertices as G and two vertices of different color are joined in $\overline{G}$ if and only if they are not joined in G). Let α_i $(i=1, 2, \cdots, \mu)$ be the edges of $\overline{G}$. Our aim is to prove that $\mu \leqq (n-k)(m-l+1)$. Since G is a graph with property $(k+1, l)$, in $\overline{G}$ we can certainly choose, for each edge α_i some set G_i of $k+1$ blue and l green vertices such that α_i is the only edge in G_i. Let the blue vertices of G_i be $x_0^i, x_1^i, \cdots, x_k^i$, where α_i is incident with the vertex x_0^i. We say that α_i (or G_i) *assigns a weight* $n-k$ to x_0^i and a weight 1 to x_j^i $(j=1, 2, \cdots, k)$ for every $i=1, 2, \cdots, \mu$.

(a) We shall show that the α_i's altogether assign a weight of at most $(n-k)(m-l+1)$ to an arbitrary blue vertex a.

Suppose that r edges are joined to a in $\overline{G}$. Now $r \leqq m-l+1$, for a obviously must be joined to at least $l-1$ green vertices in G. These edges assign a weight of $r(n-k)$ to a.

The other edges of $\overline{G}$ assigning weight to a cannot be joined to these r edges for otherwise there would be at least two edges in some G_i; so they must be in an $(n-1, m-r)$ bipartite subgraph of $\overline{G}$, say G^*. Moreover if an edge α_i of G^* assigns a weight (and consequently weight 1) to a, every vertex, except a of G_i must be in G^*. Hence for each edge α_i of G^* there is a (k, l) bipartite graph of G^*, the only edge of which is α_i. Hence by the induction hypothesis the number of edges α_i of G^* is at most $(n-k)(m-r-l+1)$.

Consequently the weight assigned to a is at most $r(n-k)+(n-k)(m-r-l+1)$ $=(n-k)(m-l+1)$.

(b) Now each α_i $(i=1, 2, \cdots, \mu)$ assigns a weight n to the set of blue vertices. Thus by (a):

$$n\mu \leqq n(n-k)(m-l+1).$$

So $\mu \leqq (n-k)(m-l+1)$, and this proves the theorem.

It is easily seen that this result is best possible. One of the many extremal graphs is the following.

Let $x_1, x_2, \cdots, x_n$ be the blue and $y_1, y_2, \cdots, y_m$ the green vertices. Connect the vertices $x_1, x_2, \cdots, x_{k-1}$ with every green vertex and the other blue vertices with $y_1, y_2, \cdots, y_{l-1}$. It is almost obvious that this n by m graph has property (k, l), and it has $(k-1)m+(l-1)n-(k-1)(l-1)$ edges. We do not make an attempt to enumerate all the extremal graphs.

We may restate the theorem in a slightly weaker form: If an n by m bipartite graph does not contain a complete (k, l) graph but upon adding any new edge it loses this property, then the graph has at least $(k-1)m+(l-1)n-(k-1)(l-1)$ edges. This statement can be considered as the dual of a problem of Zarankiewicz (cf. [2]), who proposed the problem of determining the graphs with this property which have the maximal number of edges.

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SOME CONGRUENCES FOR k LINE PARTITIONS OF A NUMBER

J. M. GANDHI, University of Alberta, Edmonton

In this note we derive the following interesting congruences for $t_k(n)$, the k line partitions of a number n:

$$t_2(2n) + t_2(2n+1) \equiv 0 \pmod 2, \tag{1}$$

$$t_3(3n+4) \equiv t_3(3n+3) \pmod 3 \tag{2}$$

$$t_4(4n) \equiv t_4(4n+2) \equiv t_4(4n+1) \pmod 2 \tag{3}$$

$$t_4(4n+3) \equiv 0 \pmod 2 \tag{4}$$

$$t_5(5n+1) \equiv t_5(5n+3) \pmod 5 \tag{5}$$

$$t_5(5n+2) \equiv t_5(5n+4) \pmod 5, \tag{6}$$

where $t_k(n)$ is defined as

$$\prod_m (1-x^m)^{-\min(m,k)} = \sum_{n=0}^{\infty} t_k(n)x^n. \tag{7}$$

Define $p_\gamma(n)$ as

$$\left[\prod_m (1 - x^m)\right]^\gamma = \sum_{n=0}^{\infty} p_\gamma(n)x^n. \tag{8}$$

Then from the identities

$$t_2(n) = p_{-2}(n) - p_{-2}(n-1) \tag{9}$$

and

$$t_3(n) = p_{-3}(n) - 2p_{-3}(n-1) + 2p_{-3}(n-3) - p_{-3}(n-4). \tag{10}$$

Cheema and Gordon [1] proved that

$$t_2(n) \equiv 0 \pmod 5 \quad \text{if } n \equiv 3 \text{ or } 4 \pmod 5 \tag{11}$$

and

$$t_3(n) \equiv 0 \pmod 3 \quad \text{if} \quad n \equiv 2 \pmod 3. \tag{12}$$

Using the following two theorems of Gandhi [2] namely

THEOREM 1. *If* $(k/n) = (m/t) = (m, t) = 1$ *then* $p_{\alpha k}(n) \equiv 0 \pmod m$ *for* $\alpha = \pm 1$.

THEOREM 2. *If* R *is a prime, then* $p_{kR}(mR) \equiv p_k(m) \pmod R$ *the congruences* (11) *and* (12) *immediately follow from the identities* (9) *and* (10).

Now replacing n by $2n$ and $2n+1$ respectively in (9) we get

$$t_2(2n) = p_{-2}(2n) - p_{-2}(2n-1), \tag{13}$$

$$t_2(2n+1) = p_{-2}(2n+1) - p_{-2}(2n). \tag{14}$$

Adding (13) and (14) we get (1).

In view of Theorems 1 and 2, we have

$$p_{-2}(2n) \equiv p_{-1}(n) \pmod 2 \quad \text{and} \quad p_{-2}(2n+1) \equiv 0 \pmod 2. \tag{15}$$

Using (15), from (13) and (14) we get

$$t_{2n} \equiv p_{-1}(n) \pmod 2 \tag{16}$$

$$t_2(2n+1) \equiv p_{-1}(n) \pmod 2. \tag{17}$$

From which we conclude that for a particular value of n, $t_2(2n)$, $t_2(2n+1)$ and $p_{-1}(n)$ have the same parity. It is to be noted that with the definition of $p_\gamma(n)$ as given by (8), $p_{-1}(n)$ denotes the unrestricted partitions of a number.

Now in identity (10), replace n by $3n+1$ and $3n+3$ respectively to get

$$t_3(3n+1) = p_{-3}(3n+1) - 2p_{-3}(3n) + 2p_3(3n-2) - p_{-3}(3n-3), \tag{18}$$

$$t_3(3n+3) = p_{-3}(3n+3) - 2p_{-3}(3n+2) + 2p_{-3}(3n) - p_{-3}(3n-1). \tag{19}$$

In view of Theorems 1 and 2, we have $p_{-3}(3n+1)$, $p_{-3}(3n-2)$,

$$p_{-3}(3n+2), \qquad p_{-3}(3n-1) \text{ each} \equiv 0 \pmod 3$$

and

$$\begin{aligned} p_{-3}(3n) &\equiv p_{-1}(n), \qquad p_{-3}(3n+3) \equiv p_{-1}(n+1) \\ p_{-3}(3n-3) &\equiv p_{-1}(n-1) \pmod 3. \end{aligned} \tag{20}$$

Using (20) from (18) and (19) we get

$$t_3(3n+1) \equiv p_{-1}(n) - p_{-1}(n-1) \bmod 3), \tag{21}$$

$$t_3(3n+3) \equiv p_{-1}(n+1) - p_{-1}(n) \pmod 3. \tag{22}$$

Replacing n by $n+1$ in (21) and subtracting it from (22), we get congruence (2).

Congruences (4)–(6) can be proved exactly on the same lines as that of congruence (2) and follow from the same technique and observing that

$$\begin{aligned} \sum_{n=0}^{\infty} t_4(n)x^n &= \left\{ \sum_{n=0}^{\infty} p_{-4}(n)x^n \right\} (1-x)^3(1-x^2)^2(1-x^3) \\ &= \sum_{n=0}^{\infty} p_{-4}(n)x^n \left\{ \sum_{i=0}^{10} a_i x^i \right\} \end{aligned}$$

which implies that $t_4(n) = \sum_{i=0}^{10} a_i p_{-4}(n-i)$ and a similar expression for $t_5(n)$, namely

$$t_5(n) = \sum_{i=0}^{20} b_i p_{-5}(n-i).$$

For future reference we give below the values of the coefficients a_i and b_i.

$$\begin{aligned} &a_0 = +1, \quad a_1 = -3, \quad a_2 = +1, \quad a_3 = +4, \quad a_4 = -2, \quad a_5 = -2, \\ &b_0 = +1, \quad b_1 = -4, \quad b_2 = +3, \quad b_3 = +6, \quad b_4 = -7, \quad b_5 = -2, \\ &b_6 = -4, \quad b_7 = +10, \quad b_8 = +6, \quad b_9 = -10, \quad b_{10} = +2. \end{aligned}$$

The remaining a_i's and b_i's can be calculated from the formulae $a_{10-i} = a_i$ and $b_{20-i} = b_i$.

Added in Proof: We have obtained a large number of congruences for $t_k(n)$, where $k = 6, 7, 8$ and 9, which will be published soon.

Acknowledgments. The author is grateful to Prof. L. Carlitz and Prof. H. Gupta for their kind encouragement and to Prof. M. F. Soonawala for helpful discussions.

The author is on leave from University of Rajasthan, Jaipur (India).

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A GENERATING FUNCTION FOR THE GENERALIZED BESSEL POLYNOMIALS

MRINAL KANTI DAS, c/o Bangabasi College, Calcutta, India

1. In [1] Burchnall remarked that his operational methods enabled him to repair one omission in [2], namely the failure to supply a generating function for the generalized Bessel polynomials $y_n(x; a, b)$. The generating function for the polynomials $y_n(x; a, b)$ supplied by Burchnall is

$$[\tfrac{1}{2} + \tfrac{1}{2}(1 - 2xt)^{1/2}]^{2-a}(1 - 2xt)^{-1/2} \exp\left[\frac{b}{2x}\{1 - (1 - 2xt)^{1/2}\}\right] = \sum_{n=0}^{\infty} \frac{(b/2)^n y_n(x; a, b)}{n!} t^n. \tag{1.1}$$

The object of this note is to present a very simple proof of (1.1) based on Lagrange's theorem.

2. The generalized Bessel polynomials $y_n(x; a, b)$ can be defined by

$$y_n(x; a, b) = b^{-n}x^{2-a}e^{b/x}D^n(x^{2n+a-2}e^{-b/x}). \tag{2.1}$$

Now Lagrange's theorem states that if $u = x + t\phi(u)$ where, when u is many-valued, that branch is taken which converges to x when $t \to 0$, then

$$f(u) = f(x) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{dx^{n-1}}\left[\{\phi(x)\}^n \frac{d}{dx} f(x)\right]. \tag{2.2}$$

From (2.2) we easily deduce that

$$\frac{\partial}{\partial x} f(u) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n[\{\phi(n)\}^n \cdot f'(x)]. \tag{2.3}$$

We remark that (2.3) may be conveniently applied to find the generating functions of those orthogonal polynomials whose Rodrigues' formulae are such that Taylor's formula, viz., $e^{tD}f(x) = f(x+t)$, fails to supply the desired generating functions.

Now let us rediscover the generating function (1.1) by use of Lagrange's theorem. For this purpose we take $\phi(x) = x^2$

$$f'(x) = x^{a-2}e^{-b/x} \quad \text{and} \quad u = x + (t/2)u^2$$

so that

$$u = \frac{1}{t}(1 - \sqrt{(1 - 2xt)}),$$

since we are to have $u = x$, when $t \to 0$. Thus it at once follows from (2.3) that

$$f'(u)\frac{\partial u}{\partial x} = \sum \frac{(t/2)^n}{n!} D^n(x^{2n} \cdot x^{a-2}e^{-b/x}). \tag{2.4}$$

Now we obtain from (2.1) and (2.4)

$$u^{a-2}e^{-b/u}(1-2xt)^{-1/2} = x^{a-2}e^{-b/x}\sum_{n=0}^{\infty}\frac{(b/2)^n y_n(x;\,a,\,b)}{n!}\,t^n$$

whence we derive

$$[1-\tfrac{1}{2}(1-\sqrt{(1-2xt)})]^{2-a}(1-2xt)^{-1/2}\exp[(b/2x)(1-\sqrt{(1-2xt)})] = \sum_{n=0}^{\infty}\frac{(b/2)^n y_n(x;\,a,\,b)}{n!}\,t^n.$$

I thank Dr. S. K. Chatterjea for his kind help in the preparation of this note.

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COMMUTING MAPS LACKING COMMUTING EXTENSIONS

D. R. Anderson and D. C. Kay, University of Wyoming

The following problem was recently proposed by De Marr [1]: Let X be a topological space which has the property that every continuous function (map) from a closed subset X_0 of X to itself may be extended to a map of X. Further, let f and g be two commuting maps of X_0 into X_0, that is, $f(g(x))=g(f(x))$ for all $x\in X_0$. Then, do there exist continuous extensions of f and g to the whole space X which commute? The answer to this is in the negative even for the case $X=[0,\,1]$, as we shall show.

Let $X=[0,\,1]$ and suppose that X_0 denotes a closed subset of X, with f and g commuting maps of X_0 into X_0. In the cases (a) $g=f$, (b) $g=i$ (identity), (c) $g=f=i$ on the boundary of X_0, and (d) $g=C$ (constant), and $f(C)=C$, it is easy to provide extensions to X which commute. In (c) one may take $F=G=i$ on $X-X_0$ and $F=f$, $G=g$ on X_0. The continuity and commutativity of F and G readily follow. Consider, however, the example:

$$\left.\begin{aligned} f(x) &= g(x) = 1 \\ f(x) &= x/2 - 1/3 \\ g(x) &= -x/2 + 2/3 \end{aligned}\right\} \quad \begin{aligned} &\text{for } 0 \leqq x \leqq 1/3, \\ &\text{for } 2/3 \leqq x \leqq 1. \end{aligned}$$

Here, $X_0=[0,\,\frac{1}{3}]\cup[\frac{2}{3},\,1]$.

Let F, G be continuous extensions of f and g, respectively, and suppose that $x_0=\inf\{x\colon F(x)=\frac{2}{3}\}$ and $x_1=\inf\{x\colon G(x)=\frac{2}{3}\}$. Since F and G are continuous, $F(x_0)=G(x_1)=\frac{2}{3}$ and, if $x_0\leqq x_1$ then $G(x_0)\geqq\frac{2}{3}$, with equality only when $x_0=x_1$. Hence

$$G(F(x_0)) = G(2/3) = 1/3,$$

while

$$F(G(x_0)) \leqq 1/6.$$

Thus $G(F(x_0)) \neq F(G(x_0))$. Similarly, if $x_1 < x_0$ then we may prove that $F(G(x_1)) = 0$ and $G(F(x_1)) \geqq \frac{1}{6}$. Therefore no continuous extensions of f and g have the commuting property, even though f and g themselves commute.

We observe that in the above example X_0 was not connected. This is necessary, for otherwise X_0 would be a closed interval and in that event one may always extend commuting maps f and g over X_0 to commuting maps over X. Perhaps the connectivity of X_0 is of some importance to the general problem. Another feature which may be significant in more general spaces is the monotonicity of the functions f and g. In our example, g was monotonic (decreasing) while f was not monotonic.

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FORBIDDEN REGIONS ARE CONVEX

DAVID SINGMASTER AND GERALDA SINGMASTER, American University of Beirut

Ogilvy [1] has raised the following problem: Consider a triangle T of area A. Let S_k be the set of points such that if $p \in S_k$, then there is no line through p which cuts the triangle into pieces of area kA and $(1-k)A$. Is S_k simply connected?

Hoggatt [2] has presented a solution to this problem through use of oblique coordinates. He describes S_k as a "forbidden region" and attributes the problem to Free Jamison [3].

In this note, we shall present an elementary proof that the analogous sets S_k are convex for any region T of finite n-measure in n-space ($n \geqq 2$), using only ideas of continuity.

We may assume, without loss of generality, that T has measure 1. We shall call a hyperplane in n-space simply a plane. We let $V(S)$ be the n-dimensional Lebesgue measure of S which we will call the volume or measure of S.

DEFINITIONS. *A k-plane P is a plane which divides T into two regions having measure k and $1-k$, where we take $0 \leqq k \leqq \frac{1}{2}$.*

A k-point p is a point on some k-plane.

$$S_k = \{p \mid p \text{ is not a } k\text{-point}\}.$$

The following Lemma is an n-dimensional analog of the well-known fact that if T is a set of finite measure in the reals, then the function $f(x) = V(T \cap (-\infty, x))$ is continuous in x.

LEMMA. *If a plane P rotates about an axis A in P (i.e., A is an $(n-2)$-dimensional subset of P), then the measures $V_1(\theta)$, $V_2(\theta)$, of the subsets T_1, T_2 of T which lie on each side of P, vary continuously with θ.*

Proof. Take any $\epsilon>0$. From the definition of Lebesgue measure in n-space, there is a bounded set B such that $V(B\cap T)>1-\epsilon/2$. Let $d(x, y)$ be Euclidean distance. Let $R=\sup\{d(A, t)\,|\,t\in B\cap T\}$ and let $D=\operatorname{diam}(B\cap T)=\sup\{d(s, t)\,|\,s, t\in B\cap T\}$. Then if P rotates through an angle δ, it sweeps through a region of $B\cap T$ which is contained in two sections of an n-dimensional right circular "cylinder" whose axis is A, whose $(n-2)$-dimensional "height" is D^{n-2}, whose radius is R and whose angle is $|\delta|$. Hence the changes in volume of $B\cap T_1$ and $B\cap T_2$ are at most $2(\frac{1}{2}R^2|\delta|)D^{n-2}=R^2D^{n-2}|\delta|$. Consequently, if we take $|\delta|<\epsilon/2R^2D^{n-2}$, then these changes are less than $\epsilon/2$. Adding in the changes which occur in $T\sim B$, which are also less than $\epsilon/2$, we see that $V_1(\theta)$ and $V_2(\theta)$ change by no more than ϵ.

THEOREM 1. *Let p be a k-point. Then p is a j-point for all j, $k\leqq j\leqq\frac{1}{2}$.*

Proof. Let P be a k-plane through p and let A be any axis in P, through p. As P rotates from its original position through an angle θ, let $V_1(\theta)$ and $V_2(\theta)$ be the volumes of the regions T_1 and T_2 lying on each side of P. We may assume that $V_1(0)\leqq V_2(0)$, so that $V_1(0)=k$ and $V_2(0)=1-k$. But when $\theta=\pi$, the regions T_1 and T_2 have been interchanged, so that $V_1(\pi)=1-k$. Since $V_1(\theta)$ is continuous, it takes on every value j such that $k\leqq j\leqq\frac{1}{2}(\leqq 1-k)$.

COROLLARY. $0\leqq k\leqq j\leqq\frac{1}{2}$ *implies* $S_j\subset S_k$.

COROLLARY. *Every point is a $\frac{1}{2}$-point, i.e.,* $S_{1/2}=\phi$.

COROLLARY. $S_k\subset S_0\subset T'$, *where T' is the convex closure of T.*

The last corollary depends on the observation that if $p\notin T'$, then there is a plane through p which does not meet T' and hence, is a 0-plane. It is easy to see that $S_0\neq T'$ can hold. It is clear that S_0 is the interior of T when T is convex, but $S_0\neq\operatorname{int}(T')$ can hold.

THEOREM 2. S_k *is convex for all* k, $0\leqq k\leqq\frac{1}{2}$.

Proof. Let p, q be in $S_k(p\neq q)$, and suppose that r is a point on the line segment pq and $r\notin S_k$. Then r is a k-point and there is a k-plane P through r. Neither p nor q can be on P so the line pq meets P in the unique point r. Therefore p and q lie on opposite sides of P. From the symmetry of the situation, we may suppose that p lies on the same side of P as does the region T_1 of smaller volume $V_1=k$, of the two regions into which P divides T. Consider the plane P' through p and parallel to P. Let T_1' be the region of T_1 which lies on the *outside* of P' (i.e. on the other side from P). Clearly the volume V_1' of T_1' is less than $V_1=k$, since $T_1'\subset T_1$. Let $V_1'=j$; then p is a j-point and $j\leqq k$. Consequently, p is a k-point by Theorem 2, contrary to hypothesis. Hence r cannot be a k-point and S_k must be convex.

COROLLARY. *S_k is simply connected.*

Note that the restriction throughout that $0 \leqq k \leqq \frac{1}{2}$ does not involve any loss of generality since $S_k = S_{1-k}$.

A natural problem suggested by this work is to find $k_0 = \inf\{k \mid S_k = \phi\}$. Hoggatt's work shows that $k_0 = 4/9$ for a triangle in 2-space.

The first author was an NSF Graduate Fellow at the University of California at the time of writing this paper.

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ABSTRACT ALGEBRAS WITH A SINGLE OPERATION AND GROUP-LIKE AXIOMS

D. J. ALLEN, Bennington, Vermont

Abstract algebras and abstract operations are defined in [1]. It is revealing to study the structure of an abstract algebra by focussing attention on a single operation and axioms generalized from group theory. It is assumed throughout that the operations are of finite rank $n > 2$. The structures resulting from the imposition of these axioms are called suggestively hypersemigroups and hypergroups. The main result of this paper is to show that these axioms are necessary and sufficient conditions for n-products in hypersemigroups with identity and hypergroups to be rewritten as iterated binary products derived from the abstract operation. In other words, with either of these two structures, it is always possible to find a plane section of its "Cayley multiplication table" which determines the entire table.

AXIOM I. Associativity:

$$\begin{aligned}(a_1a_2 \cdots a_{n-1}(a_n \cdots a_{2n-1})) &= (a_1a_2 \cdots a_{n-2}(a_{n-1} \cdots a_{2n-2})a_{2n-1}) \\ &= \cdots = ((a_1a_2 \cdots a_n)a_{n+1} \cdots a_{2n-1}).\end{aligned}$$

AXIOM II. Identity: There exists a unique $e \in A$ such that

$$(ae \cdots e) = (eae \cdots e) = \cdots = (e \cdots ea) = a$$

for all $a \in A$.

AXIOM III. Inverse: For each $a \in A$, there exists a unique $(n-1)$-tuple $b_1b_2 \cdots b_{n-1}$, $(b_i \in A)$ such that $(ab_1b_2 \cdots b_{n-1}) = (b_1b_2 \cdots b_{n-1}a) = e$.

DEFINITION I. *A hypersemigroup with identity S_n^1 is a pair composed of a nonempty set S_n^1 and an n-ary operation which satisfies Axioms I and II.*

DEFINITION II. *A hypergroup G_n is a hypersemigroup with identity which also satisfies Axiom III.*

It follows immediately from Axioms I and II that in any hypersemigroup with identity S_n^1 the identity commutes with all elements of S_n^1 in the sense that

$$(ea_1a_2 \cdots a_{n-1}) = (a_1ea_2 \cdots a_{n-1}) = \cdots$$
$$= (a_1a_2 \cdots a_{m-1}ea_m \cdots a_{n-1}) = \cdots = (a_1a_2 \cdots a_{n-1}e).$$

We have, in fact:

LEMMA I. *In any hypersemigroup with identity S_n^1, the value of an n-product $(a_1 \cdots a_n)$ remains unchanged under all permutations which leave the order of the elements $a_i \neq e$ unchanged.*

THEOREM I. *If S_n^1 is an abstract algebra with one operation and if there exists a semigroup with identity S^1 defined on the same set as S_n^1 and whose binary operation is derived from the operation in S_n^1 and such that each n-product in S_n^1 can be rewritten as an iteration of the binary operation in S^1, then it is necessary and sufficient that S_n^1 be a hypersemigroup with identity.*

Proof. It is obvious that the identity in S^1 is also an identity in S_n^1. Likewise, the operation in S_n^1 is associative since S^1 is a semigroup. On the other hand, we can define a binary operation by the relation $(abe \cdots e) = a * b$. The operation $(*)$ is associative since $(a(bce \cdots e)e \cdots e) = ((abe \cdots e)ce \cdots e) \Rightarrow a * (b * c) = (a * b) * c$. We also have $e * a = a * e = a$. Thus the pair $(S_n^1, *)$ is a semigroup with identity S^1. Let $(a_1a_2 \cdots a_n)$ be any n-product in S_n^1. Now $a_1 * a_2 = (a_1a_2e \cdots e)$ and $a_1 * a_2 * a_3 = (a_1a_2a_3e \cdots e)$ as is easily verified. If this process is continued, the product $a_1 * a_2 * \cdots * a_n = (a_1a_2 \cdots a_n)$ can be constructed, which completes the proof.

We now turn to an arbitrary hypergroup G_n.

LEMMA II. *For each of the equations*

$$(xa_1 \cdots a_{n-1}) = b_1$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$(a_1 \cdots a_mxa_{m+1} \cdots a_{n-1}) = b_{m+1}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$(a_1 \cdots a_{n-1}x) = b_n$$

there exists a unique solution in G_n.

Proof. Let A_{n-1}^{-1} represent the $(n-1)$-tuple inverse to a_{n-1}. Then

$$((xa_1 \cdots a_{n-1})A_{n-1}^{-1}) = (xa_1 \cdots a_{n-2}(a_{n-1}A_{n-1}^{-1})) = (xa_1 \cdots a_{n-2}e)$$
$$= (xea_1 \cdots a_{n-2}) = b_1A_{n-1}^{-1}.$$

Repeating this process leads to $x = b_1A_{n-1}^{-1} \cdots A_1^{-1} = c$. Since the A_i^{-1} are each unique, the solution is unique. The other cases are proven in a similar manner.

COROLLARY. *For every* $(n-1)$*-tuple* $a_1 \cdot \cdot \cdot a_{n-1}$ *in* G_n *there exists a unique* $b \in G_n$ *such that* $(ba_1 \cdot \cdot \cdot a_{n-1}) = e$.

THEOREM II. *If* G_n *is an abstract algebra with one operation and if there exists a group* G *defined on the same set as* G_n *and whose binary operation is derived from the operation in* G_n *and such that each n-product in* G_n *can be rewritten as an iteration of the binary operation in* G, *then it is necessary and sufficient that* G_n *be a hypergroup.*

Proof. We know from Theorem I that G_n is a hypersemigroup with identity. For any $a \in G_n$ there exists a unique $(n-1)$-tuple $a^{-1}e \cdot \cdot \cdot e$ such that $(a^{-1}e \cdot \cdot \cdot ea) = (aa^{-1}e \cdot \cdot \cdot e) = e$, since G is a group. On the other hand, we define a binary operation $(*)$ in the same way as in Theorem I. This results in a semigroup with identity G. We see from the corollary to Lemma II that if $ae \cdot \cdot \cdot e$ is an $(n-1)$-tuple in G_n, there exists a unique a^{-1} such that $(a^{-1}ae \cdot \cdot \cdot e) = a^{-1} * a = e$, so that the pair $(G_n, *)$ is a group. The rest of the proof follows as in Theorem I.

I wish to thank the referee for a very valuable suggestion.

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CLASSROOM NOTES

THE SCARCITY OF CROSS PRODUCTS ON EUCLIDEAN SPACES

BERTRAM WALSH, University of California, Los Angeles

When first introduced to the dot product on $\mathbf{R}^n$ (n arbitrary) and the cross product on $\mathbf{R}^3$, students are not unlikely to inquire about the definition of a cross product for $\mathbf{R}^n$, $n \neq 3$. This note points out that by proving the elementary propositions and theorem below, students may answer this question themselves. Namely, they may show that if reasonable demands are made of "cross products" (that they satisfy the axioms for cross products in Apostol's recent Calculus text [2]), then only on $\mathbf{R}^1$, $\mathbf{R}^3$ and $\mathbf{R}^7$ can cross products exist—the one on $\mathbf{R}^1$ being trivial and the ones on $\mathbf{R}^7$ somewhat pathological. We indicate the elementary proofs, for whose rediscovery students will need knowledge of the fact that orthonormal (o.n.) sets in $\mathbf{R}^n$ have cardinalities $\leqq n$, with equality if and only if the o.n. set is a basis; orthogonal projection is required a couple of times, but projection is onto a subspace for which an orthonormal basis has already been constructed, so that the projection can be written out explicitly, componentwise.

The axioms [2, p. 275] are

(5.28) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

(5.29) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

(5.30) $\lambda(\mathbf{a} \times \mathbf{b}) = (\lambda\mathbf{a}) \times \mathbf{b}$

(5.31) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

(5.32) $|\mathbf{a} \times \mathbf{b}| = [|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2]^{1/2}$

for all scalars λ and all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. One can easily verify that in the presence of the other four axioms, (5.32) is equivalent to the statement that the cross product of two orthogonal unit vectors is also a unit vector, so that in spite of its complicated appearance the axiom is natural.

Our principal tools are the identities

(1) $$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -[\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})]$$

and

(2) $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 2(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c};$$

the first of these, which expresses the fact that the linear transformation $\mathbf{b} \to (\mathbf{a} \times \mathbf{b})$ is skew-symmetric, is easily derived from (5.31) by setting $\mathbf{a}+\mathbf{c}$ where $\mathbf{a}$ occurs, using linearity and symmetry, and canceling what one already knows about. The second is derived by two applications of the "polarization" trick: writing (5.32) as

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2$$

and setting $\mathbf{b}+\mathbf{d}$ where $\mathbf{b}$ is, using bilinearity and symmetry to get an expansion of both sides and canceling what one already knows about, one gets

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{d}).$$

This can be written, using (1) and the linearity of the dot product, as

$$-[\mathbf{a} \times (\mathbf{a} \times \mathbf{b})] \cdot \mathbf{d} = [(\mathbf{a} \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a}] \cdot \mathbf{d}$$

and since $\mathbf{d} \in \mathbf{R}^n$ is arbitrary, this gives

(3) $$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b},$$

a useful special case of (2). Now setting $\mathbf{a}+\mathbf{c}$ wherever $\mathbf{a}$ is in (3), expanding by linearity, canceling what one already knows about and using the skew-symmetry of the cross product (i.e. (5.28)) leads directly to (2). A useful special case, incidentally, is

(4) For mutually orthogonal $\mathbf{a}, \mathbf{b}, \mathbf{c}$, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$

which one sees by observing that the right side of (2) is zero whenever $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are mutually orthogonal. The identities (1), (2), (3) and (4) hold for any cross product satisfying the axioms, on any $\mathbf{R}^n$.

Given such a cross product on $\mathbf{R}^n$, we make two definitions:

(I) A vector subspace A of $\mathbf{R}^n$ is *closed under* $\times$ if whenever $\mathbf{a}_1 \in A$ and $\mathbf{a}_2 \in A$, then also $\mathbf{a}_1 \times \mathbf{a}_2 \in A$.

(II) A subspace B of $\mathbf{R}^n$ is *stable under* $A\times$, where $A \subseteq \mathbf{R}^n$, if whenever $\mathbf{a} \in A$ and $\mathbf{b} \in B$, then $\mathbf{a} \times \mathbf{b} \in B$.

So A is closed under $\times$ iff stable under $A\times$. We now have

PROPOSITION 1. *Let $A \subseteq \mathbf{R}^n$, B be a subspace of $\mathbf{R}^n$. If B is stable under $A\times$, then so is its orthogonal complement $B^\perp = \{\mathbf{c} \mid \mathbf{c} \in \mathbf{R}^n, \mathbf{b}\cdot\mathbf{c} = 0$ for all $\mathbf{b} \in B\}$.*

Proof. If $\mathbf{c} \in B^\perp$, then for any $\mathbf{a} \in A$ and $\mathbf{b} \in B$

$$\mathbf{b}\cdot(\mathbf{a} \times \mathbf{c}) = -[(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}] = 0 \quad \text{since } \mathbf{a} \times \mathbf{b} \in B.$$

Thus for any $\mathbf{b} \in B$, $\mathbf{b} \perp (\mathbf{a}\times\mathbf{c})$, and so $\mathbf{a}\times\mathbf{c} \in B^\perp$. ∎

PROPOSITION 2. *Let A be a subspace of $\mathbf{R}^n$ which is closed under $\times$ and possesses an orthonormal basis $\{\mathbf{f}_1, \cdots, \mathbf{f}_k\}$. Let $\mathbf{b} \in A^\perp$. Then the vectors $\{\mathbf{b}, \mathbf{f}_1\times\mathbf{b}, \cdots, \mathbf{f}_k\times\mathbf{b}\}$ lie in $A^\perp$ and are mutually orthogonal and of the same length as $\mathbf{b}$. In particular if $\mathbf{b}$ is a unit vector they form an orthonormal set of $k+1$ elements in $A^\perp$.*

Proof. That $\{\mathbf{b}, \mathbf{f}_1\times\mathbf{b}, \cdots\} \subseteq A^\perp$ follows from Proposition 1. The orthogonality relations and lengths follow from $\mathbf{b}\cdot(\mathbf{f}_i\times\mathbf{b}) = 0$ and

$$\begin{aligned}(\mathbf{f}_i \times \mathbf{b})\cdot(\mathbf{f}_j \times \mathbf{b}) &= (\mathbf{b} \times \mathbf{f}_i)\cdot(\mathbf{b} \times \mathbf{f}_j)\\ &= -[\mathbf{b} \times (\mathbf{b} \times \mathbf{f}_i)\cdot\mathbf{f}_j] \quad \text{by (1)}\\ &= -[(\mathbf{b}\cdot\mathbf{f}_i)\mathbf{b} - (\mathbf{b}\cdot\mathbf{b})\mathbf{f}_i]\cdot\mathbf{f}_j = (\mathbf{f}_i\cdot\mathbf{f}_j)(\mathbf{b}\cdot\mathbf{b}).\end{aligned}$$ ∎

It is easy to verify that the only cross product on $\mathbf{R}^1$ is identically zero, and that there is no cross product on $\mathbf{R}^2$: it couldn't be zero, and there aren't enough dimensions for it to be nonzero. Suppose we have a cross product on $\mathbf{R}^n$, then, $n \geqq 3$. If $\mathbf{e}_1$ and $\mathbf{e}_2$ are an orthonormal pair of elements of $\mathbf{R}^n$, then $\mathbf{e}_1\times\mathbf{e}_2$ is a unit (by (5.32)) vector normal to both $\mathbf{e}_1$ and $\mathbf{e}_2$; set $\mathbf{e}_3 = \mathbf{e}_1\times\mathbf{e}_2$. Let H be the linear subspace of $\mathbf{R}^n$ spanned by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Regardless of what n may be,

PROPOSITION 3. *H is closed under $\times$.*

Proof. Bilinearity of $\times$ insures that it suffices to check this on basis vectors. By identity (3)

$$\begin{aligned}\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_2 \times (\mathbf{e}_1 \times \mathbf{e}_2) &= -[\mathbf{e}_2 \times (\mathbf{e}_2 \times \mathbf{e}_1)]\\ &= -[(\mathbf{e}_2\cdot\mathbf{e}_1)\mathbf{e}_2 - (\mathbf{e}_2\cdot\mathbf{e}_2)\mathbf{e}_1] = \mathbf{e}_1\end{aligned}$$

and similarly $\mathbf{e}_3\times\mathbf{e}_1 = \mathbf{e}_2$. ∎

Thus the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ has the same multiplication table as the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for $\mathbf{R}^3$ with the usual cross product.

One can resolve any vector $\mathbf{a} \in \mathbf{R}^n$ into

$$\mathbf{a} = \sum_{i=1}^{3} (\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}_i + \left[\mathbf{a} - \sum_{i=1}^{3} (\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}_i\right]$$

with the first component in H and the second in $H^\perp$. Thus either $H=\mathbf{R}^n$, in which case $n=3$, or else it is possible to find a unit vector $\mathbf{m}\in H^\perp$. Suppose the latter. Then by Proposition 2, the set $\{\mathbf{m}, \mathbf{e}_1\times\mathbf{m}, \mathbf{e}_2\times\mathbf{m}, \mathbf{e}_3\times\mathbf{m}\}$ is an orthonormal set in $H^\perp$, and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{m}, \mathbf{e}_1\times\mathbf{m}, \mathbf{e}_2\times\mathbf{m}, \mathbf{e}_3\times\mathbf{m}\}$ is an orthonormal set in $\mathbf{R}^n$, so $n\geqq 7$. Let C denote the subspace of $\mathbf{R}^n$ spanned by this last set.

PROPOSITION 4. *C is closed under $\times$.*

Proof. One need only check the five types of products which can occur:

(i) $\mathbf{e}_i \times \mathbf{e}_j$. Belongs to $H \subseteq C$, by Proposition 3.

(ii) $\mathbf{e}_i \times \mathbf{m}$. Belongs to C by definition.

(iii) $\mathbf{e}_i \times (\mathbf{e}_j \times \mathbf{m}) = 2(\mathbf{e}_i\cdot\mathbf{m})\mathbf{e}_j - (\mathbf{e}_j\cdot\mathbf{m})\mathbf{e}_i - (\mathbf{e}_i\cdot\mathbf{e}_j)\mathbf{m} + \mathbf{m}\times(\mathbf{e}_i\times\mathbf{e}_j)$
$= -(\mathbf{e}_i\times\mathbf{e}_j)\times\mathbf{m} - (\mathbf{e}_i\cdot\mathbf{e}_j)\mathbf{m} \in C.$

(iv) $\mathbf{m}\times(\mathbf{e}_i\times\mathbf{m}) = -\mathbf{m}\times(\mathbf{m}\times\mathbf{e}_i) = (\mathbf{m}\cdot\mathbf{m})\mathbf{e}_i - (\mathbf{m}\cdot\mathbf{e}_i)\mathbf{m} = \mathbf{e}_i \in C.$

(v) $(\mathbf{e}_i\times\mathbf{m})\times(\mathbf{e}_j\times\mathbf{m}) = (\mathbf{m}\times\mathbf{e}_i)\times(\mathbf{m}\times\mathbf{e}_j)$
$= \mathbf{e}_j \times [(\mathbf{m}\times\mathbf{e}_i)\times\mathbf{m}]$ by (4)
$= \mathbf{e}_j \times [\mathbf{m}\times(\mathbf{e}_i\times\mathbf{m})]$
$= \mathbf{e}_j \times \mathbf{e}_i$ by (iv) above. ∎

One can resolve any vector $\mathbf{a}\in\mathbf{R}^n$ into

$$\mathbf{a} = \left[\sum_{i=1}^{3} (\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}_i + (\mathbf{a}\cdot\mathbf{m})\mathbf{m} + \sum_{i=1}^{3} (\mathbf{a}\cdot(\mathbf{e}_i\times\mathbf{m}))(\mathbf{e}_i\times\mathbf{m})\right]$$
$$+ \left[\mathbf{a} - \sum_{i=1}^{3} (\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}_i - (\mathbf{a}\cdot\mathbf{m})\mathbf{m} - \sum_{i=1}^{3} (\mathbf{a}\cdot(\mathbf{e}_i\times\mathbf{m}))(\mathbf{e}_i\times\mathbf{m})\right]$$

with the first component in C and the second in $C^\perp$. Thus either $C=\mathbf{R}^n$, in which case $n=7$, or it is possible to find a unit vector $\mathbf{n}\in C^\perp$.

THEOREM. *If there exists a cross product on $\mathbf{R}^n$ which satisfies Apostol's axioms, then $n=1$, 3 or 7. Conversely, there exist cross products on these three spaces.*

Proof. We have seen that for $n\leqq 3$, cross products exist precisely when $n=1$ or 3, the cross product on $\mathbf{R}^1$ being identically zero and the two cross products on $\mathbf{R}^3$ being the classical ones, and that if $\mathbf{R}^n$ has a cross product for $n>3$, then $n\geqq 7$. If $n>7$, then one can find a unit vector $\mathbf{n}\in C^\perp$, where C is constructed as above. We shall show, however, that the existence of such a vector would lead to a contradiction. Indeed, should such an $\mathbf{n}$ exist, then $\mathbf{m}\times\mathbf{n}$ is also a unit vector in $C^\perp$; set $\mathbf{p}=\mathbf{m}\times\mathbf{n}$. Just as in Proposition 3, we have $\mathbf{n}\times\mathbf{p}=\mathbf{m}$ and $\mathbf{p}\times\mathbf{m}=\mathbf{n}$. Let us compute some products: we find that, for $i\neq j$,

$$
\begin{aligned}
&(\mathbf{e}_i \times \mathbf{m}) \times (\mathbf{e}_j \times \mathbf{n}) \\
&\quad = (\mathbf{e}_i \times \mathbf{m}) \times (\mathbf{e}_j \times (\mathbf{p} \times \mathbf{m})) \quad \text{by (4)} \\
&\quad = (\mathbf{e}_i \times \mathbf{m}) \times (\mathbf{m} \times (\mathbf{e}_j \times \mathbf{p})) \quad \text{by (4)} \\
&\quad = (\mathbf{e}_j \times \mathbf{p}) \times ((\mathbf{e}_i \times \mathbf{m}) \times \mathbf{m}) \\
&\quad = (\mathbf{e}_j \times \mathbf{p}) \times (\mathbf{m} \times (\mathbf{m} \times \mathbf{e}_i)) = (\mathbf{e}_j \times \mathbf{p}) \times [(\mathbf{m}\cdot\mathbf{e}_i)\mathbf{m} - (\mathbf{m}\cdot\mathbf{m})\mathbf{e}_i] \\
&\quad = -(\mathbf{e}_j \times \mathbf{p}) \times \mathbf{e}_i = \mathbf{e}_i \times (\mathbf{e}_j \times \mathbf{p}) = \mathbf{p} \times (\mathbf{e}_i \times \mathbf{e}_j).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(\mathbf{e}_j \times \mathbf{n}) \times (\mathbf{e}_i \times \mathbf{m}) &= \\
(\mathbf{e}_j \times \mathbf{n}) \times (\mathbf{e}_i \times (\mathbf{n} \times \mathbf{p})) &= \\
-(\mathbf{e}_j \times \mathbf{n}) \times (\mathbf{e}_i \times (\mathbf{p} \times \mathbf{n})) &= \quad \text{by (4)} \\
-(\mathbf{e}_j \times \mathbf{n}) \times (\mathbf{n} \times (\mathbf{e}_i \times \mathbf{p})) &= \quad \text{by (4)} \\
-(\mathbf{e}_i \times \mathbf{p}) \times ((\mathbf{e}_j \times \mathbf{n}) \times \mathbf{n}) &= \cdots \text{by (3)} \\
&= -(\mathbf{e}_i \times \mathbf{p}) \times (-\mathbf{e}_j) \\
&= \mathbf{p} \times (\mathbf{e}_i \times \mathbf{e}_j).
\end{aligned}
$$

(The computations indicated by the ellipsis are straightforward.) But putting these together gives $(\mathbf{e}_i \times \mathbf{m}) \times (\mathbf{e}_j \times \mathbf{n}) = (\mathbf{e}_j \times \mathbf{n}) \times (\mathbf{e}_i \times \mathbf{m})$; this contradicts the skew-symmetry of $\times$ unless this particular product is zero. However, $\mathbf{p} \times (\mathbf{e}_i \times \mathbf{e}_j)$ is the product of two perpendicular unit vectors and thus has unit norm. This contradiction shows that the case $n > 7$ cannot arise.

It remains to exhibit a cross product on $\mathbf{R}^7$ which satisfies the axioms. Such a product can be constructed from the familiar one on $\mathbf{R}^3$ as follows: $\mathbf{R}^7$ corresponds 1-1 with the set of all triples $(\mathbf{a}, \lambda, \mathbf{b})$ where $\mathbf{a}$ and $\mathbf{b}$ are in $\mathbf{R}^3$ and $\lambda \in \mathbf{R}$ under the correspondence

$$a_1\mathbf{i}_1 + \cdots + a_7\mathbf{i}_7 \to (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},\ a_4,\ a_5\mathbf{i} + a_6\mathbf{j} + a_7\mathbf{k}).$$

It is easy to see that this correspondence turns addition and scalar multiplication in $\mathbf{R}^7$ into component-by-component addition and scalar multiplication of triples; defining the dot product of triples by

$$(\mathbf{a}_1, \lambda_1, \mathbf{b}_1)\cdot(\mathbf{a}_2, \lambda_2, \mathbf{b}_2) = (\mathbf{a}_1\cdot\mathbf{a}_2) + \lambda_1\lambda_2 + (\mathbf{b}_1\cdot\mathbf{b}_2)$$

we see easily that the dot product of two triples equals the dot product of the two elements of $\mathbf{R}^7$ to which they correspond. We actually define the cross product on these triples: if one recalls what Propositions 3 and 4 told one the multiplication table of C should be, and if one thinks of $(\mathbf{a}, \lambda, \mathbf{b})$ as representing a vector whose component in H is $\mathbf{a}$, whose component along $\mathbf{m}$ is λ, and whose component in the space spanned by $\{\mathbf{e}_1 \times \mathbf{m}, \mathbf{e}_2 \times \mathbf{m}, \mathbf{e}_3 \times \mathbf{m}\}$ is $\mathbf{b} \times \mathbf{m}$, then one is led to the definition

$$(\mathbf{a}_1, \lambda_1, \mathbf{b}_1) \times (\mathbf{a}_2, \lambda_2, \mathbf{b}_2) = ([\lambda_1\mathbf{b}_2 - \lambda_2\mathbf{b}_1 + (\mathbf{a}_1 \times \mathbf{a}_2) - (\mathbf{b}_1 \times \mathbf{b}_2)], [-(\mathbf{a}_1 \cdot \mathbf{b}_2) + (\mathbf{b}_1 \cdot \mathbf{a}_2)], [\lambda_2\mathbf{a}_1 - \lambda_1\mathbf{a}_2 - (\mathbf{a}_1 \times \mathbf{b}_2) - (\mathbf{b}_1 \times \mathbf{a}_2)]).$$

(Note that the cross products inside the brackets in the triple are products in $\mathbf{R}^3$ and thus well defined.) Knowing that the cross product in $\mathbf{R}^3$ satisfies the axioms, one easily verifies that this cross product on $\mathbf{R}^7$ (via identification of $\mathbf{R}^7$ with the triples) satisfies the axioms also. ∎

Remark: The pathological behavior of the cross product on $\mathbf{R}^7$ begins with its lack of anything near uniqueness: in $\mathbf{R}^3$ there are only two choices for the cross product of a given orthonormal pair of vectors, while in $\mathbf{R}^7$ our construction can be modified to yield cross products which choose any of the unit vectors in the five-dimensional space orthogonal to the given pair. The cross products on $\mathbf{R}^7$ also fail to satisfy the Lie identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = 0;$$

indeed

$$\mathbf{m} \times (\mathbf{e}_1 \times \mathbf{e}_2) + \mathbf{e}_2 \times (\mathbf{m} \times \mathbf{e}_1) + (\mathbf{e}_1 \times (\mathbf{e}_2 \times \mathbf{m})) = -3(\mathbf{e}_3 \times \mathbf{m}).$$

Any number of other standard identities also fail in $\mathbf{R}^7$.

Postscript. The discussion above has concentrated on the geometric problem of defining a cross product on a real inner product space, and has attacked this problem via reasoning processes close to those of three-dimensional vector geometry. The theorem it produces is actually equivalent to a classical theorem of Hurwitz which states that a bilinear "multiplication" operation (denoted by juxtaposition) with the property $|\mathbf{xy}| = |\mathbf{x}|\,|\mathbf{y}|$ can be introduced on a finite-dimensional real inner product space if and only if the space has dimension 1, 2, 4 or 8, and that the operation is then "essentially" the multiplication of the reals, complexes, Hamilton quaternions, or Cayley numbers, and actually is one of those multiplications if an identity is present. Indeed, assuming Hurwitz's theorem and being given an $\mathbf{R}^n$ with a cross product as before, write $\mathbf{R}^{n+1}$ as (isometric to) $\mathbf{R}\mathbf{e} \oplus \mathbf{R}^n$ where $\mathbf{e} \in \mathbf{R}^{n+1}$, and define a multiplication on $\mathbf{R}^{n+1}$ by

$$(\lambda\mathbf{e} + \mathbf{a})(\mu\mathbf{e} + \mathbf{b}) = (\lambda\mu - \mathbf{a} \cdot \mathbf{b})\mathbf{e} + (\mu\mathbf{a} + \lambda\mathbf{b} + \mathbf{a} \times \mathbf{b});$$

then it is easy to verify, using the cross product axioms, that this bilinear multiplication has the property $|\lambda\mathbf{e}+\mathbf{a}|\,|\mu\mathbf{e}+\mathbf{b}| = |(\lambda\mathbf{e}+\mathbf{a})(\mu\mathbf{e}+\mathbf{b})|$ and hence $n+1 = 2$, 4 or 8, $n = 1$, 3, or 7. On the other hand, Hurwitz's theorem follows quite readily from ours: given a real inner product space A with a multiplication satisfying $|\mathbf{xy}| = |\mathbf{x}|\,|\mathbf{y}|$ and a left-and-right identity $\mathbf{e}$, one finds that $(\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y}) = (\mathbf{xy} \cdot \mathbf{xy})$ leads, after the usual couple of polarizations, to

$$2(\mathbf{z} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{w}) = (\mathbf{zy} \cdot \mathbf{xw}) + (\mathbf{zw} \cdot \mathbf{xy}), \tag{5}$$

of which we shall need only the cases

$$(6,\ 6') \qquad (\mathbf{e}\cdot\mathbf{x})(\mathbf{y}\cdot\mathbf{y}) = (\mathbf{y}\cdot\mathbf{xy}), \qquad (\mathbf{x}\cdot\mathbf{x})(\mathbf{e}\cdot\mathbf{w}) = (\mathbf{x}\cdot\mathbf{xw})$$

and

$$(7) \qquad 2(\mathbf{z}\cdot\mathbf{e})(\mathbf{y}\cdot\mathbf{e}) = (\mathbf{zy}\cdot\mathbf{e}) + (\mathbf{z}\cdot\mathbf{y}).$$

Now let V denote the subspace $\mathbf{Re}^{\perp}$ of A, and define an operation $\times$ on V by $\mathbf{a}\times\mathbf{b} = \mathbf{ab} + (\mathbf{a}\cdot\mathbf{b})\mathbf{e}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{e} = (\mathbf{ab}\cdot\mathbf{e}) + (\mathbf{a}\cdot\mathbf{b}) = 2(\mathbf{a}\cdot\mathbf{e})(\mathbf{b}\cdot\mathbf{e}) = 0$ by (7), $\times$ maps V bilinearly into itself, i.e. satisfies (5.29) and (5.30), and the version of (5.30) with λ applied to $\mathbf{b}$. It follows easily from (6) and (6′) that $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0 = \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})$, so (5.31) holds; moreover, it is not difficult to show from these two versions of (5.31) that (5.28) also holds. Finally, (5.32) follows quickly from (7). Thus by our theorem (which, it is easy to check, does not rest on finite-dimensionality hypotheses) V is 7, 3, 1 or possibly 0-dimensional, A is 8, 4, 2 or 1-dimensional, and quite easily seen to be isomorphic to the Cayley numbers, quaternions, complexes or reals. An elegant algebraic treatment of Hurwitz's theorem is to be found in [3].

Hurwitz's classical theorem suggests generalizations in various directions. We have noted in passing that (in the presence of an identity) finite dimensionality is a useless hypothesis; Wright [4] has shown that a nonassociative real normed division algebra with $|xy| = |x|\,|y|$ must actually have an inner-product norm, thus be one of our four standard algebras. A "nonassociative Gelfand-Mazur theorem," with merely $|xy| \leqq |x|\,|y|$, seems still to be lacking. In the situation of the classical Hurwitz theorem, the unit sphere of $\mathbb{R}^n$ ($n = 1, 2, 4$ or 8) comes equipped with a continuous product operation, with identity; a long series of profound topological results, culminating in [1], has shown that these values of n are the only ones for which such a continuous product on the unit sphere can exist.

The author thanks the referee for his suggestion that material be appended which would explicitly point out the relation of the preceding material to the Hurwitz theorem and to the recent topological results.

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A COUNTEREXAMPLE TO TWO CONJECTURES IN IDEAL THEORY

R. W. GILMER, JR., Florida State University

This paper contains an example of an indecomposable one-dimensional ring T with identity which is not an integral domain and in which every ideal with prime radical is primary. (A commutative ring R is *n-dimensional* if there is a chain $P_1 \subset \cdots \subset P_{n+1}$ of prime ideals ($\neq R$) of R, but no such chain of $n+2$ prime ideals. Thus a field is 0-dimensional and the domain Z of ordinary integers is 1-dimensional.) The ring T provides a counterexample to a conjecture of Gilmer (page 78 of [1]). We also show that the zero ideal of T provides a counterexample to the following conjecture of Krull mentioned in [2; p. 733] and [3; p. 16]: If A is an ideal of the commutative ring R and if P is a maximal prime belonging to A, then P is a maximal prime ideal belonging to $A(P)$. (See [4; pp. 110, 187] for terminology and notation.) Nagata has also given a counterexample to this conjecture in [5].

To obtain the ring T, we let K be a field and we let $\{X_a\}_{a\in Q}$ be a set of indeterminates over K, where Q denotes the set of rational numbers. We set $D=K[\{X_a\}]$ and we let B be the ideal of D generated by the set $S=\{X_aX_b-X_a$ such that $a<b\}$. We show that $T=D/B$ is the required example.

T is a commutative ring with identity and we first show T is one-dimensional. If P is a proper prime ideal of D containing B there are two cases to consider:

CASE 1. Given $r\in Q$, $X_r\in P$ or $X_r-1\in P$. In this case P is maximal since $D/P\simeq K$.

CASE 2. There exists $r\in Q$ such that $X_r\notin P$, $X_r-1\notin P$. Then if $a<r$ we have $X_a(X_r-1)\in B\subseteq P$ so that $X_a\in P$ since P is prime. Similarly, if $a>r$, $X_a-1\in P$. It follows that

$$P \supseteq P_r = (N_r), \quad \text{where } N_r = \{\{X_a\}_{a<r}, \{X_a - 1\}_{a>r}\};$$

but $D/P_r\simeq K[X_r]$, and proper prime ideals of $K[X_r]$ are maximal. It then follows from elementary ideal theoretic properties of a ring homomorphism (for example, see Theorem 11, p. 151 of [6]) that either $P=P_r$ or P is maximal in D.

We conclude that T is one-dimensional and every nonmaximal prime ideal of T is of the form P_r/B for some $r\in Q$.

To show that every ideal of T with prime radical is primary it suffices to show that for $p\in P_r$, there exists $q_p\in P_r$ such that $p-pq_p\in B$ [1; p. 76]. First, if $p\in N_r$ then either $p=X_b$ where $b<r$, in which case we can take $q_p=X_d$ where $b<d<r$, or $p=X_b-1$ where $b>r$ and we take $q_p=1-X_d$ where $r<d<b$. Now observe that if $p_i-sp_i\in B$ for $1\leqq i\leqq k$ and if $p_{k+1}-tp_{k+1}\in B$, then $p_i-up_j\in B$ for $1\leqq j\leqq k+1$ where $u=s+t-st$.

It now follows by mathematical induction that if N is a finite subset of N_r, then there is an element $v\in P_r$ such that $p-pv\in B$ for each $p\in N$. Because N_r generates P_r, this condition also holds for arbitrary $p\in P_r$.

Finally, we show that the ring T is indecomposable; that is, we must show that 0 and 1 are the only idempotents of T. Because any algebraic relation $v^2 - v \in B$, where $v \in D$, can involve only finitely many of the indeterminates $\{X_a\}$, it suffices to show that if $S' = K[X_1, \cdots, X_n]$ and if B' is the ideal of S' generated by $\{X_i - X_j X_i\}_{i<j}$, then the ring $T' = S'/B'$ is indecomposable. This will follow if we can show B' is not representable as the intersection of two relatively prime ideals properly containing it. [6; p. 178]. We shall prove this on the basis of the following fact, which follows easily by induction:

If in S', P_i' is the prime ideal generated by

$$\{X_1, \cdots, X_{i-1}, X_{i+1} - 1, \cdots, X_n - 1\}, \quad \text{then } B' = \bigcap_{i=1}^{n} P_i'.$$

Now if we had $B' = C' \cap D' = C'D'$, where C' and D' are relatively prime ideals of S', then P_1' would contain one of C' or D', say $P_1' \supseteq C'$. Also $P_2' \supseteq C'$ or D'. But $P_1' + P_2' \subset S'$ and $C' + D' = S'$. Hence $P_2' \supseteq C'$. Then because $P_2' + P_3' \subset S'$, $P_3' \supseteq C'$, etc., so that $C' \subseteq B' \subseteq C'$. This completes the proof that the ring T', and therefore the ring T, is indecomposable.

Hence the ring T provides a counterexample to the conjecture of Gilmer cited previously.

We next note that if $r \in Q$, the ideal M_r generated by the set

$$\{X_a\}_{a \leqq r} \cup \{X_a - 1\}_{a > r}$$

is a maximal ideal of D containing P_r. Further, if

$$u = \sum_{i=1}^{t} b_i X_{a_i} + \sum_{i=1}^{s} c_i (X_{d_i} - 1) \in M_r$$

where $a_1 < \cdots < a_t \leqq r < d_1 < \cdots < d_s$ and if $d \in Q$ is such that $r < d < d_1$, then $X_d^2 - X_d \notin B$ (since D/B is indecomposable) and $u(X_d^2 - X_d) \in B$. This shows that M_r is a maximal prime belonging to B. But $B(M_r) = P_r$ since we always have $B(M_r) \subseteq P_r$ and in this case we have previously proved that given $p \in P_r$, there exists $q_p \in P_r$ such that $p(q_p - 1) \in B$. Now $q_p \in P_r$ implies $q_p - 1 \notin M_r$ so that $p \in B(M_r)$. Consequently $P_r = B(M_r)$ and M_r is not a prime belonging to $B(M_r)$. These observations show that the ideal B of D or the ideal (0) of T provides a counterexample to Krull's conjecture mentioned earlier.

Our proof above shows clearly what conditions are necessary in order that Krull's conjecture hold in a one-dimensional ring R in which each ideal with prime radical is primary. Namely, it is necessary and sufficient that every maximal ideal of height one contain a regular element. This statement then represents a partial converse to Theorem 2 of [1].

As a final remark, we observe that if M' is the ideal of D generated by $\{X_a\}_{a \in Q}$, M' is a minimal prime ideal of B. Yet we have already shown $B = \sqrt{B} = \bigcap_{r \in Q} P_r$.

Hence B is semi-prime but has imbedded prime components. Nagata gave an example of a semi-prime ideal with this property in [5].

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PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

COLLABORATING EDITORS: J. BARLAZ, Rutgers—The State University; L. CARLITZ, Duke University; HASKELL COHEN, Louisiana State University; H. EVES, University of Maine; M. S. KLAMKIN, Ford Scientific Laboratory; R. C. LYNDON, University of Michigan; and A. WILANSKY, Lehigh University.

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P. O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted legibly on separate, signed sheets and should be mailed before June 30, 1967. Any contributor wishing acknowledgement of his solution should include a self-addressed, stamped postcard. (Solvers outside the United States may omit the stamp.)

E 1955. *Proposed by D. J. Newman, Yeshiva University*

Suppose $x_{n+1}=x_n+1/(x_1+x_2+\cdots+x_n)$ and that $x_1=1$ (thus $x_2=2$, $x_3=7/3$, $x_4=121/48$, $\cdots$). Prove that $x_n\sim\sqrt{2\log n}$.

E 1956. *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India*

Let a, b, c be the sides of a triangle ABC, and let R, r, r_1, r_2, r_3 be the circumradius, inradius, and exradii, and h_i $(i=1, 2, 3)$ be the altitudes from A, B, C to the opposite sides. Prove that

1. $$\frac{h_1}{bcr_1}+\frac{h_2}{car_2}+\frac{h_3}{abr_3}=\frac{1}{2R\cdot r},$$

2. $$\frac{h_1+r_1}{bcr_1}+\frac{h_2+r_2}{car_2}+\frac{h_3+r_3}{abr_3}=\frac{1}{R\cdot r}.$$

E 1957. *Proposed by C. C. Lindner, Coker College, Hartsville, S. C.*

Can the accompanying table be completed to a group table without placing x_1 again on the principal diagonal, where the ith row and ith column are both headed by x_i?

	x_1	x_2	x_3	x_4	$\cdots$	x_n
x_1	x_1					
x_2		x_1	x_4			
x_3		x_5				
x_4						
$\vdots$						
x_n						

E 1958. *Proposed by S. R. Petrick, Air Force Cambridge Research Laboratories*

Find a point P in the plane of the triangle ABC such that

$$\min\{|PA-PB|, |PB-PC|, |PC-PA|\}$$

is maximized and determine the value of this maximum.

E 1959. *Proposed by R. A. Struble, North Carolina State University*

If $f(x, y)\in C^1$ and is periodic in x with least (positive) period $P(y)$, then the partial derivative $f_y(x, y)$ is periodic if and only if $P'(y)=0$.

E 1960. *Proposed by Erwin Just, Bronx Community College*

Prove that the numerator of the fraction determined by $\sum_{k=0}^{2^n-1} 1/(2k+1)^m$, where m is an odd integer, is divisible by 2^{n+1}.

E 1961. *Proposed by L. M. Graves, Chicago, Ill.*

Note that $18\cdot 6=108$, $15\cdot 7=105$, $45\cdot 9=405$. Find all bases α other than 10 for which there exist triples (a, b, c) of digits with a corresponding property: $(\alpha a+b)c=\alpha^2 a+b$, where $0<a<\alpha$, $1<b<\alpha$, $1<c<\alpha$. Also find, for each α, the number of such triples, or a lower bound for the number.

E 1962. *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India*

If $\phi(n)$ denotes Euler's phi-function and $\tau(n)$ is the number of divisors of n, prove that $\phi(n)\tau(n)\geqq n$.

E 1963. *Proposed by Jack C. Abad, City College of San Francisco*

For whole numbers q and n define $S_q(0)=1$, $S_q(n+1)=q^{S_q(n)}$. Find the greatest power of 2 that will divide $S_3(n+1)-S_3(n)$.

E 1964. *Proposed by Erwin Just, Bronx Community College*

Let $\{m_i\}$ be the set of $\phi(m)$ positive integers which are less than m and relatively prime to m. If $m>2$, prove that $\sum_{i=1}^{\phi(m)} 1/m_i$ is not an integer.

SOLUTIONS OF ELEMENTARY PROBLEMS

Notice by the Editors. Six months of effort have failed to produce the files of correspondence regarding Solutions of Elementary Problems sent to the Editors since about March 25, 1966. It will be greatly appreciated if readers (including the proposers) who submitted solutions for problems E 1835 through E 1894 would find it possible to send duplicate solutions as soon as possible directly to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060.

Ring Axioms

E 1812 [1965, 782]. *Proposed by Samuel Stern, State University College, Buffalo, N. Y.*

Let R be a set satisfying all ring axioms, omitting commutativity of addition. If R has a right multiplicative identity prove that R is a ring.

Solution by David M. Cohen, Tenth grade student, Central High School, Philadelphia, Pa. Let e be the right multiplicative identity. We have

$$0 = b[e + (-e)] = b(e) + b(-e) = b + b(-e). \tag{1}$$

Therefore $-b=b(-e)$.

$$0=(-b)+(-a)+a+b=b(-e)+a(-e)+a+b=(b+a)(-e)+a+b. \tag{2}$$

Therefore $a+b=-[(b+a)(-e)]$.

$$(b+a)+(b+a)(-e)=(b+a)(e)+(b+a)(-e)=(b+a)[(e)+(-e)]=0. \tag{3}$$

Therefore $b+a=-[(b+a)(-e)]$.

Hence $a+b=b+a$, completing the proof.

Also solved by R. G. Albert, George Alberts, L. R. Anderson, R. A. Avelsgaard, P. N. Bajaj, T. L. Bartlow, D. A. Blaeuer, A. K. Bose, J. V. Bou, H. G. Bray, F. S. Brenneman, J. L. Brown, Jr., S. I. Brown, A. B. Carter, P. L. Chabot, Gregory Cherlin, D. I. A. Cohen, J. K. Cole, C. R. Conniff, C. G. Cullen, R. D. Davis, M. J. DeLeon, J. F. Dillon, C. W. Dodge, J. W. Duke, E. W. Ewing, J. L. Faix, R. A. Feinman, R. W. Feldmann, Neal Felsinger, N. J. Fine, K. G. Fischer, John Ganci, P. K. Garlick, Tina Handelman, R. B. Hardin, Jr., W. E. Hoff, G. T. Hogan, J. E. Homer, Jr., Colonel Johnson, Jr., Erwin Just, P. C. Kainen, B. C. Klein, Max Klicker, Kenneth Kramer, Daryl Kreiling, E. S. Langford, Milton Legg, R. N. Leggett, Jr., Steven Ligh, C. C. Lindner, G. J. Lubrano, R. E. Maas, C. F. Marion, D. C. B. Marsh, J. J. Martinez, W. E. Mastrocola, J. T. Mathis, C. J. Maxson, J. M. McGuinness, M. R. Meck, Helen Merkel, Michael Merscher, J. J. More, G. S. Musser, Barbara W. Nason, R. W. Neufeld, John Nichols, L. D. Olson, C. B. A. Peck, Harsh Pittie, Richard Poss, Bob Prielipp, Alfred Raws III, J. T. Renfrow, Henry Ricardo,

Azriel Rosenfeld, Nina M. Roy, B. D. Rudin, L. O. Rush, P. S. Schnare, R. N. Schneider, Claude Schochet, Gerald Schrag, G. F. Schumm, Robin Sibson (England), David A. Smith, John M. Smith, LaRon Smith, Al Somayajulu, L. E. Spence, Robert Spira, Preston Stein, Margaret W. Taft, Elaine Tatham, T. K. Teague, E. W. Wallace, F. T. Walsh, A. Wilansky, C. R. Williams, Dale Woods, Paul Yang, K. L. Yocom, and the proposer.

C. J. Maxson points out that the problem was solved by Berman and Silverman [*Near Rings*, this MONTHLY, 66 (1959) 23–34] requiring (instead of the existence of e) only that every element of R be a product of two elements of R. R. A. Feinman shows that the conclusion still holds when the hypothesis is relaxed by permitting non-associative multiplication and asking only that R contain an element which is not a right divisor of zero. Other readers found essentially the same problem in seven current textbooks.

A Row Semi-magic Square

E 1813 [1965, 782]. *Proposed by Milton Legg, University of Minnesota, Duluth*

Let A be a row semi-magic square and let A_j denote the matrix obtained by replacing the jth column of A by a column of 1's. Show that $\det(A_j)$ is independent of j.

Solution by J. Wesley, Union College, Barbourville, Ky. In A_i let all columns except the ith be added to the jth, $j \neq i$. This gives for the jth column of A_i the elements $k-a_{1i}, k-a_{2i}, \cdots, k-a_{ni}$, where k is the magic number. Expanding by this column splits A_i into two determinants, one of them vanishing. In the other, transposing the ith and jth columns gives $\det(A_i)=\det(A_j)$.

Also solved by R. F. Jackson, E. S. Langford, Simeon Reich (Israel), Robin Sibson (England), Sidney Spital, and K. L. Yocom.

Fourteen other correspondents added to the jth column of A all the other columns, obtaining $\det(A)=k \det(A_j)$, and concluded that $\det(A_j)$ is constant. This proof is inconclusive in case $k=0$.

Sums of Two Squares

E 1814 [1965, 782]. *Proposed by R. C. Thompson, University of California, Santa Barbara*

Show that there exist infinitely many sets of three consecutive integers, each of which is a sum of two squares of positive integers.

I. *Solution by W. J. Blundon, Memorial University of Newfoundland.* There is at least one n (e.g. $n=72$) such that n, $n+1$, $n+2$ are each sums of two positive squares. From any such set we may obtain another, larger, set. In fact, if $n=a^2+b^2$, $n+1=c^2+d^2$, $n+2=e^2+f^2$, we have

$$n^2+2n=(af-be)^2+(ae+bf)^2,$$

$$n^2+2n+1=(c^2-d^2)^2+(2cd)^2,$$

$$n^2+2n+2=(n+1)^2+1^2.$$

Iteration of this process will produce as many sets as desired.

II. *Solution by P. N. Nagara, Kasetsart University, Bangkok, Thailand.* It is easily verified that

$$(17b - 1)^2 + (34b - 2), \qquad (15b)^2 + (8b)^2, \qquad (17b)^2 + 1^2$$

are consecutive integers. The problem is solved for all b such that $34b - 2 = (2n)^2$ or $b = (2n^2+1)/17$. Now b will be integral when $n = 17a \pm 5$ for all a.

III. *Solution by Maxey Brooke, Sweeny, Texas.* Math. Quest. Educ. Times (2), 3 (1903) 41–43 shows that $(2n)^2+(2n)^2$, $8n^2+1$, and $(2n-1)^2+(2n+1)^2$ are such a triple if n is any triangular number $n = (m^2+m)/2$. See also Dickson, *History of the Theory of Numbers*, II, p. 252.

Also solved by Leon Bankoff, E. Beck (Germany), M. T. L. Bizley (England), D. I. A. Cohen, G. C. Dodds, R. B. Eggleton (Australia), N. J. Fine, Michael Goldberg, M. L. Goodman, Cornelius Groenewoud, Wallace Growney, M. A. Jacobs, Sam Kravitz, Sidney Kravitz, Kent Kromarek, R. C. Lyness (England), C. F. Marion, D. C. B. Marsh, J. J. Martinez, G. F., Melendy, Norman Miller, F. D. Parker, Walter Penney, Robin Sibson (England), D. L. Silverman, G. C. Thompson, E. W. Trost (Switzerland), Benjamin Volk, J. Wesley, J. W. Wilson, Dale Woods, K. L. Yocom, and the proposer.

Note that, of any four consecutive integers, at least one is not representable as the sum of two squares, since it will be of form $4k-1$.

A Pairing Problem

E 1815 [1965, 902]. *Proposed by D. M. Paine, Wells College, Aurora, N. Y.*

Can $2n$ people be paired differently on $2n-1$ occasions, so that each is paired with each other on exactly one occasion?

Solution by Maxey Brooke, Sweeny, Texas. This is a modification of Kirkman's Schoolgirl Problem (*Ladies' and Gentlemen's Diary*, 1850, p. 48). One solution is to put 1 in the center of a circle and the other $2n-1$ numbers equally spaced around the circumference. Draw a diameter through one number and pair the other numbers with chords perpendicular to this diameter. The points located on the diameter and chords are one arrangement. Distinct arrangements are obtained corresponding to each such diameter. M. Kraitchek, *Mathematical Recreations*, Dover, 1942, pp. 226–227, discusses this problem in detail.

Also solved by E. P. Berger & W. J. Conover, L. J. Burton, P. J. Campbell, Gary Chartrand, Jane W. Di Paola, P. M. Ellis, R. P. Farrell, R. P. Fiske, Michael Goldberg, M. L. Goodman, Dale Griffith, D. E. Helton, J. E. Homer, Jr., Erwin Just, Kenneth Kramer & Joel Spruck, R. N. Leggett, Jr., F. W. Lozier, Charles McCracken & J. C. Koegler, R. E. Maas, Lieselotte Miller, J. J. More, P. J. Nyikos, F. D. Parker, C. B. A. Peck, Walter Penney, Harsh Pittie, Walter Sadler, J. S. Scandale, Jr., R. W. Shoemaker, Walter Stromquist, R. G. Thompson, Simon Vatriquant (Belgium), R. J. Walker, J. J. Weinkam, John Wessner, and the proposer.

For other references see J. E. Freund, *Round Robin Mathematics*, this MONTHLY, 63(1956) 112–114; O. Ore, *Graphs and their Uses*, p. 50. Farrell would like to know how many distinct solutions are there of the problem. For a similar problem see E 208 [1938, 479].

An Inequality Related to Stirling's Formula

E 1816 [1965, 902]. *Proposed by J. Garfunkel, Brooklyn, N. Y.* For $n>1$, prove

$$(n!)! > [(n-1)!]^{n!}.$$

Solution by Harsh Pittie, Swarthmore College. The inequality

$$\sum_{k=2}^{m} \log k > \int_1^m \log x dx, \quad m > 2$$

gives us $m! > m^m e^{1-m}$, so that $n^m m! > m^m e(n/e)^m$. Replacing m by $n!$ we obtain the sharpened inequality

$$(n!)! > [(n-1)!]^{n!} \cdot e(n/e)^{n!}.$$

Also solved by Larry Armi & Larry Martin, R. A. Bell, D. G. Beverage, N. P. Bhatia, L. J. Burton, P. J. Campbell, John Christopher, D. D. Culbertson, J. A. Donaldson, R. V. Esperti, R. S. Farrell, J. A. Faucher, Hyman Gabai, P. K. Garlick, Jerry Goodman, D. M. Hancasky, Bill Helliwell, Erwin Just, Kenneth Kramer & Joel Spruck, D. C. B. Marsh, J. J. More, Barbara W. Nason, P. J. Nyikos, C. B. A. Peck, L. A. Ringenberg, A. W. Roberts, J. S. Scandale, Jr., P. S. Schnare, R. Shantaram, Richard Sinkhorn, Al Somayajulu, R. P. Soni, G. P. Speck, Sidney Spital, Preston Stein, Raymond Struble, C. Van de Vyle (Belgium), John Vinson & Bob Juola, Benjamin Volk, J. J. Weinkam, J. Wesley, Dean West, L. R. Wirak, So Chul Yang, K. L. Yocom, Leonard Yonis, and the proposer.

Bhatia, Juola and Vinson obtain the result in the form

$$(n!)! > [(n-1)!]![(n-1)!]^{n!}, \quad n > 1.$$

Campbell makes use of E 1777 [1965, 420]. Marsh uses Mitrinović, *Elementary Inequalities*, 3.16, p. 85–86.

A Divisibility Problem

E 1817 [1965, 902]. *Proposed by C. F. Blackmore, University of Arkansas*

Let $r=(2n+1)^k$, where n and k are nonnegative integers. Show that, for any integer a, $a^r+(a+1)^r+\cdots+(a+2n)^r$ is always divisible by $(2n+1)^{k+1}$.

Solution by D. C. B. Marsh, Colorado School of Mines. This is a consequence of a result from number theory (easily proved by induction on k): For every nonnegative integer k, $a \equiv b \pmod{m}$ implies

$$a^{m^k} \equiv b^{m^k} \pmod{m^{k+1}}. \tag{1}$$

For $r=(2n+1)^k$ and $f(a)=a^r+(a+1)^r+\cdots+(a+2n)^r$, we have

$$f(a+1)-f(a) = (a+2n+1)^r - a^r \equiv 0 \pmod{(2n+1)^{k+1}} \tag{2}$$

by (1). With r odd, $f(-n)=(-n)^r+(-n+1)^r+\cdots+(n-1)^r+n^r=0$ is divisible by $(2n+1)^{k+1}$, whence (2) implies that $f(a)$ is divisible by $(2n+1)^{k+1}$ for every integer a.

Also solved by L. J. Burton, Carey Jensen, Erwin Just, John Keesee, Felix Magnotta, and C. Van de Vyle (Belgium).

An Equiangular Octagon

E 1818 [1965, 902]. *Proposed by I. J. Schoenberg, University of Wisconsin*

Let $A_0A_1 \cdots A_7A_0$ be a convex octagon in the plane having all its angles equal. Show that, if all the sides A_jA_{j+1} ($j=0, 1, \cdots, 7$; $A_8=A_0$) are commensurable, then opposite sides must be equal in pairs, i.e. $A_jA_{j+1}=A_{j+4}A_{j+5}$ ($j=0, 1, 2, 3$).

Solution by Michael Goldberg, Washington, D. C. Let the lengths of the sides be a, b, c, d, e, f, g, h. Then the components parallel to the side a must satisfy the equation

$$h/\sqrt{2} + a + b/\sqrt{2} = f/\sqrt{2} + e + d/\sqrt{2},$$

since each angle is 145°. Hence $(h+b-f-d)/\sqrt{2}=e-a$. If the sides are commensurable, then each member of the equation must be zero since $\sqrt{2}$ is irrational. Hence $e=a$. By symmetry, $f=b$, $g=c$, $h=d$.

Also solved by P. N. Bajaj, Mary B. Bridgham, John Burslem, D. M. Hancasky, Ned Harrell, Robert Maas, D. C. B. Marsh, C. B. A. Peck, L. A. Ringenberg, Sidney Spital, Walter Stromquist, C. Van de Vyle (Belgium), Simon Vatriquant (Belgium), William Wernick, and the proposer.

Probability in an Arrangement of Two Kinds of Elements

E 1819 [1965, 902]. *Proposed by V. K. Rohatgi, The University, Sheffield, England*

Let p, q, k be positive integers such that $pk<q$. Consider a sequence consisting of p plus ones and q minus ones in a random arrangement. Find the probability that at some point in the sequence the number of minus ones up to that point is less than k times the number of plus ones up to that point.

Solution by the proposer. There are $\binom{p+q}{q}$ possible arrangements in all. It suffices to count all possible arrangements in which the number of minus ones at some point is at least k times the number of plus ones up to that point. Noting that the last element in such arrangements could either be a plus one or a minus one it can be shown by induction that there are $r\binom{p+q}{q}$ such arrangements, where $r=(q+1-kp)/(q+1)$. Hence the required probability is $1-r=kp/(q+1)$.

This problem is a variant of the famous Bertrand Ballot Theorem [1, 66] generalized in [2]. It is also a direct generalization of an urn problem solved in [3] where a short historical account may be found.

1. W. Feller, *An Introduction to Probability Theory and its Applications*, vol. I (2nd ed.), 1957.

2. L. Takács, *A generalisation of the ballot problem and its application in the theory of queues*, J. of the Amer. Statistical Assn., 57 (1962) 327–337.

3. SIAM Review, vol. 7, No. 1 (1965) 136.

Also solved by Sidney Spital who employs Theorem 2 of A. Dvoretzky and T. Motzkin, *A problem of arrangements*, Duke Math. J., 14 (1947) 308.

II. *Solution by Ted Aldrich, Western Reserve University.* Define $a_0 = -\infty$, $a_{n+1} = +\infty$, then for k such that $a_{k-1} < x \leqq a_k$, $k \in \{1, 2, \cdots, n+1\}$, we may take

$$f(x) = \tfrac{1}{2}(f_k(x) + b_k) + \tfrac{1}{2}(-1)^{2H}(f_k(x) - b_k),$$

where the argument of H is $\prod_{i=1}^{n}(x - a_i)$.

Also solved by J. T. Burke, George Cherlin, J. A. Donaldson, R. B. Eggleton (Australia), P. K. Garlick, R. L. Farrell, F. N. Fritsch, Michael Goldberg, F. W. Keene, Kenneth Kramer, J. E. McKenna, Beatriz Margolis, D. C. B. Marsh, Steven Minsker, C. B. A. Peck, Donald Quiring, G. S. Rogers, Walter Sadler, P. A. Scheinok, John Vinson, C. Van de Vyle (Belgium), and Anders Vretblad (Sweden).

A Determinant of Binomial Expansions

E 1823 [1965, 903]. *Proposed by W. C. Brady, Washington and Jefferson College*

Show that

$$\begin{vmatrix} (a_0 + b_0)^n & (a_0 + b_1)^n \cdots (a_0 + b_n)^n \\ (a_1 + b_0)^n & (a_1 + b_1)^n \cdots (a_1 + b_n)^n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (a_n + b_0)^n & (a_n + b_1)^n \cdots (a_n + b_n)^n \end{vmatrix} = (-1)^{n(n+1)/2} \prod_{j=0}^{n} \binom{n}{j} \prod_{\substack{j<i \\ i=0}}^{n} (a_i - a_j)(b_i - b_j).$$

Solution by P. G. Kirmser, Kansas State University. Let the determinant to be evaluated by Δ, and replace its ijth element with $\sum_{k=0}^{n} \binom{n}{k} a_{i-1}^{k} b_{j-1}^{n-k}$. Then, by the multiplication rule we can write

$$\Delta = \begin{vmatrix} 1 & a_0 & a_0^2 \cdots a_0^n \\ 1 & a_1 & a_1^2 \cdots a_1^n \\ \cdots & \cdots & \cdots\cdots \\ 1 & a_n & a_n^2 \cdots a_n^n \end{vmatrix} \cdot \begin{vmatrix} \binom{n}{0} b_0^n & \binom{n}{0} b_1^n & \cdots \binom{n}{0} b_n^n \\ \binom{n}{1} b_0^{n-1} & \binom{n}{1} b_1^{n-1} & \cdots \binom{n}{1} b_n^{n-1} \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ \binom{n}{n} b_0^0 & \binom{n}{n} b_1^0 & \cdots \binom{n}{n} b_n^0 \end{vmatrix}.$$

If we reverse the order of columns in the first of these determinants, thereby multiplying it by $(-1)^{n(n+1)/2}$, and remove the binomial coefficient factors in the rows of the second, we have the well-known Vandermonde determinants. Hence the result as stated.

Also solved by P. N. Bajaj, P. K. Garlick, J. E. Homer, Jr., D. C. B. Marsh, Sidney Spital, Benjamin Volk, and the proposer.

An Easy Consequence of Fermat's Theorem

E 1824 [1965, 903]. *Proposed by Douglas Lind, Falls Church, Va.*

Let $p_1, p_2, \cdots, p_n$ be n primes, not necessarily distinct, let $p^{(n)}$ be the product of these primes, and let p' be the integer

$$p' = p^{(n)}(1 - 1/p_1 - 1/p_2 - \cdots - 1/p_n).$$

Prove that

$$\frac{x^{p_n}}{p_n} + \frac{x^{p_n \cdot 1}}{p_{n-1}} + \cdots + \frac{x^{p_1}}{p_1} + \frac{p'x}{p^{(n)}}$$

is an integer for all integral x.

Solution by Harsh Pittie, Swarthmore College. Substituting the expression for p', the sum becomes

$$x + \sum_{k=1}^{n} \frac{x(x^{p_k-1}-1)}{p_k}.$$

Each term in the summation is an integer; for, either $p_k \mid x$, or by Fermat's theorem, $x^{p_k-1} \equiv 1 \pmod{p_k}$. If x is a negative integer no difficulty is encountered.

Also solved by Joseph Arkin, José Asseo, Homer Bechtell, John Burslem, John Christopher, J. A. Donaldson, Ragnar Dybvik (Norway), R. B. Eggleton (Australia), P. K. Garlick, Ray Glenn, Michael Goldberg, Jerry Goodman, A. S. Howard, Erwin Just, Kenneth Kramer, Marijo LeVan, D. C. B. Marsh, J. T. Mathis, Donald Quiring, P. A. Scheinok, Al Somayajulu, Sidney Spital, Preston Stein, A. M. Vaidya (India), C. Van de Vyle (Belgium), Simon Vatriquant (Belgium), John Vinson, Benjamin Volk, J. J. Weinkam, K. L. Yocom, David Zeitlin, and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of advanced problems in this issue should be submitted on separate, signed sheets and should be mailed before August 31, 1967. Any contributor wishing acknowledgement of his solution should include a self-addressed stamped postcard. (Solvers outside the United States may omit the stamp.)

5460. *Proposed by Adriano Behrmann, Sao Paulo, Brazil*

Prove the following series expansion for the Euler constant:

$$C = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{a_n}{(2n-2)(2n-1)(2n)},$$

where a_n satisfies $2^{a_n} < 2n-1 < 2^{a_n+1}$. Compare 4353 [1951, 116].

5461. *Proposed by M. S. Klamkin, Ford Scientific Laboratory*

Show that it is possible in E_n to have $n+1$ mutually orthogonal spheres. What is the maximum number of such spheres?

5462. *Proposed by W. A. McWorter, University of British Columbia*

A finite ring is a field if and only if it has at most one nilpotent and at most two idempotents.

5463. *Proposed by W. G. Spohn, Jr., Applied Physics Laboratory, Johns Hopkins University*

For $n=2, 4, 6, \cdots$, define $f_n(x)=\log(1+x^n)$. Prove $f_n^{(n)}(-1)=0$.

5464. *Proposed by C. M. Petty and W. E. Johnson, Lockheed Aerospace Sciences Laboratory*

If A is a normal matrix (i.e., A commutes with its conjugate transpose), then the characteristic roots of A form a symmetric set with respect to the origin in the complex plane (i.e. if Z is a characteristic root of multiplicity r then $-Z$ is a characteristic root of multiplicity r) if and only if the trace $(A^{2k+1})=0$ for $k=0, 1, 2, \cdots$.

5465. *Proposed by M. S. Klamkin, Ford Scientific Laboratory*

It is a known result that an equilateral triangle cannot be imbedded in a square lattice but that it can be done in a cubic lattice. If in the latter case, one of the vertices is at the origin (0, 0, 0), determine all possible coordinates for the other pair of vertices.

5466. *Proposed by Benjamin Volk, Yeshiva University*

Discuss the convergence, as $x\to\infty$, of

$$\sum_{n\leq x}\frac{1}{n}\left(\sin\frac{x}{n}+\sin\frac{n}{x}\right).$$

Cf. 5203 [1965, 559].

5467. *Proposed by M. V. Subbarao, University of Alberta*

If $|x|<1$, prove that

$$x\left(\frac{1+x}{1-x}\right)+2x^4\left(\frac{1+x^2}{1-x^2}\right)+3x^9\left(\frac{1+x^3}{1-x^3}\right)+\cdots+nx^{n^2}\left(\frac{1+x^n}{1-x^n}\right)+\cdots$$
$$=x\frac{1-x}{(1-x)^2}+x^2\frac{(1-x^4)}{(1-x^2)^2}+x^3\frac{(1-x^9)}{(1-x^3)^2}+\cdots+x^n\frac{(1-x^{n^2})}{(1-x^n)^2}+\cdots.$$

5468. *Proposed by S. W. Williams, Lehigh University*

A subset S of a linear space is called *radial at a point* x if and only if S contains a line segment through x in every direction. A set S is *radially open* if it is radial at each of its points. Let T be the collection of radially open sets in the plane R^2. Show that the topological space (R^2, T) formed by T is separable, but neither Lindelöf, first countable, nor normal. Is it regular?

5469. *Proposed by Charles Green, University of Maine*

Let Z_m be the ring of integers modulo the composite integer m, and let k be a proper factor of m. Then $I=\{x\in Z_m: \bar{k}x=\bar{0}\}$ is an ideal in Z_m. Find necessary and sufficient conditions that I be a field; if it is, characterize the identity.

SOLUTIONS OF ADVANCED PROBLEMS

Correction: Problem 5333 [1966, 1022]. In the last line p. 1023, replace $x-g$ by xg.

Linearly Independent Sets over Countable Subfields

5344 [1965, 1135]. *Proposed by Ben Fitzpatrick, Jr., Auburn University, Alabama*

In proving that a subset of E_n is n-dimensional if and only if it contains a nonempty open subset of E_n, Hurewicz and Wallman (*Dimension Theory*, p. 45) use the intuitively appealing statement that if X and Y are two countable subsets of E_n, "it is always possible to choose the coordinate system so that the axes are in general position (no parallel to a coordinate hyperplane contains more than one point x or one point y) since this condition requires the axes to avoid at most a countable number of directions." Verify the following proposition and show that it implies their statement.

If K is a countable subfield of the field of complex (real) numbers and n is a positive integer, $n>1$, then there exists an $n\times n$ unitary (orthogonal) matrix P such that for each $i=1, \cdots, n$, the elements of the ith row of P are linearly independent over K.

Solution by the proposer. Let S denote the space of all $n\times n$ unitary (orthogonal) matrices, with metric

$$d(A, B) = \left(\sum_{i,j=1}^{n} | a_{ij} - b_{ij} |^2 \right)^{1/2}.$$

Then (S, d) is a complete metric space. Let G denote the collection to which the n-term sequence $\alpha=(\alpha_1, \cdots, \alpha_n)$ belongs if and only if $\alpha_i\in K$ for each $i=1, \cdots, n$, and $\sum_{i=1}^{n}|\alpha_i| \neq 0$. The collection G is countable. For each $\alpha\in G$ and each $i=1, \cdots, n$, denote by $U(\alpha, i)$ the set of all matrices P of S such that $\sum_{j=1}^{n}\alpha_j p_{ij}\neq 0$. It follows that each $U(\alpha, i)$ is open and dense in S. Then $\cap U(\alpha, i)$, the intersection being taken over all integers $i=1, \cdots, n$ and over all elements of G, is the intersection of countably many dense open sets in a complete metric space and thus is dense in S. Let P be an element of this intersection. Then P is the desired unitary (orthogonal) matrix.

Now suppose X and Y are countable subsets of E_n, as is also $X\cup Y$. Let L denote the set of all real numbers α such that $\alpha=s_i-t_i$ for some $i=1, \cdots, n$ and some s and t in $X\cup Y$. Then L is countable and is thus a subset of a countable subfield K of the reals. Consider an orthogonal matrix P as in the proposi-

tion above. Now if the images of two distinct points s and t under the orthogonal transformation P have the same ith coordinate for some $i=1, \cdots, n$, we have a linear combination of the elements of the ith row of P equaling 0, where the scalar multipliers are in L and hence in K.

About $\Sigma 1/n$

5346 [1965, 1136]. *Proposed by Louis Comtet, Boulogne (Seine), France*

Let $s(n)=1+\frac{1}{2}+\frac{1}{3}+\cdots+1/n$; for each $x\geqq 1$, define $n(x)$ by: $s(n(x))\leqq x<s(n(x)+1)$. With the notations: $E(x)=\exp(x-\gamma)-\frac{1}{2}$, ($\gamma=0.57\cdots$ is the Euler constant), $\alpha(x)=1/(\exp(x-1)-1)$, $[u]=$integral part of u; if $x\geqq 2$, prove that

$$\left[E(x)-\frac{3}{2}\alpha(x)\right]\leqq n(x)\leqq\left[E(x)+\frac{1}{12}\alpha(x)\right].$$

(N.B., if $x\geqq 2$, then $\frac{1}{12}\alpha(x)+\frac{3}{2}\alpha(x)<1$, and the formula gives at most two possible values for $n(x)$.)

Solution by the proposer. From the Euler-Maclaurin formula we obtain

$$\log(n)+\gamma+\frac{1}{2n}-\frac{1}{12n^2}\leqq s(n)\leqq\log(n)+\gamma+\frac{1}{2n};$$

consequently, with $n=n(x)$,

$$\log(n)+\gamma+\frac{1}{2n}-\frac{1}{12n^2}\leqq s(n)\leqq x<s(n+1)\leqq\log(n+1)+\gamma+\frac{1}{2(n+1)}.$$

It therefore follows that

$$n\Phi(n)\equiv n\exp\left(\frac{1}{2n}-\frac{1}{12n^2}\right)<\exp(x-\gamma)$$
$$<n\exp\left(\frac{3}{2}\cdot\frac{1}{n}-\frac{4}{4}\cdot\frac{1}{n^2}+\frac{5}{6}\frac{1}{n^3}-\cdots\right)\equiv n\exp[\psi(n)].$$

It is now easy to prove that, if $n\geqq 2$ (corresponding to $x\geqq\frac{3}{2}$), then

$$n\exp(\psi(n))<n+\frac{3}{2}+\frac{3}{2n}\cdot\qquad\text{and}\qquad n\Phi(n)>n+\frac{1}{2}-\frac{1}{12n}\cdot$$

Now, use the well-known inequalities $\log(n+1)<s(n)<1+\log(n)$, which imply $\exp(x-1)-1<n=n(x)<\exp(x)-1$, to obtain the final result.

Determinants in a Matrix over $GF(p)$

5358 [1966, 89]. *Proposed by H. F. Mattson and E. F. Assmus, Jr. Sylvania Electronic Systems, Waltham, Mass.*

Let A be an $m\times n$ matrix over the integers with $2<m+1<n$. Prove that if

p is a prime such that $p<\max(m, n-m)$ then p divides one of the $m\times m$ determinants in A.

I. *Solution by Robert Breusch, Amherst College.* We reduce all the entries mod p. Multiplication of a whole row by a number $a\not\equiv 0 \pmod{p}$ will not change the determinants' divisibility or non-divisibility by p, nor will any of the legitimate row operations. Moreover we allow $p\leqq\max(m, n-m)$.

Now consider the matrix formed by the first m columns. Either its determinant is $\equiv 0 \pmod{p}$, in which case there is nothing more to prove; or it is possible, by a combination of the previously mentioned operations, to bring this matrix into the form of the identity matrix (mod p), so that the whole matrix A has the form

$$\begin{Vmatrix} 1 & 0 & \cdots & 0 & a_{1,m+1} & \cdots & a_{1,n} \\ 0 & 1 & \cdots & 0 & a_{2,m+1} & \cdots & a_{2,n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 & a_{m,m+1} & \cdots & a_{m,n} \end{Vmatrix}.$$

The matrix has at least two rows, and has at least two columns beyond the mth.

(1) Let $p\leqq n-m$. Consider three cases:

(a) $a_{1,r}\equiv 0 \pmod{p}$ for some $r(m+1\leqq r\leqq n)$. Then columns 2, 3, $\cdots$, m, r form a matrix whose determinant is divisible by p.

(b) $a_{2,r}\equiv 0 \pmod{p}$ for some r with $m+1\leqq r\leqq n$; then the determinant of the matrix formed by columns 1, 3, $\cdots$, m, r is divisible by p.

(c) All the $a_{1,r}$ and $a_{2,r}$ are $\not\equiv 0 \pmod{p}$, for $m+1\leqq r\leqq n$. Then (mod p), each of the ratios $a_{1,r}/a_{2,r}$ is one of the numbers 1, 2, $\cdots$, $p-1$; and since there are at least p of these ratios, two of them must be equal to each other, say, for $r=s$ and $r=t$ $(s<t)$. Then the determinant of the matrix formed by columns 3, $\cdots$, m, s, t is $\equiv 0 \pmod{p}$.

(2) If $p\leqq m$, then we work with columns number $m+1$ and $m+2$, and reasoning completely analogous to that of (1) shows again that one of the $m\times m$ matrices must have a determinant which is divisible by p.

II. *Solution by Robert Silverman, Ohio State University.* A stronger result may be proved: Let $L(q, n, m)$ denote a set of q^m ordered n-tuples on q objects, such that no two of the n-tuples agree in as many as m components. Such combinatorial configurations may be viewed as systems of $n-m$ m-dimensional orthogonal latin squares on q letters. They have been studied in [1] where it is proved that for $m\geqq 2$, $q>m$ is implied by $q>n-m$ and $n>m+1$.

Let q be a prime power and consider an $m\times n$ matrix A over $GF(q)$, such that every set of m columns is linearly independent. The row space of A forms an $L(q, n, m)$ configuration, and the result of the problem now follows immediately.

1. Robert Silverman, A metrization for power-sets with application to combinatorial analysis, Canad. J. Math., 12 (1960) 158–176.

For another viewpoint on this problem see also:

2. Carl Maneri and Robert Silverman, A vector-space packing problem, J. of Algebra, to appear.

Also solved by L. Carlitz, M. G. Greening (Australia), and the proposers.

Some Dirichlet Multiplications

5360 [1966, 205]. *Proposed by R. B. Eggleston, Avondale College, Australia*

If μ is the Möbius function, evaluate:

$$S(a) = \sum_{d|a} \mu(d)\mu(a/d) \qquad \text{and} \qquad S^*(a) = \sum_{d|a} \sum_{m|d} \mu(m)\mu(d/m).$$

Solution by H. S. Zuckerman, University of Washington. The problem is the special case $k=2$ of the following

THEOREM. *Let*

$$S_k(n) = \sum_{a_1 \cdots a_k = n} \mu(a_1)\mu(a_2) \cdots \mu(a_k), \qquad S_k^*(n) = \sum_{d|n} S_k(d).$$

Then

$$S_k(n) = \prod_{p^e \| n} (-1)^e \binom{k}{e} \qquad \text{and} \qquad S_k^*(n) = S_{k-1}(n).$$

Proof. It is clear that $S_k(n)$ is multiplicative. The summands in $S_k(p^e)$ are zero unless each a_i is 1 or p, in which case the summands are all $(-1)^e$. There are $\binom{k}{e}$ of these nonzero summands. This proves the first formula. We have also

$$\begin{aligned} S_k(n) &= \sum_{a_1 \cdots a_k = n} \mu(a_1) \cdots \mu(a_{k-1})\mu(n/a_1 \cdots a_{k-1}) \\ &= \sum_{d|n} \sum_{a_1 \cdots a_{k-1} = d} \mu(a_1) \cdots \mu(a_{k-1})\mu(n/d) \\ &= \sum_{d|n} S_{k-1}(d)\mu(n/d). \end{aligned}$$

Applying Möbius' inversion theorem we get $S_{k-1}(n) = \sum_{d|n} S_k(d) = S_k^*(n)$. Note that, in particular, we have $S_1(n) = \mu(n)$,

$$S_2(n) = \begin{cases} 0 & \text{if } n \text{ is not cube-free} \\ (-2)^r & \text{if } n = p_1 p_2 \cdots p_r p_{r+1}^2 p_{r+2}^2 \cdots p_{r+t}^2. \end{cases}$$

Also solved by Anders Bager (Denmark), Gabriel Bastien, Robert Breusch, C. C. Cadogan (West Indies), M. S. Demos, E. P. Emerson, J. D. Featherstone, Addison Fischer, F. Göbel (Netherlands), M. G. Greening (Australia), Emil Grosswald, E. C. Hook, F. T. Howard, N. Johnston & E. M. Horadam (Australia), Betty Kvarda, E. S. Langford & D. H. Wagner, R. L.

McFarland, Lieselotte Miller, C. B. A. Peck, Stanton Philipp, H. E. Reinhardt, M. S. Sarma (India), Margaret B. Seay, R. Sivaramakrishnan (India), David A. Smith, Al Somayajulu, Stephen Spindler, Sidney Spital, N. M. Stephens (England), M. V. Subbarao, M. L. Tepley, E. W. Trost (Switzerland), A. M. Vaidya (India), C. S. Venkataraman (India), C. S. Venkataraman & R. Sivaramakrishnan, Dieter Wolke (Germany), and the proposer.

Fischer achieves the above generalization by setting $\mu_1 = \mu$ and iterating

$$\mu_{m+1}^{(n)} = \sum_{d|n} \mu(d)\mu_m(n/d), \qquad \mu_m(n) = \sum_{d|n} \mu_{m+1}(d),$$

noting that μ_k is identical to S_k.

Smith notes that the calculation of $S(a)$ ($\equiv S_2$) may be found in Cashwell and Everett, *The ring of number-theoretic functions*, Pacific J. Math., 9 (1959) 979.

Derivative of Even Polynomials

5361 [1966, 205]. *Proposed by W. E. Sewell, Duke University*

Let $P_n(z)$ be a polynomial of degree n in $z = re^{i\theta}$ with only even powers of z, and let $|P_n(z)| \leqq M$ for z on the curve C whose polar equation is $r = |\cos \frac{1}{2}\theta|$. Show that, for z on C, $|P_n'(z)| \leqq 3Mn$.

Solution by R. K. Meany, Iowa State University. Let $P_n(z) = \sum_{k=0}^{n} c_k z^k$. If $z = re^{i\theta}$, $r = |\cos \frac{1}{2}\theta| = [(1+\cos\theta)/2]^{1/2}$, and $P_n(z)$ contains only even powers of z, then, on C, $P_n(z)$ becomes a trigonometric polynomial in θ of order $3n/2$, say $Q(\theta)$, and

$$Q'(\theta) = \sum_{k=0}^{n} c_k \left(kr^{k-1}e^{ki\theta}\frac{dr}{d\theta} + kir^k e^{ki\theta} \right) = \left(\frac{dr}{d\theta} + ir \right) e^{i\theta} P_n'(z).$$

Now, $\min|dr/d\theta + ir| = \min[\frac{1}{4}\sin^2\frac{1}{2}\theta + \cos^2\frac{1}{2}\theta]^{1/2} = \frac{1}{2}$, and from Bernstein's inequality (see e.g., S. N. Bernstein, *Leçons sur les propriétés extrémales et la meilleur approximations*, Paris, 1926),

$$\max |Q'(\theta)| \leqq (3n/2) \max |Q(\theta)| \leqq 3nM/2.$$

Hence $|P_n'(z)| = |Q'(\theta)| / |dr/d\theta + ir| \leqq 3nM$.

Also solved by I. N. Baker (England), and the proposer.

On Lexicographic Ordering in an Orthonormal Basis

5362 [1966, 205]. *Proposed by John de Pillis, San Francisco State College*

Let $\mathfrak{X}$ be the n-dimensional vector space of n-tuples with complex entries. Suppose $n = p \cdot q$ for some integers p and q. The elements X of $\mathfrak{X}$ can then be represented by the n-tuple

$$X = (x_{11}, x_{12}, x_{13}, \cdots, x_{1q}, x_{21}, x_{22}, \cdots, x_{2q}, \cdots, x_{p1}, \cdots, x_{pq}).$$

The ordering is lexicographic by subscripts. If every $X \in \mathfrak{X}$ is identified with its $p \times q$ matrix (the ijth entry $= x_{ij}$) then for any orthonormal basis $\{X^1, X^2 \cdots, X^n\}$ of $\mathfrak{X}$, show that $\sum_{i=1}^{n} X^i X^{i*} = q \cdot I_p$, where I_p is the $p \times p$ identity matrix and X^{i*} is the adjoint (conjugate transpose) of X^i.

Solution by Walter Noll, Carnegie Institute of Technology. Let $\{Y^1, \cdots, Y^n\}$ be the set of $p \times q$ matrices whose entries are 0 except for a single 1. It is easily verified that this set is identified with an orthonormal basis of $\mathfrak{X}$ and that

$$(*) \qquad \sum_i Y^i Y^{i*} = qI_p.$$

If $\{X^1, \cdots, X^n\}$ is an arbitrarily orthonormal basis of $\mathfrak{X}$ then there exists a $n \times n$ unitary matrix $\|u_{ij}\|$ such that

$$X^i = \sum_j u_{ij} Y^j, \qquad X^{i*} = \sum_k \bar{u}_{ik} Y^{k*}.$$

We find that

$$\sum_i X^i X^{i*} = \sum_{j,k} Y^j Y^{k*} \left(\sum_i u_{ij} \bar{u}_{ik} \right) = \sum_j Y^j Y^{j*},$$

which, together with (*), yields the desired result.

Also solved by M. G. Greening (Australia), A. S. Householder, D. G. Kabe, Marvin Marcus, J. T. Rosenbaum, and the proposer.

Unitary Operators on a Hilbert Space

5363 [1966, 205]. *Proposed by Joseph Kitchen, Duke University*

Let S be an isometry of a Hilbert space H into itself. Prove that there exists a closed subspace M and a unitary operator U on H such that (a) M^0 (the orthogonal complement of M) lies in the range of S, (b) U acts like S on M, and (c) U acts like S^{-1} on M^0.

Solution by T. Kriete, University of Virginia. Denote the range of S by R, and assume R a proper subspace of H. S^{-1} makes sense on R since if $y \in R$, $y = Sx$ for at least one $x \in H$, and if $Sx_1 = Sx_2$, $\|S(x_1 - x_2)\| = \|x_1 - x_2\| = 0$, so there is only one such x.

A theorem of P. R. Halmos (*Shifts on Hilbert Spaces*, Jour. Reine Angew. Math., 208 (1961) pp. 102 ff.) states that H may be written as the orthogonal direct sum

$$H = N \oplus R^\perp \oplus SR^\perp \oplus S^2R^\perp \oplus S^3R^\perp \oplus \cdots,$$

where S restricted to N is unitary. The solution follows easily from this representation. Let

$$M = R^\perp \oplus S^2R^\perp \oplus S^4R^\perp \oplus \cdots \text{ and } P = SR^\perp \oplus S^3R^\perp \oplus S^5R^\perp \oplus \cdots = SM.$$

Then S^{-1} makes sense on P and $S^{-1}P = M$. Let $M^0 = P \oplus N$. Clearly $H = M \oplus M^0 = M \oplus N \oplus P$. If $f \in H$, write $f = m + n + p$ where $m \in M$, $n \in N$, $p \in P$. Since S is isometric, all of the subspaces which have appeared are closed. Now

$$Uf = Um + Un + Up = Sm + S^{-1}n + S^{-1}p$$

is the described unitary operator.

Also solved by M. D. Mavinkurve, J. T. Rosenbaum, and the proposer.

Collinear Arrays of Lattice Points $\leqq x^a$

5364 [1966, 205]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College*

Let $f(x)$ be an integral valued function defined for all positive integers with $0 \leqq f(x) \leqq x^a$, $0 < a < 1$. (a) Prove that the graph of $f(x)$ contains at least k collinear points for any positive integer k. (b) Is the same conclusion true when $0 \leqq f(x) \leqq x$?

Partial Solution by Dan Marcus, Adelphi University. With $0 < a < 1$ we may show that the graph of $f(x)$ contains, for each k, a horizontal configuration of k points. For suppose to the contrary that each horizontal line contains at most $k-1$ points. Then the number of points of $y=f(x)$ in the region bounded by $y=0$, $y=x^a$, $x=T$ does not exceed $(k-1)(T^a+1)$. But the number of points of the graph in this region must be at least T—a contradiction for sufficiently large T.

Also solved (part (a)) by the proposers. No solution for part (b) was received.

Nonsingular Decompositions of the Identity Matrix

5365 [1966, 205]. *Proposed by A. Himmelfarb, Fordham University*

Let R be a ring with identity and n a natural number greater than 1. Show that there exist two nonsingular matrices whose sum is the identity matrix.

I. *Solution by J. L. Brenner, Stanford Research Institute.* The case n even is taken care of by the decomposition

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},$$

and the case $n>1$ odd by the additional decomposition

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

A solution for any $n>1$ is obtained as the direct sum of matrices of these types.

II. *Solution by M. D. Mavinkurve, Siddharth College, India.* Let $A=(a_{ij})$ be the $n \times n$ matrix with $a_{i,i+1}=1$ for $i=1, \cdots, n-1$, $a_{11}=a_{n1}=1$ and $a_{ij}=0$ otherwise. $B=I-A$. These are nonsingular as may be seen by shifting rows.

Also solved by G. A. Heuer, E. S. Langford & D. H. Wagner, Azriel Rosenfeld, W. R. Scott, and the proposer.

Differentiable Functions with Prescribed Discontinuities

5366 [1966, 206]. *Proposed by H. D. Keesing, University of Wisconsin*

Find an explicit function f on $[0, 1]$ to the real numbers, such that f' is defined on $[0, 1]$ and discontinuous on the rationals in $[0, 1]$.

Solution by L. A. Steen, St. Olaf College. Let $f(x)=x^2 \sin(1/x)$; then f is differentiable everywhere, and f' is discontinuous only at $x=0$. Let $r_1, r_2, \cdots$ be an enumeration of the rationals in $[0, 1]$; define

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Then $g'(x)=\sum_{n=1}^{\infty} 2^{-n} f'(x-r_n)$, since both series are uniformly convergent. Then g' exists and is discontinuous at each rational in $[0, 1]$.

Also solved by Daniel Asimov, Robert Breusch, D. A. Hejhal, H. D. Keesing, E. S. Langford, Jaroslav Lukeš (Czechoslovakia), Dan Marcus, R. E. Moyer, and Michael Schulz.

A variety of problems of this sort have been treated in the literature. For an excellent summary see the paper by A. M. Bruckner and J. L. Leonard on Derivatives in no. 11 of the Slaught Memorial Papers, *Papers in Analysis*, 73 (1966) No. 4, Pt. II, this MONTHLY. Specifically compare the paper by Hailperin referred to on p. 27.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Correspondence about Reviews will be welcome.

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (Teacher Training), L (Library purchase), 15 (junior level)—18 (second graduate year).

Applications in natural science and engineering

Mathematical Methods for Physicists. By George Arfken. Academic Press, New York, 1966. xvi+654 pp. $12.75. Vector analysis, coordinate systems, tensor analysis, matrices and determinants, series, complex variable, second order differential equations, Sturm-Liouville theory, Gamma, Bessel, Legendre and special functions, Fourier series, transforms, integral equations, variations. T (15–17), S, P.

The Matrix Analysis of Vibration. By R. E. D. Bishop, G. M. L. Gladwell, and S. Michaelson. Cambridge, London, 1965. x+403 pp. $19.50. S (15–16), P.

Mathematics and Statistics for Chemists. By C. J. Brookes, I. G. Betteley and S. M. Loxton. Wiley, London and New York, 1966. vii+418 pp. $10.00. ". . . primarily for students of Chemistry, Chemical Engineering and Chemical Technology . . . covers the mathematics and statistics required in most undergraduate courses." T (13), S, P.

to be mathematically sophisticated." Homotopy, the fundamental group, covering spaces and fibrations, polyhedra, homology, products, general cohomology theory and duality, homotopy theory, obstruction theory, spectral sequences and homotopy, groups of spheres. T (year plus course, 17–18), P, L.

Topological Structures. By Wolfgang J. Thron, Holt, Rinehart and Winston, New York, 1966. xi+240 pp. $9.50. For a year course presupposing "baby real variables." Besides "standard material," covers nets, filters, separation axioms, compactness, uniform structure, proximity spaces, topological groups. T (16–17).

Homotopy Theory. By George W. Whitehead. M.I.T. Press, Cambridge, Mass., 1966. viii+124 pp. $6.00. ". . . notes for a second-year graduate course in advanced topology given by Professor Whitehead at M.I.T." T (17–18), S, P.

Max Jeger, Transformation Geometry. English version by A. W. Deicke and A. G. Howson. Wiley, New York, 1966. $3.50 (paperback).

The book contains the essentials of the geometry course which the author regularly gives to his classes at the Oberrealschule (a high level science high-school) at Lucerne, Switzerland. The translation has been done very competently.

In the opinion of the reviewer, a group theoretical treatment of elementary geometry based on the Euclidean groups and affine transformations gives the students a much better insight into the working of both geometry and group theory than the usual foundations course based on projective geometry. The present booklet is by far the best available treatment of this kind for a one-quarter course in Euclidean geometry for students with a minimum of preparation. It is really teachable, in contrast to I. M. Yaglom's Geometric Transformations. It deals with the fundamentals of group theoretic geometry in a sound way. It has many interesting problems, though not enough homework problems by American standards.

On this occasion, the reviewer wants to issue a plea to the publishers of reprints and translations for a new issue, preferably in English translation, of the grandfather of all books on Euclidean transformation geometry, Hadamard's *Géometrie plane.* While that book is pre-Hilbert and therefore completely outdated from the axiomatic point of view, it still can serve as a source of much inspiration in group theoretical thinking and in geometrical problems. Any teacher wanting to supplement the problems in Jeger's book will find an ample supply in Hadamard. Many problems treated in both Yaglom and Jeger can in fact be traced back to Hadamard. Hadamard's example also shows that you can become a famous analyst by teaching geometry.

H. GUGGENHEIMER, University of Minnesota

Functions of a Complex Variable: Theory and Technique. By George F. Carrier, Max Krook, Carl E. Pearson. McGraw-Hill, New York, 1966. ix+438 pp. $10.50.

This book, in spite of its title, is primarily concerned with topics of the technique and application of complex variable theory. It treats conformal mapping, contour integration, some special functions (gamma, hypergeometric, Legendre, and Bessel functions), asymptotic expansions, and transform methods. The final chapter deals with the Wiener-Hopf method and with related material on integral equations. The authors provide many illustrative examples and a well-selected collection of interesting problems. The student who works a substantial number of these will have a good grasp of *applied* function theory. But he will need an experienced mentor! For the *theory* is not treated so effectively. The hints for the problems and those parts of the explanations that are "readily seen," are, by no means, always on the surface.

The methods here exposed will be understandable to the user only if he has had previous acquaintance with the underlying justifications, or if his study is directed by someone in a position to clarify and to tighten a good number of the arguments. But the reader with an introductory course in function theory behind him will find here a large variety of uses of this theory. They are explained lucidly, and there are, in most cases, some allusions to the conditions under which the various techniques are applicable.

This work is in no sense a handbook of special tricks; it is a serious text for a substantial year course in the topics that it treats.

E. C. SCHLESINGER, Connecticut College

Integral Transforms and Operational Calculus. By V. A. Ditkin and A. P. Prudnikov. Translated by D. E. Brown, translation edited by I. N. Sneddon. Pergamon Press, Oxford, 1965. (International series of Monographs in Pure and Applied Mathematics, Volume 78.) xii+526 pp. $15.50.

This book contains mountains of information but no proofs. Weighted more than two-to-one in favor of "tables" over "theory," the work touches on Fourier, Laplace, Mellin, Hankel, Meijer, Kontorovitch-Lebedev, Mehler-Fock, and Hilbert transforms. The Laplace transform receives, along with a table of over four hundred entries, a somewhat fuller theoretical treatment, including discussions of the inversion problem and of several applications. The "operational calculus" in the title refers to a chapter on Mikusinskii's theory. One innovation is the authors' replacement of the usual convolution by its derivative; this device makes convolution with a constant function coincide with ordinary multiplication by the constant. The chapter concludes with a brief but valuable sketch of an analogous calculus for the operator $d/dt(t(d/dt))$, in which the role of the Laplace transform is played by a certain Bessel transform.

The reader looking for a rigorous development of integral transform theory will find little here other than references to the bibliography; it is a book for those who wish to write down solutions. A beginner would find the exposition rather terse, and had better go into the tables armed with a stout heart and a copy of Whittaker and Watson. An expert might value the book as a compendious reference.

R. C. RIDDELL, University of British Columbia

Vector Analysis. By Barry Spain. Van Nostrand, London, 1965. 114 pages. $5.85.

This book contains a development of vector calculus, assuming familiarity with the calculus of real-valued functions of two and three variables. Coordinate-free definitions are given and, where there are proofs, they are coordinate-free. The book is not intended as a first introduction to the subject. The author assumes that the topics are motivated by previous acquaintance from, for example, a physics course. This reviewer concludes that the book intends to provide a mathematical foundation for the techniques of vector calculus. If this is the aim, then the book is unsuccessful. Definitions and theorems are not clearly stated. There has been an apparently arbitrary choice of what is proved and what is left unproved and many "proofs" are simply unjustified manipulations with the magic notation of Leibniz. There are some serious errors. For example, on page 33 the author concludes that a vector-valued function of one real variable is equal to its Taylor series if one only assumes that (for some n) the $(n+1)$st derivative is continuous. On page 62, an example is presented to show that curl $\mathbf{a}=0$ on a multiply-connected region does not imply that the line integral is independent of the path. This example is fallacious and, moreover, the calculation given to support it (a blindly performed change of variable in an integral) should cause the reader to question many other "proofs" which hinge on similar calculations.

H. E. CHRESTENSON, Reed College

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of California, Berkeley: Professor L. A. Henkin has accepted the position of Acting Chairman of the Department of Mathematics; Associate Professor R. S. Lehman has been promoted to Professor and has accepted the position of Vice Chairman of the Department of Mathematics; Professors J. L. Kelley and P. E. Thomas have been appointed Research Professors in the Miller Institute for Basic Research, Berkeley; Dr. David Gale, Brown University, has been appointed Professor of Industrial Engineering and Mathematics; Associate Professor Hewitt Kenyon, George Washington University, has been appointed Visiting Associate Professor; Associate Professor L. E. Dubins has been promoted to Professor.

Central Missouri State College: Mr. C. H. Best and Mr. L. S. Dilley have been promoted to Assistant Professors.

Harvey Mudd College: Associate Professor C. S. Coleman has been promoted to Professor; Assistant Professors A. F. Pixley and A. M. White have been promoted to Associate Professors.

Louisiana State University in New Orleans: Assistant Professor M. P. Berri, Tulane University, has been appointed Visiting Associate Professor; Associate Professor Diran Sarafyan has been promoted to Professor; Dr. Jeong Sheng Yang, University of Miami, has been appointed Assistant Professor.

State University of New York at Buffalo: Assistant Professor G. R. Blakley, University of Illinois, has been appointed Associate Professor; Assistant Professor K. D. Magill, Jr., has been promoted to Associate Professor.

Dr. R. A. Alo, Pennsylvania State University, has been appointed Assistant Professor at Carnegie Institute of Technology.

Associate Professor J. A. Ferling, Claremont Men's College, has been promoted to Professor.

Dr. C. H. Franke, Bell Telephone Laboratory, Whippany, has been appointed Assistant Professor at Seton Hall University.

Assistant Professor T. K. Frutiger, Lycoming College, has been appointed Assistant Professor at Colgate University.

Associate Professor R. F. Gabriel, Seton Hall University, has been promoted to Professor.

Dr. R. W. Goodman, Massachusetts Institute of Technology, has been promoted to Assistant Professor.

Dr. W. E. Gould, Washington College, has been appointed Associate Professor at Bradley University.

Associate Professor Arnold Grudin, Denison University, has been promoted to Professor.

Associate Professor Melvin Hausner, New York University, has been promoted to Professor.

Assistant Professor R. C. N. Hourston, Knox College, has been promoted to Associate Professor.

Associate Professor R. B. Jackson, Jr., Davidson College, has been promoted to Professor.

Assistant Professor W. A. Kirby, Bowling Green State University, has been promoted to Associate Professor.

Assistant Professor J. A. Lutts, Loyola College, has been appointed Assistant Professor at the University of Massachusetts at Boston.

Assistant Professor E. M. Mandelbaum, West Chester State College, has been promoted to Associate Professor.

Mr. W. E. Mastrocola, Colgate University, has been promoted to Assistant Professor.

Assistant Professor R. F. McCoart, Jr., Georgetown University, has been appointed Associate Professor at Loyola College.

Mr. Stephen Milles, State University of New York at Buffalo, has been appointed Assistant Professor at Canisius College.

Mrs. Dixie Moore, Morehead State University, has been promoted to Assistant Professor.

Mr. J. G. Moser, Daniel Wagner Associates, has been appointed Associate Professor at West Chester State College.

Mrs. Janet K. Myhre, Claremont Men's College, has been promoted to Assistant Professor.

Professor F. D. Parker, SUNY at Buffalo, has been appointed Professor at the St. Lawrence University.

Dr. R. J. Plemmons, National Security Agency, has been appointed Associate Professor at the University of Mississippi.

Professor G. E. Reves has been appointed Head of the Mathematics Department at The Citadel.

Associate Professor R. A. Roberts, Denison University, has been promoted to Professor.

Assistant Professor J. I. Rosenthal, Nassau Community College, has been appointed Assistant Professor at Saint Peter's College.

Mr. R. D. Salmon, Louisiana Polytechnic Institute, has been promoted to Assistant Professor.

Assistant Professor C. S. Smith, Lake Forest College, has been promoted to Associate Professor.

Associate Professor R. A. Stokes, University of Mississippi, has been promoted to Professor.

Associate Professor W. A. Thompson, Jr., Florida State University, has been promoted to Professor and appointed Acting Head of the Mathematics Department.

Mr. Robert Vivona, Fordham University, has been appointed Assistant Professor at Marist College.

Dr. L. H. Williams, Florida State University, has been appointed Associate Professor and Director of the Computer Center at Auburn University.

Assistant Professor P. R. Young, Stanford University, has been appointed Assistant Professor at Purdue University.

Professor Emeritus P. A. Caris, University of Pennsylvania, died on July 5, 1966. He was a member of the Association for forty-three years.

Professor Bernard Friedman, University of California, Berkeley, died on September 12, 1966. He was a member of the Association for six years.

Lt. Col. C. A. Rupp, Defense Department, Washington, D.C., died in October, 1966. He was a member of the Association for forty-three years.

Associate Professor J. E. Vollmer, Western Michigan University, died on July 7, 1966. He was a member of the Association for seventeen years.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The forty-ninth annual meeting of the Rocky Mountain Section of the MAA was held at Colorado State University, Fort Collins, Colorado, on Friday and Saturday, May 13 and 14, 1966.

There were 122 people registered for the meeting, including Dean W. E. Briggs, Sectional Governor and Professor F. M. Stein, Section Chairman.

An invited address, "Algebraic Topology for Undergraduates(?)" was given on Friday afternoon by Professor A. B. Willcox, Amherst College, Second Vice-President of the Mathematical Association of America. On Saturday morning Professor George Seifert of Iowa State University delivered the SIAM invited address. His title was "Almost Periodic Solutions for Nonautonomous Systems of Ordinary Differential Equations."

At the banquet Friday night Professor M. L. Madison of Colorado State University presided. The Section was welcomed by President W. E. Morgan of Colorado State University. Following the banquet mathematical films were shown and card tables were available.

The business meeting was held on Saturday morning, at 8:45 A.M., with Professor Stein presiding.

Professor R. E. Doutt reported for the Meeting Committee that the 1967 meetings would be held at Western State College in Gunnison, Colorado. He announced that the University of Denver had extended an invitation to meet there in 1968. It was moved, seconded and carried that this invitation be accepted.

It was announced that a letter had been received from Southern Colorado State College at Pueblo requesting that that institution be listed on the rotation list for future meetings. The motion was made, seconded and carried that Southern Colorado State be added to the list of those institutions in District D.

The following officers were elected for 1966–67: Chairman, W. E. Dorgan, Western State College; Vice-Chairman, Kenneth Noble, University of Denver; and Secretary-Treasurer, W. N. Smith, University of Wyoming.

The chairman announced the appointment of Professor R. L. Eisenman of the Air Force Academy, and Professor Neville Hunsaker of Utah State University, to serve on the Meeting and Nomination Committees, respectively. These standing committees are now:

Meeting Committee: R. E. Doutt, South Dakota School of Mines; Chairman: R. W. Ellingwood, University of Colorado; R. L. Eisenman, Air Force Academy.

Nomination Committee: F. N. Fisch, Colorado State College; Chairman: W. E. Dorgan, Western State College; Neville Hunsaker, Utah State University.

Professor E. R. Deal of Colorado State University, Contest Chairman for the Annual High School Mathematics Contest, reported that 140 schools had participated—18 from Wyoming, 24 from Utah and 98 from Colorado.

The following 17 papers were presented at the meeting:

1. *Hilbert space with an indefinite inner product*, by R. W. McKelvey, University of Colorado.

An indefinite inner product $[x, y]$ on a vector space V is a symmetric bilinear functional such that $[x, x]$ may be positive, negative or zero. A Nevanlinna space is a Hilbert space with positive definite inner product (x, y) and a second indefinite inner product given by $[x, y]=(Jx, y)$ where J and J^{-1} are bounded self-adjoint operators. The talk is a brief exposition of the theory of Nevanlinna space: subspaces, orthogonal projectors, Cartesian sum decompositions, and the spectral resolution for a self-adjoint operator.

2. *The generalized Jordan canonical form*, by D. W. Robinson, Brigham Young University.

One of the topics usually considered in a course in linear algebra is the study of various matrix representations for a linear transformation on a vector space over a field. If the field is algebraically closed, then the most useful and well-known representation is the Jordan canonical form. However, it is not as well known that, by means of a slight extension, this form may essentially be used over a much larger class of fields. The purpose of this paper is to suggest a way to bring this "generalized Jordan canonical form" into the classroom.

3. *Approximate continuity*, by J. E. Kimber, Jr., Utah State University.

4. *Trilinear equations in a finite field*, by A. Duane Porter, University of Wyoming.

Let $F=GF(q)$ be the finite field of $q=p^r$ elements, p arbitrary, and let $N(n_1a)$ denote the number of solutions in F of $a_1x_1y_1z_1+\cdots+a_nx_ny_nz_n=a$. Also, let $N(n, a_i, b_i, a, b)$ denote the number of solutions in F of the system $a_1x_1y_1z_1+\cdots+a_nx_ny_nz_n=a$, $b_1x_1y_1z_1+\cdots+b_nx_ny_nz_n=b$, where all coefficients are from F. Explicit formulas for both $N(n, a)$ and $N(n, a_i, b_i, a, b)$ are obtained. To evaluate $N(n, a_i, b_i, a, b)$, two cases are considered. First, when $a_i, b_i\neq 0$, $1\leqq i\leqq n$, a, b arbitrary, and second, when a_i, b_i are all arbitrary.

5. *On Ramanujan's sum*, by G. S. Donovan, University of Colorado.

6. *Compound stochastic processes*, by S. A. Patil and M. M. Siddiqui, Colorado State University.

7. *A simple remark on Waring-type problems and linear Diophantine equations*, by A. J. Kempner, University of Colorado.

"Obvious, but not trivial": $1x_1+2x_2+3x_3+\cdots+kx_k+\cdots=n$, $n, x_1, x_2, x_3, \cdots$ integers $\geqq 0$, has for all n solutions $3\geqq x_1\geqq x_2\geqq x_3\geqq\cdots$ (0 from same point on). Similarly $1x_1+3x_2+5x_3+\cdots+(2k-1)x_k+\cdots=n$ has for all n solutions $4\geqq x_1\geqq x_2\geqq x_3\geqq\cdots$ (0 from same point on). Corresponding statements are given for Waring's general theorem, for Fermat's $x^n+y^n=F^n$, etc.

8. *A problem on integral operators*, by G. H. Meisters, University of Colorado.

Because everywhere-defined linear transformations of Hilbert space are bounded whenever they (1) are closed, (2) have adjoints with dense domains, (3) have a matrix representation—and for other reasons, the author conjectures first (vaguely) that linear transformations which are "constructively" defined everywhere on Hilbert space (or any B-space) are necessarily bounded, and second (precisely) that everywhere-defined (absolutely convergent Lebesgue) integral operators on Hilbert space are necessarily bounded. The author proves that if there exists a measurable set $E_1\subset E(=\text{measurable }\subset R^n)$ such that $m(E-E_1)=0$ and that for all $x\in E_1$ and all $f\in L_2(E)$, $Kf(x)=\int_E k(x, y)f(y)dy$ exists and belongs to $L_2(E)$, where k is measurable, then $K:L_2(E)\to L_2(E)$ is bounded.

9. *Differential inequalities with exceptional sets*, by J. W. Bebernes and G. H. Meisters, University of Colorado.

The following two theorems are slightly generalized versions of results of G. H. Meisters and this author. THEOREM 1: If $D_+u(t)\leqq f(t, u(t))$ a.e., and $D_+u(t)<+\infty$ nearly everywhere on $[a, b]$ (n.e. allows a countable exceptional set) where u and f are continuous, then any maximal solution ϕ_m of $x'=f(t, x)$ with $\phi_m(a)\geqq u(a)$ satisfies $u(t)\leqq\phi_m(t)$ on its interval of existence. THEOREM 2: Suppose there is a nonnegative continuous function $h(t, u)$ such that $u'=h(t, u)$, $u(t_0)=0$, has zero as its unique (right) solution and that $x_1\leqq x_2$ implies $f(t, x_2)-f(t, x_1)\leqq h(t, x_2-x_1)$ a.e. on $[a, b]$. If $D^+u(t)<+\infty$, $D^+v(t)>-\infty$ n.e., and if $D^+u(t)-f(t, u(t))\leqq D^+v(t)-f(t, v(t))$ a.e., for continuous u and v with $u(a)\leqq v(a)$, then $u(t)\leqq v(t)$ on $[a, b]$.

10. *The approximate solution of Riccati's equation*, by Ronald Huffstutler and F. M. Stein, Colorado State University.

This paper discusses the approximate solution of the Riccati equation $L(y)\equiv y'-P(x)y-Q(x)y^2=R(x)$ over $[0, 1]$ by a sum of zero-th order Bessel functions $S_n(x)=\sum_{m=1}^{n}B_mJ_0(\lambda_m x)$,

satisfying $y_0=y(x_0)$, that is the *best approximation* in the sense that $\int_0^1|R(x)-L[S_n(x)]|^m dx$, $m>0$, is a minimum. Particular use is made of the fact that $J_0'(x)=-J_1(x)$, and thus both the solution $y(x)$ and its derivative $y'(x)$ can be uniformily approximated by $S_n(x)$ and $S_n'(x)$ respectively throughout $[0, 1]$.

11. *On triples of quasi-conjugate matrices*, by L. S. Johnson and V. J. Varineau, University of Wyoming.

The concept of quasi-conjugate n-tuples of matrices is re-examined and some of its most interesting properties are discussed. A similarity relation for quasi-conjugate n-tuples is defined. The set of quasi-conjugate triples over a finite field is examined with respect to the similarity partition and the number of triples in each class is determined. The number of similarity classes for a finite field with m elements is found to be m.

12. *Geometry and vision: A plea for perspective geometry in senior high school*, by A. J. Kempner, University of Colorado.

Few of our students realize that we live our daily lives in two mutually contradictory worlds of geometry: the tactile (Euclidean) and the optical (perspective). An understanding of this situation—apparently totally lacking under our present high-school training—would be of marked scientific, cultural and philosophical value. To mention only one aspect: it would prepare the ground for psychological acceptance of the various non-Euclidean geometries and of the Einstein-Minkowski geometry of relativity, etc. A course including a satisfactory foundation of projective geometry could well be fitted into our present high school set-up.

13. *The statistics program at Colorado State University*, by J. S. Williams, Colorado State University.

14. *Implementation of CUPM Recommendations, Levels I and III*, Panel on Teacher Training, by J. J. Fisher, Colorado State Department of Education.

15. *The junior college mathematics curriculum*, by T. D. Cavanagh, Colorado State College.

The speaker discussed a study he had made of the junior college mathematics curriculum. The study was primarily a questionnaire study. A survey of the catalogues available from the junior colleges in the sample was included as a part of the study. The speaker discussed the present mathematics offerings of the junior colleges, the ways in which these offerings are changing, and the factors which influence such changes. He also made some recommendations for change in the junior college mathematics curriculum.

16. *Use of modern language techniques in the teaching of mathematics*, by Miss Ann Pape, Lakewood, Colorado.

Over a four year period, the audio-lingual methods used in the teaching of languages, including extensive use of tape recorders, have been applied in mathematics classes. Postulates and theorems are taught like grammar drills in a language situation. There are a number of advantages to this method. Necessary repetition is attained with less effort; slow learners and chronic absentees are helped without infringing upon teacher time; novelty and variety are added. The progress of groups using audio-lingual methods shows a significant improvement in accomplishment over control groups.

17. *Information feedback for mathematics student teachers*, by J. M. Moser, University of Colorado.

The paper discussed a study of the mathematics student teaching experience. The study attempted to minimize the subjective nature of evaluations of student teaching through the medium of feeding back objective information to the student teacher by means of audio tape recordings and discussions of teaching behavior matrices which are part of the Minnesota System of Interaction Analysis. Observations made during a year indicated that student teachers tend to become fairly

rigid in the pattern of their teaching behavior once they have found one which suits their personality. The greatest amount of student participation and teacher-student interaction was found in those classes which used SMSG or UICSM text materials.

W. N. SMITH, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Forty-eighth Summer Meeting, University of Toronto, Toronto, Ontario, Canada, August 28–30, 1967.

Fifty-first Annual Meeting, San Francisco, California, January 25–27, 1968.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1967.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1967.

INDIANA, Wabash College, Crawfordsville, May 13, 1967.

IOWA, Drake University, Des Moines, April 21, 1967.

KANSAS, Fort Hays State College, Hays, April 22, 1967.

KENTUCKY, Murray State University, Murray, April 1, 1967.

LOUISIANA-MISSISSIPPI, Jung Hotel, New Orleans, Louisiana, March 4–5, 1967.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, April 22, 1967.

METROPOLITAN NEW YORK, Long Island University, Brooklyn Division, March 18, 1967.

MICHIGAN, University of Michigan, Ann Arbor, March 18, 1967.

MINNESOTA, St. John's University, Collegeville, May 6, 1967.

MISSOURI, Northeast Missouri State Teachers College, Kirksville, April 29, 1967.

NEBRASKA, University of South Dakota, Vermillion, May 6, 1967.

NEW JERSEY

NORTHEASTERN, Mt. Allison University, Sackville, New Brunswick, June 23–24, 1967.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 22, 1967.

OKLAHOMA-ARKANSAS, Northeastern State College, Tahlequah, Oklahoma, April 1–2, 1967.

PACIFIC NORTHWEST, University of Montana, Missoula, June 16–17, 1967.

PHILADELPHIA, University of Delaware, Newark, November 18, 1967.

ROCKY MOUNTAIN, Western State College of Colorado, Gunnison, May 12–13, 1967.

SOUTHEASTERN, Florida Presbyterian College, St. Petersburg, Florida, March 31–April 1, 1967.

SOUTHERN CALIFORNIA, San Diego State College, San Diego, March 11, 1967.

SOUTHWESTERN, University of Arizona, Tucson, March 31–April 1, 1967.

TEXAS, Austin College, Sherman, April 14–15, 1967.

UPPER NEW YORK STATE, State University College, Plattsburgh, May 20, 1967.

WISCONSIN, St. Norbert College, DePere, May 6, 1967.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, New York, N. Y., December 26–31, 1967.

AMERICAN MATHEMATICAL SOCIETY, Toronto, Ontario, Canada, Aug. 29–Sept. 1, 1967.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Michigan State University, June 19–23, 1967.

ASSOCIATION FOR COMPUTING MACHINERY, Sheraton-Park, Washington, D. C., August 29–31, 1967.

ASSOCIATION FOR SYMBOLIC LOGIC

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 23–25, 1967.

INSTITUTE OF MATHEMATICAL STATISTICS

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Convention Center, Las Vegas, Nevada, April 16–20, 1967.

OPERATIONS RESEARCH SOCIETY OF AMERICA, New York Hilton Hotel, May 31–June 2, 1967.

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SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D. C., June 12–15, 1967 (Symposium on applied probability and fluid dynamics).

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A MODERN INTRODUCTION TO GEOMETRIES
by Annita Tuller, Hunter College of the City University of New York. **Just Published.** 312 pages; $7.50. Recommended for the basic undergraduate course.

CALCULUS WITH ANALYTIC GEOMETRY
by Albert Fadell, State University of New York at Buffalo. 1964; 705 pages; $9.75. Recommended for the first year of calculus.

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by John G. Kemeny and J. Laurie Snell, both of Dartmouth College, and Anthony W. Knapp, Massachusetts Institute of Technology. 1966; 452 pages; $12.50.

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by Leo Sario, University of California, Los Angeles, and Kiyoshi Noshiro, Nagoya University. 1966; 256 pages; $7.50.

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TOPICS IN THE THEORY OF FUNCTIONS ON ONE COMPLEX VARIABLE 12
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by Lennart Carleson, University of Upsala. Available March, 1967; about 160 pages; approximately $2.95 (paper).

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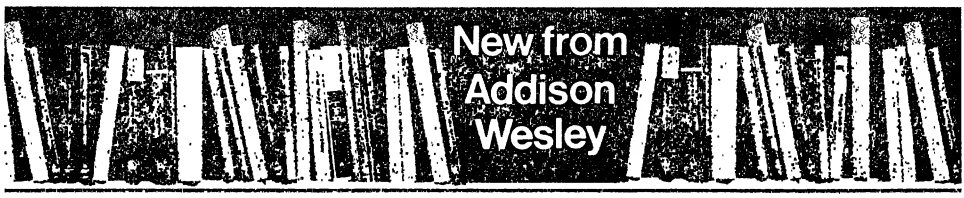
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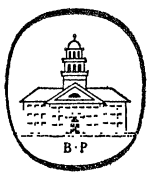
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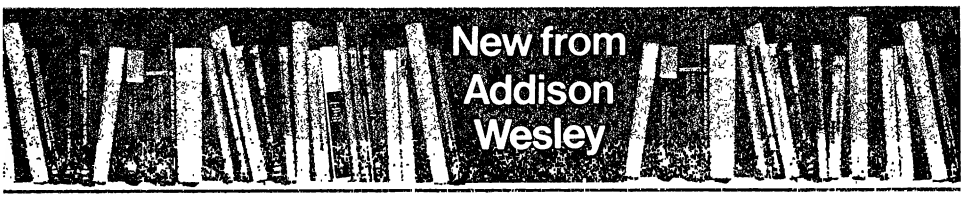
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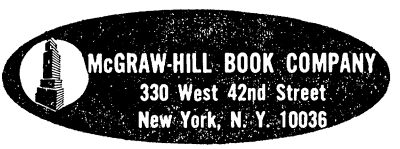
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MARCH 1967

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(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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RECENT ADVANCES IN DIFFERENCE SETS

H. B. MANN, Mathematics Research Center, U. S. Army, University of Wisconsin

A difference set (v, k, λ) in a finite group G of order v is a set of elements $a_1, \cdots, a_k$ of G such that the equation

$$a_i a_j^{-1} = g$$

has exactly λ solutions for every $g \in G$, $g \neq 1$. It is easy to check that the blocks $B_g = (a_1 g, \cdots, a_k g)$ form a balanced incomplete block design (bib) with parameters v, k, λ. Hence we have

$$\lambda(v - 1) = k(k - 1). \tag{1}$$

According to Schuetzenberger [5] we also know, that *if v is even then $k-\lambda$ is a square.*

The incidence matrix of a (bib) with parameters $(4N-1, 2N-1, N-1)$ can be used to construct Hadamard matrices of order $4N$. To do this one replaces the zeros of the incidence matrix by -1 and borders the resulting matrix by a row and a column of 1's. Other Hadamard matrices may be obtained directly (without bordering) from (bib) designs with parameters $4N^2$, $2N^2-N$, N^2-N. Hadamard matrices have been used in the construction of binary codes [4], and there is no reason why other (bib) designs should not prove advantageous especially in asymmetric channels. It seems also reasonable to expect that codes constructed from groups and especially from Abelian groups will be relatively easy to implement. Apart from its intrinsic interest as a problem in combinatorial analysis, therefore, the construction of difference sets and the question of their existence for certain parameter combinations is of interest in the theory of error correcting codes.

We shall restrict ourselves here principally to Abelian groups because not much is known about difference sets in non-Abelian groups. The difference sets with $k = v$ and with $k = v - 1$ are called trivial and will not be considered here. It is easy to see that the complement of every difference set is a difference set so that we may always assume $k < v/2$. The research here goes in two directions: The construction of new sets, and the proof of the impossibility of solutions for certain parameter combinations. While progress in construction of difference sets was comparatively slow, an impressive number of theorems ruling out many infinite classes of parameter combinations have been obtained in recent years. These impossibility proofs sometimes refer to all Abelian groups and sometimes to specific groups only. Cyclic and elementary Abelian groups have received special attention. A complete result has been obtained for groups of order 2^m. It has been shown [3] that the only possible solutions for v, k, λ are $v = 2^{2m}$, $k = 2^{2m-1} - 2^{m-1}$, $\lambda = 2^{2m-2} - 2^{m-1}$. Difference sets for these parameter combinations can be constructed easily in groups which are the direct products of four-groups and of cyclic groups of order 4. According to R. Turyn [7] one can take

all elements of a direct product of m of these groups for which an odd number of components are unit elements. These difference sets yield Hadamard matrices directly. Very few solutions are known in other groups of order 2^{2m}. At any rate, by Corollary 2.3, the exponent of such a group is at most 2^{m+1} so that there are no cyclic solutions except the trivial ones.

Another set of solutions likely to be useful in coding are the difference sets of Stanton and Sprott [6]. If $p^n+2=q^m$, p, q primes then we consider the additive group of the direct product R of G.F.$(p^n)\times$G.F.(q^m). Let x be a primitive root of G.F.(p^n) and y a primitive root of G.F.(q^m). Write the elements of R as pairs (a, b), $a\in$G.F.(p^n), $b\in$G.F.(q^m) and put $z=(x, y)$, $s=p^n$. Then

$$z^0, z^1, \cdots, z^{(s^2-3)/2}, (0, 0), (1, 0), \cdots, (x^{s-2}, 0)$$

is a difference set for the additive group of R with parameters $v=p^nq^m$, $k=(v-1)/2$, $\lambda=(v-3)/4$. These sets also lead to Hadamard matrices.

Another difference set that yields an Hadamard matrix consists of the squares of G.F.(p^m) where $p^m\equiv 3(4)$.

In 1956 Marshall Hall [1] published an enumeration of all cyclic difference sets with $k\leqq 50$, which left 12 cases undecided. It is a rather curious fact that only in 1962 these problems were attacked, and simultaneously by four researchers. The most satisfactory solution was obtained by K. Yamamoto [8] who disposed at once of all of Hall's 12 cases by one theorem on congruence relations in cyclotomic fields.

To formulate Yamamoto's theorem consider a function $C(i)$ which vanishes except when i is an integer. Such a function will be called a number-theoretic function. We define

$$\Delta(\rho) = C(i+\rho) - C(i).$$

We shall denote the field of rational numbers by R, and ζ_m will denote a primitive mth root of unity. Yamamoto's theorem runs as follows:

THEOREM 1 (Yamamoto). *Let $N=p_1^{e_1}\cdots p_r^{e_r}$ be the prime power decomposition of N. Let m be relatively prime to N, C a periodic number-theoretic function with period N whose values $C(i)$ are integers of $R(\zeta_m)$ and $f(x)=\sum_{i=0}^{N-1} C(i)x^i$. Moreover let d be a divisor of N and α an integer of $R(\zeta_m)$.*

Then, in order that $f(\zeta_N^t)\equiv 0(\alpha)$ for all divisors t of d, it is necessary and sufficient that

$$p_1^{t_1}\cdots p_r^{t_r}\Delta(Np_1^{-t_1-1})\Delta(Np_r^{-t_r-1})C(i) \equiv 0(\alpha) \tag{2}$$

for all i and for all $t_1, \cdots, t_r$ such that $d\equiv 0\ (p_1^{t_1}\cdots p_r^{t_r})$.

Applying Yamamoto's theorem to the case $d=1$ and $m=1$ one easily proves by induction:

COROLLARY 1.1. *If u is a rational integer, $\sum C(i)\zeta_N^i\equiv 0(u)$, $\sum C(i)\zeta_N^{(i)}\neq 0$ and if $|C(i)|\leqq M$ then $u\leqq 2^rM$, if $0\leqq C(i)\leqq M$ then $u\leqq 2^{r-1}M$.*

To apply the corollary to Abelian groups in general we consider the group ring $R(G)$ over a ring R of an Abelian group G. The group ring $R(G)$ consists of expressions of the form $\sum_{g\in G} a_g g$, $a_g \in R$. We define

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g)g, \qquad \sum_{g\in G} a_g g \sum b_g g = \sum_{g^*\in G} g^* \sum_{gg'=g^*} a_g b_{g'}.$$

If D is any set of G we shall write $D(t) = \sum_{g\in D} g^t$. If D is a difference set then

$$D(1)D(-1) = n + \lambda G, \tag{3}$$

where $n = k - \lambda$ and n stands for n times the identity of G.

A character χ of an Abelian group G is a mapping of G into a group of roots of unity. The character χ given by $\chi(g) = 1$, $g \in G$ is called the principal character and will be denoted by χ_0. If $g_1, \cdots, g_t$ is a basis of G, the character χ is defined when $\chi(g_i)$ is defined and we may choose for $\chi(g_i)$ any root of unity whose order divides the order of g_i. We define the product of two characters χ, χ' by

$$\chi\chi'(g) = \chi(g)\chi'(g).$$

Under this definition the characters themselves form a group isomorphic to G. Now let R be a ring whose elements are numbers. We extend the definition of the characters χ to the group ring. If

$$A = \sum_{g\in G} a_g g$$

then we define $\chi(A) = \sum_{g\in G} a_g \chi(g)$.

A character maps the group ring homomorphically into a cyclotomic field. One easily establishes the inversion formula: *If v is the order of G and*

$$A = \sum a_g g \tag{4}$$

then

$$a_g = \frac{1}{v} \sum_{\chi} \chi(A)\chi(g^{-1}). \tag{5}$$

Now let R be the ring of integers of a number field F and let $\mathfrak{M}$ be an ideal in R. From (5) one easily gets

LEMMA 1. *If $(\mathfrak{M}, v) = 1$ and $\chi(A) \equiv 0(\mathfrak{M})$ for all characters χ then $a_g \equiv 0(\mathfrak{M})$. If $\chi(A) \equiv 0(\mathfrak{M})$ for all nonprincipal characters χ then*

$$A = \mu G(1) + M, \qquad \mu \in R, \qquad M \in R(G), \tag{6}$$

where all coefficients of M are in $\mathfrak{M}$.

We now proceed to prove a theorem of Turyn [7].

THEOREM 2 (TURYN). *Let v be the order of G and D any subset of G, m a rational integer, χ_1 a character of order v_1, r the number of distinct prime factors of v_1, $\hat{H}$ a*

group of characters of G of order v_2, $(m, v_2)=1$, $\chi_1^j \notin \hat{H}$ if $\chi_1^j \neq \chi_0$.

If $r \geqq 0$, $\chi\chi_1(D) \neq 0$ for at least one χ and $m \mid \chi\chi_1(D)$ for all $\chi \in \hat{H}$ then

$$2^{r-1}v \geqq mv_1v_2. \tag{7}$$

If $r \geqq 0$, $\chi\chi_1(D) \neq 0$ for at least one nonprincipal χ and $m \mid \chi\chi_1(D)$ for all nonprincipal characters $\chi \in \hat{H}$ then

$$2^r v \geqq mv_1v_2. \tag{7a}$$

We first prove

LEMMA 2. *Let H be the kernel of $\hat{H}$ and set*

$$\psi(g_1) = \sum_{g \in Hg_1} y_g \chi_1(g), \tag{8}$$

where $D(1) = \sum_{g \in G} y_g g$. If $m \mid \chi\chi_1(D)$ for all $\chi \in \hat{H}$ then for all $g \in G$

$$\psi(g) \equiv 0(m). \tag{9}$$

If $m \mid \chi\chi_1(D)$ for all nonprincipal $\chi \in \hat{H}$ then for any pair g_1, g_2 in G

$$\psi(g_1) \equiv \psi(g_2)(m). \tag{10}$$

Proof. The characters χ of $\hat{H}$ can be considered as the group of characters of the factor group $G/H = H^*$ consisting of the cosets Hg of H. Consider the element

$$A = \sum_{h^* \in H^*} h^* \sum_{g \in h^*} y_g \chi_1(g)$$

of the group ring of H^* over the integers of $R(\zeta_{v_1})$. For any character χ in $\hat{H}$ we then have

$$\chi(A) = \sum_{h^*} \sum_{g \in h^*} y_g \chi\chi_1(g) = \sum_{g \in G} y_g \chi\chi_1(g) = \chi\chi_1(D). \tag{11}$$

Lemma 2 now follows from Lemma 1 and equation (11).

We now prove the theorem of Turyn. Let

$$\psi(g_1) = \sum_{g \in Hg_1} y_g \chi_1(g) = \sum_{i=0}^{v_1-1} Y_i \zeta_{v_1}^i.$$

If $g \in Hg_1$, $\chi_1(g) = \zeta_{v_1}^i$, $\chi \in \hat{H}$ for fixed i then $\chi\chi_1(g) = \chi(g_1)\zeta_{v_1}^i$. Hence all these g lie in the same coset modulo the kernel of $\hat{H} \times \{\chi_1^j\}$. But the assumptions of our theorem imply that the order of this kernel is v/v_1v_2. Hence $Y_i \leqq v/v_1v_2$. From (11) it also follows that $\chi(A) \neq 0$ for at least one χ, hence $\psi(g) \neq 0$ for at least one g in the first case and $\psi(g_1) \neq \psi(g_2)$ for at least one pair g_1, g_2 in the second case. Theorem 2 now follows from Lemma 2 and Corollary 1.1.

We now return to our difference sets. From (3) we have for all nonprincipal characters χ of G

$$\chi(D(1))\chi(D(-1)) = n. \tag{12}$$

Now let $v \equiv 0(v_2)$, let χ be a nonprincipal character whose order is a divisor of v_2 and assume that

$$n \equiv 0(p^j), \qquad n \not\equiv 0(p^{j+1}), \tag{13}$$

where $v_2 = p^\alpha v_1$, $\alpha \geqq 0$ and $p^\beta \equiv -1(v_1)$ for some $\beta \geqq 0$. Then the automorphism $\zeta_{v_2} \to \zeta_{v_2}^{-1}$ of $R(\zeta_{v_2})$ leaves all prime factors of p invariant so that 12 implies that

$$\chi(D) \equiv 0(p^{j/2}). \tag{14}$$

A prime p for which $v_2 = p^\alpha v_1$, $p^\beta \equiv -1(v_1)$ will be called self-conjugate mod v_2. If all prime factors of an integer m are self conjugate mod v_2 then m is called self-conjugate mod v_2.

We now derive the following corollaries of Theorem 2.

COROLLARY 2.1. *If there is in G a difference set (v, k, λ), if $p \mid n$ and p is a prime self conjugate* mod v^* *where v^* is the exponent of G, then the highest power of p dividing n is even. If $p^{2s} \mid n$, the p exponent of G is p^e and $v = p^t w$, $(p, w) = 1$ then*

$$p^{s+e} \leqq p^t. \tag{15}$$

(This means in particular that the p component of G cannot be cyclic.)

Proof. If $w > 1$ then the first part of the theorem is true because p has no multiple factors in $R(\zeta_w)$. If $w = 1$ then it follows from the equation $\lambda p^t = k^2 - n$.

Now in Theorem 2 let $\hat{H}$ be the subgroup of characters of order w and let χ_1 be any character of order p^e. Then if $\chi_1 \neq \chi_0$ we have $\chi\chi_1(D) \equiv 0(p^s)$ for all $\chi \in \hat{H}$ and if $\chi_1 = \chi_0$ then $\chi\chi_1(D) \equiv 0(p^s)$ for all nonprincipal $\chi \in \hat{H}$ whence $p^s p^e w \leqq p^t w$, $p^{s+e} \leqq p^t$.

COROLLARY 2.2. *Let D be a difference set in a group which has a character of order w. Let $m^2 \mid n$ and let m be self conjugate* mod w. *If $(m, w) > 1$ then $wm \leqq 2^{r-1}v$, where r is the number of distinct prime factors of (m, w). If $(m, w) = 1$ then $wm \leqq v$.*

Proof. Let $w = v_1 v_2$ where v_1 is a product of prime powers dividing (m, w), (if there are any) and $(v_2, m) = 1$. Let χ_1 be a character of order v_1 and χ_2 of order v_2. In Theorem 2 let $\{\chi_2^j\} = \hat{H}$ and the corollary follows.

An Hadamard matrix can be constructed from a difference set $4p^{2s}$, $2p^{2s} - p^s$, $p^{2s} - p^s$. We shall refer to such sets as Hadamard sets. We have

COROLLARY 2.3. *Let D be an Hadamard set in a group G. If $v = 4p^{2s}$, p a prime, and if p^e is the p exponent of G, then $e \leqq s+2$ if $p = 2$ and $e \leqq s$ if $p \neq 2$.*

For proof we may, in Theorem 2, put $v_1 = 2^e$, $v_2 = 1$ for $p = 2$ and $v_1 = p^e$, $v_2 = 2$ for $p > 2$.

Corollaries 2.1 and 2.2 suffice to dispose of all of Marshall Hall's [1] unsolved cases, disposing in a number of cases not only of cyclic difference sets but of other Abelian sets and sometimes of all Abelian sets as well. Corollary 2.3 shows

that there are no circulant Hadamard matrices when n is a prime-power unless $n=1$. (The conjecture that there are no circulant Hadamard matrices unless $v=4$ is still unproved.)

About two years ago I investigated difference sets in elementary Abelian groups. We already mentioned that the Hadamard sets $v=2^{2m+2}$, $k=2^{2m+1}-2^m$, $\lambda=2^{2m}-2^m$ are the only nontrivial sets in elementary Abelian groups of order 2^t for any t. It is also easy to check that the squares of G.F.(m) form a difference set $(m, (m-1)/4, (m-3)/4)$ in the additive group of G.F.(m) if $m\equiv -1(4)$. For difference sets in elementary Abelian groups of order p^m, $p\equiv 1(2)$ the following theorem holds. We recall that the number t is multiplier of D if $D(t)=gD(1)$ for some $g\in G$.

THEOREM 3. *If the quadratic residues* mod p *are multipliers of the elementary Abelian difference set D with parameters p^m, k, λ, then the equation*

$$x^2 + py^2 = 4n \tag{16}$$

is solvable. Let $(x_1, y_1), \cdots, (x_l, y_l)$ be the solutions of (16) *then*

$$k + \tfrac{1}{2}(p-1)(x_1z_1 + \cdots + x_lz_l) = \epsilon p^m, \tag{17}$$

where $z_1, \cdots, z_l$ are nonnegative integers and ϵ may take either the value 0 *or the value* 1 *(but not both) and*

$$z_1 + \cdots + z_l = (p^m - 1)/(p - 1). \tag{18}$$

COROLLARY 3.1. *If $l=1$ then $k=(p^m-1)/2$.*

COROLLARY 3.2. *If $p>3$ then n is not a prime.*

A family of cases in which the condition of Theorem 3 holds is given by

THEOREM 4. *If $p=2q+1$, q a prime and if no prime divisor of n is congruent to* 1 mod p *then the quadratic residues* mod p *are multipliers of any elementary Abelian difference set (p^m, k, λ) and n is not a prime if such a difference set exists.*

A search for noncyclic elementary Abelian difference sets with $v\leqq 2500$, $v\equiv 1(2)$ failed to produce any solutions except for the parameters p^m, $(p^m-1)/2$, $(p^m-3)/4$. Corollary 2.1 together with Theorem 3 and its corollaries rule out all but 9 of the other solutions of (1). Since the completion of my paper, 6 of the nine cases have been ruled out by R. J. Turyn and one by A. Gleason. The only open cases with $v\leqq 2500$ are 31^2, 256, 68 and 3^6, 273, 102. This gives some support to the conjecture that no elementary noncyclic Abelian set exists unless either $v=2^{2m+2}$, $n=2^{2m}$ or $v=p^m$, $k=(p^m-1)/2$.

This work was sponsored by the Math. Research Center, U. S. Army, Madison, Wisconsin.

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ON CONVOLUTED NUMBERS AND SUMS

DAVID ZEITLIN, University of Minnesota

1. Preliminaries. Recently, Arkin [1], defining numbers $C_n^{(k)}$ by the identity

$$(1.1) \qquad (1 - x - x^2)^{-k} = \sum_{n=0}^{\infty} C_n^{(k)} x^n \quad (k = 1, 2, \cdots),$$

gave four relations involving $C_n^{(k)}$, where for $k=1$, $C_n^{(1)} = F_{n+1}$, the Fibonacci sequence defined by $F_{n+2} = F_{n+1} + F_n$, $n = 0, 1, \cdots$, with $F_0 = 0$ and $F_1 = 1$. It should be noted that the numbers $C_n^{(k)}$ are not new; they have appeared previously (but in a different notation) in the paper by Riordan [2, pp. 11–12], who refers to $C_n^{(k)}$ as the "kth convoluted Fibonacci number," because the left side of (1.1) is the kth power of the generating function for Fibonacci numbers. Thus, we obtain from (1.1) the following convoluted sum identity:

$$(1.2) \qquad C_n^{(k)} = \sum_{i_1+i_2+\cdots+i_k=n} F_{1+i_1} F_{1+i_2} \cdots F_{1+i_k},$$

where the ordered set of indices $i_1, i_2, \cdots, i_k$ assume the values of all permutations of all sets of k nonnegative integers whose sum is n. Alternatively, one could write (1.2) as

$$C_n^{(k)} = C_n^{(1)} * C_n^{(k-1)} = \sum_{j=0}^{n} C_j^{(1)} C_{n-j}^{(k-1)} = C_n^{(1)} * C_n^{(1)} * \cdots * C_n^{(1)},$$

where $H_n * G_n$ is the convolution (defined above) of the sequences $(H_0, H_1, \cdots)$ and $(G_0, G_1, \cdots)$.

Recalling that

$$(1 - y)^{-k} = \sum_{j=0}^{\infty} \binom{k-1+j}{k-1} y^j,$$

we see that for $y = x + x^2$,

$$(1.3)\qquad \begin{aligned}(1-x-x^2)^{-k} &= \sum_{j=0}^{\infty}\binom{k-1+j}{k-1}(x+x^2)^j \\ &= \sum_{j=0}^{\infty}\binom{k-1+j}{k-1}\sum_{i=0}^{j}\binom{j}{i}x^{i+j} \\ &= \sum_{n=0}^{\infty}x^n\sum_{m=0}^{[n/2]}\binom{k+n-1-m}{k-1}\binom{n-m}{m}.\end{aligned}$$

Comparison of (1.3) with (1.1) gives

$$(1.4)\qquad C_n^{(k)} = \sum_{m=0}^{[n/2]}\binom{k+n-1-m}{k-1}\binom{n-m}{m} \quad (k=1,2,\cdots).$$

For $k=1$, (1.4) reduces to a well-known identity for Fibonacci numbers.

Associated with the identity

$$(1.5)\qquad \delta_{nk} = \sum_{j=0}^{n-k}(-1)^j\binom{n-j}{j}\frac{k}{(2n-2j-k)}\binom{2n-2j-k}{n-j},$$

where δ_{nk} is the Kronecker delta, one has the inverse relations (see [3])

$$(1.6)\qquad a_n = \sum_{j=0}^{[n/2]}\binom{n-j}{j}b_{n-j}, \qquad b_n = \sum_{j=0}^{n}(-1)^j\frac{n-j}{n+j}\binom{n+j}{j}a_{n-j}.$$

An application of (1.6) with $a_n=C_n^{(k)}$ and $b_n=\binom{k+n-1}{k-1}$ gives the companion formula to (1.4) as

$$(1.7)\qquad \binom{k+n-1}{n} = \sum_{j=0}^{n}(-1)^j\frac{n-j}{n+j}\binom{n+j}{j}C_{n-j}^{(k)} \quad (k=1,2,\cdots).$$

For $k=1$, (1.7) gives

$$(1.8)\qquad 1 = \sum_{j=0}^{n}(-1)^j\frac{n-j}{n+j}\binom{n+j}{j}F_{n+1-j} \quad (n=0,1,\cdots).$$

One aim of this paper is to generalize (1.1) and the related identities, (1.2), (1.4), and (1.7), by replacing the Fibonacci sequence by a general linear recurring sequence of order two with real constant coefficients and arbitrary initial conditions. We will also obtain an explicit formula for these generalized kth convoluted numbers, which appear naturally in the evaluation of convoluted sums for linear recurring sequences of order two and higher.

2. A generalization of (1.1). The first generalization of (1.1) is given by

$$(2.1)\qquad (1-ax-bx^2)^{-k} = \sum_{n=0}^{\infty}C_n^{(k)}(a,b)x^n \quad (k=1,2,\cdots),$$

where a and $b\neq 0$ are real numbers. It should be noted that the numbers

$C_n^{(k)}(a, b)$ are not new, for they have appeared previously (but in a different notation) in the paper by Carlitz [4, p. 522]. Expanding (2.1) just as above on (1.3), we readily find that

$$(2.2)\quad C_n^{(k)}(a, b) = \sum_{m=0}^{[n/2]} b^m a^{n-2m} \binom{k+n-1-m}{k-1} \binom{n-m}{m} \quad (k = 1, 2, \cdots).$$

If we multiply both sides of (2.2) by $a^n b^{-n}$ and then apply (1.6) with

$$a_n = a^n b^{-n} C_n^{(k)}(a, b), \qquad b_n = a^{2n} b^{-n} \binom{k+n-1}{k-1},$$

we obtain, after simplification, the companion formula to (2.2) as

$$(2.3)\quad a^{2n} \binom{k+n-1}{n} = \sum_{j=0}^{n} (-1)^j \frac{n-j}{n+j} \binom{n+j}{j} a^{n-j} b^j C_{n-j}^{(k)}(a, b),$$

which gives (1.7) for $a = b = 1$.

We note that $C_n^{(k)}(a, b)$ and Gegenbauer polynomials, $P_n^{(k)}(t)$, are closely related. The generating function for the latter (see [5], pp. 81–85) is

$$(2.4)\quad (1 - 2tz + z^2)^{-k} = \sum_{n=0}^{\infty} P_n^{(k)}(t) z^n \quad (k = 1, 2, \cdots).$$

Hence

$$(2.5)\quad P_n^{(k)}(t) = \sum_{m=0}^{[n/2]} (-1)^m \binom{k+n-1-m}{k-1} \binom{n-m}{m} (2t)^{n-2m},$$

and

$$(2.6)\quad C_n^{(k)}(a, b) = b^{n/2} i^n P_n^{(k)}(-ia/(2\sqrt{b})) \quad (i^2 = -1;\ k = 1, 2, \cdots).$$

Since $C_n^{(k)} = C_n^{(k)}(1, 1) = i^n P_n^{(k)}(-i/2)$, the relations I and III in [1], p. 276 are instances of two well-known identities for Gegenbauer polynomials (see [5], p. 83, (4.7.17); p. 84, (4.7.27)).

If we multiply both sides of (2.5) by $(-1)^n (2t)^n$ and then apply (1.6) with

$$a_n = (-1)^n (2t)^n P_n^{(k)}(t), \qquad b_n = (-1)^n (2t)^{2n} \binom{k+n-1}{k-1},$$

we obtain the companion formula to (2.5),

$$(2.7)\quad (2t)^{2n} \binom{k+n-1}{n} = \sum_{j=0}^{n} \frac{n-j}{n+j} \binom{n+j}{j} (2t)^{n-j} P_{n-j}^{(k)}(t).$$

Identity (2.7) is of special interest, since $P_n^{(k)}(t)$, for $k = \frac{1}{2}, 1$, reduces, respectively, to Legendre polynomials and Chebyshev polynomials, $U_n(t)$, of the second kind. Since (see [5], p. 82, (4.7.8))

$$\lim_{k\to 0} k^{-1}\cdot P_m^{(k)}(t) = (2/m)T_m(t) \quad (m = 1, 2, \cdots), \tag{2.8}$$

where $T_m(t)$ is the Chebyshev polynomial of the first kind, and noting that

$$\binom{k+n-1}{n} = (k/n)\binom{k+n-1}{n-1},$$

we obtain from (2.7), upon using (2.8) with $m=n-j$, the identity

$$(2t)^{2n} = \sum_{j=0}^{n-1} \frac{2n}{n+j}\binom{n+j}{j}(2t)^{n-j}T_{n-j}(t) \quad (n = 1, 2, \cdots). \tag{2.9}$$

As already indicated, we obtain from (2.7) with $k=1$

$$(2t)^{2n} = \sum_{j=0}^{n-1} \frac{n-j}{n+j}\binom{n+j}{j}(2t)^{n-j}U_{n-j}(t) \quad (n = 1, 2, \cdots). \tag{2.10}$$

An application of (2.9) and (2.10) will be given in the next section.

Inverse relations (2.9) and (2.10) may be contrasted, respectively, to the following well-known inverse relations (see problem 4b of [6], Chapter 8, p. 223):

$$(2t)^n = 2\sum_{j=0}^{[n/2]}\binom{n}{j}T_{n-2j}(t) \quad (n = 1, 2, \cdots), \tag{2.11}$$

$$(2t)^n = \sum_{j=0}^{[n/2]}\left[\binom{n}{j} - \binom{n}{j-1}\right]U_{n-2j}(t) \quad (n = 1, 2, \cdots). \tag{2.12}$$

Inverse relation (2.7) may also be contrasted to the known result (see [7], p. 282 (32))

$$(1-t)^n 2^{-n-1} = n!(k+\tfrac{1}{2})_n \sum_{j=0}^{n}(-1)^j \frac{(k+j)}{(n-j)!(2k)_{n+j+1}} P_j^{(k)}(t), \tag{2.13}$$

where $(p)_n = p(p+1)\cdots(p+n-1)$.

3. A generalized k-th convoluted number for recurring sequences of order two. For complete generality, we recall from the theory of generating functions that if W_0, W_1, a, and $b\neq 0$ are real numbers, and if

$$W_{n+2} = aW_{n+1} + bW_n \quad (n = 0, 1, \cdots), \tag{3.1}$$

then for $k=1, 2, \cdots$, the identity

$$\begin{aligned}[W_0 + (W_1 - aW_0)x]^k(1 - ax - bx^2)^{-k} &= \left(\sum_{n=0}^{\infty} W_n x^n\right)^k \\ &= \sum_{n=0}^{\infty} A_n^{(k)}(a, b; W_0, W_1)x^n,\end{aligned} \tag{3.2}$$

defines the kth convoluted numbers $A_n^{(k)}(a, b; W_0, W_1) \equiv A_n^{(k)}$. Since $(1-ax-bx^2)\cdot(W(x))^k = [W_0+(W_1-aW_0)x](W(x))^{k-1}$, where $W(x) = W_0+W_1x+W_2x^2+\cdots$, the $A_n^{(k)}$ have the recurrence

$$(3.3)\quad A_n^{(k)} = aA_{n-1}^{(k)} + bA_{n-2}^{(k)} + W_0A_n^{(k-1)} + (W_1 - aW_0)A_{n-1}^{(k-1)} \quad (n, k = 2, 3, \cdots),$$

where $A_n^{(1)}(a, b; W_0, W_1) = W_n$, $n = 0, 1, \cdots$. We define the Lucas functions (of n) associated with (3.1) as Z_n and V_n, where $Z_n = A_n^{(1)}(a, b; 0, 1)$ and $V_n = A_n^{(1)}(a, b; 2, a)$, $n = 0, 1, \cdots$. Using (3.2), we note that $A_n^{(k)}(a, b; W_0, aW_0) = W_0^kC_n^{(k)}(a, b)$, $C_n^{(k)}(a, b) = A_n^{(k)}(a, b; 1, a)$, $n = 0, 1, \cdots$, and $A_n^{(k)}(a, b; 0, W_1) = W_1^kC_{n-k}^{(k)}(a, b)$, $n = k, k+1, \cdots$.

If we multiply both sides of (2.1) by $(W_0+(W_1-aW_0)x)^k$ and recall (3.2), we obtain the following result:

$$(3.4)\quad A_n^{(k)}(a, b; W_0, W_1) = \sum_{s=0}^{n} C_s^{(k)}(a, b)\binom{k}{n-s}(W_1 - aW_0)^{n-s}W_0^{s+k-n} \quad (n = 0, 1, \cdots).$$

If $k = 1$, we obtain from (3.4) the relations

$$(3.5)\quad Z_n = C_{n-1}^{(1)}(a, b) = \sum_{m=0}^{[(n-1)/2]} \binom{n-1-m}{m} b^m a^{n-1-2m} \quad (n = 1, 2, \cdots),$$

$$(3.6)\quad W_n = W_0C_n^{(1)}(a, b) + (W_1 - aW_0)C_{n-1}^{(1)}(a, b) \equiv W_0Z_{n+1} + (W_1 - aW_0)Z_n \quad (n = 1, 2, \cdots),$$

$$(3.7)\quad V_n = 2C_n^{(1)}(a, b) - aC_{n-1}^{(1)}(a, b) \equiv 2Z_{n+1} - aZ_n \quad (n = 1, 2, \cdots).$$

If we define

$$(3.8)\quad G_n(y) = \sum_{j=1}^{n} A_{n-j}^{(j)}(a, b; W_0, W_1)y^j \quad (n = 1, 2, \cdots),$$

then from (3.3), with $k = j$, both sides multiplied by y^j, and n replaced by $n+2-j$, we obtain by summation on j from 1 to $n+2$

$$(3.9)\quad G_{n+2}(y) = (a + W_0y)G_{n+1}(y) + [b + (W_1 - aW_0)y]G_n(y) \quad (n = 1, 2, \cdots),$$

where $G_1(y) = W_0y$ and $G_2(y) = W_1y+W_0^2y^2$. If we define

$$(3.10)\quad G(x, y) = \sum_{n=0}^{\infty} G_{n+1}(y)x^n,$$

then, from (3.2) with $k = 1$, we obtain the generating function

$$(3.11)\quad G(x, y) = y[W_0+(W_1-aW_0)x]/[1-(a+W_0y)x-(b+(W_1-aW_0)y)x^2];$$

and if $w = W_1/W_0$, $W_0 \neq 0$, then

$$G(x(a+y)^{-1}, yW_0^{-1}) = y[1+(w-a)(a+y)^{-1}x] \\ /[1-x-(b+wy-ay)(a+y)^{-2}x^2] \\ = y[1+(w-a)(a+y)^{-1}x]K(x,(b+wy-ay)(a+y)^{-2}), \tag{3.12}$$

where

$$K(x,u) = (1-x-ux^2)^{-1} = \sum_{n=0}^{\infty} K_n(u)x^n. \tag{3.13}$$

Thus, from (3.12), with $z=(b+wy-ay)(a+y)^{-2}$, we obtain

$$G_{n+1}(y/W_0) = y(a+y)^n[K_n(z)+(w-a)(a+y)^{-1}K_{n-1}(z)] \quad (n=0,1,\cdots). \tag{3.14}$$

Since $K_n(u)=C_n^{(1)}(1,u)$, we obtain from (2.2) that

$$K_n(u) = \sum_{m=0}^{[n/2]} \binom{n-m}{m} u^m \quad (n=0,1,\cdots), \tag{3.15}$$

and (3.15) can be used in (3.14) to obtain $G_{n+1}(y)$ as a sum. We note that

$$(1-ax-bx^2)^{-1} = K(ax, b/a^2). \tag{3.16}$$

For $h=\sqrt{1+4u}$, we have the known identity

$$K_n(u) = \begin{cases} 2^{-n-1}h^{-1}[(1+h)^{n+1}-(1-h)^{n+1}], & h \neq 0; \\ 2^{-n}(n+1), & h=0, \end{cases} \tag{3.17}$$

(see problem 5b of [6], Chapter 8). Substitution of (3.17) with $u=z$ into (3.14) gives $G_{n+1}(y)$ explicitly.

If $W_1=aW_0$, then $w=a$, and (3.14), with $g_{n+1}(y)\equiv G_{n+1}(y/W_0)$ reduces to

$$g_{n+1}(y) = y(a+y)^n K_n(b/(a+y)^2) = \sum_{j=1}^{n+1} C_{n+1-j}^{(j)}(a,b)y^j \quad (n=0,1,\cdots). \tag{3.18}$$

If $a=b=y=1$, then (3.18), using (3.17) with $h=\sqrt{2}$, gives relation IV in [1]. Using (3.15), we obtain from (3.18)

$$g_{n+1}(y) = y\sum_{m=0}^{[n/2]} \binom{n-m}{m} b^m(a+y)^{n-2m} \quad (n=0,1,\cdots). \tag{3.19}$$

An interesting example of (3.18) occurs when $b<0$ and $b/(a+y)^2=-\frac{1}{4}$. Let $b=-p^2$, $p>0$. Choosing $a+y=2p$, we obtain from (3.18) (since $h=0$) the identity

$$\sum_{j=1}^{n+1} (2p-a)^{j-1}\cdot C_{n+1-j}^{(j)}(a,-p^2) = (n+1)p^n \quad (p>0;\ n=0,1,\cdots). \tag{3.20}$$

We note that the right-hand side of (3.20) is independent of a. Since (see (2.2))

$$(3.21)\quad C_{2n+1}^{(k)}(0, b) = 0, \qquad C_{2n}^{(k)}(0, b) = b^n \binom{n + k - 1}{n} \quad (n = 0, 1, \cdots),$$

we obtain from (3.20) with $a=0$ the following identities for $n=2m+1$ and $n=2m$, respectively:

$$(3.22)\quad \sum_{r=1}^{m+1} (-1)^{m+1-r} \binom{m + r}{2r - 1} 4^r = 4m + 4 \quad (m = 0, 1, \cdots),$$

$$(3.23)\quad \sum_{r=1}^{m+1} (-1)^{m+1-r} \binom{m + r - 1}{2r - 2} 4^r = 8m + 4 \quad (m = 0, 1, \cdots).$$

Since $P_n^{(k)}(y) = (-1)^n P_n^{(k)}(-y)$ (see (2.5)), we obtain from (2.6) that $C_n^{(k)}(2x, -1) = P_n^{(k)}(x)$. Thus, from (3.20) with $a=2x$ and $p=1$, we obtain an identity for Gegenbauer polynomials,

$$(3.24)\quad \sum_{j=1}^{n+1} (2 - 2x)^{j-1} P_{n+1-j}^{(j)}(x) = n + 1 \quad (n = 0, 1, \cdots).$$

If we multiply both sides of (3.19) by $(a+y)^n b^{-n}$ and then apply (1.6) with

$$a_n = (a + y)^n b^{-n} g_{n+1}(y) \quad \text{and} \quad b_n = y b^{-n} (a + y)^{2n},$$

we obtain the following companion formula to (3.19):

$$(3.25)\quad y(a + y)^{2n} = \sum_{j=0}^{n} (-1)^j \frac{n - j}{n + j} \binom{n + j}{j} b^j (a + y)^{n-j} g_{n+1-j}(y).$$

It is of interest to note that the Lucas functions, V_n and Z_n, associated with (3.1) can be expressed in terms of Chebyshev polynomials as follows:

$$(3.26)\quad V_n = 2(-b)^{n/2} T_n(a/(2i\sqrt{b})) \qquad (n = 0, 1, \cdots),$$

$$(3.27)\quad Z_{n+1} = (-b)^{n/2} U_n(a/(2i\sqrt{b})) \qquad (n = 0, 1, \cdots).$$

To prove (3.26), set $x=a/(2i\sqrt{b})$ in the identity $T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$, and then multiply both sides by $2(-b)^{(n+2)/2}$. Thus, using (3.26), we have $V_0=2$, $V_1=a$, and $V_{n+2}=aV_{n+1}+bV_n$, $n=0, 1, \cdots$. A similar proof using $U_n(x)$ can also be given for (3.27).

If we set $t=a/(2i\sqrt{b})$ in (2.9) and (2.10) and then simplify by using (3.26) and (3.27), we obtain, respectively,

$$(3.28)\quad a^{2n} = \sum_{j=0}^{n-1} (-1)^j \frac{n}{n + j} \binom{n + j}{j} a^{n-j} b^j V_{n-j} \qquad (n = 1, 2, \cdots),$$

$$(3.29)\quad a^{2n} = \sum_{j=0}^{n-1} (-1)^j \frac{n - j}{n + j} \binom{n + j}{j} a^{n-j} b^j Z_{n+1-j} \quad (n = 1, 2, \cdots).$$

The companion formula to (3.29) is given by (3.5).

If $a=b=1$, then (3.1) defines the Fibonacci sequence with $Z_n \equiv F_n$ and $V_n \equiv L_n$, the Lucas sequence. Thus, from (3.28), we obtain the identity

$$1 = \sum_{j=0}^{n-1} (-1)^j \frac{n}{n+j}\binom{n+j}{j} L_{n-j} \quad (n = 1, 2, \cdots); \tag{3.30}$$

and from (3.29) we obtain a previous result, (1.8).

If $a=2$ and $b=-1$, then (3.1) defines sequences where $Z_n \equiv n$ and $V_n \equiv 2$. Thus, from (3.28) and (3.29), we obtain, respectively,

$$2^{2n-1} = \sum_{j=0}^{n-1} \frac{n}{n+j}\binom{n+j}{j} 2^{n-j}, \tag{3.31}$$

$$2^{2n} = \sum_{j=0}^{n-1} \frac{n-j}{n+j}\binom{n+j}{j} 2^{n-j}(n+1-j). \tag{3.32}$$

From (3.5), we obtain

$$n = \sum_{m=0}^{[(n-1)/2]} (-1)^m 2^{n-1-2m}\binom{n-1-m}{m}. \tag{3.33}$$

Moreover, since

$$(1-x)^{-2k} = \sum_{n=0}^{\infty}\binom{2k+n-1}{n} x^n,$$

we have

$$C_n^{(k)}(2, -1) = \binom{2k+n-1}{n}$$

(see (2.1)). Thus, from (2.3), we obtain the identity

$$2^{2n}\binom{k+n-1}{n} = \sum_{j=0}^{n} \frac{n-j}{n+j}\binom{n+j}{j}\binom{2k+n-j-1}{n-j} 2^{n-j}. \tag{3.34}$$

4. Evaluation of convoluted sums for recurring sequences of order three. Let Q_1, Q_2, m_1, m_2, and $m_3 \neq 0$ be real numbers and let Q_n, with $Q_0 = 0$, satisfy

$$Q_{n+3} + m_1 Q_{n+2} + m_2 Q_{n+1} + m_3 Q_n = 0 \quad (n = 0, 1, \cdots). \tag{4.1}$$

The generating function of Q_n is given by

$$\sum_{n=0}^{\infty} Q_n x^n = x(Q_1 + (Q_2 + m_1 Q_1)x)/(1 + m_1 x + m_2 x^2 + m_3 x^3). \tag{4.2}$$

Let $f(x) \equiv 1 + m_1 x + m_2 x^2 + m_3 x^3 = (1 - rx)(1 - ax - bx^2)$, where $r^{-1} \neq 0$ is a real root of $f(x) = 0$. Then $m_1 = -a - r$, $m_2 = ar - b$, and $m_3 = br$; and thus $b = m_3/r$

and $a=-m_1-r$. Recalling that

$$(1-x)^{-k}=\sum_{j=0}^{\infty}\binom{k-1+j}{k-1}x^j,$$

we obtain from (4.2), with $W_0=Q_1$ and $W_1-aW_0=Q_2+m_1Q_1$,

$$\begin{aligned}\left(\sum_{n=0}^{\infty}Q_nx^n\right)^k&=\left(\frac{x}{1-rx}\right)^k\left(\frac{W_0+(W_1-aW_0)x}{1-ax-bx^2}\right)^k\\&=\sum_{n=k}^{\infty}r^{n-k}\binom{n-1}{k-1}x^n\sum_{j=0}^{\infty}A_j^{(k)}(a,b;W_0,W_1)x^j\\&=\sum_{n=k}^{\infty}\left(\sum_{j=0}^{n-k}r^{n-j-k}\binom{n-j-1}{k-1}A_j^{(k)}(a,b;W_0,W_1)\right)x^n.\end{aligned}\tag{4.3}$$

Thus, from (4.3), we obtain an identity for the kth convoluted sum (or number) for a recurring sequence Q_n of order three, namely,

$$\begin{aligned}&\sum_{j=0}^{n-k}r^{n-k-j}\binom{n-1-j}{k-1}A_j^{(k)}(-m_1-r,m_3/r;Q_1,Q_2-rQ_1)\\&\qquad=\sum_{i_1+i_2+\cdots+i_k=n}Q_{i_1}Q_{i_2}\cdots Q_{i_k}\quad(n\geqq k).\end{aligned}\tag{4.4}$$

We note that the left-hand side of (4.4) is a linear combination of kth convoluted numbers generated by a recurring sequence of order two.

As an example of (4.4), it is readily verified that $Q_n=Z_nZ_{n+1}$, $n=0,1,\cdots$, satisfies (4.1) with $m_1=-b-a^2$, $m_2=-ba^2-b^2$, and $m_3=b^3$. Since $Q_1=a$ and $Q_2=a^3+ab$, we have $Q_2+m_1Q_1=0$. Thus, from (4.2) we obtain the generating function

$$\sum_{n=0}^{\infty}Z_nZ_{n+1}x^n=\frac{x}{1+bx}\cdot\frac{a}{1-(a^2+2b)x+b^2x^2}.\tag{4.5}$$

Since $r=-b$, we have $-m_1-r=a^2+2b$, $m_3/r=-b^2$, $Q_2-rQ_1=a^3+2ab$, and so $Q_2-rQ_1=(-m_1-r)Q_1$. We obtain from (4.4) the identity

$$\begin{aligned}&a^k\sum_{j=0}^{n-k}(-b)^{n-k-j}\binom{n-1-j}{k-1}C_j^{(k)}(a^2+2b,-b^2)\\&\qquad=\sum_{i_1+i_2+\cdots+i_k=n}\prod_{s=1}^{k}Z_{i_s}Z_{1+i_s}\quad(n\geqq k).\end{aligned}\tag{4.6}$$

For the special case $a=2$ and $b=-1$, we have, as before,

$$Z_n\equiv n\quad\text{and}\quad C_j^{(k)}(2,-1)=\binom{2k+j-1}{j}.$$

Thus, from (4.6), we obtain the identity

$$(4.7)\quad 2^k \sum_{j=0}^{n-k} \binom{n-1-j}{k-1}\binom{2k+j-1}{j} = \sum_{i_1+i_2+\cdots+i_k=n} \prod_{s=1}^{k} i_s(1+i_s) \quad (n \geqq k).$$

Since $C_0^{(k)}(x, y)=1$ and $C_1^{(k)}(x, y)=xk$, then (4.6), with $k=n-1$, gives the identity

$$(4.8)\quad a^{n-1}(a^2+b)(n-1) = \sum_{i_1+i_2+\cdots+i_{n-1}=n} \prod_{s=1}^{n-1} Z_{i_s}Z_{1+i_s} \quad (n=2, 3, \cdots).$$

An identity for Chebyshev polynomials can be obtained from (4.8), for when $a=2x$ and $b=-1$, we have $Z_n \equiv U_{n-1}(x)$, $n=0, 1, \cdots$.

REMARKS. The above method may be used to evaluate convoluted sums for recurring sequences of order four and higher. As an example, we note that $R_n=F_nF_{n+1}F_{n+2}$ or $F_{n+1}^2F_{n+2}$ satisfies $R_{n+4}-3R_{n+3}-6R_{n+2}+3R_{n+1}+R_n=0$ for $n=0, 1, \cdots$. Thus, we have the following generating functions:

$$(4.9)\quad \sum_{n=0}^{\infty} F_nF_{n+1}F_{n+2}x^n = \frac{2x}{1-3x-6x^2+3x^3+x^4} = \frac{2x}{1+x-x^2}\cdot\frac{1}{1-4x-x^2},$$

$$(4.10)\quad \sum_{n=0}^{\infty} F_{n+1}^2F_{n+2}x^n = \frac{1-x}{1+x-x^2}\cdot\frac{1}{1-4x-x^2}.$$

Recalling (2.1), and (3.2) we obtain from (4.9)

$$(4.11)\quad \left(\sum_{n=0}^{\infty} F_nF_{n+1}F_{n+2}x^n\right)^k = \sum_{n=0}^{\infty} A_n^{(k)}(-1, 1; 0, 2)x^n \sum_{j=0}^{\infty} C_j^{(k)}(4, 1)x^j,$$

and from (4.11) the identity for $n \geqq k$,

$$(4.12)\quad 2^k \sum_{j=k}^{n} C_{n-j}^{(k)}(4, 1)C_{j-k}^{(k)}(-1, 1) = \sum_{i_1+i_2+\cdots+i_k=n} \prod_{s=1}^{k} F_{i_s}F_{1+i_s}F_{2+i_s}.$$

Using (2.1) and (3.2), we obtain from (4.10)

$$(4.13)\quad \left(\sum_{n=0}^{\infty} F_{n+1}^2F_{n+2}x^n\right)^k = \sum_{j=0}^{\infty} A_j^{(k)}(-1, 1; 1, -2)x^j \sum_{n=0}^{\infty} C_n^{(k)}(4, 1)x^n;$$

and from (4.13), we obtain the identity

$$(4.14)\quad \sum_{j=0}^{n} A_j^{(k)}(-1, 1; 1, -2)C_{n-j}^{(k)}(4, 1) = \sum_{i_1+i_2+\cdots+i_k=n} \prod_{s=1}^{k} F_{1+i_s}^2F_{2+i_s}.$$

5. Additional convoluted sums. Since

$$(1-ax-bx^2)^{-1} = \sum_{j=0}^{\infty} Z_jx^{j-1},$$

we obtain after one differentiation (for $k=1$)

$$\left(\sum_{n=0}^{\infty}(n+1)Z_{n+2}x^n\right)^k = \left(\frac{a+2bx}{(1-ax-bx^2)^2}\right)^k$$

$$= (1-ax-bx^2)^{-k}\cdot\left(\frac{a+2bx}{(1-ax-bx^2)}\right)^k \tag{5.1}$$

$$= \sum_{j=0}^{\infty} C_j^{(k)}(a,b)x^j \sum_{n=0}^{\infty} A_n^{(k)}(a,b;a,a^2+2b)x^n,$$

and from (5.1) we obtain the identity

$$\sum_{j=0}^{n} C_{n-j}^{(k)}(a,b)A_j^{(k)}(a,b;a,a^2+2b) = \sum_{i_1+i_2+\cdots+i_k=n}\ \prod_{s=1}^{k}(1+i_s)Z_{2+i_s}. \tag{5.2}$$

For $a=2$ and $b=-1$,

$$A_n^{(k)}(2,-1;2,2) = 2^k\binom{k-1+n}{k-1},$$

and thus (5.2) gives

$$2^k\sum_{j=0}^{n}\binom{2k+n-1-j}{2k-1}\binom{k-1+j}{k-1} = \sum_{i_1+i_2+\cdots+i_k=n}\ \prod_{s=1}^{k}(1+i_s)(2+i_s). \tag{5.3}$$

For $a=2x$ and $b=-1$, (5.2) gives an identity for Gegenbauer polynomials,

$$\sum_{j=0}^{n} P_{n-j}^{(k)}(x)A_j^{(k)}(2x,-1;2x,4x^2-2) = \sum_{i_1+i_2+\cdots+i_k=n}\ \prod_{s=1}^{k}(1+i_s)U_{1+i_s}(x). \tag{5.4}$$

For $k=1$, (5.4) simplifies to

$$2\sum_{j=0}^{n} U_{n-j}(x)T_{j+1}(x) = (n+1)U_{n+1}(x). \tag{5.5}$$

Recalling (3.26) and (3.27), we obtain from (5.5) for $x=a/(2i\sqrt{b})$ the identity

$$(n+1)Z_{n+2} = \sum_{j=0}^{n} Z_{n+1-j}V_{j+1} \tag{5.6}$$

for the Lucas functions associated with (3.1).

For $W_n\equiv Z_n$, (3.1) is of special interest when $b=-c^2$, $c\neq 0$. Since $x/(1-ax+c^2x^2)=\sum_{n=0}^{\infty}Z_nx^n$, we obtain after one differentiation (for $k=1$)

$$\left(\sum_{n=0}^{\infty}(n+1)Z_{n+1}x^n\right)^k = \left(\frac{1+cx}{1-ax+c^2x^2}\right)^k\left(\frac{1-cx}{1-ax+c^2x^2}\right)^k, \tag{5.7}$$

and from (5.7) we obtain the identity

$$\sum_{j=0}^{n} A_j^{(k)}(a, -c^2; 1, a+c)A_{n-j}^{(k)}(a, -c^2; 1, a-c) = \sum_{i_1+i_2+\cdots+i_k=n} \prod_{s=1}^{k} (1+i_s)Z_{1+i_s}. \tag{5.8}$$

If $a=2$, $c=1$, then $Z_n \equiv n$, with

$$\sum_{n=0}^{\infty} A_n^{(k)}(2, -1; 1, 1)x^n = (1-x)^{-k} = \sum_{j=0}^{\infty} \binom{k-1+j}{j} x^j,$$

and

$$\sum_{j=0}^{\infty} A_j^{(k)}(2, -1; 1, 3)x^j = (1+x)^k(1-x)^{-2k} = \sum_{j=0}^{\infty} \left(\sum_{s=0}^{j} \binom{k}{j-s} \binom{2k-1+s}{s} \right) x^j.$$

Thus, from (5.8), we obtain the identity

$$\sum_{j=0}^{n} \binom{n+k-1-j}{k-1} \sum_{s=0}^{j} \binom{k}{j-s} \binom{2k-1+s}{s} = \sum_{i_1+i_2+\cdots+i_k=n} \prod_{s=1}^{k} (1+i_s)^2. \tag{5.9}$$

Added in proof. Additional simplification of (4.7) and (5.3) occurs by noting, respectively, that

$$\sum_{j=0}^{n-k} \binom{n-1-j}{k-1} \binom{2k+j-1}{j} = \binom{2k+n-1}{n-k}, \tag{6.1}$$

$$\sum_{j=0}^{n} \binom{2k+n-1-j}{2k-1} \binom{k-1+j}{k-1} = \binom{3k+n-1}{n}. \tag{6.2}$$

Acknowledgment. I wish to thank the referee for his constructive review of my paper, and especially for his suggestion of (1.6), which appears to have many interesting applications.

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THE GROUP OF AUTOMORPHISMS OF THE GAME OF 3-DIMENSIONAL TICKTACKTOE

ROLAND SILVER, MITRE Corporation

1. The automorphisms of a board-game. The player of the ordinary game of Ticktacktoe recognizes intuitively that an initial play in the upper-left hand corner square is essentially the same as an initial play in any other corner. He also will admit that the following eight positions are essentially the same, all consisting of "an X in a corner square, and an O in an adjacent side square."

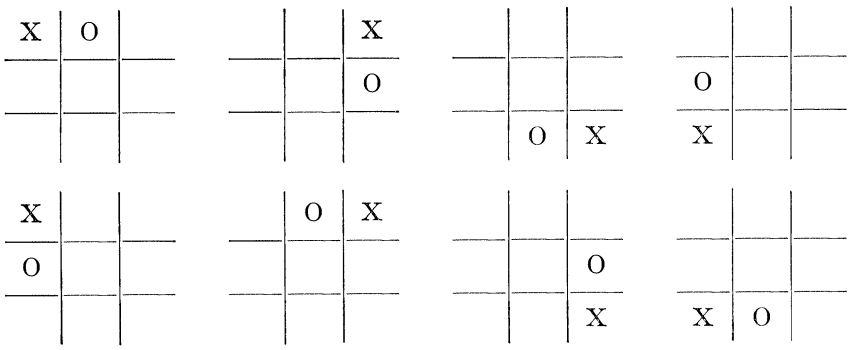


FIG. 1. Equivalent positions.

After some thought, he will probably agree that no other position is essentially the same as one of these.

Behind this notion of the "essential sameness" of two positions P_1 and P_2 is a formal idea: There is a one-one mapping F of the board (the set of squares) onto itself, such that:

(1) $P_2 = P_1F$ (P_2 is the image of P_1 under the mapping F).

(2) For any sequence of legal moves $m_1 \cdot \cdot \cdot m_k$ made starting from position P_1 and leading to position P_3, the corresponding sequence of moves $m_1F \cdot \cdot \cdot m_kF$ are legal starting from position P_2 and lead to position P_3F.

(3) P_3 is won for X, won for O, drawn, or undecided, according as P_3F is won for X, won for O, drawn, or undecided, respectively.

Looking at it another way, if a player has a strategy for playing from position P_1, the transformation F gives him a recipe for transforming that strategy into one for playing from position P_1F. The essential property that F possesses is that it *preserves 3 in a row*: Three distinct points a, b, c of the board are collinear if and only if aF, bF, cF are collinear. An *automorphism* of Ticktacktoe is a mapping of the board onto itself that preserves 3 in a row.

The set $\overline{T}_2$ of mappings of the Ticktacktoe board onto itself is a set of 9! elements. Given U, V in $\overline{T}_2$, we define the *product* UV to be the mapping which takes a square s of the board into the square $(sU)V$. The set $\overline{T}_2$ forms a group

under this product operation; in fact it is isomorphic to S_9, the symmetric group on 9 elements.

The set of automorphisms $\overline{A}_2$ of the game is a subgroup of $\overline{T}_2$; in fact it is just the group of symmetries (rotations and reflections) of the square, having 8 elements. The elements of $\overline{A}_2$ map the position P into the 8 positions shown in Figure 1, which perhaps makes clearer why these 8 positions and only these are equivalent to P.

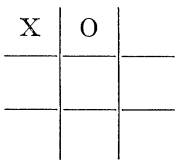


FIG. 2. A position.

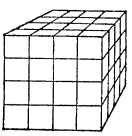


FIG. 3.

2. The board and rules of 3 DT (3-dimensional Ticktacktoe). 3 DT (3-Dimensional Ticktacktoe) is played by two players on a board consisting of 64 unit cubes (*cells*) stacked into a larger $4\times4\times4$ cube. The first player chooses any cell of the board, and places his mark ("X") in it. The second player then chooses any other cell, and places his mark ("O") in it. The game proceeds as the players alternate marking previously unmarked cells, until one of two things happens:

(1) The mover wins by making a cell which creates four-in-a-row, i.e., there are 4 cells, all marked by him, whose centers are collinear.

(2) The board is completely filled, but neither player has 4-in-a-row, constituting a tie.

Just as in Ticktacktoe, there are one-one mappings of the 3DT board which preserve the "essential sameness" of positions. We will define these automorphisms, and show that they form a group of 192 elements. This group will be seen to contain some peculiar mappings which scramble the cells of the board while preserving 4-in-a-row.

3. Points and lines of 3DT. We choose a coordinate system in which the center of each board cell has integer coordinates 0, 1, 2, or 3. We will refer to a triple (x, y, z) with $0\leqq x\leqq3$, $0\leqq y\leqq3$, $0\leqq z\leqq3$ as a *point* of the board. For clarity

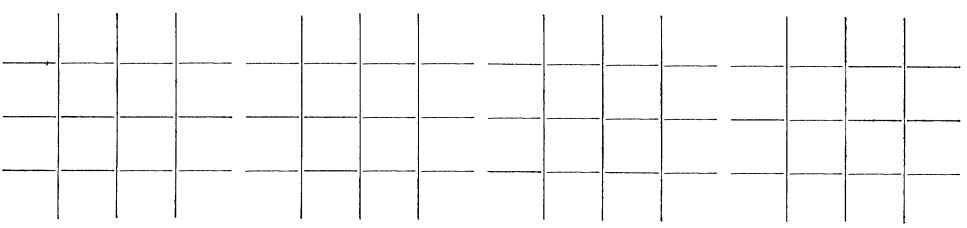


FIG. 4. A representation of the 3DT board.

we shall depict the board as four 4×4 squares representing the cross-sections $Z=0$, $Z=1$, $Z=2$, $Z=3$, from left to right.

We partition the 64 points of the board into two classes: *rich* points and *poor* points. The 16 rich points are marked "r" in the following diagram; the other 48 points are poor. A point (x, y, z) is *rich* if each of x, y, and z is 0 or 3, or if each of x, y, and z is 1 or 2; otherwise it is poor.

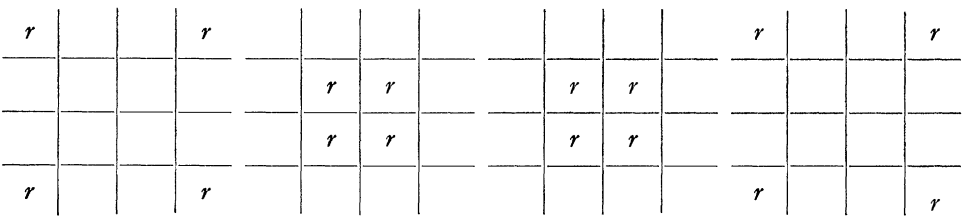


FIG. 5. The 16 rich points.

A *line* is a set of four distinct points of the board which lie in a straight line in the 3-dimensional space in which the points of the board are embedded. We will speak of a point being *on* a line, and of two lines *intersecting* in a point, in the usual sense. Two or more distinct points $p_1, \cdots, p_n$ are *collinear* if they all lie on some line. There are just 76 lines, which we divide into four classes:

3.1. *Full lines*—A *full* line consists of four rich points. There are 4 full lines: the "space" diagonals (see the line marked "f" in Figure 6).

3.2. *Empty lines*—An *empty* line consists of 4 poor points. There are 24 empty lines: the lines parallel to the coordinate axes which do not pass through rich points (see the lines marked "e_1," and "e_2" in Figure 6).

3.3. *Diagonal line*—A *diagonal* line consists of 2 rich and 2 poor points, and passes through two opposite edges of the cube. There are 24 diagonal lines (see the lines marked "d_1" and "d_2" in Figure 6).

3.4. *Ranking lines*—A *ranking* line consists of 2 rich and 2 poor points and is parallel to a coordinate axis. There are 24 ranking lines (see the line marked "r_1" and "r_2" in Figure 6).

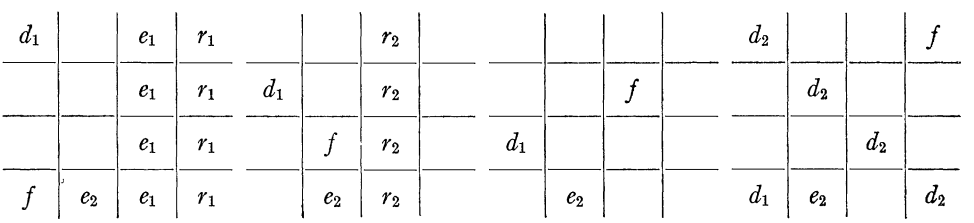


FIG. 6. Examples of the four classes of lines.

We note some interesting properties of these classifications:

3.5. Each rich point lies on 7 lines: 1 rich, 3 diagonal, and 3 ranking.

3.6. Each poor point lies on 4 lines: 2 poor, 1 diagonal, and 1 ranking.

Let us define the *star* of a line L to be the set of all points not on L, but collinear with some point on L.

3.7. Each diagonal line has 34 points in its star (see Figure 7).

3.8. Each ranking line has 36 points in its star (see Figure 8).

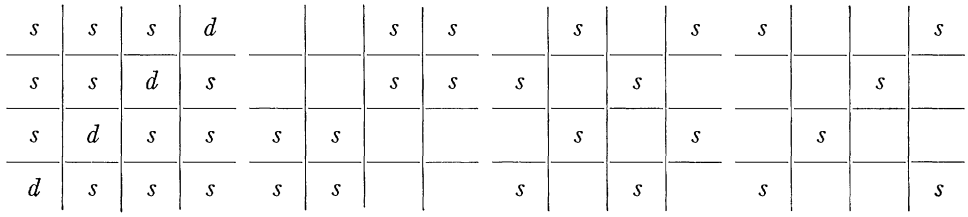


FIG. 7. The star of a diagonal line.

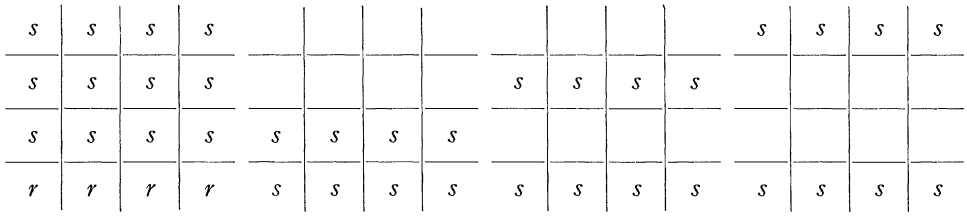


FIG. 8. The star of a ranking line.

4. Automorphisms of 3DT; Invariants. The set $\overline{T}_3$ of all one-one mappings of the 3DT board onto itself has 64! elements. Given a mapping $T \in \overline{T}_3$ and a line $L = \{p_1, p_2, p_3, p_4\}$, we define LT to be $\{p_1T, p_2T, p_3T, p_4T\}$. Given U, V in $\overline{T}_3$, we define the *product* UV to be the mapping: $p \to (pU)V$. Under this product operation, the set $\overline{T}_3$ forms a group isomorphic to S_{64}, the symmetric group on 64 elements.

An *automorphism* A of 3DT is a member of $\overline{T}_3$ which preserves lines, that is, for any line $\{p_1, p_2, p_3, p_4\}$, the four points p_1A, p_2A, p_3A, p_4A are collinear. Let $\overline{A}_3$ be the set of automorphisms of $\overline{T}_3$.

4.1. It is obvious that the product of any two automorphisms is an automorphism. It follows that $\overline{A}_3$ is a group.

4.2. For any automorphism A, A^{-1} is an automorphism, so that p_1, p_2, p_3, p_4 are collinear if and only if p_1A, p_2A, p_3A, p_4A are. Clearly an automorphism maps distinct lines into distinct lines. A set of points $\overline{P}$ is *invariant* if for any A in $\overline{A}_3$, p in $\overline{P}$ implies pA in $\overline{P}$.

A set of lines $\overline{L}$ is *invariant* if for any $A \in \overline{A}_3$, $L \in \overline{L}$ implies LA in $\overline{L}$.

4.3. It is clear that the following sets are invariant: rich points (by 3.5), poor points (by 3.6), full lines (by 3.1), empty lines (by 3.2), diagonal lines (by 3.7), ranking lines (by 3.8).

These classifications will be useful further on in proving that we have caught all the members of $\bar{A}_3$. It will turn out that the points and lines cannot be parti-

tioned more finely than we have done, in the sense that none of the six sets of 4.3 contains an invariant proper subset. In fact, $\overline{A}_3$ will turn out to be transitive over each of these sets.

5. The group $\overline{R}$ of rotations of the 3DT board. There are 24 rotations which carry the cube into itself:

5.1. The identity.

5.2. Rotation by 90°, 180°, 270° about each of three axes through the centers of opposite faces of the cube.

5.3. Rotation by 180° about each of six axes bisecting opposite edges of the cube.

5.4. Rotation by 120° and 240° about each of 4 axes through opposite corners of the cube.

These 24 rotations form a subgroup $\overline{R}$ of $\overline{T}_3$. They are obviously automorphisms.

6. The evisceration mapping E and the scramble mapping S. Consider the mapping E: $(x, y, z)\rightarrow(xe, ye, ze)$ where $0e=1$, $1e=0$, $2e=3$, $3e=2$. The mapping E turns the board inside out, mapping the 8 interior rich points onto the 8 exterior rich points, and *vice versa*.

Also, consider S: $(x, y, z)\rightarrow(xs, ys, zs)$, where $0S=0$, $1S=2$, $2S=1$, $3S=3$. S scrambles the interior rich points, mapping each into its opposite, while leaving the 8 exterior rich points fixed.

It is not too hard to see that both E and S are automorphisms.

7. The "flexible" group $\overline{F}$ of the 3DT board. Consider e and s as permutations on the set $\{0, 1, 2, 3\}$. Letting "i" denote the identity element, we have

$$\begin{aligned} e &= (01)(23) & e^2 &= i \\ s &= (12) & s^2 &= i \\ es &= (0231) \\ se &= (0132) \\ ese &= (03) \\ ses &= (03)(12) \\ eses &= sese = (03)(12). \end{aligned}$$

Thus s and e generate the 8 element group $\{i, e, s, es, se, ese, ses, eses\}$. In an analogous way, S and E generate the 8 element group $\overline{F}=\{I, E, S, ES, SE, ESE, SES, ESES\}$, which is evidently a subgroup of $\overline{A}_3$. For example

$$(x, y, z)ESE = (xe, ye, ze)SE = (xes, yes, zes)E = (xese, yese, zese).$$

In fact, $\overline{F}$ is isomorphic to the group of symmetries of the square!

8. The group $\overline{R}\times\overline{F}$. The set $\overline{R}\overline{F}$ of all elements of the form RF, with R in $\overline{R}$ and F in $\overline{F}$ is a direct product if

(1) $\overline{R}\cap\overline{F}=\{I\}$, and (2) $\overline{R}$ and $\overline{F}$ commute.

8.1. It is obvious that no member of $\overline{F}$ except I is a rotation.

8.2. Let us define a to be $ses = (03)(12)$, and define the following mappings in $\overline{T}_3$:

$$(x, y, z)T_x = (xa, y, z) \qquad (x, y, z)T_{xy} = (y, x, z)$$

$$(x, y, z)T_y = (x, ya, z) \qquad (x, y, z)T_{yz} = (x, z, y)$$

$$(x, y, z)T_z = (x, y, za) \qquad (x, y, z)T_{xz} = (z, y, x).$$

Every rotation in R can be expressed as a product of some of these 6 elements. To show that $\overline{R}$ and $\overline{F}$ commute, it will suffice then to show that each of these six elements commutes with E and with S.

- We have $ae = esese = eeses = ea$, and $as = esess = seses = sa$. Now:

$$(x, y, z)T_xE = (xae, ye, ze) = (xea, ye, ze) = (x, y, z)ET_x; \quad \text{so } T_xE = ET_x.$$

$(x, y, z)T_{xy}E = (y, x, z)E = (ye, xe, ze) = (xe, ye, ze)T_{xy} = (x, y, z)ET_{xy}$, so $T_{xy}E = ET_{xy}$. The other 10 commutation relations are demonstrated similarly. We have thus that the product $\overline{RF}$ is in fact a direct product.

This group $\overline{R}\times\overline{F}$ is a subgroup of $\overline{A}_3$ containing $24\times 8 = 192$ elements.

9. Completeness of $\overline{R}\times\overline{F}$. We want to show that in fact $\overline{R}\times\overline{F} = \overline{A}_3$. This will follow if we can show two things:

9.1. Given any automorphism A, there is an X in $\overline{R}\times\overline{F}$ such that AX leaves the four points (0, 0, 0), (1, 0, 0), (3, 0, 0), (0, 3, 0) fixed.

9.2. If an automorphism A leaves the four points (0, 0, 0), (1, 0, 0), (3, 0, 0), (0, 3, 0) fixed, then $A = I$.

Let us proceed: $P = (0, 0, 0)A$ is a rich point. If p is an interior point, then let $X_1 = E$, otherwise let $X_1 = I$. Thus, pX_1 is an exterior rich point. There is some rotation X_2 which takes pX_1 into (0, 0, 0), so (0, 0, 0) is fixed under AX_1X_2.

Now consider the line L through (0, 0, 0) and (3, 0, 0). L is a ranking line, so LAX_1X_2 is a ranking line L' through (0, 0, 0). There is clearly a rotation X_3 which leaves (0, 0, 0) fixed, and for which $L'X_3 = L$. Thus $AX_1X_2X_3$ fixes (0, 0, 0) and maps L into L. Since (3, 0, 0) is the other rich point on L, $AX_1X_2X_3$ fixes (3, 0, 0). $AX_1X_2X_3$ takes the ranking line through (0, 0, 0) and (0, 3, 0) into either the ranking line through (0, 0, 0) and (0, 3, 0) or the ranking line through (0, 0, 0) and (0, 0, 3). In the former case, let $X_4 = I$. In the latter case, consider the mapping R: $(x, y, z)R = (xa, za, ya)$. R is clearly the rotation about the bisectors of the edges $\{(0, 0, 3), (3, 0, 3)\}$ and $\{(0, 3, 0), (3, 3, 0)\}$. Let $X_4 = RESES$. Thus X_4 is in $\overline{R}\times\overline{F}$.

We have $(0, 0, 0)X_4 = (0, 0, 0)$, $(3, 0, 0)X_4 = (3, 0, 0)$, $(0, 0, 3)X_4 = (0, 3, 0)$. Thus X_4 leaves (0, 0, 0) and (3, 0, 0) fixed, and maps $(0, 3, 0)AX_1X_2X_3$ into (0, 3, 0). Therefore $AX_1X_2X_3X_4$ leaves the three points (0, 0, 0), (3, 0, 0) and (0, 3, 0) fixed. Now $(1, 0, 0)AX_1X_2X_3X_4$ is a poor point on the line through (0, 0, 0) and (3, 0, 0), hence it is either (1, 0, 0) or (2, 0, 0). Let $X_5 = I$ or S accordingly. Both S and I leave (0, 0, 0), (3, 0, 0) and (0, 3, 0) fixed.

Thus, $AX_1X_2X_3X_4X_5$ leaves the four points fixed. Letting $X = X_1X_2X_3X_4X_5$, we have proven 9.1.

Let A be an automorphism. We have evidently:

9.3. If A fixes three points on a line, then it fixes the fourth also.

9.4. If A leaves a diagonal (ranking) line invariant, and fixes one of its rich points, then it fixes its other rich point also.

9.5. If A leaves each of two different intersecting lines invariant, then it fixes their intersection.

9.6. If A fixes two points on a line, then it leaves the line invariant.

9.7. Let p be a rich point, and let p_1, p_2, p_3 be the rich points on the three diagonal (ranking) lines through p. If A fixes p, p_1 and p_2, then it fixes p_3.

9.8. Let p be a rich point, and let p_1, p_2, p_3 be the rich points on diagonal (ranking) lines through p. If A fixes p_1, p_2, p_3, then A fixes p.

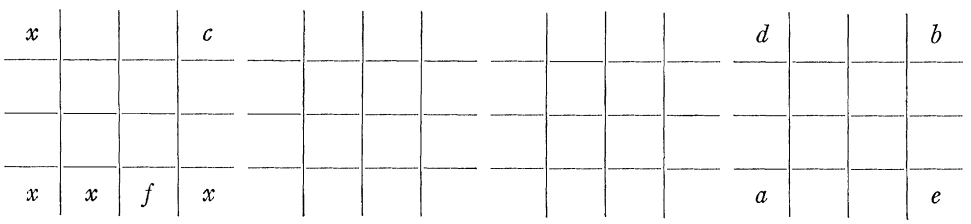


FIG. 9. Fixed points of A.(1) a and b are fixed, by 9.7; c, d and e are fixed, by 9.8; f is fixed, by 9.3.

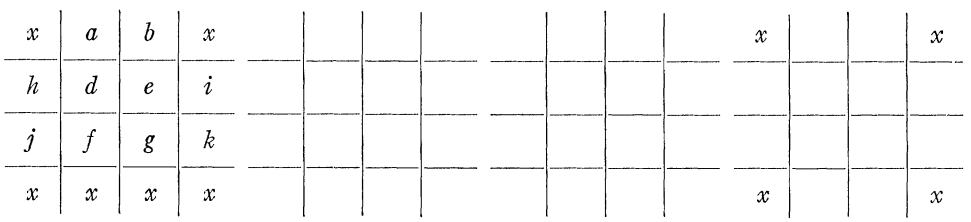


FIG. 10. Fixed points of A.(2).

Now suppose that A fixes (0, 0, 0), (1, 0, 0), (3, 0, 0) and (0, 3, 0): a is on the invariant (9.6) line through (0, 3, 0) and (3, 3, 0), hence aA is one of the poor points a or b. But a is collinear with (1, 0, 0) while b is not, and (1, 0, 0) is fixed, so $aA = a$. b is fixed (9.3).

The line through a and d is invariant (9.6), as is the line through g and d (9.6) so d is fixed (9.5). Similarly, e, f and g are fixed.

The line through d and e is invariant, as is the line through (0, 0, 0) and (0, 3, 0), so h is fixed. Similarly i, j and k are fixed.

The sequence of arguments above showed that from the six fixed points of the left square of Figure 10 we could deduce that all the points of that square are fixed. Each of the other faces of the cube can be treated similarly. Thus, all of the exterior points of the cube are fixed. But each interior point is on the intersection

of two lines each of which intersects the surface, hence 9.6 and 9.5 show the interior points fixed as well.

This completes the proof of 9.2, and thus $\overline{R}\times\overline{F}=\overline{A}_3$.

10. Transitivity. We have succeeded in discovering the group of automorphisms $\overline{A}_3=\overline{R}\times\overline{F}$ of the 3 DT board. It turns out to be isomorphic to the direct product of the group of rotations of the cube by the group of symmetries of the square.

We will now show that $\overline{A}_3$ is transitive over each of the invariant sets of 4.3. First note that if $\overline{A}_3$ is transitive over the set $\overline{S}$, and also over the set $\overline{T}$, and if some A in $\overline{A}_3$ maps a member of $\overline{S}$ into a member of $\overline{T}$, then $\overline{A}_3$ is transitive over $\overline{S}\cup\overline{T}$.

10.1. *Rich Points.* Clearly $\overline{R}$ is transitive over the set $\overline{S}$ of exterior rich points and the set $\overline{T}$ of interior rich points. But E maps a point of $\overline{S}$ into a point of $\overline{T}$. Therefore, $\overline{A}_3$ is transitive over $\overline{S}\cup\overline{T}$, the rich points.

10.2. *Poor points.* Clearly $\overline{R}$ is transitive over the set $\overline{S}$ of poor edge points, and also over the set $\overline{T}$ of poor points on a face but not on an edge. But E maps $\overline{S}$ points into $\overline{T}$ points, so $\overline{A}$ is transitive over $\overline{S}\cup\overline{T}$, the poor points.

Full lines. Clearly $\overline{R}$ is transitive over the full lines.

Empty lines. Clearly $\overline{R}$ is transitive over the empty lines.

Diagonal lines. The diagonal lines can be partitioned into two classes of 12 lines each: the set $\overline{S}$ of these that lie on a face of the cube, and the set $\overline{T}$ of these that do not. Clearly $\overline{R}$ is transitive over each of these sets, while E maps the $\overline{S}$-line $\{(0, 0, 0), (1, 1, 0), (2, 2, 0), (3, 3, 0)\}$ into the $\overline{T}$-line $\{(0, 0, 1), (1, 1, 1), (2, 2, 1), (3, 3, 1)\}$.

Ranking lines. The ranking lines can be partitioned into two classes of 12 lines each: the set $\overline{S}$ of those that are edges of the cube, and the set $\overline{T}$ of those that are not. Clearly $\overline{R}$ is transitive over $\overline{S}$ and $\overline{T}$, while E maps the $\overline{S}$-line. $\{(0, 0, 0), (1, 0, 0), (2, 0, 0), (3, 0, 0)\}$ into the $\overline{T}$-line $\{(0, 1, 1), (1, 1, 1), (2, 1, 1), 3, 1, 1)\}$.

11. Applications to the game of 3DT. Of the 192 automorphisms $\overline{A}_3$ of the 3DT board, 48 are obvious (the symmetries of the cube), while the others are not so obvious. This is an extraordinarily large group of automorphisms for a board game: contrast the order of the automorphism group for checkers (1), chess (1), checkers with kings only (4), Go and Go-moku (8), Ticktacktoe (8). In any attempt to analyze the game, or to provide a heuristic computer program for playing it, this group will provide a powerful method for pruning the move tree by eliminating positions equivalent to one already considered. For example, it should be evident from 10 that there are essentially two opening moves: playing on a rich point, and playing on a poor point.

The research reported in the paper was sponsored by the Air Force Electronic Systems Division, Air Force Systems Command, under Contract AF 19(628)2390. This paper will also be obtainable as Technical Documentary Report Number 64-146. Further reproduction is authorized to satisfy needs of the U. S. Government. The MITRE Corporation has obtained approval for the release of the information contained in this paper.

INFINITELY DIVISIBLE DISTRIBUTIONS ON CYCLIC GROUPS

LEOPOLD FLATTO, Yeshiva University and P. A. SCHEINOK, Hahnemann Medical College

I. Introduction. The theory of infinitely divisible distributions on the real line, their characterization and application to the limit theorems of the Theory of Probability (see [3], [4], and [5]) although not formulated until the mid-nineteen-thirties is now classic fare in the basic graduate curriculum in Probability and Statistics. At the same time, in the past fifteen years probabilists have been trying to extend many of the classic results in the Theory of Probability relating to the real line to abstract spaces having certain algebraic relations defined on them. They have thus been investigating such structures as groups, semigroups, topological vector spaces, and algebras to see what the counterpart of the classic Theory of Probability is within such individual frameworks. For a fairly recent summation of results in this area see Grenander [2]. One of the subjects which is being studied in the setting of such algebraic structures is that of the characterization of infinitely divisible distributions. In [1] W. Böge proved a general representation theorem which characterizes the infinitely divisible distributions on finite groups, and answered certain uniqueness questions relating to such a representation. Although Böge's generalized treatment can nontrivially be particularized to the situation discussed below, the authors have wished to proceed from first principles in order to see what concrete analytic representation one can give for very simple groups. In the ensuing, the cyclic groups of order three and four will be examined and characterized. (The case $n=2$ is trivial.) The complexity of the case $n=4$ leads us to believe that this method cannot be generalized to arbitrary values of n. Still, the cases $n=3, 4$ seem to be most interesting in their own right, and present us with a very elegant solution, which we believe to be pedagogically useful.

II. Basic definitions. Classically [3], the definition of infinite divisibility is as follows.

DEFINITION. *The random variable X is infinitely divisible if for every natural number n it can be represented as the sum*

$$X = Y_{n1} + Y_{n2} + \cdots + Y_{nn}$$

of n independent identically distributed random variables $Y_{n1}, Y_{n2}, \cdots, Y_{nn}$.

Let $F_X(x)$ be the distribution function of X, and let $\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} dF_X(x)$ be its characteristic function (Fourier-Stieltjes transform of $F_X(x)$). Then by the well-known properties of characteristic functions, X will be infinitely divisible if and only if for every natural number n, $\phi_X(t)$ is the nth power of some characteristic function $\phi_{Y_{ni}}(t)$. This characteristic function $\phi_{Y_{ni}}$ is naturally dependent on n.

Many of the famous classic probability distributions are infinitely divisible,

which of course is what led the impetus to research in this area. We shall see several examples of this.

Example 1: Let X have the Gaussian distribution with mean μ and variance $\sigma^2(X\sim N(\mu, \sigma^2))$, then for every natural number $n>0$, $X=\sum_{i=1}^{n} Y_{ni}$ where each $Y_{ni}\sim N(\mu/n, \sigma^2/n)$, $i=1, \cdots, n$.

Example 2: Let X have the Poisson distribution with mean λ, $(X\sim P(\lambda))$, then for every natural number $n>0$, $X=\sum_{i=1}^{n} Y_{ni}$, where each $Y_{ni}\sim P(\lambda/n)$, $i=1, \cdots, n$.

It is easiest to use the properties of characteristic functions in order to verify these examples.

In our situation, consider a random variable X which assumes the values $0, 1, \cdots, n-1$. Suppose addition of random variables is done modulo n, and let $p(k)=\Pr\{X=k\}$ $0\leqq k\leqq n-1$. We wish to find criteria for p to be infinitely divisible. To do this we shall consider the transform $\hat{p}$, and our criteria will be on the real and imaginary parts of $\hat{p}$. Let $\zeta=e^{2\pi i/n}$, then one defines the Fourier transform as

$$\hat{p}(j) = \sum_{k=0}^{n-1} p(k)\zeta^{kj} \tag{2.1}$$

and one has the inversion formula

$$p(k) = \frac{1}{n}\sum_{j=0}^{n-1} \hat{p}(j)\zeta^{-jk}. \tag{2.2}$$

If there exists a probability distribution $g(k)$ such that $p=g*\cdots*g$, an n-fold convolution, then this is equivalent to $\hat{p}=\hat{g}^n$. We also note that the conditions for p to be a probability distribution

$$p(k) \geqq 0 \qquad (0 \leqq k \leqq n-1) \tag{2.3}$$

$$\sum_{k=0}^{n-1} p(k) = 1 \tag{2.4}$$

are equivalent respectively to the conditions on the transform,

$$\sum_{j=0}^{n-1} \hat{p}(j)\zeta^{-jk} \geqq 0 \qquad (0 \leqq k \leqq n-1) \tag{2.3'}$$

$$\hat{p}(0) = 1. \tag{2.4'}$$

To characterize the infinitely divisible $p(k)$ one wishes to find conditions on $\hat{p}(j)$ $(0\leqq j\leqq n-1)$ satisfying (2.3') and (2.4') which will guarantee that for each integral $m\geqq 0$, $\hat{p}^{1/m}$ will also satisfy (2.3') and (2.4'). We shall examine in detail the cases $n=3, 4$. We note that

$$\hat{p}(n-k) = \overline{\hat{p}(k)} \qquad (k = 0, 1, 2, \cdots, n-1) \tag{2.5}$$

a fact very useful in the sequel.

III. The case $n=3$. In this case one notes that the condition (2.1) implies that the points $\hat{p}(0)$, $\hat{p}(1)$, $\hat{p}(2)$ lie inside or on the triangle with vertices at 1, ω, ω^2 in the complex plane, where ω represents one of the cube roots of unity. We have $\hat{p}(0)=1$, $\hat{p}(2)=\overline{\hat{p}(1)}$, and we see that $\hat{p}$ is infinitely divisible if and only if $[\hat{p}(1)]^{1/m}$ lies inside the triangle for every positive integer $m\geqq 1$.

Write $p(1)=re^{i\theta}$, $0\leqq|\theta|\leqq\pi$. The result we seek is embodied in the following theorem.

THEOREM. *For the case $n=3$, $p(k)$ is an infinitely divisible law if and only if $\hat{p}(1)=re^{i\theta}$ is such that $r\leqq e^{-\sqrt{3}|\theta|}$, $0\leqq|\theta|\leqq\pi$.*

Proof. Suppose $p(k)$ is infinitely divisible. Because of symmetry it suffices to restrict oneself to points $\hat{p}(1)$ in the upper half of the complex plane. By hypothesis, the point $[\hat{p}(1)]^{1/m}=r^{1/m}e^{i(\theta+2\pi k)/m}$ must lie inside the triangle $(1, \omega, \omega^2)$ for some value of k, $k=0, 1, \cdots, m-1$. If one lets $m=3j$, $j=1, 2, \cdots$, then it is clear that if there is an mth root which lies inside the triangle, there will always be one in the sector $|\theta|\leqq\pi/3$ which will lie inside the triangle, and since all the mth roots are on the circle $|Z|=r^{1/m}$, it is clear that the root $r^{1/m}e^{i\theta/m}$ which is closest to the axis, must also be inside the triangle. Thus, if we consider the line $r=\frac{1}{2}\sec(\theta-\pi/3)$ which connects the points 1 and ω, we demand that $\hat{p}(1)=re^{i\theta}$ be such that

$$r^{1/m}\leqq\tfrac{1}{2}\sec\left(\frac{\theta}{m}-\frac{\pi}{3}\right)$$

for an infinity of m. In the limit as $m\to\infty$, we have

$$\left[\frac{1}{2}\sec\left(\frac{\theta}{m}-\frac{\pi}{3}\right)\right]^m\to e^{-\sqrt{3}\theta}.$$

Thus for an infinitely divisible law $r\leqq e^{-\sqrt{3}\theta}$, and by symmetry $r\leqq e^{-\sqrt{3}|\theta|}$, $0\leqq\theta\leqq\pi$. Conversely, let $r\leqq e^{-\sqrt{3}\theta}$. Then we have the following result.

Contention: Let

$$f(\theta)=\begin{cases}\frac{1}{2}\sec(\theta-\pi/3) & 0\leqq\theta\leqq 2\pi/3\\ -\frac{1}{2}\sec\theta & 2\pi/3\leqq\theta\leqq\pi.\end{cases}$$

Then $f^m(\theta/m)\epsilon\downarrow$ (is nonincreasing).

Proof: $f^m(\theta/m)\epsilon\downarrow$ if and only if $m\log f(\theta/m)\epsilon\downarrow$ which would follow from $(1/\theta u)\cdot\log f(\theta u)\epsilon\uparrow$ in $0\leqq u\leqq 1$. Now consider $1/x\log f(x)$. This expression $\epsilon\uparrow$ in $0\leqq x\leqq 1$ if and only if

$$\frac{1}{x}[\log f]'-\frac{1}{x^2}[\log f]\geqq 0.$$

This is equivalent to showing that $[\log f]'\geqq\log f/x$ which would follow from $(\log f)''=(ff''-f'^2)/f\geqq 0$. Direct computation, however, shows that for $f(\theta)$ defined above, $ff''-f'^2\geqq 0$, so that the contention is proved.

We note that the latter contention establishes the converse, since $r \leqq e^{-\sqrt{3\theta}}$ implies that $r^{1/m} \leqq f(\theta/m)$ for every m, so that there is an mth root inside the triangle $(1, \omega, \omega^2)$ for $m = 1, 2, \cdots$ and $p(k)$ is thus infinitely divisible.

IV. The case $n = 4$. In this situation we have $\hat{p}(0) = 1$, $\hat{p}(2)$ is real, and $\overline{\hat{p}(1)} = \hat{p}(3)$. The condition (2.3′) then becomes

$$\begin{array}{ll} \text{(a)}\ 1 + \hat{p}(1) + \hat{p}(2) + \overline{\hat{p}(1)} \geqq 0 & \text{(c)}\ 1 - \hat{p}(1)i + \hat{p}(2) - \hat{p}(1) \geqq 0 \\ \text{(b)}\ 1 - \hat{p}(1)i - \hat{p}(2) + \overline{\hat{p}(1)i} \geqq 0 & \text{(d)}\ 1 + \hat{p}(1)i - \hat{p}(2) - \hat{p}(1)i \geqq 0. \end{array} \tag{4.1}$$

One notes that (a) and (c) are equivalent to

$$| \operatorname{Re} \hat{p}(1) | \leqq \frac{1 + \hat{p}(2)}{2}, \tag{4.2a}$$

while (b) and (d) are equivalent to

$$| \operatorname{Im} \hat{p}(1) | \leqq \frac{1 - \hat{p}(2)}{2}. \tag{4.2b}$$

Geometrically, this means that $-1 \leqq \hat{p}(2) \leqq 1$, and for fixed $\hat{p}(2)$, $\hat{p}(1)$ is in shaded rectangle in Figure 1.

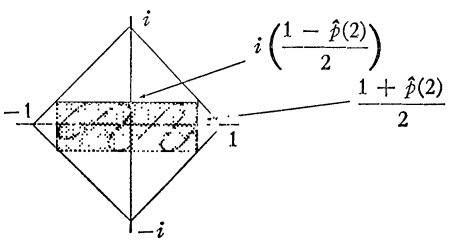


FIG. 1

To say that $p(k)$ has an mth root is equivalent to the conditions

$$| \operatorname{Re} [\hat{p}(1)]^{1/m} | \leqq \frac{1 + [\hat{p}(2)]^{1/m}}{2}, \tag{4.3a}$$

and

$$| \operatorname{Im} [\hat{p}(1)]^{1/m} | \leqq \frac{1 - [\hat{p}(2)]^{1/m}}{2}, \tag{4.3b}$$

where we take the mth root to be positive. For m even, we might have taken the mth root to be negative, in which case we would have the inequalities

$$| \operatorname{Re} [\hat{p}(1)]^{1/m} | \leqq \frac{1 - [\hat{p}(2)]^{1/m}}{2}, \tag{4.3′a}$$

and

$$(4.3'b) \qquad |\operatorname{Im} [\hat{p}(1)]^{1/m}| \leqq \frac{1 + [\hat{p}(2)]^{1/m}}{2}.$$

If $\hat{p}(2)$ is to have a real mth root for m even, then clearly $0 \leqq \hat{p}(2) \leqq 1$. Since the rectangle in Figure 1 is symmetric about x-axis we can again write $\hat{p}(1) = re^{i\theta}$, where $0 \leqq \theta \leqq \pi$. Also write $\hat{p}(2) = B$.

LEMMA 1. $\lim_{m\to\infty} ((1+B^{1/m})/2)^m = \sqrt{B}$.

Proof: For $B=0$ or 1 this is obvious. Assume then that $0<B<1$ so that $B=e^{-b}$ where $0<b<\infty$.

$$\frac{1 + B^{1/m}}{2} = \frac{1 + e^{-b/m}}{2} = 1 - \frac{b}{2m} + O\left(\frac{1}{m^2}\right)$$

$$\therefore \quad \log\left(\frac{1 + B^{1/m}}{2}\right)^m = m \log\left(\frac{1 + B^{1/m}}{2}\right)$$

$$= m \log\left(1 - \frac{b}{2m} + O\left(\frac{1}{m^2}\right)\right) = -\frac{b}{2} + O\left(\frac{1}{m}\right).$$

Thus $\lim_{m\to\infty} ((1+B^{1/m})/2)^m = e^{-b/2} = \sqrt{B}$.

LEMMA 2. $((1+B^{1/m})/2)^m \epsilon \downarrow$ for $m=1, 2, \cdots, (B>0)$.

Proof: $((1+B^{1/m})/2)^m \epsilon \downarrow$ is equivalent to $m \log ((1+B^{1/m})/2) \epsilon \downarrow$ which follows from $f(u) = 1/u \log ((1+B^u)/2) \epsilon \uparrow$ for $0 \leqq u \leqq 1$. Let $g = (1+B^u)/2$. Then $f(u) \epsilon \uparrow$ if $f' = [(1/u) \log g]' \geqq 0$. Thus one must show that $\log g + u[\log g]' \geqq 0$ which is guaranteed by $-[\log g]' + [u(\log g)']' \geqq 0$, or simply by $[\log g]'' \geqq 0$. But $[\log g]'' = (gg'' - g'^2)/g^2$, and $gg'' - g'^2 = \frac{1}{4}(\log B)^2 B^u \geqq 0$ thereby proving the lemma.

Let d be the semi-diagonal of the rectangle in Fig. 1. Then $d = [(1+B^2)/2]^{1/2}$. It also follows from the equations (4.3) that if p is to be infinitely divisible for every m, then

$$r^{1/m} \leqq \left(\frac{1 + B^{2/m}}{2}\right)^{1/2} \quad \text{or} \quad r \leqq \left(\frac{1 + B^{2/m}}{2}\right)^{m/2}.$$

Thus $r \leqq \sqrt{B}$ if we have infinite divisibility. Now by Lemma 2, $r \leqq \sqrt{B}$ implies that $r \leqq (1+B^{1/m}/2)^m$, or $r^{1/m} \leqq (1+B^{1/m}/2)$. This tells us that the mth roots of $\hat{p}(1)$ lie on a circle whose radius is less than or equal to the length of the rectangle $[1+(\hat{p}(2))^{1/m}]/2$. Thus if there is a root inside the rectangle there has to be one in that portion of the rectangle in the right half-plane, and the closest one to the x-axis will again have the argument θ/m. Hence, from (4.3) one gets for odd m that

$$(4.4a) \qquad r^{1/m} \cos\frac{\theta}{m} \leqq \frac{1 + B^{1/m}}{2}, \qquad (4.4b) \qquad r^{1/m} \sin\frac{\theta}{m} \leqq \frac{1 - B^{1/m}}{2}.$$

From (4.4a) no new information is gained. From (4.4b) multiplication of both sides by m and letting $m\to\infty$ through the odd numbered integers leads to

$$\lim_{m\to\infty} m\left(\frac{1-B^{1/m}}{2}\right) = \lim_{m\to\infty} m\left(\frac{1-e^{-b/m}}{2}\right) = \frac{b}{2} = \log\frac{1}{\sqrt{B}}\,.$$

Conversely, let $r\leqq\sqrt{B}$, $\theta\leqq\log 1/\sqrt{B}$. Then we note that (4.4a) is guaranteed, since $r^{1/m}\cos\theta/m\leqq r^{1/m}\leqq(1+B^{1/m})/2$ by Lemma 2. In addition, we can show that (4.4b) holds as

$$(1-B^{1/m})\frac{r^{-1/m}}{\sin^{\theta/m}}\,\epsilon\downarrow\,.$$

The latter implies that

$$\frac{1-B^{1/m}}{2}\,\frac{r^{-1/m}}{\sin^{\theta/m}}\geqq 1$$

or that (4.4b) holds. We shall use the following lemma.

LEMMA 3: *Let* $U(0)=V(0)=0$. *Let* $U(x)$, $V(x)\geqq 0$, $U''(x)\geqq 0$, $V''(x)\leqq 0$. *Then* $U/V\epsilon\uparrow$.

Proof: $(U/V)'=(VU'-UV')/V^2$. We wish to show that $VU'-UV'\geqq 0$. But $VU'-UV'=0$ at $x=0$. Hence $(VU'-UV')'=VU''-UV''\geqq 0$ which proves our lemma.

We now note that

$$\frac{(1-B^{1/m})r^{-1/m}}{\sin\theta/m} = \left(\frac{B^{1/2}}{r}\right)^{1/m}\frac{B^{-1/2m}-B^{1/2m}}{\sin\theta/m}\,.$$

By hypothesis $B^{1/2}/r\geqq 1$, so that $(B^{1/2}/r)^{1/m}\epsilon\downarrow$. Thus it suffices to show that $[(B^{-1/2m}-B^{1/2m})/(\sin\theta/m)]\epsilon\downarrow$ which would follow from

$$\frac{B^{-u/2}-B^{u/2}}{\sin\theta u}\,\epsilon\uparrow \quad \text{for } 0\leqq u\leqq 1.$$

If we let $U=B^{-u/2}-B^{u/2}$, $V=\sin\theta u$ we see that the hypotheses of Lemma 3 are satisfied, and the conclusions of the lemma prove our contention.

We have thus proved the following theorem for the case $n=4$.

THEOREM. *For* $n=4$, $p(k)$ *is an infinitely divisible law if and only if* $\hat{p}(1)=re^{i\theta}$, $\hat{p}(2)=B$ (*real*), *are such that* (i) $r\leqq\sqrt{B}$ *and* (ii) $\theta\leqq\log 1/\sqrt{B}$, $0\leqq B\leqq 1$.

To conclude the discussion on $n=4$, one notes that one gets the following pictorial representation of the transforms of the infinitely divisible laws. In each case the rectangle's semi-length along the x-axis is $(1+B)/2$, while the semi-length along the y-axis is $(1-B)/2$, where $B=\hat{p}(2)$.

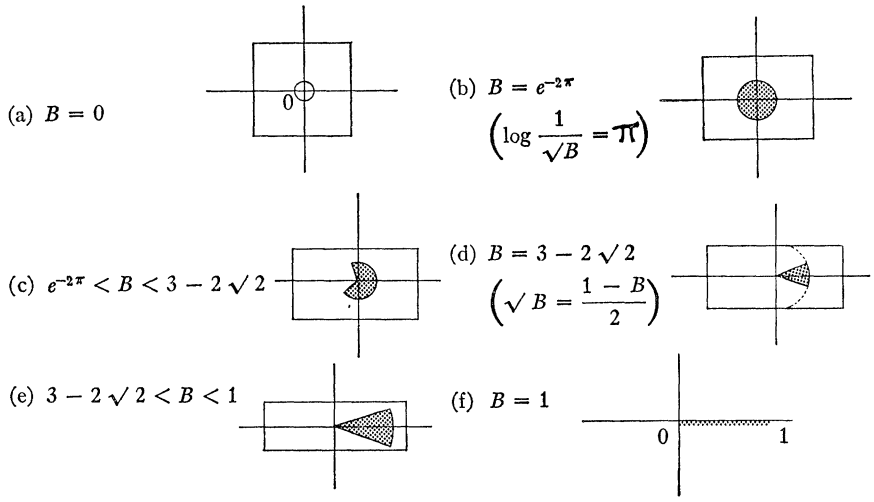


FIG. 2

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BETWEEN T_1 AND T_2

ALBERT WILANSKY, Lehigh University

1. There has been an outbreak recently of discussion in the MONTHLY on the subject of the Hausdorff separation axiom T_2. Specifically, it has been noted that various consequences of T_2 are or are not sufficient to imply it. There is a considerable amount of duplication in these results. See [3], [5], [6], [8], [10], [12], [13], [14], [15]. The object of this article is to introduce some systemization into the discussion, to point out its importance, and to show some surprising contact (Theorem 5) with a concept, k space, which has been introduced independently and for quite other purposes.

One possible importance of separation axioms weaker than T_2 lies in the fact that if X is a topological space and X^+ its one point compactification ($X^+ = X \cup \{\infty\}$, neighborhoods of ∞ are complements of compact closed subsets of X, see [7] Chapter 5, Theorem 21) then X^+ is not T_2 unless X is itself T_2 and locally compact. But one may ask what degree of separation X^+ has if X is T_2

and has some other pleasant property. An example of a result of this type is the contact with k space mentioned above. Another instance occurs in [1], Theorems 5.1, 5.3.

2. We now introduce two separation axioms, making no claim of novelty. (In [8], the name CC is applied to what we here call a compact KC space. In [1], J_1' is used for KC.)

DEFINITION. *A topological space is called a KC space if every compact set is closed, and a US space if every convergent sequence has exactly one limit to which it converges.*

In Theorem 3.1 of [1], equivalences are given between KC and separation properties of compact sets.

THEOREM 1. $T_2 \Rightarrow KC \Rightarrow US \Rightarrow T_1$; *but no converse implication holds even if the space is compact.*

The first implication is standard. See for example [11] Section 9.5, Corollary 3 or [7], Chapter 5, Theorem 7. The third implication holds because in a non-T_1 space there is a constant sequence with more than one limit. Finally, to prove the second implication we note first that KC implies T_1 since each singleton is compact. Next let X be a T_1 space which is not US. Then there exists a sequence $\{x_n\}$ which converges to two distinct limits, a, b. Now $\{b\}$ is closed; hence its complement is a neighborhood of a and there exists n_0 such that $n \geqq n_0$ implies $x_n \neq b$. Let $K = \{a, x_{n_0}, x_{n_0+1}, \cdots\}$. Then K is compact. But K is not closed, since $b \in \overline{K}$. Thus X is not a KC space.

A compact T_1 space need not be US (Example 1, below); a compact US space need not be KC (Example 4); and a compact KC space need not be T_2 (Example 2).

Note. The axiom T_{AB} of [1] lies between KC and US.

The following is a standard example.

Example 1. Let X be a countably infinite set and give X the topology in which a proper subset is closed if and only if it is finite. This is the cofinite, or minimal T_1 topology. It makes X a compact T_1 space but not a US space, since every sequence of distinct members of X converges to every member of X.

If in Example 1, X is uncountable and "finite" is replaced by "countable" we obtain a KC, non-T_2 space. This example has been cited frequently in recent issues of the MONTHLY and occurs in several text books.

For the next result see [3], [5], [6].

THEOREM 2. *For first countable spaces, $T_2 \equiv KC \equiv US$.*

The topology of a first countable space is determined by the behavior of sequences; thus $US \Rightarrow T_2$ and the result follows from Theorem 1.

DEFINITION. *A topological space X is called locally compact if each neighborhood of each point includes a compact neighborhood of that point.*

The next result, with Theorem 1, emphasizes the fact that a compact space need not be locally compact.

THEOREM 3. *For locally compact spaces,* $T_2 \equiv KC$.

If X is a locally compact space the set of compact neighborhoods of each point is a local base; hence, if, in addition, X is KC, the set of closed neighborhoods of each point is a local base. Thus X is regular, hence T_3 (by Theorem 1).

3. KC spaces have the following advantage over T_2 spaces. (For proofs see [8] Theorem 2, Example 1 and [11], p. 162, Theorem 3 and preceding discussion.)

A compact T_2 topology is maximal compact but not conversely. A compact KC topology, however, is maximal compact, and now the converse does hold—a maximal compact topology is KC. See [1], p. 201, last 6 lines.

4. As mentioned earlier, if X is T_2 but not locally compact, X^+, the one-point compactification of X, is not T_2. But X^+ may be T_1, US, KC, etc. and it is very reasonable to ask what properties of X lead to these. It is easy to check that X^+ is T_1 if and only if X is T_1. For US the situation is more complicated (see Section 5), and for KC we have Theorem 5, below, which we regard as our main result.

THEOREM 4. *If* X *is* KC, X^+ *is* US.

Let X be KC and let $\{x_n\}$ be a sequence in X^+ with $x_n \to a \neq \infty$. Since X is a neighborhood of a in X^+ we have $x_n \neq \infty$ for sufficiently large n, say $n \geqq n_0$. Let $K = \{a, x_{n_0}, x_{n_0+1}, \cdots\}$. Then K is a compact subset of X and hence, by hypothesis, a closed subset of X. Thus it is a closed subset of X^+ and since $\infty \notin K$ we have $x_n \not\to \infty$. Since X is a US space $x_n \not\to b$ if $a \neq b \in X$.

We shall see (Example 5) that if X is US, X^+ need not be.

DEFINITION. *A topological space is called a k space if it has the property that any subset* S *such that* $S \cap K$ *is closed for all closed compact* K *is itself closed.*

This concept is discussed in [7], pp. 230, 231, 234, 240, 241 where it is pointed out, for example, that a T_2 space is a k space if it is either first countable or locally compact. (Compare Theorems 2, 3 above, and 5, below.) See also [2], and [9], p. 76.

THEOREM 5. *Let* X *be a* KC *space. Then* X^+ *is* KC *if and only if* X *is a k space.*

The assumption that X is KC cannot be dropped since KC is hereditary. (See Example 3 below.)

In the following proof every topological concept referring to X^+ will have a $+$ superscript; those without this superscript refer to X.

Let X be a k space. Let S be a compact $^+$ subset of X^+. If $S \subset X$, S is compact, hence closed, thus closed $^+$. If, on the other hand, $\infty \in S$, let $F = S \setminus \{\infty\}$. Now let K be an arbitrary closed compact subset of X. Then K is closed $^+$, and so

$S \cap K$ is compact $^+$; hence compact, since it is a subset of S. Since X is a KC space, $S \cap K$ is closed, and so $F \cap K = S \cap K$ is closed. Since X is a k space it follows that F is closed. Then S is closed $^+$ because $X^+ \setminus S = X \setminus F$ is open, hence open $^+$.

Conversely, let X^+ be KC. Let F be a subset of X which obeys the test condition, namely $F \cap K$ is closed for every closed compact set $K \subset X$. Let $S = F \cup \{\infty\}$. Let Ω be a covering of S by open $^+$ sets. Choose $G \in \Omega$ with $\infty \in G$. Now $X \setminus G$ is closed $^+$, i.e. closed and compact; thus it meets F in a closed (hence compact) set, i.e. $F \setminus G$ is compact. Let $\Omega' = \{0 \setminus \{\infty\} : 0 \in \Omega\}$; then Ω' is an open cover of $F \setminus G$. Reducing Ω' to a finite cover of $F \setminus G$, restoring ∞ to each of its members and adjoining G gives a covering of S which is a finite subcover of Ω. We have now proved that S is compact $^+$. By hypothesis, it follows that S is closed $^+$. But $F = S \cap X$; thus F is closed. It follows that X is a k space.

COROLLARY 1. *Let X be a T_2 space. Then X^+ is always US, X^+ is KC if and only if X is a k space, X^+ is T_2 if and only if X is locally compact.*

COROLLARY 2. *Let X be a first countable T_2 space. Then X^+ is a KC space.*

We may also use Theorem 5 to give an alternate proof of the following result of Aull [1], Theorem 5.3.

COROLLARY 3. *Let X be T_2 and X^+ first countable. Then X is locally compact.*

Since X is first countable it is a k space; hence by Theorems 5 and 2, X^+ is T_2.

Example 2. To obtain a compact KC space which is not T_2 we simply take X^+ where X is first countable, T_2, but not locally compact. The special case $X =$ rationals was given in [8], Example 1.

Example 3. Let X be as in Example 1. Then every subset of X is compact; hence X is a k space. X^+ is, of course, not a KC space.

Example 4. To obtain a compact US space which is not KC, let X be a non-compact T_2 space which is not a k space, (Example 7). It now follows from Theorems 4, 5 that X^+ is compact US but not KC.

5. In this section we consider X^+ when X is US.

THEOREM 6. *Let X be a US space. Then X^+ is US if and only if, in X, every convergent sequence has a relatively compact subsequence.*

If X^+ is US, and $x_n \to a$ in X, then $x_n \nrightarrow \infty$ in X^+. Thus there is an open $^+$ neighborhood $^+$ G of ∞ such that $x_{k(n)} \notin G$ for some sequence $\{k(n)\}$. Since the complement of G is a compact closed subset of X, the closure of $\{x_{k(n)}\}$ is compact.

Conversely, let the condition hold and let $\{x_n\}$ be a sequence in X^+ with $x_n \to a \neq \infty$. Let K be the closure (in X) of a relatively compact subsequence. Then $X^+ \setminus K$ is a neighborhood $^+$ of ∞ and it is false that x_n belongs to it eventually. Thus $x_n \nrightarrow \infty$.

Example 5. Theorem 6 leaves open the question of whether there actually exists a US space X such that X^+ is not US. We now construct one. Let N be the positive integers and I the set of all infinite subsets of N. For any $S\in I$, the cardinality of $\overline{S}$, the closure of S in βN, (the Cech compactification of N) is 2^c ([4], Theorem 9.2). For each such S, let A_S be a countably infinite subset of $\overline{S}\backslash N$. Let $B=N\cup(\cup A_S\colon S\in I)$ and $X=B\cup\{t\}$, $t\in N^+\backslash N$, i.e. t is adjoined to B and assigned as neighborhoods every set which includes t and all of N except for a finite subset of N. Then X is a US space since B is T_2 and the only sequences in B which converge to t are sequences of members of N; they converge to no other member of B. (See [11], Section 14.3, Problem 20.) However, the condition of Theorem 6 fails in that $n\to t$ but $\{n\}$ has no relatively compact subsequence. To see this, let S be any infinite subset of N and S_X its closure in X. Now $S_X\backslash N^+$ is a subset of $B\backslash N$ which includes A_S, hence its cardinality k satisfies $\aleph_0\leqq k\leqq c$. By [4], Theorem 9.3 and top p. 134, $S_X\backslash N^+$ is not compact. But it is a closed subset of S_X, since N^+ is open in X; thus S_X is not compact.

Example 6. Theorem 6 also leaves open the question of whether X must be KC if X^+ is US, i.e. whether or not the condition of Theorem 6 is sufficient for X to be KC. (By Theorem 4, it is necessary.) We give an example (with X not compact to avoid triviality) in which X is not KC but X^+ is US. Let N, βN be as in Example 5. Let $b\in\beta N\backslash N$, $t\in N^+\backslash N$, $X=(\beta N\backslash\{b\})\cup\{t\}$. Now X is not KC since $N\cup\{t\}$ is dense and compact (a convergent sequence and its limit). However, we shall show that X satisfies the criterion of Theorem 6. Let $\{x_n\}$ be a convergent sequence. If $x_n\to a\neq t$, then $\{a, x_1, x_2, \cdots\}$ is compact and closed, (or it could be observed that $\{x_n\}$ has finite range). If $x_n\to t$, we must have $x_n\in N$ eventually. Now any infinite subset of N must have an infinite subset S with $b\notin\overline{S}$, the closure of S in βN, (otherwise N would not be C^* embedded in βN, see [11], Section 14.3, Application 2 and Problem 20). Then S is relatively compact in $\overline{S}$.

Example 7. A noncompact T_2 space which is not a k space. Let $X=N\cup\{b\}$, where $b\in\beta N\backslash N$. If S is any infinite subset of N, neither S nor $S\cup\{b\}$ is closed in βN, hence neither is compact. Thus X is pseudo-finite, i.e. all its compact sets are finite. If X were a k space, it would, being pseudo-finite, be discrete.

6. The following open question appears to be interesting. Must a locally compact US space be T_2? (The definition of local compactness is given in Section 2.)

7. The condition on 0-sequences given by Robertson and Franklin (this MONTHLY, 72 (1965) 510) is strictly between US and T_1.

8. An exhaustive study of the relationships among T_2, US, first and second countability, and countability, is given by W. Sierpinski, Soc. Sci. Lett. de Varsovie, C.R. Classe III (40 (1957) 66–78. (See Math. Reviews 12 (1951) p. 195.) The results overlap slightly with parts of Theorems 1 and 2.

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DENSENESS AND COMPLETENESS IN CERTAIN FUNCTION SPACES

HENRY SHARP, JR., Emory University

1. Introduction. Let E^n denote euclidean n-dimensional space and let S^n denote the unit n-dimensional sphere embedded as a subspace in E^{n+1} centered at the origin. Consider the function space

$$T = \{f: f \text{ any function on } S^n \text{ into } S^n\}$$

metrized by the familiar formula

$$\rho(f, g) = \sup d(f(x), g(x)),$$

where $f, g \in T$ and d is the metric in S^n. Note that the members of T need not be continuous; we use the word "mapping" when continuity is implied.

We shall investigate a number of subspaces of T. The two main results (Theorems 3 and 6) are known, but the general development leads to several interesting by-products. An application of Theorem 3, together with an outline of the proof, appears in [4, p. 561]; a broader version of Theorem 6 is discussed in [3, Chapter 7].

Repeated use will be made of the facts that a closed subset of a complete space is complete and that a G_δ-subset of a complete space can be assigned a topologically equivalent metric under which it is complete. The space T, above, is complete under the metric ρ [5, pp. 27–29]. Throughout the following, we fix "p" as the north pole of S^n, i.e. $p=(0, \cdots, 0, 1)$, and if X is a subset of T then X_p will denote those functions in X which leave p fixed.

2. Mappings on S^n. In this section we consider the subspaces

$$M = \{f \in T: f \text{ continuous on } S^n\}$$
$$G = \{f \in M: f \text{ is onto}\}$$
$$H = \{f \in M: f \text{ is one-to-one}\},$$

as well as M_p, G_p, and H_p.

The space M is a closed, hence complete, subspace of T [3, p. 227]. The topology on M induced by the metric ρ is identical with the compact-open topology [3, p. 230]. It is an immediate consequence that the subspaces M_p and G are closed, hence complete, for in each case the complement is a member of the sub-base of the *c-o* topology.

The first theorem below is an immediate consequence of the fact that S^n is not homeomorphic with a proper subset of itself. Here, however, we derive this result from Theorem 1, which is proved from the Borsuk-Ulam theorem: *if f maps S^n into E^n, then f maps a pair of antipodal points into the same point.* [1, p. 50].

THEOREM 1. $H \subset G$.

Proof. If $f \in M-G$, then there is a point $q \in S^n - f(S^n)$. Let π denote a homeomorphism on $S^n - \{q\}$ onto E^n. Then πf: $S^n \to E^n$ is continuous and by the Borsuk-Ulam theorem there is a point $x \in S^n$ such that $\pi f(x) = \pi f(-x)$, where x and $-x$ are antipodal points. It follows that f is not one-to-one.

Since the space H consists of the homeomorphisms on S^n (one-to-one maps on a compact space to a Hausdorff space), Theorem 1 implies

COROLLARY 1. *If h: $S^n \to S^n$ is a homeomorphism, then h is onto.*

Corresponding to any $f \in M$ and any $\epsilon > 0$, it is easy to find a function $g \in T-M$ such that $\rho(f, g) < \epsilon$; thus $T-M$ is dense in T. Furthermore, the example of a rotation through an arbitrarily small angle shows that the set $M-M_p$ is dense in M. On the other hand, it is a consequence of the Borsuk-Ulam theorem that the set $M-G$ is not dense in M. For if f is any map in $M-G$ there is an x such that $f(x)=f(-x)$; thus $\rho(f, \text{identity}) > 1$, so the identity map is an interior point of G.

With attention restricted now to subspaces of M_p, it follows as above that G_p is closed in M_p and that G_p has nonempty interior. By Theorem 1, we may consider H_p a subspace of G_p, and again it is easy to construct examples which show, in this case, that G_p-H_p is dense in G_p but that H_p is not closed in G_p.

Nevertheless, using an argument involving ϵ-mappings [2, p. 57], it follows that H_p is a G_δ-subset of G_p hence is topologically complete.

THEOREM 2. *Let $f: E^n \to E^n$ be a homeomorphism. Then f is onto iff the image of each unbounded sequence is unbounded.*

Proof. Suppose first that $f(E^n) = E^n$, that $\{x_i\}$ is an unbounded sequence, and that the image sequence $\{y_i\}$ is bounded. We may assume that $\{x_i\}$ has no cluster point. There exists a subsequence $\{y_i'\}$ of $\{y_i\}$ converging to a point y and a point x such that $y = f(x)$. But f^{-1} is continuous at y, hence $\{x_i\}$ must have a cluster point, contrary to the assumption. Next, suppose that the image of each unbounded sequence is unbounded. By Brouwer's theorem on invariance of domain [2, p. 95], $f(E^n)$ is open in E^n. If y is any limit point (in E^n) of $f(E^n)$, then there is a bounded sequence $\{y_i\}$ of distinct points in $f(E^n)$ converging to y. If $x_i = f^{-1}(y_i)$, then $\{x_i\}$ is bounded and hence contains a convergent subsequence with limit x. But f is continuous at x, hence $y \in f(E^n)$. Since $f(E^n)$ is both open and closed in E^n, $f(E^n) = E^n$.

THEOREM 3. *The set Φ of all homeomorphisms on E^n onto itself can be topologized as a complete metric space homeomorphic to H_p.*

Proof. By Theorem 2 there is a natural one-to-one correspondence, via the stereographic projection $\pi: S^n - \{p\} \to E^n$ with center p, between Φ and H_p. For suppose $f: S^n \to S^n$ is a function which is onto, one-to-one, and leaves p fixed. Then if f is continuous at all $x \neq p$, f is continuous also at p and hence $f \in H_p$. The correspondence is then $f = \pi^{-1}\phi\pi$, where $\phi \in \Phi$ and $f \in H_p$.

3. Functions on S^n. In this section we consider the subspaces

$$T_p = \{f \in T : f(p) = p\},$$
$$F_p = \{f \in T_p : f \text{ continuous at each } x \neq p\},$$
$$Q_p = \{f \in F_p : f(S^n) \text{ is dense in } S^n\}.$$

In addition, we establish certain relations between these spaces and those considered in Section 2.

If $f \in T - T_p$ and if $d(p, f(p)) = \epsilon$, then $\rho(f, g) < \epsilon$ implies $g \in T - T_p$; hence T_p is closed in T. Again, the example of a rotation through a sufficiently small angle shows that $T - T_p$ is dense in T. It follows by equally simple arguments that $T_p - F_p$ is both open and dense in T_p.

The space M_p (of Section 2) is a subset of F_p. By extending slightly the useful notion of *compact mapping* [6, p. 90], this subspace has a simple characterization. Define $f \in T$ to be *compact* iff for each set K closed in S^n the set $f^{-1}(K)$ is closed in S^n.

THEOREM 4. *Suppose $f \in F_p$. Then $f \in M_p$ iff f is compact.*

Proof. If $f \in M_p$ then f is continuous on S^n, hence is compact. On the other hand, if $f \in F_p - M_p$ there is a sequence $\{x_i\}$ such that $x_i \to p$, $f(x_i) \neq p$, and p

is not a cluster point of $\{f(x_i)\}$. If K is the closure of the set $\{f(x_i)\}$ then K is closed but $f^{-1}(K)$ is not since it fails to contain p.

Since M_p is a closed subset of T, it follows that M_p is closed in F_p. The example of a small rotation except for a discontinuity at p (which is fixed) shows that $F_p - M_p$ is dense in F_p.

THEOREM 5. *The space $F_p - Q_p$ is both open and dense in F_p.*

Proof. If $f \in F_p - Q_p$ then there is a spherical neighborhood of radius $\epsilon > 0$ contained in $S^n - f(S^n)$. For each $g \in F_p$, if $\rho(g, f) < \epsilon/3$ then $S^n - g(S^n)$ contains a spherical neighborhood, hence $g \in F_p - Q_p$. Now let $s(\epsilon)$ denote a spherical neighborhood about p of radius ϵ and let g be defined by

$$g(x)\begin{cases} = p \text{ if } x = p, \\ = x \text{ if } x \in S^n - s(\epsilon), \\ \text{uniformly shrinks } s(\epsilon) - \{p\} \text{ onto the annular region } s(\epsilon) - \overline{s(\epsilon/2)}. \end{cases}$$

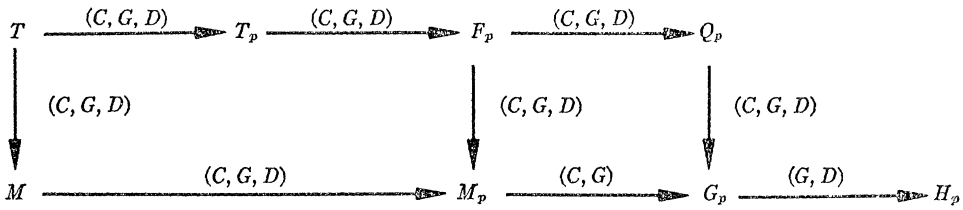
Then if $f \in Q_p$, $\rho(f, gf) < \epsilon$ but $gf \in F_p - Q_p$.

COROLLARY. *The space $F_p - (M_p \cup Q_p)$ is open and dense in F_p.*

The diagram below summarizes much of the information in Sections 2 and 3. Except for H_p, each space is complete in the metric ρ, and H_p is complete in a topologically equivalent metric. In the symbol

$$A \xrightarrow{(C, G, D)} B$$

C means that B is a closed subset of A, G means that B is a G_δ-subset of A, and D means that $A - B$ is dense in A. Absence of a letter means failure of the corresponding property.



4. Mappings on E^n. As indicated in Section 2, there is a one-to-one correspondence between H_p and the set of all homeomorphisms on E^n onto itself. One might be tempted to think that a similar one-to-one correspondence exists between F_p and the set of all mappings on E^n into itself, but the argument below shows that an additional restriction must be imposed. Again let $\pi: S^n - \{p\} \to E^n$ denote the stereographic projection with center p. Let $g: E^n \to E^n$ be a map and define $f: S^n \to S^n$ by

$$f(x) = \begin{cases} \pi^{-1} g \pi(x) & \text{if } x \in S^n - \{p\}, \\ p & \text{if } x = p. \end{cases}$$

Then $f\in F_p$ and note that $f(x)\neq p$ if $x\neq p$. Now let $f\in F_p$ and define a mapping g by

$$g(y) = \pi f\pi^{-1}(y).$$

If for some $x\in S^n-\{p\}$, $f(x)=p$, then the domain of g is not E^n but is some proper subset. If, however, $f(x)\neq p$ for $x\neq p$, then the domain of g is E^n. Define

$$K_p = \{f\in F_p: f^{-1}(p) = p\}.$$

This argument shows that there is a one-to-one correspondence between K_p and the set of all mappings on E^n into itself.

THEOREM 6. *The space K_p is complete under a metric topologically equivalent to ρ.*

Proof. We need to show that K_p is a G_δ-subset of the complete space F_p. For each positive integer j, let $s(j)$ denote the spherical neighborhood about p of radius $1/j$, and let $K_p(j)=\{f\in F_p: f^{-1}(p)\subset s(j)\}$. Let f be a limit point of $F_p-K_p(j)$; of course, $f\in F_p$ and we wish to show that $f\notin K_p(j)$. There is a sequence $\{f_i\}$ such that $f_i\in F_p-K_p(j)$ and $\rho(f_i, f)\to 0$. For each i, there is a point $x_i\in S^n-s(j)$ such that $f_i(x_i)=p$. The sequence $\{x_i\}$ has a cluster point $x\in S^n-s(j)$ and a subsequence $\{x_i'\}$ converging to x. Corresponding to $\epsilon>0$ there exist integers N_1 and N_2 such that if $i>N_1$ then $\rho(f, f_i)<\epsilon/2$, and if $i>N_2$ then $d(f(x), f(x_i'))<\epsilon/2$. Now there is an integer $m>N_1$ such that $x_m=x_i'$ for some $i>N_2$, hence

$$d(f(x), p) \leqq d(f(x), f(x_m)) + d(f(x_m), f_m(x_m)) < \epsilon.$$

Since ϵ is arbitrary, $f(x)=p$ and $f\in F_p-K_p(j)$. Thus $K_p(j)$ is open for each j, and $K_p=\bigcap_j K_p(j)$.

Again, simple examples suffice to show that F_p-K_p is not open but is dense in F_p. Furthermore, as in the proof of Theorem 5, K_p-Q_p is both open and dense in K_p. From this it follows that $K_p\cap Q_p$ is closed, hence complete, in K_p. Whether or not a similar result holds for the subspace of *onto* mappings is unknown to the author, except in the special case $n=1$.

THEOREM 7. *The space of mappings on E^1 onto E^1 is topologically complete.*

Proof. By the remark preceding Theorem 6, it will be sufficient to show that (when $n=1$) $K_p\cap Q_p$ consists in fact of the onto mappings. If $f\in K_p\cap Q_p$ then $f(S^1-\{p\})$ is both connected and dense in S^1. Hence $f(S^1-\{p\})\supset S^1-\{q\}$ for some point $q\in S^1$. If $q\neq p$ there is a point $x\neq p$ such that $f(x)=p$, which contradicts $f\in K_p$. Therefore, $q=p$ and it follows that

$$f(S^1 - \{p\}) = S^1 - \{p\}.$$

The author is indebted to the referee for several helpful comments.

This work was completed during the author's tenure as a National Science Foundation Science Faculty Fellow (1964–65).

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THE MEANS OF ORDER t, AND THE LAWS OF THERMODYNAMICS

E. D. CASHWELL AND C. J. EVERETT,
Los Alamos Scientific Laboratory, University of California

Introduction. If an isolated system of n materials, with masses m_j, specific heats $c_j(T)$, and initial temperatures T_j not all equal, are placed in thermal contact, then, according to the laws of thermodynamics, the final equilibrium temperatures F_j approached must satisfy the energy-conservation equation (1st law):

$$E_2 - E_1 = \Delta E = \sum \int_{T_j}^{F_j} c_j(T) m_j dT = 0 \tag{1}$$

and moreover, must maximize the entropy change (2nd law):

$$S_2 - S_1 = \Delta S = \sum \int_{T_j}^{F_j} c_j(T) m_j dT/T. \tag{2}$$

Mathematically, these conditions imply that all $F_j = M$, the uniform, mean temperature defined by (1):

$$\sum \int_{T_j}^{M} c_j(T) m_j dT = 0,$$

and that the corresponding maximum entropy change in (2) is positive:

$$\sum \int_{T_j}^{M} c_j(T) m_j dT/T > 0.$$

The treatment of this very simple instance of the thermodynamic laws by Mahan [2] suggests an *implicit* definition of a mean $M(T_j, w_j)$ of n positive numbers T_j, relative to a set of *functions* $w_j = w_j(T)$. Curiously enough, this approach allows a completely trivial proof of a fundamental inequality (I) for such means, which not only establishes the above implications, but also yields the monotonicity property of the whole hierarchy of classical means $M(t)$ of

order t [1]. Generalization to the case of a continuous distribution $T(x)$ is straightforward, and is indicated without proof.

1. The means $M(T_j, w_j)$. Let $w_j = w_j(T)$, $j = 1, \ldots, n$, be positive, continuous functions for $T > 0$, and let T_j be arbitrary positive numbers with min $T_j \equiv T_a$, max $T_j \equiv T_b$. The function

$$e(T) \equiv e(T; T_j, w_j) \equiv \sum \int_{T_j}^{T} w_j(T')dT'$$

is obviously continuous and increasing for $T > 0$. If $T_a = T_b$, then $e(T_a) = 0$, whereas, if $T_a < T_b$, $e(T_a) < 0 < e(T_b)$. Hence *there exists a unique mean* $M \equiv M(T_j, w_j)$ *of the* T_j *relative to the* w_j, *such that* $e(M) = 0$, $M > 0$. Moreover, $M = T_a$ when all T_j are equal; otherwise, $T_a < M < T_b$.

Now, if $f = f(T)$ is positive, continuous, and *decreasing* for $T > 0$, then the mean $M_1 \equiv M(T_j, w_j f)$ of the T_j relative to the functions $w_j(T)f(T)$ is also uniquely defined, by $e(M_1; T_j, w_j f) = 0$, and we have at once the

THEOREM.

(IA) $$e(M; T_j, w_j f) \geqq 0, \quad \textit{where } M \equiv M(T_j, w_j)$$

or, equivalently,

(IB) $$M_1 \equiv M(T_j, w_j f) \leqq M(T_j, w_j) \equiv M.$$

Equality holds, in either case, if and only if all T_j *are equal.*

Proof. If $T_a = T_b$, then $M = T_a = M_1$, and equality in both instances is manifest. Otherwise ($T_a < T_b$), let l, u range over the subscripts j for which $T_j \leqq M$, $M < T_j$, respectively, and note that

$$\sum_l \int_{T_j}^{M} w_j f dT > f(M) \sum_l \int_{T_j}^{M} w_j dT = f(M) \sum_u \int_{M}^{T_j} w_j dT > \sum_u \int_{M}^{T_j} w_j f dT,$$

which yields strict inequality in (IA). Since the function $e(T; T_j, w_j f)$ is continuous and increasing for $T > 0$, with unique "zero" M_1, the equivalence of (IA) and (IB) is clear.

COROLLARY 1. *If* $m_j > 0$ *have sum* $\sum_1^n m_j = 1$, *and* $w_j \equiv m_j T^{t-1}$, *where* t *is real, then the corresponding mean* $M(T_j, w_j)$ *is the classical "mean of order* t*":*

$$M(t) = \prod T_j^{m_j} \ (t = 0); \qquad M(t) = \left(\sum m_j T_j^t \right)^{1/t} \quad (t \neq 0).$$

If $s < t$, *and* $f(T) = T^{-d}$, *where* $d = t - s > 0$, *then* $M(T, w_j f) = M(s)$, *and* (IB) *reads* $M(s) \leqq M(t)$, *strict inequality holding unless all* T_j *are equal.*

This is the monotonicity property of the means of order t for the T_j weighted by the m_j. The identification referred to above rests on explicit integration. Indeed, for $w_j(T) = m_j W'(T)$, one has $W(M) = \sum W(T_j) m_j$.

2. The maximum property. If $w_j(T), f(T)$ are given as before, together with $T_j>0$, not all equal, their mean $M\equiv M(T_j, w_j)$ provides the unique solution to the maximum problem stated in

COROLLARY 2. *For every set of $F_j>0$, satisfying the condition*

$$\text{(E)} \qquad \sum \int_{T_j}^{F_j} w_j dT = 0$$

we have

$$\text{(S)} \qquad \sum \int_{T_j}^{F_j} w_j f dT \leqq \sum \int_{T_j}^{M} w_j f dT$$

(the latter being positive), with equality if and only if all $F_j=M$.

Proof. The condition (E) is equivalent to

$$\sum \int_{T_j}^{M} w_j dT = \sum \int_{F_j}^{M} w_j dT.$$

Thus, relative to the w_j, the T_j and the F_j have the *same* mean M. But

$$\sum \int_{T_j}^{F_j} w_j f dT = \sum \int_{T_j}^{M} w_j f dT - \sum \int_{F_j}^{M} w_j f dT$$

and the result follows immediately from (IA) of the Theorem.

Note: The implications of the laws of thermodynamics cited in the introduction derive from the corollary if we set $w_j=m_j c_j(T)$, and $f(T)=1/T$.

3. The continuous case. Generalizations in many different directions are indicated. We mention one of the simplest and most interesting. Let $w(T, x)$ be a positive, continuous function of T, x, for $T>0$ and $0\leqq x\leqq 1$ (the range of x being immaterial), and $T(x)$ a positive, continuous function, with $T_a\equiv\min T(x)$, $T_b\equiv\max T(x)$ on the latter interval. If $T_a<T_b$, suppose further that, for every T properly between the two, $T(x)=T$ for only a finite set of x. Then

$$e(T) \equiv e(T; T(x), w) \equiv \int_0^1 \int_{T(x)}^{T} w(T', x) dT' dx$$

is continuous increasing for $T>0$, with a unique positive "zero" $M\equiv M(T(x), w)$, just as before. If $f(T)$ is positive, continuous, and decreasing on $T>0$, then the analogue of the Theorem holds for the means $M(T(x), wf)\leqq M(T(x), w)$. (The proof follows the same lines, with the range of x-integration split into a finite number of open intervals on which $T(x)<M$, $T(x)>M$, resp.)

The generalization of Corollary 1 then concerns a positive weighting function $m(x)$ continuous on $[0, 1]$, with $\int_0^1 m(x)dx=1$, and the function $w(T, x)\equiv T^{t-1}m(x)$. With these identifications, one obtains the monotonicity of the *integral* means of order t [1], namely

$$M(t) = \exp \int_0^1 (\log T(x))m(x)dx \quad (t = 0)$$

$$M(t) = \left(\int_0^1 T^t(x)m(x)dx \right)^{1/t} \quad (t \neq 0).$$

The analogue of Corollary 2 also follows immediately, for $F(x)$ positive and continuous. Its physical application involves a rod of density $m(x)$ per unit length, specific heat $c(T, x)$, and initial nonuniform temperature distribution $T(x)$, with the same thermodynamic implications as before; the final entropy-maximizing temperature distribution is uniform, with value $F(x) \equiv M(T(x), w)$, where $w = c(T, x)m(x)$.

Work performed under the auspices of the U. S. Atomic Energy Commission.

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MATHEMATICAL NOTES

LINEAR TRANSFORMATIONS WITH INVARIANT CONES

GARRETT BIRKHOFF, Harvard University

In three dimensions, geometrical intuition readily suggests that if a linear transformation T maps into itself a closed convex cone C with vertex at the origin, then T must leave fixed some direction (ray) in C. The Brouwer fixed point theorem, applied to the convex disc of all such directions (which is homeomorphic to a sphere of one lower dimension) asserts that this is also true in n dimensions. Choosing for C the hyperoctant of all real vectors $\mathbf{x} = (x_1, \cdots, x_n)$ with all $x_i \geqq 0$ (i.e., of all "nonnegative" vectors), there follows after a little further consideration the following classic result of Perron: *If $A = \|a_{jk}\|$ is an $n \times n$ matrix with all $a_{jk} > 0$, then A has a positive eigenvector with positive eigenvalue.* Perron proved more: that this eigenvalue is simple, and greater in magnitude than any other eigenvalue of A.

In this note, a somewhat more general result will be established by a more constructive method, which uses the Jordan canonical form for complex matrices instead of Brouwer's fixed point theorem. This result is the following

THEOREM. *Let T be any linear transformation of real n-space $\boldsymbol{R}^n = V$ such that $CT \subset C$ for some proper closed convex C with nonvoid interior. Then C contains an eigenvector ω whose eigenvalue is the spectral radius of T.*

Proof. We have assumed that C satisfies

$$C \cap -C = \mathbf{0}, \quad \overline{C} = C, \quad C - C = V, \quad C + C \subset C. \tag{1}$$

From (1) there follows obviously

$$\mathbf{x}_k \in C,\ w_k \geqq \mathbf{0}, \text{ and } \sum_{k=1}^{s} w_k \mathbf{z}_k = \mathbf{0} \text{ imply that every } w_k \mathbf{z}_k = \mathbf{0}: \tag{1'}$$

since C is proper, a sum of nonzero vectors of C cannot be the zero vector $\mathbf{0}$.

We can uniquely extend V and T to a complex n-space and choose a Jordan canonical basis in this complex space, such that for each basis vector ε_j:

$$\varepsilon_{j+k} T = \lambda_j \varepsilon_{j+k} + \varepsilon_{j+k+1}, \quad k = 0, \cdots, m+1, \tag{2}$$

$$\varepsilon_{j+m} T = \lambda_j \varepsilon_{j+m}, \quad m = m(j). \tag{2'}$$

By induction on r, one can then verify that

$$\varepsilon_j T^r = \sum_{k=0}^{m} \lambda_j^{r-k} \binom{r}{k} \varepsilon_{j+k}, \quad m = m(j), \tag{3}$$

where the binomial coefficient $\binom{r}{k}$ is a polynomial of degree k in r.

Now let $\rho = \max |\lambda_j|$ be the *spectral radius* of T. The case $\rho = 0$ (i.e., all $\lambda_j = 0$) of nilpotent T is trivial. For, in this case, $CT^r = \mathbf{0}$ for some minimal r, and for this r some nonzero vector $\omega = \mathbf{x}T^{r-1} \in C$ satisfies $\omega T = \mathbf{0} = \rho\omega$.

Otherwise $\rho > 0$, and if M is the largest $m(j)$ with $|\lambda_j| = \rho$, then

$$\left(\sum c_j \varepsilon_j \right) T^r = \rho^r r^m \left(\sum_J c j e^{ir\theta_j} \varepsilon_{j+m} + 0(1) \right), \tag{4}$$

where J is the set of integers j such that

$$m(j) = M, \qquad \lambda_j = \rho e^{i\theta_j}, \qquad \varepsilon_{j+M} T = \lambda_j \varepsilon_{j+M}. \tag{4'}$$

On the other hand, the nonvoid open *interior* of C must contain vectors $\mathbf{z} = \sum c_j \varepsilon_j$ with all $c_j \neq 0$. For any such $\mathbf{z}$, consider the sequence $\mathbf{z}T^r$ of its iterated transforms. Clearly, no $\mathbf{z}T^r = \mathbf{0}$, as otherwise (4) would fail; hence every $\mathbf{z}T^r$ (which is in C since $CT \subset C$) has a *direction* in the set of directions in C. This set is *compact* since C is closed; hence the set of limits of subsequences of this sequence of directions (which may be called "ω-limit directions") is a *nonvoid* (closed) subset of C. By (4'), it contains only vectors ω of the form

$$\omega = \sum d_k \varepsilon_k, \qquad \varepsilon_k T = \rho e^{ir\theta_k} \varepsilon_k, \qquad \theta_k \text{ real}. \tag{5}$$

Such vectors ω, if nonzero, will be called *ω-vectors*. We now state without proof an elementary result.

LEMMA. *Unless the complex number $\alpha = \rho e^{i\theta}$ is positive, there exists a finite sequence of positive constants w_r such that $\sum w_r \alpha^r = 0$.*

For any $\theta_l \not\equiv 0 \pmod{2\pi}$—i.e., any nonpositive λ_l,—we can therefore construct a vector $\omega' = \sum w_r \omega T^r$ in C, which will be nonzero by (1'), such that in

$$\omega' = \sum \xi d_k' \varepsilon_k, \qquad d_k' = \sum w_r \lambda_k{}^r d_k, \tag{5'}$$

$d_k = 0$ implies $d_k' = 0$, *and* $d_l' = 0$. Hence, for *any* nonzero ω-limit vector of the form (5) in C which has a *minimum* number of nonzero coefficients d_k in (5), $d_k \neq 0$ must imply $\lambda_k = \rho$, and so the vector is an eigenvector in C with eigenvalue ρ. This completes the proof.

A NOTE ON MATRIX COMMUTATORS

R. C. THOMPSON, University of California at Santa Barbara

The purpose of this note is to offer a strengthened form and simplified proof of a theorem due to I. Sinha [1] and, as a consequence, to supply a very short proof of another theorem due to Sinha published in this MONTHLY [2]. We let I_n be the n-square identity matrix and the characteristic polynomial of matrix A is $\det(\lambda I_n - A)$.

THEOREM. *Let A, B, C be n-square nonsingular matrices with elements in an algebraically closed field such that*

$$C = ABA^{-1}B^{-1}, \qquad CA = AC, \qquad CB = BC. \tag{1}$$

Then

(i) *each eigenvalue of C is a root of unity*;

(ii) *if r is the least common multiple of the orders of the roots of unity which appear as eigenvalues of C, then $C^t - I_n$ is nilpotent if and only if r divides t*;

(iii) *if $\gamma_1, \cdots, \gamma_s$ are the distinct eigenvalues of C where γ_i has order r_i and multiplicity e_i, $1 \leqq i \leqq s$, then r_i divides e_i and the characteristic polynomials of A and B each have the form*

$$\prod_{i=1}^{s} \prod_{j=1}^{e_i/r_i} (\lambda^{r_i} - a_{ij});$$

(iv) *if γ is an eigenvalue of C then C has at least two elementary divisors of the form $(\lambda - \gamma)^h$, $h > 0$, belonging to the eigenvalue γ, with one exceptional case: $\gamma = 1$ may be a simple eigenvalue of C belonging to the single elementary divisor $\lambda - 1$*;

(v) *C is never nonderogatory when $n > 1$*;

(vi) *if $\gamma = 1$ is not an eigenvalue of C then both A and B have trace zero, hence are additive commutators.*

Proof. Replacing C, A, B with SCS^{-1}, SAS^{-1}, SBS^{-1}, respectively, throughout (1), we obtain a new equation easily derived from (1). Hence we may throw C into Jordan canonical form and assume $C = C_1 + \cdots + C_s$ (direct sum), where C_i is e_i-square and has γ_i for its only eigenvalue, $1 \leqq i \leqq s$. Then it is well known and easy to prove that $CA = AC$, $CB = BC$ imply that $A = A_1 + \cdots + A_s$,

COROLLARY 1. *If B does not have among its eigenvalues all roots of any polynomial $\lambda^u - a$ with $u > 0$, then* (1) *implies that $C - I_n$ is nilpotent.*

Proof. The hypothesis implies that each $r_i = 1$, hence $r = 1$, and (ii) supplies the result. This is Sinha's theorem in [1].

COROLLARY 2. *Over the complex field, if the eigenvalues of B lie in a cone in the complex plane with apex at the origin and apex angle $< \pi$, then* (1) *implies that $C - I_n$ is nilpotent.*

Proof. If $r_i > 1$, then the roots of (4) cannot be contained in any such cone.

COROLLARY 3. *If $n = 2$ and* (1) *holds then $C = I_2$ or $C = -I_2$.*

Proof. Since C has roots of unity of order $\leqq 2$ as eigenvalues, the eigenvalues of C are 1 or -1. If $\gamma = -1$ appears as an eigenvalue of C then $C = -I_2$, since $\gamma = -1$ belongs to at least two elementary divisors. If $\gamma = -1$ does not appear then $\gamma = 1$ is the only eigenvalue and is not simple, hence belongs to at least two elementary divisors, forcing $C = I_2$. This is Sinha's theorem in [2].

We note that explicit formulas for all solutions of (1) are given in [4], when $C = fI_n$ and f is a primitive root of unity of order n.

Added in proof: Additional results on the problem considered here have recently appeared in [5]. Still further results will appear in [6], where the problem is solved of obtaining necessary and sufficient conditions that C must satisfy in order to be representable in the form (1).

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RADIAL FUNCTIONS OF CONVEX AND STAR-SHAPED BODIES

F. A. TORANZOS, University of Washington and University of Buenos Aires

The fact that the radial function of a convex body is Lipschitzian belongs to the folklore of convexity. In Theorem 1 we compute the best Lipschitz constant valid for all the convex bodies whose boundaries are contained in an annular region. In Theorem 2 we extend this result to a similar class of star-shaped bodies, answering a problem stated by Valentine ([1], p. 168, Problem 3.2). The

Lemma that precedes Theorem 2, despite its triviality, seems to be a natural and unexploited way to attack some problems involving star-shaped bodies. Clearly it is valid in any real linear space, regardless of its dimension or topology.

NOTATION. All the sets considered are in E^d, the d-dimensional euclidean space. The symbol θ denotes the origin, B the unit ball and U the unit sphere centered at θ. The closed segment joining a and b is denoted by $[a, b]$ and $\overrightarrow{ab} = \{x = \mu a + (1-\mu)b\colon \mu \leqq 1\}$. The distance between any two points x, $y \in U$, denoted by $d(x, y)$, is the measure of the angle formed by $\overrightarrow{\theta x}$ and $\overrightarrow{\theta y}$. The family of all convex compact sets having θ as an interior point is denoted by $\mathcal{K}$ and if r and R are two positive numbers, $\mathcal{K}_r = \{K \in \mathcal{K}\colon rB \subset K\}$, $\mathcal{K}^R = \{K \in \mathcal{K}\colon RB \supset K\}$ and $\mathcal{K}_r^R = \mathcal{K}_r \cap \mathcal{K}^R$.

The *radial function* of $K \in \mathcal{K}$ is the function $\rho_K\colon U \to R^+$ defined by $\rho_K(x) = \|(\overrightarrow{\theta x} \cap \text{bdry } K)\|$.

THEOREM 1. *If ρ is the radial function of $K \in \mathcal{K}_r^R$ and $k = R((R/r)^2 - 1)^{1/2}$ then for all x_1, $x_2 \in U$, $|\rho(x_1) - \rho(x_2)| \leqq k d(x_1, x_2)$. Furthermore, k is the best Lipschitz constant valid for the radial functions of all the members of $\mathcal{K}_r^R$.*

Proof. Let x_1, $x_2 \in U$ and assume $\rho(x_2) \leqq \rho(x_1) = \rho_1$. The point $p_1 = \rho_1 x_1$ clearly belongs to K, whence the set $D_{\rho_1} = \text{conv}(rB \cup \{p_1\})$ is contained in K. Hence, if ρ' denotes the radial function of D_{ρ_1} then $\forall y \in U\ \rho(y) \geqq \rho'(y)$ and by construction $\rho(x_1) = \rho'(x_1)$. If $\alpha_0 = \arccos(r/\rho_1)$ and $d(x_1, y) = \alpha$, then

$$\rho'(y) = \begin{cases} r & \text{if } \alpha \geqq \alpha_0 \\ r/\cos(\alpha_0 - \alpha) & \text{if } \alpha < \alpha_0. \end{cases}$$

If $d(x_1, x_2) > \alpha_0$ let x_3 be a point of U between x_1 and x_2 such that $d(x_1, x_3) = \alpha_0$; if $d(x_1, x_2) \leqq \alpha_0$ let $x_3 = x_2$. Then, if $\beta = d(x_1, x_3)$

$$\Delta(x_3) = \frac{\rho'(x_1) - \rho'(x_3)}{d(x_1, x_3)} = \frac{\rho_1 - (r/\cos(\alpha_0 - \beta))}{\beta} = \frac{\rho_1 \cos(\alpha_0 - \beta) - r}{\beta \cos(\alpha_0 - \beta)}.$$

It is easy to see that $\Delta(x_3)$ increases as x_3 tends to x_1. An elementary computation shows that $k(\rho_1) = \lim_{x_3 \to x_1} \Delta(x_3) = \rho_1((\rho_1/r)^2 - 1)^{1/2}$. From the previous considerations it follows that

$$\frac{\rho(x_1) - \rho(x_2)}{d(x_1, x_2)} \leqq \frac{\rho'(x_1) - \rho'(x_2)}{d(x_1, x_2)} \leqq \frac{\rho'(x_1) - \rho'(x_3)}{d(x_1, x_3)} = \Delta(x_3) \leqq k(\rho_1).$$

Finally, since $\rho_1 \leqq R$ then $k(\rho_1) \leqq k(R) = k$ and the theorem follows. To see that k is the best constant, consider a point x with $\|x\| = R$ and the set $D_R = \text{conv}(rB \cup \{x\})$. Then $D_R \in \mathcal{K}_r^R$ and admits no smaller constant.

If S is a nonempty set, a *convex component* of S is a maximal convex subset of S. The *kernel* of S is the set $\ker S = \{x \in S\colon \forall y \in S\ [x, y] \subset S\}$. The set S is *star-shaped* iff $\ker S \neq \varnothing$. We define $\mathcal{S}$ as the family of all compact star-shaped sets having θ as interior point of its kernels. If R and r are two positive numbers

then

$$\mathcal{S}_r = \{S \in \mathcal{S} : rB \subset S\}, \qquad \mathcal{S}^R = \{S \in \mathcal{S} : RB \supset S\} \quad \text{and} \quad \overset{R}{\mathcal{S}}_r = \mathcal{S}_r \cap \mathcal{S}^R.$$

It is clear that $\mathcal{K} \subset \mathcal{S}$. If $S \in \mathcal{S}$, $\forall x \in U$ the set $(\overrightarrow{\theta x} \cap \text{bdry } S)$ consists of a single point; hence the radial function of S is well defined.

LEMMA. *The set* ker S *is the intersection of all the convex components of* S.

Proof. Let $\{K_\lambda\}_{\lambda \in L}$ be the family of convex components of S, and $K_0 = \bigcap_{\lambda \in L} K_\lambda$. From the maximal principle it follows that $S = \bigcup_{\lambda \in L} K_\lambda$. Let $x \in K_0$; then $\forall y \in S$ $\exists \lambda \in L$ such that $y \in K_\lambda$ and $[x, y] \subset K_\lambda \subset S$; hence $x \in \ker S$ and $K_0 \subset \ker S$. Conversely, let $z \in \ker S$ and K_λ be a convex component of S. Set $K'_\lambda = \bigcup_{t \in K_\lambda} [z, t]$; then K'_λ is convex, and furthermore $K_\lambda \subset K'_\lambda \subset S$. So, by the maximality of K_λ, $K'_\lambda = K_\lambda$ and $z \in K_\lambda$. Since this is true for every $\lambda \in L$, $z \in K_0$ and $K_0 = \ker S$.

COROLLARY: ker S *is convex.*

THEOREM 2. *If* ρ *is the radial function of* $S \in \mathcal{S}_r^R$ *and* $k = R((R/r)^2 - 1)^{1/2}$ *then* $\forall x_1, x_2 \in U \, |\rho(x_1) - \rho(x_2)| \leq k d(x_1, x_2)$, *and this is the best Lipschitz constant valid for all the members of* $\mathcal{S}_r^R$.

Proof. Let $\{K_\lambda\}_{\lambda \in L}$ be the family of all convex components of S. Clearly $S \in \mathcal{S}_r^R$ implies $K_\lambda \in \mathcal{K}_r^R \forall \lambda \in L$ and, since $\rho(x) = \max_\lambda \rho_\lambda(x)$ where ρ_λ is the radial function of K_λ, the result follows immediately from Theorem 1. The example in Theorem 1 shows that k is the best Lipschitz constant valid for all the members of $\mathcal{S}_r^R$.

This work has been done under a fellowship of the Consejo Nacional de Investigaciones Cientificas y Técnicas de la Argentina.

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ON COUNTABLY COMPACT NONLOCALLY COMPACT SPACES

T. B. RUSHING, Wake Forest College, (Now at the University of Georgia)

The purpose of this note is to establish the existence of countably compact nonlocally compact spaces, to give general methods for constructing such spaces, and to consider properties which they may or may not enjoy. A nonlocally compact space certainly must be noncompact and one can easily find examples of countably compact noncompact spaces in the literature covering lower spaces (e.g. the long line, see [1] p. 55); however, these seem invariably to be locally compact.

A topological space X is said to be *countably compact* if and only if every infinite subset of X has at least one limit point in X. We will call the space X

locally compact if and only if each point of X is contained in a compact neighborhood. (Our results would not be as general if we defined a space to be locally compact when each point of the space is contained in a neighborhood whose closure is compact.) A subcollection of a base $\mathfrak{B}$ for the topology of the space X is called a *distinguished collection* relative to $\mathfrak{B}$ if and only if each set of the collection contains at least one point of X not in any other element of $\mathfrak{B}$.

THEOREM 1. *If a countably compact space $(X, \mathfrak{I})$ has an uncountable distinguished collection relative to a base $\mathfrak{B}$, then the space $(X, \mathfrak{I}^*)$, where B^* is an element of the base $\mathfrak{B}^*$ of $\mathfrak{I}^*$ if and only if B^* can be expressed as the union of all but a denumerable number of elements of $\mathfrak{B}$, is countably compact but not locally compact.*

(The reader can prove that $\mathfrak{I}^*$ is the collection of all sets which are the union of all but a countable number of elements of $\mathfrak{B}$.)

Proof. (1) We will first show that $\mathfrak{B}^*$ is a base for a topology of X. Clearly, $\bigcup\{B^*: B^*\in\mathfrak{B}^*\}$ covers X. Let B_1^* and B_2^* be any two elements of $\mathfrak{B}^*$. We can find subcollections $\{B_\lambda\}$ and $\{B_\eta\}$ of $\mathfrak{B}$ such that $B_1^*=\bigcup_\lambda B_\lambda$ and $B_2^*=\bigcup_\eta B_\eta$ and such that $\mathfrak{B}\setminus\{B_\lambda\}$ and $\mathfrak{B}\setminus\{B_\eta\}$ are denumerable. Then $(\mathfrak{B}\setminus\{B_\lambda\})\cup(\mathfrak{B}\setminus\{B_\eta\})$ is denumerable and each element of $\mathfrak{B}\setminus[(\mathfrak{B}\setminus\{B_\lambda\})\cup(\mathfrak{B}\setminus\{B_\eta\})]=\{B_\lambda\}\cap\{B_\eta\}$ is contained in $B_1^*\cap B_2^*$. Now let x be any point of $B_1^*\cap B_2^*$. Then x is in some element B_1 of $\{B_\lambda\}$ and x is in some B_2 of $\{B_\eta\}$. Thus, $B_1\cap B_2$ is a member of $\mathfrak{I}$ which contains x. It follows that there is an element B_x of $\mathfrak{B}$ containing x and contained in $B_1\cap B_2$; hence, contained in $B_1^*\cap B_2^*$. Consider the collection $(\{B_\lambda\}\cap\{B_\eta\})\cup\{B_x\}$. The complement of this collection with respect to $\mathfrak{B}$ is denumerable; therefore, the union of all members of this collection is an element of $\mathfrak{B}^*$ which contains x and is contained in $B_1^*\cap B_2^*$ as desired.

(2) Next we will show that $(X, \mathfrak{I}^*)$ is countably compact. Let E be any infinite subset of X. Since $(X, \mathfrak{I})$ is countably compact there is an $x\in X$ such that every open set of $\mathfrak{I}$ which contains x also contains a point of E not x. In particular, every $B\in\mathfrak{B}$ which contains x has this property. Since each element of $\mathfrak{I}^*$ is a union of elements of $\mathfrak{B}$, it follows that every element of $\mathfrak{I}^*$ which contains x contains a point of E not x. Hence, x is also a limit point of E in the space $(X, \mathfrak{I}^*)$. (Notice here that Theorem 1 would still hold even if we had chosen the common, more restrictive, definition of countable compactness.)

(3) We conclude the proof by showing that $(X, \mathfrak{I}^*)$ is not locally compact. Let x be any point of X and let N^* be any neighborhood of $(X, \mathfrak{I}^*)$ which contains x. Then N^* contains an open set T^* of $(X, \mathfrak{I}^*)$. We know that T^* is a union of a subset of $\mathfrak{B}^*$, denoted by $\{B_\lambda^*\}$. Since each member of $\{B_\lambda^*\}$ can be expressed as the union of all but a denumerable number of elements of $\mathfrak{B}$, it follows that T^* can be expressed as the union of all but a countable number of elements of $\mathfrak{B}$. By hypothesis, $(X, \mathfrak{I})$ contains an uncountable distinguished collection relative to $\mathfrak{B}$; hence, T^* also contains an uncountable distinguished collection relative to $\mathfrak{B}$. Therefore, T^* contains a denumerable distinguished collection, $\{B_n\}$, $n=1, 2, \cdots$. Now let $\{B_\rho\}$ denote the collection of all elements of $\mathfrak{B}$ not in $\{B_n\}$, i.e., $\{B_\rho\}=\mathfrak{B}\setminus\{B_n\}$. Clearly, the collection $\{B_n^*\}$, $n=1, 2, \cdots$, where $B_n^*=\{\bigcup_\rho B_\rho\}\cup B_n$, is an open covering of N^*. Furthermore, no finite sub-

collection of $\{B_n^*\}$ will cover N^* since if B_n^* is removed from the collection then no member of the remaining collection will contain the element(s) of B_n which is (are) not in any other element of $\mathcal{B}$. This completes the proof of the theorem.

We know that no countably compact non-locally compact space can be both T_1 and second axiom since countable compactness implies local compactness in such spaces. However, one may prove that any space which satisfies the hypotheses of Theorem 1 (countably compact with an uncountable distinguished collection) is not T_0; hence, no countably compact non-locally compact space generated by use of Theorem 1 will be T_0. In addition, one may prove that no such generated space is first axiom. Obviously $(X, \mathfrak{I}^*)$ is connected.

Let us now consider two examples of spaces which may be employed to obtain countably compact non-locally compact spaces by application of Theorem 1. In these examples we make use of the fact that an easy way to obtain a countably compact (non-T_0) space is to "split" all (or all but a finite number) of the points of an infinite space. Given a space X, by a space obtained through *splitting a subset* A of X we mean, any space homeomorphic to the space $(X\backslash A)\cup(A\times\{0, 1\})$ having open sets of the form $(G\backslash A)\cup(\{G\cap A\}\times\{0, 1\})$ where G is open in X. The entire base of the space of the first example is a distinguished collection; however, the entire base of the space of the second example is not. More important, both the space of the first example and the corresponding generated space are not separable; however, the space of the second example is separable from which it follows that the generated space is also separable (choose the rationals in X as the countable collection).

EXAMPLE 1. Let X be the set of real numbers and let the base $\mathcal{B}$ for the topology $\mathfrak{I}$ of $(X, \mathfrak{I})$ be defined as follows: $\{0\}\in\mathcal{B}$ and given $x\neq 0$, the set $\{x, -x\}$ is an element of $\mathcal{B}$.

EXAMPLE 2. Let X be the following set of real numbers: $[0, 1]\cup[2, 3]$. Let a base $\mathcal{B}$ for $\mathfrak{I}$ be defined as follows: (1) Given any open interval (a, b), $\{\text{all rationals in } (a, b)\cap[0, 1]\}\cup\{\text{all rationals in } (a+2, b+2)\cap[2, 3]\}$ is a member of $\mathcal{B}$. (2) Let $c>0$ be a fixed real number. Then given any irrational number x in $[0, 1]$, $\{x\}\cup\{x+2\}\cup\{\text{all rationals in } (x-c, x+c)\cap[0, 1]\}\cup\{\text{all rationals in } (x-c+2, x+c+2)\cap[2, 3]\}$ is an element of $\mathcal{B}$.

REMARK. One can define *splitting a subspace* of a given space in a fashion analogous to our definition of splitting a subset (i.e., split the portion of each open set lying on the subspace), and observe for instance that any space obtained by splitting a nonempty, proper, closed (as a subset) subspace of a connected n-manifold without boundary is a locally n-Euclidean non-T_2 space.

Whereas Theorem 1 gives a method for obtaining a countably compact nonlocally compact space when given a particular type of countably compact space, the following easily proved theorem provides a method for obtaining such a space when given any nonlocally compact space.

THEOREM 2. *If X is a nonlocally compact space, then any space X^* obtained by splitting all but a finite number of the points of X is countably compact but not locally compact.*

Although any space generated by Theorem 2 will be non-T_0, the following generated spaces are second axiom (hence separable and Lindelöf). Note that X^* of example 3 is not connected.

EXAMPLE 3. Let X be the rationals in $[0, 1]$. Then one X^* may be expressed as the set of rationals in $([0, 1]\cup[2, 3])$ with the topology induced by base elements of the form, $\{\text{all rationals in } (a, b)\cap[0, 1]\}\cup\{\text{all rationals in } (a+2, b+2)\cap[2, 3]\}$.

EXAMPLE 4. Let X be Hilbert space and form some X^*.

Acknowledgements. The author would like to thank Dr. J. Gaylord May, Dr. Peter Harley, and the referee for their suggestions.

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SEPARATION AXIOMS IN QUASI-UNIFORM SPACES

S. A. NAIMPALLY, Iowa State University

In a remarkable paper [1] Davis showed how indexed systems of neighborhoods can be introduced in general topological spaces. This enables one to compare the sizes of neighborhoods at different points in the space, as in metric spaces, and consequently concepts like Cauchy nets, completeness, etc. can be introduced in general topological spaces. Pervin [3] used a different construction to prove that every topological space has a quasi-uniformity (satisfying all the conditions of a uniformity except the symmetry condition) which induces its original topology. In this note necessary and sufficient conditions are given in order that a quasiuniform space be (a) R_0 or (b) regular.

We describe Pervin's construction briefly. Let (X, T) be a topological space with topology T. Let $G\in T$.

(i) Let $S_G=G\times G\cup((X-G)\times X)$. It is easily shown that

(ii) $S_G \circ S_G\subset S_G$.

(iii) $\Delta=\{(x, x)\,|\,x\in X\}\subset S_G$.

So $\{S_G\,|\,G\in T\}$ is a subbase for a quasi-uniformity $\mathfrak{U}$ for X satisfying the conditions (iv)–(vii).

(iv) For each $U\in\mathfrak{U}$, $\Delta\subset U$.

(v) For each U, $V\in\mathfrak{U}$, $U\cap V\in\mathfrak{U}$.

(vi) For each $U\in\mathfrak{U}$, there exists a $V\in\mathfrak{U}$, such that $V\circ V\subset U$.

(vii) For each $U\in\mathfrak{U}$, if $U\subset V$ then $V\in\mathfrak{U}$.

(viii) $G\in T$ iff for each $x\in G$, there exists a $U\in\mathfrak{U}$ such that $U[x]\subset G$.

It is interesting to compare Pervin's construction with that of Davis. In a quasi-uniformity "triangle-inequality" (vi) is always satisfied, while in an indexed system of neighborhoods a local "triangle-inequality" is satisfied only for regular spaces. A symmetry condition need not hold in the former but is satisfied in the latter for R_0-spaces.

DEFINITION. *(X, T) is an R_0-space iff for each $x \in G \in T$, $\{\bar{x}\} \subset G$,* [1].

1. THEOREM. *(X, T) is an R_0-space if and only if X has a quasi-uniformity $\mathfrak{U}$ satisfying*: (ix) *for each $U \in \mathfrak{U}$ and $x \in X$ there exists a symmetric $V \in \mathfrak{U}$ such that $V[x] \subset U[x]$.*

Proof. Let (X, T) be an R_0-space and let $U \in \mathfrak{U}$, $x \in X$. Let $G = U(x) \in T$. Let $V = G \times G \cup (X - \{\bar{x}\}) \times (X - \{\bar{x}\})$. Obviously V is symmetric and it is easy to show that $S_G \cap S_{X-\{\bar{x}\}} \subset V$. Hence, by (vii), $V \in \mathfrak{U}$ and $V[x] = G$.

On the other hand let $\mathfrak{U}$ satisfy (ix). Let $x \in G \in T$. Then by (viii) there exists a $U \in \mathfrak{U}$ such that $U[x] \subset G$ and by (ix) there exists a symmetric $V \in \mathfrak{U}$ such that $V[x] \subset U[x]$. If $y \in \{\bar{x}\}$ then $x \in V[y]$. Since V is symmetric, $y \in V[x] \subset G$. So $\{\bar{x}\} \subset G$ and (X, T) is R_0.

Proof of the following theorem is similar to the above and is used in [2].

2. THEOREM. *(X, T) is a regular space if and only if $\mathfrak{U}$ satisfies the following condition*: (x) *for each $U \in \mathfrak{U}$ and $x \in X$ there exists a symmetric $V \in \mathfrak{U}$ such that $V \circ V[x] \subset U[x]$.*

Proof. Let (X, T) be regular. Then for each $U \in \mathfrak{U}$ and $x \in X$ there exist open neighborhoods H, I of x such that $I \subset \bar{I} \subset H \subset \bar{H} \subset G = U[x]$. Let $W_1 = G \times G \cup (X - \bar{H}) \times (X - \bar{H})$, $W_2 = H \times H \cup (X - \bar{I}) \times (X - \bar{I})$. We can show that $W_1, W_2 \in \mathfrak{U}$ (for example $S_G \cap S_H \cap S_{X-\bar{H}} \subset W_1$). If $V = W_1 \cap W_2$ then V is symmetric and $V \circ V[x] = G$.

Conversely, let $\mathfrak{U}$ satisfy (x). Let $x \in G \in T$. Then by (viii) there exists a $U \in \mathfrak{U}$ such that $U[x] \subset G$ and by (x) there exists a symmetric V such that $V \circ V[x] \subset U[x]$. If $y \in \overline{V[x]}$ then there exists a $z \in V[x] \cap V[y]$. Hence $y \in V[z] \subset V \circ V[x] \subset U[x]$; $V[x] \subset G$ and (X, T) is regular.

The author is at present at the University of Alberta, Edmonton, Canada.

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THE CONVOLUTION RING OF SEQUENCES

R. G. BUSCHMAN, SUNY at Buffalo

Recently Louis Brand [2] has discussed the imbedding of the convolution ring of complex valued sequences in a division algebra. The product in this ring is defined by

$$ab(n) = \sum_{k=0}^{n} a(k)b(n - k).$$

We will show that the division ring has a related representation.

Consider the set of complex valued, two sided infinite sequences which have only a finite number of nonzero terms of negative index. Such a sequence can be denoted by

$$a = \{\bar{0}, a(-N), \cdots, a(-1); a(0), \cdots, a(n), \cdots\}.$$

(The bar over the first or last subset of elements indicates repetition to the left or right, respectively.) A two-sided convolution of such sequences can be defined by

$$ab(n) = \sum_{k=-\infty}^{+\infty} a(k)b(n-k) = \sum_{k=-Na}^{n+Nb} a(k)b(n-k),$$

where from the latter form we note that the summation index on the convolution is of finite range. This also shows that the product sequence is in the set. If further we take

$$(a+b)(n) = a(n) + b(n)$$

for the sum, then it can be verified by methods similar to those of Brand [2] that we have a commutative ring with unity, I: $\{\bar{0}; 1, \bar{0}\}$. The proof that there are no divisors of zero carries through with merely a change of notation. This ring we denote by R'; the ring of Brand, by R.

To verify that R' is a division ring we first show that the shifting sequence $s = \{\bar{0}; 0, 1, \bar{0}\}$ has an inverse. Consider $as = I$ from which we obtain

$$\sum_{k=-Na}^{n-1} a(k)s(n-k) = 1 \quad \text{if } n = 0; = 0 \text{ otherwise},$$

which reduces to $a(n-1)s(1) = 1$ if $n = 0$; $= 0$ otherwise, since $s(n-k) = 0$ unless $n-k = 1$. Hence, as expected, we have

$$s^{-1}: \{\bar{0}, 1; \bar{0}\}$$

which is a unique element of R'. Since $s^{-r} = (s^{-1})^r = (s^r)^{-1}$ is also in R' and since every sequence except the zero sequence has a first nonzero component, we can write any sequence a in the form $a = s^r b$, where r is chosen so that $b(0)$ is the first nonzero component of b. This condition $b(0) \neq 0$ is necessary and sufficient for the existence of b^{-1} in R, hence in R'. Thus $a^{-1} = s^{-r}b^{-1}$ exists in R'. Let $a, b \in R \subset R'$ with $b \neq 0$ and consider the equation $bc = a$. If $b(0) \neq 0$ then there exists a unique solution $c \in R$, but in any case there exists a unique solution c in R'. Hence we can set up the isomorphism between the equivalence class (a/b) of pairs of elements of R and the element $c \in R'$.

Lothar Berg [1] has also considered an operational calculus of sequences. The product is defined differently, that is

$$(a \cdot b)(n) = \sum_{k=0}^{n-1} a(k)[b(n-k) - b(n-1-k)] + a(n)b(0),$$

where we use "$\cdot$" to distinguish multiplications. The form is equivalent in the following sense to that which we have been considering. Let

$$\sigma: \{\,;\bar{1}\}, \rho: \{n+1\} = \{\,;1, 2, 3, \cdots\}.$$

From these special sequences we have

$$a \cdot b = \sigma ab, \qquad ab = \rho \cdot a \cdot b,$$

so that the field of "operators" of Berg is also equivalent to the ring R'.

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COMPLEMENTS IN FINITE GROUPS

J. D. DIXON, University of New South Wales, Australia

The object of this note is to give a new proof of a useful theorem of I. Schur. The original theorem of Schur was later generalized by H. Zassenhaus, who proved the following theorem.

Let G be a finite group, and let H be a normal Hall subgroup of G (i.e. the order $|H|$ of H is relatively prime to the index $|G: H|$ of H in G). Then:

(a) *There exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. (K is called a complement of H in G.)*

(b) *If L is also a complement of H in G, and either H or G/H is solvable, then L is conjugate to K in G.*

Schur proved (a) in the case that H is abelian, and Zassenhaus used an induction argument (based on Schur's result) to prove the general case, and he also proved the conjugacy condition (b). (See [1] pages 162–163.) Since Feit and Thompson have now proved the long-standing conjecture that all groups of odd order are solvable, and since either H or G/H has odd order (because H is a Hall subgroup), the conjugacy condition (b) always holds.

Zassenhaus' derivation of his theorem from Schur's result is quite easy (although ingenious), but the classical proof of Schur's theorem requires the machinery of Schreier's theory of group extensions. (For example, see [1], [2] or [3].) More recently in [4], H. Wielandt showed how Schur's result could be derived by calculations with monomial matrices. Here we shall give a proof of Zassenhaus' theorem in case H is abelian. This will include a proof of Schur's theorem which appears to be briefer and more elementary than any given to date, and uses only a minimum of knowledge of the monomial representation of groups. The key to the method is to use the conjugacy (b) to prove the existence (a).

$\cap DG^*$, we have $DP = SD \cap DG^* = DG^*$ and $D \cap P = 1$.

Thus P is a complement of D in DG^*.

Because of (ii), (iii) and (iv), we may apply (2) to the group DG^* to conclude that $c^{-1}Pc \subseteq G^*$ for some $c \in D$. Then $c^{-1}Pc$ is clearly a complement of A^* in G^*. Because of the isomorphism between G and G^*, it follows that A has a complement in G.

Note. Instead of the properties of the monomial representation which we used here, we could have used the "universal embedding theorem" of the wreath product. (For example, see [5] Theorem 3.5.) It is possible to give a similar proof of the theorem of Gaschütz quoted in [4]. (Part of this is proved in [2] page 246.) I plan to give more general results in a later paper.

Added in Proof: The paper referred to will appear in Proc. London Math. Soc., (3) 17 (1967). It is titled "Complements of normal subgroups in infinite groups."

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A NOTE ON EQUILATERAL METRIC TRIPLES

B. W. Huff and W. A. Kirk, University of California, Riverside

Let M denote a metric space and $S(a, b)$ a metric segment of M with endpoints a, b. A point f of $S(a, b)$ is said to be a *foot* of $p \in M$ on $S(a, b)$ if $pf = \inf[px: x \in S(a, b)]$. The following result is found in Blumenthal [1, p. 47].

(1) *If f is a foot of p on $S(a, b)$ and if* $\min[af, bf] > 3pf > 0$, *then there exist points q and r of $S(a, b)$ such that p, q, r form an equilateral triple* ($pq = qr = pr$).

This observation is used to characterize the metric segment among all closed, compact and convex metric spaces (which contain at least two points) by the absence of equilateral triples [1, p. 48]. (Also, see Blumenthal and Robinson [2]).

It is natural to ask whether 3 is the smallest multiplier of pf for which (1) is valid. In this note we sharpen (1) by showing that the number 3 of (1) may be replaced by 2 in general (i.e., if f is merely assumed to be any point of $S(a, b)$), and by 1 if f is a foot of p on $S(a, b)$. These numbers are the best possible.

Our terminology is that of [1].

Theorem 1. *If f is a foot of p on $S(a, b)$ and if*

$$\min[af, bf] \geq pf > 0,$$

then there exist points q, r of $S(a, b)$ such that p, q, r form an equilateral triple.

Proof. The segment $S(a, b)$ is isometric with a real line segment of length ab,

so we introduce coordinates in $S(a, b)$ in the obvious way, taking f to be the origin. Letters denoting points of $S(a, b)$ will also be interpreted as numbers, and we suppose $a<0$ and $b>0$. However, the symbol xy will denote the distance $|x-y|$ for $x, y\in S(a, b)$.

For $x\in[a, 0]$, let $F(x)=px$. Since $bf\geqq pf$, it is easily seen that there is a unique point $y(x)\in[x, b]$ such that $xy(x)=px$. Let $S(x)=y(x)p$. Since f is a foot of p on $S(a, b)$ we have

$$(*) \qquad F(0) = pf \leqq S(0).$$

Let x_0 be the point of $[a, 0]$ such that $x_0f=pf$, and let $y_0=y(x_0)$. There are two cases to consider.

Case 1. Suppose $y_0\in[0, b]$. Then

$$S(x_0) = py_0 \leqq pf + fy_0 = x_0f + fy_0 = x_0y_0 = x_0p = F(x_0).$$

This, together with (*) and the fact that both $F(x)$ and $S(x)$ are continuous, implies there is a point $X\in[a, 0]$ such that $F(X)=S(X)$. Then X, $y(X)$, p form an equilateral triple.

Case 2. If $y_0\notin[0, b]$ then, since $y(x)$ is a continuous function of x, and $y(x_0)<0$ while $y(0)=pf>0$, there is a point $X\in[x_0, 0]$ such that $y(X)=0$. Hence

$$S(X) = pf \leqq Xp = F(X).$$

The conclusion again follows as in Case 1.

THEOREM 2. *If f is any point of $S(a, b)$ such that* $\min[af, bf]\geqq 2pf$ *and if p is not on $S(a, b)$, then there exist points q, r of $S(a, b)$ such that p, q, r form an equilateral triple.*

Proof. For convenience we assume coordinates have been introduced and the metric "normalized" so that $S(a, b)\supseteq[0, 4]$, $f=2$ and $pf=1$. We may also assume that a foot f_p of p on $[0, 4]$ is in $[0, 2]$. (There is such a foot either in $[0, 2]$ or $[2, 4]$, cf. [1, p. 46].)

For $x\in[0, 2]$, let $F(x)=px$. A simple application of the triangle inequality yields a unique point $y(x)$ in $[x, 4]$ such that $xy(x)=F(x)$. Let $py(x)=K(x)$.

$F(0)=1+\eta$, $\eta\geqq 0$, since by the triangle inequality $pf+F(0)=1+F(0)\geqq 2$.

If $F(0)\geqq K(0)$ then, since $F(f_p)\leqq K(f_p)$, there is a number $X\in[0, f_p]$ such that $F(X)=K(X)$, and we reach the desired conclusion.

Suppose $F(0)<K(0)$. Let $G(x)=x+px$. Then $K(0)>1+\eta$ while $G(0)=1+\eta$. By the triangle inequality, $F(1)+\eta\geqq K(0)>1+\eta$. Thus $F(1)>1$, say $F(1)=1+\xi$, $\xi>0$. This implies

$$G(1) = 1 + F(1) = 2 + \xi$$

and therefore, again using the triangle inequality,

$$K(1) \leqq F(2) + \xi = 1 + \xi = F(1).$$

As before, since $F(0) \leqq K(0)$, there is a point $X \in [0, 1]$ such that $F(X) = K(X)$, and the proof is completed.

It might be noted that if $(p) + S(a, b) \subset E_2$, Theorem 1 is valid for the multiplier $1/\sqrt{3}$ (which is *less* than 1). In general, however, 1 is the smallest multiplier of pf for which Theorem 1 is valid. This can be seen by considering the Minkowski space $M_2^{(1)}$. The elements of this space are ordered pairs of real numbers with distance of $x = (x_1, x_2)$ and $y = (y_1, y_2)$ taken as

$$xy = |x_1 - y_1| + |x_2 - y_2| .$$

Let $S(a, b)$ be contained in the interval joining $(-1, 0)$ and $(1, 0)$ and let $p = (0, 1)$. It is easily seen that if p, x, y form an equilateral triple with x, $y \in S(a, b)$ then x and y *must* be the points $(-1, 0)$ and $(1, 0)$. Hence $S(a, b)$ is the interval joining these two points; the foot f of p on $S(a, b)$ is the origin, and $\min[af, bf] = pf$.

A slightly more complicated example can be given, also in the $M_2^{(1)}$, which shows that 2 is the smallest multiplier of pf for which Theorem 2 is valid.

The research of B. W. Huff was supported in part by the U. S. Army Research Office, Grant No. DA-ARO(D)-31-124-G383.

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LOCATION OF THE ZEROS OF POLYNOMIALS

Q. G. Mohammad, Jammu and Kashmir University, India

The following theorem is due to Montel and Marty [1, p. 107].

Theorem A. *All the zeros of the polynomial*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$$

lie in $|z| \leqq \max(L, L^{1/n})$ *where* L *is the length of the polygonal line joining in succession the points* $0, a_0, a_1, \cdots, a_{n-1}, 1$; *i.e.*

$$L = |a_0| + |a_1 - a_0| + \cdots + |a_{n-1} - a_{n-2}| + |1 - a_{n-1}| .$$

We prove

Theorem 1. *All the zeros of the polynomial of Theorem A lie in* $|z| \leqq R = \max(L_p, L_p^{1/n})$ *where*

$$L_p = n^{1/q} \left(\sum_{i=0}^{n-1} |a_i|^p \right)^{1/p}, \quad p^{-1} + q^{-1} = 1.$$

The bound is sharp.

Proof. We have

$$(1)\quad |p(z)| \geqq |z|^n\left(1-\sum_{i=1}^{n}\frac{|a_{n-i}|}{|z|^i}\right) \geqq |z|^n\left\{1-n^{1/q}\left(\sum_{i=1}^{n}\frac{|a_{n-i}|^p}{|z|^{ip}}\right)^{1/p}\right\}.$$

If $L_p \geqq 1$, $\max(L_p, L_p^{1/n}) = L_p$. Let $|z| \geqq 1$. Then $1/|z|^{ip} \leqq 1/|z|^p$ $(i=1, 2, \cdots, n)$. Hence (1) implies that if $|z| > L_p$ then

$$|p(z)| \geqq |z|^n\left\{1-\frac{n^{1/q}}{|z|}\left(\sum_{i=0}^{n-1}|a_i|^p\right)^{1/p}\right\} = |z|^n\left(1-\frac{L_p}{|z|}\right) > 0.$$

Again if $L_p \leqq 1$, $\max(L_p, L_p^{1/n}) = L_p^{1/n}$. Let $|z| \leqq 1$. Then $1/|z|^{ip} \leqq 1/|z|^{np}$ $(i=1, 2, \cdots, n)$. Hence, by (1), if $|z| > L_p^{1/n}$ then

$$\begin{aligned}|p(z)| &\geqq |z|^n\left\{1-\frac{n^{1/q}}{|z|^n}\left(\sum_{i=1}^{n}|a_{n-i}|^p\right)^{1/p}\right\}\\ &= |z|^n\left\{1-\frac{n^{1/q}}{|z|^n}\left(\sum_{0}^{n-1}|a_i|^p\right)^{1/p}\right\}\\ &= |z|^n\left(1-\frac{L_p}{|z|^n}\right) > 0.\end{aligned}$$

Hence $p(z)$ does not vanish for $|z| > \max(L_p, L_p^{1/n})$ and the theorem follows. The limit in Theorem 1 is attained by

$$p(z) = z^n - \frac{1}{n}(z^{n-1} + z^{n-2} + \cdots + z + 1)$$

since

$$L_p = n^{1/q}\left(\sum_{0}^{n-1}\frac{1}{n^p}\right)^{1/p} = n^{1/q}\left(\frac{n}{n^p}\right)^{1/p} = \frac{n^{1/p}\cdot n^{1/q}}{n} = 1$$

and 1 is a zero of $p(z)$.

Letting $q \to \infty$, it follows that all the zeros of $p(z)$ lie in

$$(2)\qquad |z| \leqq \max(L_1, L_1^{1/n}) \quad \text{where } L_1 = \sum_{i=0}^{n-1}|a_i|.$$

Applying this result to $(1-z)p(z)$ we obtain the theorem of Montel and Marty mentioned above.

THEOREM 2. *If* $0 < a_{i-1} \leqq ka_i$, $k > 0$, *then all the zeros of* $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ *lie in* $|z| \leqq \max(M, M^{1/n})$ *where*

$$M = \frac{(a_0 + a_1 + \cdots + a_{n-1})}{a_n}(k-1) + k.$$

Proof. Consider

$$F(z) = (k - z)P(z) = (k - z)(a_0 + a_1z + \cdots + a_nz^n)$$
$$= ka_0 + (ka_1 - a_0)z + (ka_2 - a_1)z^2 + \cdots + (ka_n - a_{n-1})z^n - a_nz^{n+1}.$$

Applying (2) to the polynomial $F(z)/a_n$ we find that

$$L_1 = \frac{\sum_{i=0}^{n} |ka_i - a_{i-1}|}{a_n} = \frac{k(a_0 + a_1 + \cdots + a_n) - (a_0 + a_1 + \cdots + a_{n-1})}{a_n}$$
$$= \frac{(k - 1)(a_0 + a_1 + \cdots + a_{n-1})}{a_n} + k = M$$

and the theorem follows. Putting $k=1$ in Theorem 2 we get the following result due to Kakeya [1, p. 106].

THEOREM B. *If* $0<a_0\leqq a_1\leqq \cdots \leqq a_n$, *then all the zeros of the polynomial* $a_0+a_1z+a_2z^2+ \cdots +a_nz^n$ *lie in* $|z| \leqq 1$.

This follows from the fact that $M=1$, hence $\max(M, M^{1/n})=1$.

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A NOTE CONCERNING FERMAT'S CONJECTURE

W. E. CHRISTILLES, St. Mary's University, San Antonio, Texas

This paper introduces some elementary results related to the famous unsolved conjecture of Fermat, that there exists no nontrivial solution in integers of the equation

$$x^n + y^n + z^n = 0 \tag{1}$$

for n an odd integer >2. It is sufficient to consider the equation

$$x^p + y^p + z^p = 0, \tag{2}$$

for p an odd prime. Theorem 2 (below) is a new proof of Stone's Theorem 1 [1]. In addition an extension of Stone's Theorem will be stated and proven.

Assume that equation (2) has a solution $x=a$, $y=b$, $z=c$. The following restrictions result in no loss of generality.

(3.1) $abc \neq 0$

(3.2) $|abc| \neq 1$

(3.3) $|a|$ and $|b|$ are not both unity.

(3.4) $(a, b,) = 1$, $(a, c) = 1$, and $(b, c) = 1$.

(3.5) $c < 0 < a < b < |c|$.

The first restriction excludes the trivial case. The restriction that $|abc| \neq 1$ is permissible since if $|abc| = 1$, with p an odd prime, then $a^p+b^p+c^p \neq 0$, a contradiction. If $|a|$ and $|b|$ are both unity, with p an odd prime and c a nonzero integer (see Restriction 3.1), then a, b, c is not a solution of equation (2). Hence the case in which $|a|$ and $|b|$ are both unity can be excluded. Moreover, if any two of a, b, and c have a common nonunit factor, then all of a, b, and c must contain the common factor. The common factor can be removed, resulting in the restrictions on a, b, and c that $(a, b) = 1$, $(a, c) = 1$, and $(b, c) = 1$. Finally, with p an odd prime and a, b, and c integers satisfying the first four restrictions, a, b, and c can be chosen such that $c < 0 < a < b < |c|$.

THEOREM 1. *Let a, b, c be integers satisfying restrictions* (3.1) *through* (3.5). *Then equation* (2) *has a solution if, and only if,*

$$x^{3p} + y^{3p} + z^{3p} - 3x^p y^p z^p = 0, \tag{4}$$

has a solution.

Proof. Consider the following identity:

$$x^{3p}+y^{3p}+z^{3p}-3x^p y^p z^p = (x^p+y^p+z^p)(x^{2p}+y^{2p}+z^{2p}-x^p y^p-x^p z^p-y^p z^p). \tag{5}$$

If $a^p+b^p+c^p=0$, then $a^{3p}+b^{3p}+c^{3p}-3a^p b^p c^p=0$. Conversely, suppose that $a^{3p}+b^{3p}+c^{3p}-3a^p b^p c^p=0$. Then either

$$a^p + b^p + c^p = 0 \quad \text{or} \quad a^{2p} + b^{2p} + c^{2p} - a^p b^p - a^p c^p - b^p c^p = 0. \tag{6}$$

But $a^{2p}>0$, $b^{2p}>0$, and $c^{2p}>0$. Moreover, due to restrictions (3.1) through (3.5) $c^{2p} > a^p b^p > 0$, $-a^p c^p > 0$, and $-b^p c^p > 0$. Thus the second equation in (6) is impossible. Hence $a^p+b^p+c^p=0$. This completes the proof of Theorem 1.

THEOREM 2. (Stone's Theorem, [1]). *Let a, b, c, and p be integers satisfying restrictions* (3.1) *through* (3.5), *with both p and $2p+1$ odd primes, and such that* $(abc, 2p+1) = 1$. *Then equation* (2) *has no solution.*

Proof. By Theorem 1, (2) has a nontrivial solution only if there exists a solution of (4), satisfying restrictions (3.1) through (3.5). Since $(abc, 2p+1) = 1$, (4) implies the congruence

$$a^{3p} + b^{3p} + c^{3p} - 3^p a^p c^p \equiv a^p + b^p + c^p - 3a^p b^p c^p \equiv -3a^p b^p c^p \equiv 0 \pmod{2p+1}.$$

But this is impossible since $(3abc, 2p+1) = 1$. This contradiction completes the proof of Theorem 2.

THEOREM 3. *Let a, b, c, and p be integers with p an odd prime, satisfying restrictions* (3.1) *through* (3.5). *Equation* (2) *has a solution only if*

$$x^{5p} + y^{5p} + z^{5p} - 5(x^p y^p z^p)(z^{2p} - x^p y^p) = 0 \tag{7}$$

has a solution.

Proof. Assume that (2) has a solution $x=a$, $y=b$, and $z=c$ satisfying restrictions 3.1 through 3.5. Transposing c^p, (in $a^p+b^p+c^p=0$) to the right side of the equation, raising both sides of the resulting equation to the fifth power, and simplifying yields

$$a^{5p} + b^{5p} + c^{5p} + 5a^pb^p[a^{3p} + b^{3p} + 2a^pb^p(a^p + b^p)] = 0. \tag{8}$$

Since $a^p+b^p=-c^p$, (8) can be written as

$$a^{5p} + b^{5p} + c^{5p} + 5a^pb^p[a^{3p} + b^{3p} - 3a^pb^pc^p + a^pb^pc^p] = 0. \tag{9}$$

But, due to Theorem 1, $a^p+b^p+c^p=0$ if and only if $a^{3p}+b^{3p}-3a^pb^pc^p=-c^{3p}$. Substituting $a^{3p}+b^{3p}-3a^pb^pc^p=-c^{3p}$ into (9) and simplifying results in (7)

$$a^{5p} + b^{5p} + c^{5p} - 5a^pb^pc^p(c^{2p} - a^pb^p) = 0.$$

Hence, with the stipulation that restrictions (3.1) through (3.5) apply, (2) implies (7). This completes the proof of Theorem 3.

THEOREM 4. *Let a, b, c, and p be integers satisfying restrictions* (3.1) *through* (3.5), *with both p and $4p+1$ odd primes, and such that* $(abc, 4p+1)=1$. *Then equation* (2) *has no solution.*

Proof. From Theorem 3, (2) has a solution only if there exists a solution of (7) which satisfies restrictions (3.1) through (3.5). Since $(abc, 4p+1)=1$, (7) implies the congruence

$$\begin{aligned} a^{5p} + b^{5p} + c^{5p} - 5(abc)^p(c^{2p} - a^pb^p) &\equiv a^p + b^p + c^p - 5(abc)^p(c^{2p} - a^pb^p) \\ &\equiv -5(abc)^p(c^{2p} - a^pb^p) \equiv 0 \pmod{4p+1}. \end{aligned} \tag{10}$$

But $a^p+b^p+c^p=0$, implies that

$$2(c^{2p} - a^pb^p) = a^{2p} + b^{2p} + c^{2p}. \tag{11}$$

(This is easily seen upon transposing c^p, in $a^p+b^p+c^p=0$, to the right side, squaring both sides, adding c^{2p} to both sides, and simplifying.) Multiplying (10) by -2 and substituting (11) into (10) yields

$$-5(abc)^p(a^{2p} + b^{2p} + c^{2p}) \equiv -5(abc)^p(\pm 1 \pm 1 \pm 1) \equiv 0 \pmod{4p+1}. \tag{12}$$

Since $(abc, 4p+1)=1$, (12) can be reduced to

$$-5(\pm 1 \pm 1 \pm 1) \equiv 0 \pmod{4p+1}. \tag{13}$$

With the observation that (13) is not possible for any choice of signs, the proof of Theorem 4 is complete.

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AN EXTENSION OF A THEOREM CONCERNING EQUILATERAL TRIPLES

L. M. BLUMENTHAL, University of Missouri

Introduction. A subset L of a metric space is a *metric line* provided there is a one-to-one, distance-preserving mapping of L onto a euclidean straight line. If M_1, M_2 are metric spaces, M_1 is *contained homothetically* in M_2 provided there exists a mapping f of M_1 into M_2, and a positive constant k, such that, for each two points p, q of M_1, the distance $f(p)f(q) = k \cdot pq$.

As a basis for a metric characterization of the euclidean straight line, it has been shown [1] that if a metric space contains a metric line L and a point p not on L, then the space contains an equilateral point-triple (i.e., points q, r, s with $qr = rs = qs \neq 0$). Later, that result was obtained by the writer [2] as a corollary of the following theorem: *If a metric space contains a metric segment $S_{a,b}$ with endpoints a, b, and a point p with a foot f_p in $S_{a,b}$ such that* $\min [af_p, bf_p] > 3 \cdot pf_p > 0$, *then the space contains an equilateral triple.*

We show that the first result mentioned in the preceding paragraph admits an interesting extension (which permits an extension of the metric characterization of the straight line to which the earlier result gave rise). But, on the other hand, the theorem concerning the metric segment $S_{a,b}$, and a point p not on it, *cannot be extended in that manner.*

1. The extension. The extended result is expressed in the following theorem.

THEOREM 1. *A metric space consisting of a metric line L and a point p, not on L, contains homothetically every metric point-triple.*

Proof. Let q, r, s denote any metric point-triple with $qr = a$, $rs = b$, $qs = c$ and $a \geqq b \geqq c > 0$. If q, r, s are linear, then $a = b + c$ and L contains points q', r', s' with $q'r' = qr$, $r's' = rs$, $q's' = qs$.

Suppose, now, that q, r, s are not linear, and consequently $a < b + c$. Since L is isometric with the euclidean straight line, the points of L may be represented by real numbers x with distance $xy = |x - y|$, where the number corresponding to a foot f_p of p on L is zero. (Coordinates serve also as labels.)

Put $S(x) = px$, $x \in L$, $x \geqq 0$, and write $S(0) = pf_p = d$. Clearly $S(x) \geqq d$ and, by the triangle inequality, $|x - S(x)| \leqq d$. Define, now, the positive-valued function $k(x)$ by the equation $S(x) = a \cdot k(x)$, and let $y(x) = x - b \cdot k(x)$, $T(x) = py(x)$, $x \geqq 0$. Then

$$d \leqq T(x) \leqq d + |x - b \cdot k(x)| = d + |x - (b/a)S(x)| .$$

Define the function $R(x)$, continuous for $x \geqq 0$, by

$$R(x) = T(x)/c - S(x)/a.$$

Since $0 \leqq |x - S(x)| \leqq d$, for $x \neq 0$, $0 \leqq |1 - S(x)/x| \leqq d/x$, and consequently $\lim_{x \to \infty} S(x)/x = 1$. Now

$$R(0) = T(0)/c - S(0)/a = T(0)/c - d/a \geqq 0,$$

since $T(0) \geqq d$ and $c \leqq a$.

Case 1. $R(0) = 0$. Then the pairwise distinct points p, 0, $y(0)$ are the desired triple since $\operatorname{dist}(p, 0) = S(0) = d$, $\operatorname{dist}(p, y(0)) = T(0)$, $\operatorname{dist}(0, y(0)) = b \cdot k(0) = b \cdot S(0)/a = bd/a$; that is,

$$\frac{\operatorname{dist}(p, 0)}{a} = \frac{\operatorname{dist}(0, y(0))}{b} = \frac{\operatorname{dist}(p, y(0))}{c} = k(0).$$

Since $\operatorname{dist}(p, 0) = d \leqq \operatorname{dist}(p, y(0))$ and $a \geqq b \geqq c$, the above equalities imply $\operatorname{dist}(p, 0) = \operatorname{dist}(0, y(0)) = \operatorname{dist}(p, y(0))$ and $a = b = c$. Thus if $R(0) = 0$, the given triangle is equilateral.

Case 2. $R(0) > 0$. Since $T(x) \leqq d + |x - (b/a)S(x)|$, it follows readily that

$$R(x) \leqq d/c + x\{a\,|\,1 - (b/a)S(x)/x\,| - c \cdot S(x)/x\}/ac$$

and, since $\lim_{x \to \infty} S(x)/x = 1$,

$$\lim_{x \to \infty} \{a\,|\,1 - (b/a)S(x)/x\,| - c \cdot S(x)/x\} = a - (b + c) < 0.$$

Hence $R(x) \to -\infty$ as $x \to \infty$ and, since $R(0) > 0$, it follows from the continuity of $R(x)$ that a value x_0 exists such that

$$R(x_0) = T(x_0)/c - S(x_0)/a = 0.$$

The points p, x_0, $y_0 = y(x_0)$ are the desired three points since clearly

$$\frac{px_0}{a} = \frac{x_0y_0}{b} = \frac{py_0}{c} = k(x_0);$$

that is, $px_0 = k(x_0) \cdot qr$, $x_0y_0 = k(x_0) \cdot rs$, $py_0 = k(x_0) \cdot qs$, with $k(x_0) > 0$.

COROLLARY. *A metric space is a metric line if and only if it contains at least one metric line, and there exists at least one metric point-triple that is not homothetically contained in the space.*

Since every complete, metrically convex and externally convex metric space (containing at least two distinct points) contains a metric line, the following new metric characterization of the line is obtained.

THEOREM 2. *A metric space is congruent with the euclidean straight line if and only if the space is complete, metrically convex and externally convex, contains at least two distinct points, and at least one metric triple is not homothetically contained in the space.*

We note that the analogue of Theorem 1 in the case of a metric segment $S_{a,b}$ and a point p, not on it, is *not* valid, no matter what demand is made upon the distances pf_p, af_p, bf_p, where f_p denotes a foot of p on $S_{a,b}$. For if $(p) + S_{a,b}$ is a subset of the euclidean plane, it is easily seen that an angle α $(0 < \alpha < \pi)$ exists such that α exceeds the maximum angle of any triangle with vertices p, x, y,

where x, y denote distinct points of $S_{a,b}$. Then evidently the set $(p)+S_{a,b}$ does not contain a point-triple that is homothetic to the vertices of a euclidean triangle, one of whose angles is α.

An extension of Theorem 1, in which the metric line L is replaced by euclidean n-space E_n and the metric point-triple by the vertices of an $(n+1)$-dimensional simplex, will be published elsewhere.

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A LOWER BOUND FOR THE NUMBER OF VERTICES OF A GRAPH

M. E. Watkins, University of North Carolina at Chapel Hill

In this note we derive a lower bound for the number of vertices of a graph in terms of its diameter and connectivity. We then show that the inequality so obtained is the best possible result.

Let K_m denote the complete graph on m vertices. For other terminology, see Ore [1].

Theorem 1. *If G is a graph on n vertices with connectivity $\lambda \geqq 1$ and diameter $\delta \geqq 1$, then $n \geqq \lambda(\delta+1)-2$.*

Proof. There exist vertices u and v of G whose distance from each other is δ. Since the connectivity of G is λ, there exist at least λ arcs joining u and v having only the vertices u and v in common, by Whitney [2]. Each of these arcs contains at least $\delta-1$ vertices other than u and v. Hence, the union of these arcs contains at least $\lambda(\delta-1)+2$ vertices.

Theorem 2. *Given positive integers δ and λ, there exists a graph G on n vertices with diameter δ and connectivity λ such that $n=\lambda(\delta-1)+2$.*

Proof. Let $H_1, \cdots, H_{\delta-1}$ represent $\delta-1$ disjoint copies of K_λ. Form G as follows: join each vertex of H_i to each vertex of H_{i+1} by an edge $(i=1, \cdots, \delta-2)$; then join a new vertex u to each vertex of H_1 by an edge and similarly join a vertex v to each vertex of $H_{\delta-1}$. The resulting graph clearly satisfies the prescribed conditions.

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ON THE RATIONAL SOLUTIONS OF $m^n = n^m$ WITH $m \neq n$

SOLOMON HURWITZ, City College, New York

This paper solves, in rational numbers, a problem which might be described as "commutative exponentiation." The subject is a Diophantine equation that has evoked considerable attention since the days of Euler (see [6, pp. 578–579], [7, pp. 150–151] and especially [2, p. 687]). And interest in

$$m^n = n^m \qquad (m \neq n) \tag{1}$$

still appears to be lively. Thus, equation (1) made an appearance on both the 1960 and 1961 Putnam Prize Competition [1(a), p. 633], [1(b), p. 762]. References [3, pp. 856–861] and [4, p. 571] deal with the solutions of (1) in algebraic numbers. Then there is Banach's result that (1) has infinitely many solutions in transfinite cardinals [5, p. 151] and Jacobsthal's Theorem [5, pp. 359–362] on the general solution in transfinite ordinal numbers.

1. We first consider positive rational solutions. We shall assume that $m > n$. If $n \leqq 1$, $m^n > n^n \geqq n^m$. In looking for solutions of $m^n = n^m$ we may therefore limit ourselves to the case $m > n > 1$.

Let $m = a/b$, $n = c/d$, $(a, b) = (c, d) = 1$, $a > b$, $c > d$, $ad > bc$. If $(a/b)^{c/d} = (c/d)^{a/b}$ then

$$a^{bc} = c^{ad} \quad \text{and} \quad b^{bc} = d^{ad}, \tag{2}$$

a and c must have the same primes in their canonical representation as products of powers of primes and so must b and d. In the case of a and c these are certainly not empty products. Let

$$\begin{aligned} a &= \Pi p_i^{\alpha_i}, & c &= \Pi p_i^{\gamma_i}, \\ b &= \Pi q_j^{\beta_j}. & d &= \Pi q_j^{\delta_j}. \end{aligned} \tag{3}$$

It follows from (2) that for each value of i, $\alpha_i bc = \gamma_i ad$, hence

$$\frac{\alpha_i}{\gamma_i} = \frac{ad}{bc} = \frac{\lambda}{\mu}$$

with $(\lambda, \mu) = 1$ and $\lambda > \mu$.

Let

$$\alpha_i = \lambda\rho_i, \quad \gamma_i = \mu\rho_i \quad \text{and} \quad \Pi p_i^{\rho_i} = r \tag{4}$$

then

$$a = r^\lambda, \quad c = r^\mu \quad \text{with} \quad r > 1, \quad \lambda > 1. \tag{5}$$

Similarly

$$b = s^\lambda, \quad d = s^\mu \quad \text{with} \quad s \geqq 1. \tag{6}$$

If $b=d=1$, $s=1$ and we choose λ and μ as in (5). Hence

$$\frac{\lambda}{\mu}=\frac{r^{\lambda}s^{\mu}}{r^{\mu}s^{\lambda}}=\frac{r^{\lambda-\mu}}{s^{\lambda-\mu}} \tag{7}$$

$$\lambda = r^{\lambda-\mu}, \quad \mu = s^{\lambda-\mu}$$

$$\lambda-\mu = r^{\lambda-\mu}-s^{\lambda-\mu}.$$

But (7) can hold only if $\lambda-\mu=1$. For, suppose that $\lambda-\mu>1$. Since $r>s$,

$$r^{\lambda-\mu} \geqq (s+1)^{\lambda-\mu} > s^{\lambda-\mu}+(\lambda-\mu)s^{\lambda-\mu-1},$$

since the expansion of $(s+1)^{\lambda-\mu}$ by the binomial theorem would contain more than two terms. It follows that $r^{\lambda-\mu}-s^{\lambda-\mu}>\lambda-\mu$.

Equation (7) can be satisfied only by letting $\lambda-\mu=1$ and $r-s=1$. This means that

$$a=(1+s)^{1+s}, \qquad c=(1+s)^{s},$$

$$b=s^{1+s}, \qquad d=s^{s},$$

and $m=(1+1/s)^{1+s}$, $n=(1+1/s)^{s}$ with $s\geqq 1$. We easily verify that these values for m and n satisfy $m^n=n^m$ if s is *any* positive integer.

2. We next consider rational solutions with m and n both negative, or, we consider the equation $(-u)^{-v}=(-v)^{-u}$ with $u>v>0$. Since $(-u)^{-v}$ does not necessarily have a real value we will study the equation in the complex plane where each side of the equation will generally be many valued. Is there any value of $(-u)^{-v}$ which will be the same as some value of $(-v)^{-u}$? This is equivalent to demanding that

$$e^{-v[\log u+k_1\pi i]} = e^{-u[\log v+k_2\pi i]}$$

where $\log u$ and $\log v$ are real, and k_1 and k_2 are odd integers, or

$$e^{-v\log u} = e^{-u\log v} \tag{8}$$

and

$$k_1v-k_2u=L \tag{9}$$

with L an even integer.

Equation (8) is equivalent to $u^v=v^u$. Hence

$$u=\left(\frac{s+1}{s}\right)^{1+s}, \qquad v=\left(\frac{s+1}{s}\right)^{S}$$

with s a positive integer. Substituting in (9) and multiplying both sides by s^{1+s}, we get

$$k_1s(s+1)^{s}-k_2(s+1)^{s+1}=Ls^{1+s}. \tag{10}$$

If s were *even* the left side of (10) would be an odd integer and the right side an

even integer. Hence, if s is even, no value of $(-u)^{-v}$ is equal to a value of $(-v)^{-u}$. If s is *odd* however, we shall show that for *every* odd k_1 we can find an odd k_2 and an even L so that (10) holds. For the congruence $x(s+1) \equiv k_1 \pmod{s^s}$ is solvable since $(s+1, s^s)=1$. If the integer x_1 satisfies the congruence so will x_1+s^s. Let k_2' be the odd integer of the pair x_1, x_1+s^s and let $k_2=k_2's$. Then

$$k_1 s - (k_2' s)(s+1) = L' s^{s+1}$$

and

$$k_1 s(s+1)^s - k_2(s+1) = [L'(s+1)^s] s^{1+s}$$

and $L=L'[s+1]^s$ is even. If s is odd therefore, *every* value of $(-u)^{-v}$ is a value of $(-v)^{-u}$. One of these values will be positive as we see by setting $k_1=s^s$.

3. Finally we consider rational solutions where m and n have opposite signs, or we consider the equation $u^{-v}=(-v)^u$ with $u>0$, $v>0$. We shall show that this equation has no solution in positive rationals. Taking absolute values of both members, we must have $u^v \cdot v^u = 1$. We may assume that $u>1>v$, and letting $u=a/b$, $v=c/d$ we get $a^{bc}=d^{ad}$ and proceeding as before $a=r^\lambda$, $d=r^\mu$, $b=s^\lambda$, $c=s^\mu$, $\lambda/\mu = r^{\lambda+\mu}/s^{\lambda+\mu}$, $\lambda = r^{\lambda+\mu}$. But $r^{\lambda+\mu} > 2^\lambda > \lambda$.

4. We have established the following theorem:

All the rational solutions of $m^n=n^m$ with $|m|>|n|$ are given by

$$(1) \qquad m = \left(1+\frac{1}{s}\right)^{1+s}, \qquad n = \left(1+\frac{1}{s}\right)^s$$

with s a positive integer, and

$$(2) \qquad m = -\left(1+\frac{1}{s}\right)^{1+s}, \qquad n = -\left(1+\frac{1}{s}\right)^s$$

with s a positive odd integer.

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THE 2Ω PROPERTY OF TORSION-FREE ABELIAN GROUPS

R. C. ENTRINGER, University of New Mexico

A group G will be said to be a 2Ω-group if it has the following property: if A and B are finite nonempty subsets of G with $|A|+|B|\geq 3$, then there exist at least two g in G having a unique representation $g=a+b$, $a\in A$, $b\in B$. In other words at least two entries in the list of sums $a+b$, $a\in A$, $b\in B$ occur once only.

It is conjectured in [2] that every torsion-free group is a 2Ω-group. A similar conjecture appears in [3]. As remarked in [2], the theorem that all torsion-free abelian groups are 2Ω-groups follows immediately from the property [1] that all torsion-free abelian groups are ordered groups. It is also shown in [2] that the conjecture is true if in the definition of a 2Ω-group either A or B is required to be abelian but the proof again uses the order property of torsion-free abelian groups. The first mentioned theorem may be proved directly, however, without appeal to the axiom of choice. We prove the following equivalent form.

THEOREM. *If there exist two nonempty finite subsets A and B of an abelian group G such that $|A|+|B|\geq 3$ and, with at most one exception, each entry in the list of sums $a+b$, $a\in A$, $b\in B$ occurs at least twice, then G is not torsion-free.*

Proof. We choose notation so that $A=\{a_1, \cdots, a_n\}$, $n\geq 2$, and if some $a+b$ occurs once only in the list of sums then $a=a_n$.

For each a_i $(i=1, \cdots, n-1)$ we form a finite set of equations as follows. Choose any member b_{i1} of B. We may express $a_i+b_{i1}=a_{i2}+b_{i2}$ for some $b_{i2}\in B$, $b_{i2}\neq b_{i1}$. Then $a_i+b_{i2}=a_{i3}+b_{i3}$ for some $b_{i3}\in B$, $b_{i3}\neq b_{i2}$. Since B is finite we obtain a $b_{is}=b_{ir}$ for some $r<s$ after a finite number of repetitions of this process. If we now add the set of equations

$$\begin{aligned}
a_i + b_{ir} &= a_{i,r+1} + b_{i,r+1}\\
a_i + b_{i,r+1} &= a_{i,r+2} + b_{i,r+2}\\
&\cdots\cdots\cdots\cdots\\
a_i + b_{i,s-2} &= a_{i,s-1} + b_{i,s-1}\\
a_i + b_{i,s-1} &= a_{is} + b_{ir},
\end{aligned}$$

we obtain $(s-r)a_i=\sum_{j=1}^{s-r} a_{i,r+j}$. Since we may produce an equation of this form for each a_i $(i=1, \cdots, n-1)$ we obtain the system

$$\sum_{j=1}^{n} c_{ij}a_j = 0, \qquad i = 1, \cdots, n-1, \tag{1}$$

where

(i) $\sum_{j=1}^{n} c_{ij}=0$, $i=1, \cdots, n-1$

(ii) $c_{ij}\geq 0$, $i\neq j$

(iii) $c_{ii}<0$, $i=1, \cdots, n-1$.

We now show by induction on n that the existence of such a system implies the existence of an element of finite order in G.

If $n=2$ then (1) takes the form $c_{11}a_1+c_{12}a_2=0$, where $c_{11}=-c_{12}\neq 0$. Hence a_1-a_2 is of finite order. If $n>2$ then from (1) we may obtain the equivalent system

$$\sum_{j=1}^{n} c_{1j}a_j = 0, \qquad \sum_{j=1}^{n} (c_{i1}c_{1j} - c_{11}c_{ij})a_j = 0, \qquad i = 2, \cdots, n-1,$$

where upon setting $d_{ij}=c_{i1}c_{1j}-c_{11}c_{ij}$ the last $n-1$ equations form a system

$$\sum_{j=2}^{n} d_{ij}a_j = 0, \qquad i = 2, \cdots, n-1, \tag{1'}$$

where

(i') $\sum_{j=2}^{n} d_{ij}=0$, $i=2, \cdots, n-1$;

(ii') $d_{ij}\geqq 0$, $i\neq j$;

(iii') either (a) $d_{ii}<0$, $i=2, \cdots, n-1$ and by the induction assumption G contains an element of finite order, or (b) some $d_{kk}\geqq 0$ and hence by (i') and (ii'), $d_{kj}=0$ for $j=2, \cdots, n$.

Now $c_{k1}=0$ gives $d_{kk}=-c_{11}c_{kk}<0$ which is impossible and we may conclude that $c_{1j}=0$ for $j=2, \cdots, k-1, k+1, \cdots, n$. Thus from (i) and (iii) we have $c_{11}=-c_{1k}\neq 0$. In this case a_1-a_k is of finite order.

This completes the inductive step and the theorem follows.

The author wishes to thank the referee for his informative suggestions.

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CLASSROOM NOTES

ON COWEN'S NOTE "AN ELEMENTARY FIXED POINT THEOREM"

J. G. BÁRON, Rye, New York

Cowen considers in [1] a figure formed by a closed plane curve, and a similar figure, reduced in size, placed inside the original figure. He gives an elementary proof of the existence of a fixed point. A shorter elementary proof of this theorem is given below, and it is shown that the fixed point can be constructed by straight-edge and compass.

Any given position of the smaller figure can be considered as a result of a certain similarity transformation of the plane. In such a transformation a line through any two points, and a line through the corresponding points form a constant angle, the angle of rotation of the transformation. It is shown in ele-

mentary coordinate geometry that a similarity transformation always has a fixed point. It is the center of rotation of the transformation from which any two corresponding points are seen under the angle of rotation of the transformation.

In order to complete the proof of the theorem one has to prove that, under the conditions of the problem, the fixed point is inside of the figures. Suppose that the fixed point Q were outside of the figures.

Draw from Q two tangents to the original curve with points of tangency A_1 and B_1 in such a way that all the other points of the figure should be inside of the angle A_1QB_1. Let A_2 and B_2 be the points corresponding to A_1 and B_1. Then $\sphericalangle A_1QA_2 = \sphericalangle B_1QB_2$ both in magnitude and in sense. In other words, rotation of A_1Q toward A_2Q about Q must have the same direction as rotation of B_1Q toward B_2Q. However, this would put the ray QB_2 outside of $\sphericalangle A_1QB_1$, and therefore B_2 outside of the original figure, contrary to the assumption of the problem.

In order to construct Q select any A_1 and B_1 points of the original figure and let A_2 and B_2 be their corresponding points. Suppose first that the lines through A_1B_1 and A_2B_2 are not parallel; let N be their point of intersection. Then $\sphericalangle A_1NA_2$, $\sphericalangle A_1QA_2$ and $\sphericalangle B_1QB_2$ are all equal to the angle of rotation. Therefore the circumcircles of triangles A_1NA_2 and B_1NB_2 must intersect (besides N) in Q. On the other hand, if $A_1B_1 \| A_2B_2$, then the angle of rotation is zero, the circumcircles become straight lines through A_1A_2 and B_1B_2, and Q is their intersection. The lines A_1A_2 and B_1B_2 necessarily intersect in this case; were they also parallel, then the two figures would be congruent, contrary to the assumption of the problem.

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UNIQUENESS OF THE POLAR DECOMPOSITION

ALVIN HAUSNER, City College of New York

If τ is a non-singular linear operator on a finite dimensional complex inner-product space $\mathcal{S}$, then $\tau = \sigma\eta$ where σ is a positive definite (hermitian) operator and η is unitary. (We recall that a hermitian operator is called *positive definite* if all its eigenvalues are strictly positive numbers.) This so-called *polar decomposition* of τ is unique in the sense that $\sigma_1\eta_1 = \sigma_2\eta_2$, with positive definite σ_1, σ_2 and unitary η_1, η_2, implies $\sigma_1 = \sigma_2$ and $\eta_1 = \eta_2$. The existence of the polar decomposition is easily established by defining σ to be a positive definite square root of $\tau\tau^*$, i.e. $\sigma^2 = \tau\tau^*$, and by defining $\eta = \sigma^{-1}\tau$. The uniqueness of the decomposition is most often proved by showing that a positive definite operator has a unique positive definite square root (see [3, p. 169], [4, p. 279], [5, p. 188]. In [1, pp. 14–15] uniqueness is established by proving that the exponential of

a matrix gives a one-to-one mapping from the set of all hermitian matrices onto the set of all positive definite hermitian matrices.

In this note we present a new proof of the uniqueness of the polar decomposition. All we require is the existence of *at least one* positive definite square root of a positive definite operator. The latter fact is immediate from the diagonability of a given positive definite operator. We will denote by $\sigma^{1/2}$ a fixed positive definite square root of the positive definite operator σ.

As for the following lemma, compare the theorem and exercise in [2, p. 114].

LEMMA. *If σ_1 and σ_2 are positive definite operators on $\mathcal{S}$, then $\sigma_1\sigma_2$ has only positive eigenvalues.*

Proof. The operator $\sigma_1^{1/2}\sigma_2\sigma_1^{1/2}$ is evidently nonsingular and it is hermitian, for $(\sigma_1^{1/2}\sigma_2\sigma_1^{1/2})^* = (\sigma_1^{1/2})^*\sigma_2^*(\sigma_1^{1/2})^* = \sigma_1^{1/2}\sigma_2\sigma_1^{1/2}$. Further, for any non-zero vector X in $\mathcal{S}$, we have

$$(A) \qquad (\sigma_1^{1/2}\sigma_2\sigma_1^{1/2}X, X) = (\sigma_2(\sigma_1^{1/2}X), \sigma_1^{1/2}X) > 0$$

since $\sigma_1^{1/2}X \neq 0$ and σ_2 is positive definite. Thus, if $\sigma_1^{1/2}\sigma_2\sigma_1^{1/2}X_0 = aX_0$ for some scalar $a \neq 0$ and vector $X_0 \neq 0$ in $\mathcal{S}$, then

$$(\sigma_1^{1/2}\sigma_2\sigma_1^{1/2}X_0, X_0) = (aX_0, X_0) = a(X_0, X_0) > 0$$

by (A), and this means $a > 0$.

To complete the proof, observe that the eigenvalues of $\sigma_1\sigma_2$ are the same as those of $\sigma_1^{1/2}\sigma_2\sigma_1^{1/2}$, for $(\sigma_1^{1/2})^{-1}(\sigma_1\sigma_2)\sigma_1^{1/2} = \sigma_1^{1/2}\sigma_2\sigma_1^{1/2}$.

THEOREM. *If $\sigma_1\eta_1 = \sigma_2\eta_2$ with positive definite σ_1, σ_2 and unitary η_1, η_2, then $\sigma_1 = \sigma_2$ and $\eta_1 = \eta_2$.*

Proof. The lemma applies to $\sigma_2^{-1}\sigma_1$ which thus has positive eigenvalues. The operator $\eta_2\eta_1^{-1}$ is a unitary map and hence all its eigenvalues have unit modulus. Since $\eta_2\eta_1^{-1} = \sigma_2^{-1}\sigma_1$, the eigenvalues of $\eta_2\eta_1^{-1}$ are all positive and therefore $\eta_2\eta_1^{-1}$ has **1** as its only eigenvalue. Thus $\eta_2\eta_1^{-1} = I$, the identity mapping, and $\eta_1 = \eta_2$. It follows immediately that $\sigma_1 = \sigma_2$.

We conclude by remarking that there is a polar decomposition in real vector spaces. Any nonsingular linear mapping τ can be uniquely written as $\sigma\eta$ with nonsingular positive definite symmetric σ and orthogonal η. The proof follows from the above by simply substituting the words "symmetric" for "hermitian" and "orthogonal" for "unitary."

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ON THE ARITHMETIC MEAN-GEOMETRIC MEAN INEQUALITY

FERGUS GAINES, California Institute of Technology

The purpose of this note is to derive the arithmetic mean-geometric mean inequality from the following well-known matrix theorem of I. Schur [1].

THEOREM. *Let $X=(x_{ij})$ be an $n\times n$ matrix of complex numbers with characteristic roots $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then*

$$\sum_{i,j=1}^{n} |x_{ij}|^2 \geqq \sum_{i=1}^{n} |\lambda_i|^2$$

with equality if and only if X is normal.

Let

$$A = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $a_i \geqq 0$, $i=1, 2, \cdots, n$. The characteristic roots of A are the nth roots of the product $a_1a_2\cdots a_n$, this is easily seen by simply calculating the characteristic polynomial of A. We now apply Schur's theorem to A. Thus

$$\sum_{i=1}^{n} a_i^2 \geqq n(a_1a_2\cdots a_n)^{2/n}$$

with equality if and only if A is normal, i.e. $AA^*=A^*A$ where A^* is the conjugate transpose of A. This means we get equality if and only if

$$\operatorname{diag}(a_1^2, a_2^2, \cdots, a_n^2) = \operatorname{diag}(a_n^2, a_1^2, \cdots, a_{n-1}^2),$$

i.e. if and only if $a_1=a_2=\cdots=a_n$.

If we let $a_i=x_i^{1/2}$, $i=1, 2, \cdots, n$, we get

$$\left(\sum_{i=1}^{n} x_i\right) \Big/ n \geqq (x_1x_2\cdots x_n)^{1/n},$$

where $x_i \geqq 0$, $i=1, 2, \cdots, n$, with equality if and only if $x_1=x_2=\cdots=x_n$, which is the arithmetic mean-geometric mean inequality.

REMARK. Schur's theorem can be used to generate inequalities once we are given a matrix and all its characteristic roots. However, since most of the matrices whose characteristic roots are known are normal, the cases considered by the author have almost all resulted in trivial or well-known identities.

This work was supported by a grant from the National Science Foundation.

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THE STRUCTURE OF FUNDAMENTAL MATRICES IN THE NEIGHBORHOOD OF A SINGULARITY OF THE FIRST KIND

D. B. HILL, University of Vermont

The purpose of this note is to provide a new proof the standard theorem on the behavior of solution matrices in the vicinity of a first order pole. The proof is believed to be more direct and more easily taught than those previously available.

THEOREM. *Given the linear system* $\dot{U} = (1/z)A(z)U$, *where* $A(z) = \sum_0^\infty A_n z^n$ *is analytic in* $|z| < r$. *Then every fundamental matrix is of the form* $S(z)z^P$ *where* $S(z)$ *is analytic at* $z = 0$ *and* P *is a constant matrix. Further, if the eigenvalues of* A_0 *do not differ by a positive integer,* P *may be chosen to be* A_0.

Proof. If two eigenvalues of A_0 differ by a positive integer, we may reduce the equation to one in which the eigenvalues of the constant term do not differ by a positive integer by repeated application of the following lemma whose easy proof may be found in [1].

LEMMA 1. *Let the distinct characteristic roots of* A_0 *(disregarding their multiplicity) be* $\rho_1, \cdots, \rho_k$. *There exists a matrix function* R *of* z, *nonsingular for* $z \neq 0$ *and linear (hence analytic) in* z, *such that the substitution* $U = RV$ *transforms the equation for* V *into the form* $V = (1/z)\hat{A}(z)V$, *where* $\hat{A}_0$ *has eigenvalues* $\rho_1 - 1$, $\rho_2, \cdots, \rho_k$.

Hence it suffices to prove the theorem when A_0 has eigenvalues $\lambda_1, \cdots, \lambda_k$ such that no two differ by a positive integer. Furthermore, we may clearly assume that A_0 is in Jordan canonical form.

We will first show that there is a formal power series $S(z)$ such that $S(z)z^{A_0}$ formally satisfies the equation. Let $U = Vz^{A_0}$. Then V satisfies:

$$V = \left[\frac{A_0}{z}V - V\frac{A_0}{z}\right] + [A_1 + A_2 z + A_3 z^2 + \cdots]V. \tag{1}$$

Assume $V = \sum_0^\infty S_i z^i$. Then formally $A_0S_0 - S_0A_0 = 0$. Choose $S_0 = I$ which satisfies the equation. Now assume that $S_0, S_1, \cdots, S_{k-1}$ have been determined uniquely. For S_k one has the equation:

$$(kI - A_0)S_k + S_kA_0 = A_1S_{k-1} + A_2S_{k-2} + \cdots + A_k. \tag{2}$$

At this point we will digress from the main line of the proof to introduce an idea which will be extremely useful in what follows. Suppose that A is an $n \times n$ matrix and consider the linear transformation defined on the space of $n \times n$ matrices by $\sigma: X \to AX$. Thinking of the space of $n \times n$ matrices as an n^2 dimen-

sional vector space, by a well-known theorem in linear algebra σ can be represented as an $n^2\times n^2$ matrix denoted by $\tilde{A}$. On the other hand, the map $T: X\to XA$ can also be represented by an $n^2\times n^2$ matrix denoted by $\tilde{A}'$. Using this notation, equation (2) becomes:

$$(k\tilde{I} - \tilde{A}_0 + \tilde{A}_0')S_k = A_1S_{k-1} + \cdots + A_k \quad \text{where } k\tilde{I} - \tilde{A}_0 + \tilde{A}_0' \text{ is an } n^2 \times n^2 \tag{3}$$

matrix and S_k and the right hand side are n^2 vectors. The following lemma is a trivial special case of a theorem due to Hausner [2].

LEMMA 2. *Let X and Y be $n\times n$ matrices having $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$ for their respective eigenvalues. Then the eigenvalues of the $n^2\times n^2$ matrix $\tilde{X}+\tilde{Y}'$ are $\alpha_i+\beta_j$; $i, j=1, \cdots, n$.*

Applying this lemma to $(k\tilde{I}-\tilde{A}_0)+\tilde{A}_0'$ we find that the eigenvalues of this matrix are $k-\lambda_i+\lambda_j$ where the λ_i are the eigenvalues of A_0. ($kI-A_0$ has eigenvalues $k-\lambda_i$ since A_0 is in Jordan form). By assumption $k-\lambda_i+\lambda_j\neq 0$ for all integers k, hence $k\tilde{I}-\tilde{A}_0+\tilde{A}_0'$ is nonsingular and equation (3) has a unique solution S_k. Thus we have shown that it is possible to determine S_i, $i=0, 1, 2, \cdots$, so that $(\sum_{i=0}^{\infty} S_iz^i)z^{A_0}$ satisfies formally the equation $\dot{U}=(1/z)A(z)U$.

We must now show that $\sum_0^\infty S_iz^i$ actually converges in some neighborhood of $z=0$.

LEMMA 3. *If B_k denotes the solution of:*

$$(kI - A_0)B_k + B_kA_0 = C \tag{4}$$

there is a constant M independent of k such that $\|B_k\| \leqq M\|C\|$, where $\|\ \|$ denotes any convenient matrix norm.

Proof of Lemma 3. $B_k=(k\tilde{I}-\tilde{A}_0+\tilde{A}_0')^{-1}C$. Now

$$\det(k\tilde{I} - \tilde{A}_0 + \tilde{A}_0') = \prod_{i,j=1}^{n} (k - \lambda_i + \lambda_j).$$

Each term in $(k\tilde{I}-\tilde{A}_0+\tilde{A}_0')^{-1}$ is the cofactor of an element of $(k\tilde{I}-\tilde{A}_0+\tilde{A}_0')^T$ divided by $\det(k\tilde{I}-\tilde{A}_0+\tilde{A}_0')$. Each cofactor contains a sum of products of at most n^2-1 terms $(k-\lambda_i+\lambda_j)$; hence each term in $(k\tilde{I}-\tilde{A}_0+\tilde{A}_0')^{-1}$ is a sum of terms of the form $\Gamma/\Pi(k-\lambda_i+\lambda_j)$, where Γ is independent of k. But the expression $|k-\lambda_i+\lambda_j|$ has a minimum m, over all i, j and k as is easily seen. Hence each term of the form $\Gamma/\Pi(k-\lambda_i+\lambda_j)$ is maximized (in absolute value) by Γ/m^P where P *is* the number of factors in the product. Hence there is a matrix D independent of k which maximizes $(k\tilde{I}-\tilde{A}_0+\tilde{A}_0')^{-1}$ in norm. Then clearly $\|B_k\| \leqq \|D\|\|C\|$.

Now choose a constant N such that $\|A_i\| \leqq N^i$. (It is clearly possible to do this since $\sum A_iz^i$ converges in $|z|<r$.)

We are now in a position to prove the convergence of $\sum S_i z^i$. We will show that $\|S_i\| \leqq N^i M(M+1)^{i-1}$. For $i=1$, $(1-\tilde{A}_0+\tilde{A}_0')S_1=A_1$ so by Lemma 3, $\|S_1\| \leqq M\|A_1\| \leqq MN$. Now suppose that the statement holds for $i=1, \cdots, n-1$. Then since:

$$
\begin{aligned}
(k\tilde{I} - \tilde{A}_0 + \tilde{A}_0')S_n &= A_1S_{n-1} + \cdots + A_n \\
\|S_n\| &\leqq M\|A_1S_{n-1} + \cdots + A_n\| \\
&\leqq M\{NM(M+1)^{n-2}N^{n-1} + \cdots + N^n\} \\
&= MN^n\{M(M+1)^{n-2} + \cdots + 1\} \\
&= MN^n\{1 + M + M(M+1) + \cdots + M(M+1)^{n-2}\} \\
&= MN^n(1+M) + M^2N^n\left\{\frac{(M+1) - (M+1)^{n-1}}{1-(M+1)}\right\} \\
&= N^nM(M+1)^{n-1}
\end{aligned}
$$

as asserted. Then trivially $\sum S_nZ^n$ converges if $|z|<1/(N(M+1))$. Furthermore $\sum S_nz^n$ is nonsingular at $z=0$ and hence by continuity is nonsingular in some neighborhood of $z=0$.

REMARKS. Since the proof of the theorem clearly hangs on Lemmas 2 and 3, it is probably a good idea in presenting the proof to students to take a matrix A_0 in Jordan form and exhibit the matrices $\tilde{A}_0'$ and $\tilde{A}_0$ and $k\tilde{I}-\tilde{A}_0+\tilde{A}_0'$, since this makes the two lemmas completely obvious.

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BANACH LIMITS

LOUIS SUCHESTON, Ohio State University

The purpose of this note is didactic; we show how Banach limits, together with the identification of their extremal values [5] and a theorem of Lorentz [4] giving conditions for their uniqueness, may be simply introduced to an audience armed only with an old-fashioned version of the Hahn-Banach theorem. An elementary proof of Lorentz's theorem may be found in the book by Goffman and Pedrick [2], the chief difference of our approach being in the use of a simpler expression for the functional p.

DEFINITION. *Banach limits (or Banach-Mazur limits, or generalized limits) are linear functionals L on the space B of bounded sequences of real numbers $(x_n)_{n=0}^{\infty}$, satisfying the conditions:*

$$\text{(i)} \quad L(x_n) \geqq 0 \quad \textit{if} \quad x_n \geqq 0, \qquad n = 0, 1, \cdots,$$

(ii) $L(x_{n+1}) = L(x_n)$,

(iii) $L(1) = 1$.

A word about notation: $L(x_n)$ is of course L at $(x_n) = (x_0, x_1, \cdots)$. $L(x_{n+1})$ is L at $(x_1, x_2, \cdots)$. $L(1)$ is L at $(1, 1, \cdots)$.

THEOREM. (α) *Banach limits exist.* (β) *The maximal value of Banach limits on a sequence* (x_n) *is*

$$M(x_n) \overset{\text{def}}{=} \lim_{n\to\infty} \left(\sup_j \; n^{-1} \sum_{i=0}^{n-1} x_{i+j} \right). \tag{1}$$

(γ) *The minimal value of Banach limits on a sequence* (x_n) *is*

$$\lim_{n\to\infty} \left(\inf_j \; n^{-1} \sum_{i=0}^{n-1} x_{i+j} \right) \tag{2}$$

(δ) *A necessary and sufficient condition in order that all Banach limits on a sequence* (x_n) *agree and equal* s *is that*

$$\lim_{n\to\infty} \; n^{-1} \sum_{i=0}^{n-1} x_{i+j} = s$$

uniformly in j. *Hence on convergent sequences Banach limits agree with limits.*

Proof. First we prove that the limit as $n\to\infty$ in the expression (1) for M exists. (Another proof of this is in [5] incorporated in the proof that M equals the expression used for p in the original argument of Banach, [1], p. 34.) Set

$$c_n = \sup_j \; \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j}.$$

We are to show that $\lim c_n$ exists. While (c_n) is not monotone, for each k, m one has $c_{km} \leqq c_m$. Thus

$$(r + km)c_{r+km} \leqq rc_r + kmc_{km} \leqq rc_r + kmc_m.$$

Dividing by $r+km$ and letting $k\to\infty$ with r, m fixed, we obtain

$$\limsup_{k\to\infty} c_{r+km} \leqq c_m.$$

Since this holds for $r = 1, 2, \cdots, m$, $\limsup c_n \leqq c_m$ for each m, and hence $\limsup c_n \leqq \liminf c_m$ which implies that $\lim c_n$ exists.

THE HAHN-BANACH THEOREM ([1], p. 27). *Let* B *be a linear space and* C *a subspace of* B. *Let* p *be a sublinear functional on* B: $p(x+y) \leqq p(x)+p(y)$ *and if* $a \geqq 0$, $p(ax) = ap(x)$. *Then if* f *is a linear functional on* C *with*

$$f(x) \leqq p(x) \tag{3}$$

for $x \in C$, f *can be extended to* B *so that* (3) *holds for* $x \in B$. *If* $y \in B - C$, *then the*

process of extension may begin by ascribing to $f(y)$ any value in the closed interval with left endpoint

$$\sup_{x\in C}\,[-p(-x-y)-f(x)] \tag{4}$$

and right endpoint

$$\inf_{x\in C}\,[p(x+y)-f(x)]. \tag{5}$$

We let B be the space of bounded sequences $x=(x_n)$ and we let C be the subspace of convergent sequences. The Hahn-Banach theorem is now applied with $p=M$ and $f=\lim$ on C. It is easy to see that if $\lim x_n=s$ then the Cesàro means of x_n converge to s, i.e. $\lim_{n\to\infty} n^{-1}(x_0+\cdots+x_{n-1})=s$ and also $M(x_n)=s$.

Proof of (α). We show that linear functionals $f=L$ obtained on B by the extension procedure satisfy (i), (ii) and (iii). For (iii) this is obvious, since each L coincides with lim on C. (i) follows from

$$L(x_n) = -L(-x_n) \geqq -p(-x_n) = -M(-x_n) \geqq 0$$

if $x_n\geqq 0$ for all n.

Finally, (ii) follows from the "telescoping" property of M

$$|M(x_{n+1}-x_n)| = \left|\lim_n[\sup_j n^{-1}(x_{j+n}-x_j)]\right| \leqq \lim_n 2n^{-1}\sup_j|x_j| = 0,$$

$$|M(x_n-x_{n+1})| = 0,$$

and the relations $L(x_{n+1}-x_n)\leqq M(x_{n+1}-x_n)=0$ and

$$L(x_{n+1}-x_n) = -L(x_n-x_{n+1}) \geqq -M(x_n-x_{n+1}) = 0.$$

REMARK. If instead of M, lim sup x_n were used as p, then the generalized limits thus obtained would not satisfy (ii) because lim sup does not have the "telescoping" property. However, lim sup of Cesàro averages could be used for p, this construction yielding some, but not all, Banach limits.

Proof of (β). We first show that given a sequence (y_n), there is a Banach limit L_0 such that $M(y_n)=L_0(y_n)$. We may assume $(y_n)\in B-C$. We construct L_0 by beginning the extension procedure with (y_n), defining $L_0(y_n)$ by (5):

$$L_0(y_n) = \inf_{(x_n)\in C}\,[M(x_n+y_n)-\lim x_n]$$

which equals $M(y_n)$, because the convergence of (x_n) implies $M(x_n+y_n)=\lim x_n+M(y_n)$. We now show that conversely, for each sequence (y_n) and each Banach limit L, $L(y_n)\leqq M(y_n)$. Note that $\sup_j z_j-z_n\geqq 0$, $n=0,1,\cdots$ implies by (i) and (iii) that always $\sup_j z_j\geqq L(z_n)$. If for a fixed m we let

$$z_n = m^{-1}\sum_{i=0}^{m-1} y_{i+n},$$

we have $L(y_n)=L(z_n)\leqq\sup_j z_j$. On letting $m\to\infty$, we obtain $L(y_n)\leqq M(y_n)$, which completes the proof of (β).

Proof of (γ). Apply (β) to the sequence $(-x_n)$.

Proof of (δ). (δ) is an immediate corollary of (β) together with (γ).

Banach limits, their maximal values, and sequences (called "almost convergent") on which all Banach limits agree have found applications in Ergodic Theory (cf. [5] where also further references are given). In the opposite direction, an interesting application of Ergodic Theory to Banach limits is due to Jerison [3].

In conclusion, we wish to acknowledge our debt to the paper [4] of Lorentz, and to a conversation with Professor Alfred Rényi. The author's work is in part supported by NSF Grant GP-1458.

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A FURTHER EXTENSION OF OLIVIER'S THEOREM

HYMAN GABAI, University of Illinois, UICSM

1. The extension. In [1], Štěpánek derived an extension of Olivier's theorem [2]. In this paper we shall use Štěpánek's method of proof to derive the following extension of Štěpánek's theorem.

THEOREM. *Let* $\sum_{n=1}^{\infty} a_n$ *be a convergent series of real numbers,* $\sum_{n=1}^{\infty} 1/\alpha_n$ *a divergent series of positive numbers, and* (β_n) *a sequence such that* $(\alpha_{n+1}-\alpha_n+\beta_n)$ *is a bounded monotone sequence and one of the following conditions holds*:

(a) $\alpha_n a_n \geqq (\alpha_n+\beta_{n-1})a_{n-1}[n\geqq 2]$ or (b) $\alpha_n a_n \leqq (\alpha_n+\beta_{n-1})a_{n-1}[n\geqq 2]$.

Then $\lim \alpha_n a_n = 0$.

Proof. If the inequality (a) or the inequality (b) is multiplied by -1 it is immediately evident that we need only prove the theorem for the case (a) or for the case (b). We shall prove the theorem under the hypothesis that (a) holds.

Following Štěpánek, we define $r_1=s_1=0$, and for each integer $n\geqq 2$ we define:

$$r_n = \alpha_n a_n \text{ and } s_n = -\sum_{i=1}^{n-1}(\alpha_{i+1}-\alpha_i+\beta_i)a_i.$$

For each $n\geqq 1$ we define $h_n=r_n+s_n$.

Since $\sum_{n=1}^{\infty} a_n$ converges and $(\alpha_{n+1}-\alpha_n+\beta_n)$ is bounded and monotone,

it follows by Abel's test (see, for example, [3], page 314) that the series $\sum_{n=1}^{\infty}(\alpha_{n+1}-\alpha_n+\beta_n)a_n$ converges. Thus (s_n) is a convergent sequence.

Now, still following Štěpánek's methods, we note that the condition (a) may be written in the following form: $\alpha_n a_n \geqq \alpha_{n-1}a_{n-1}+(\alpha_n-\alpha_{n-1}+\beta_{n-1})a_{n-1}$. Therefore,

$$h_n = r_n + s_n = \alpha_n a_n - \sum_{i=1}^{n-1}(\alpha_{i+1}-\alpha_i+\beta_i)a_i$$

$$\geqq \alpha_{n-1}a_{n-1} - \sum_{i=1}^{n-2}(\alpha_{i+1}-\alpha_i+\beta_i)a_i = h_{n-1}.$$

Thus, (h_n) is monotone non-decreasing. Since $r_n = h_n - s_n$, with (h_n) an increasing sequence and (s_n) a convergent sequence, it follows that either (r_n) diverges to infinity or (r_n) converges.

If (r_n) diverges to infinity, then for some integer N,

$$n > N \Rightarrow a_n > 1/\alpha_n$$

and, since $\sum_{n=1}^{\infty} 1/\alpha_n$ is a divergent series of positive terms, it follows that $\sum_{n=1}^{\infty} a_n$ is a divergent series, contrary to our hypothesis. A similar argument shows that if (r_n) converges but does not converge to 0 then $\sum_{n=1}^{\infty} a_n$ diverges. Hence (r_n) converges to 0, and the proof is complete.

2. Applications. Štěpánek's theorem is obtained from the above theorem by replacing α_n by n, and β_n by a constant β. Other applications may be obtained by setting $\alpha_n = n^p [0<p<1]$, or by setting $\alpha_n = \log_k n$ (where k is any positive integer, and $\log_k n$ is the kth iterated logarithm of n). (See Corollary (1) and (2), below.)

Another application may be obtained by setting: $\alpha_n = n \log n \log_2 n \cdots \log_k n$ and

$$\beta_n = -\log(n+1)\log_2(n+1)\cdots\log_k(n+1) - \log_2(n+1)\cdots\log_k(n+1) - \cdots - \log_k(n+1).$$

For every positive integer k, the series $\sum_{n=1}^{\infty} 1/\alpha_n$ diverges. An inductive argument shows that for every positive integer k the sequence $(\alpha_{n+1}-\alpha_n+\beta_n)$ is monotone increasing and converges to 1. (See Corollary (3), below.)

Some of these applications are listed in the following

COROLLARY. *If* $\sum_{n=1}^{\infty} a_n$ *is a convergent series such that, for all sufficiently large* n:

(1) $a_n \leqq (1+n^{-p})a_{n-1}$ $[0 \leqq p \leqq 1]$, *then* $\lim n^p a_n = 0$;

(2) $a_n \leqq (1+(\log_k n)^{-1})a_{n-1}$, *then* $\lim (\log_k n)a_n = 0$;

(3) $a_n \leqq (1-n^{-1}-(n \log n)^{-1}-\cdots-(n \log n \cdots \log_{k-1} n)^{-1})a_{n-1}$, *then* $\lim(n \log n \log_2 n \cdots \log_k n)a_n = 0$.

(The inequalities (1), (2), and (3) may be reversed, and the same conclusions hold.)

3. Remarks. In the hypothesis of our theorem, we may replace the condition:

$$(\alpha_{n+1} - \alpha_n + \beta_n) \textit{ is a bounded monotone sequence,}$$

by the condition:

$$\sum_{n=1}^{\infty} | \alpha_{n+2} - 2\alpha_{n+1} + \alpha_n + \beta_{n+1} - \beta_n | \textit{ is a convergent series.}$$

The proof would require the application of the Dedekind test (see [3], page 315) in place of the Abel test.

We may also vary the hypothesis of the theorem by placing appropriate restrictions on the sequences $(\alpha_{n+1}-\alpha_n)$ and (β_n) individually. Thus, for example, we may state the theorem with the conditions that $(\alpha_{n+1}-\alpha_n)$ is a bounded monotone sequence and $\sum_{n=1}^{\infty} |\beta_{n+1}-\beta_n|$ is a convergent series.

The author wishes to thank Professor Herbert E. Vaughan for a number of helpful suggestions.

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2. L. O. Olivier, Remarques sur les séries infinies et leur convergence, Crelle's J. Reine Angew. Math., 2 (1827) 31–44.
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MOTIVATION OF TRUTH TABLES FROM TAUTOLOGIES

P. S. MARCUS, Shimer College and Illinois Institute of Technology

In symbolic logic, a teaching problem is presented by the conventions defining the truth table of the implication formed from two propositions p, q by the binary operator denoted by "$\Rightarrow$." One possible way to motivate these conventions is to derive them from simple tautologies which the student readily accepts on an intuitive basis. Three tautologies which can be used for this purpose are:

1. $p \Rightarrow p$ or any proposition implies itself.
2. $(p \wedge q) \Rightarrow p$ or any proposition in conjunction with another proposition still implies itself.
3. $\sim(p \wedge \sim p)$ or any contradiction is false.

From these tautologies one may derive the truth table for $p \Rightarrow q$ as follows:

If we grant that $p \Rightarrow p$ is a tautology, then in particular a false proposition implies itself. This gives us one particular example of a true implication with false hypothesis and false conclusion. But since computation of truth values depends only on truth values of components, this one special case is enough to establish the convention that an implication with false hypothesis and false conclusion is always counted as true. From the same tautology and by the same reasoning it follows that an implication with true hypothesis and true conclusion is also counted as true.

Our next step is to use the tautology $(p \wedge q) \Rightarrow p$. If we grant that $(p \wedge q) \Rightarrow p$ is a tautology, then it holds true for any two propositions p, q. In particular, it holds for the two propositions p and $\sim p$. It follows that $(p \wedge \sim p) \Rightarrow p$ is always true. But by the last of our three tautologies, the hypothesis here is always false. Hence, if p is true, we have a true implication with the false hypothesis $p \wedge \sim p$ and the true conclusion p. Again, this particular example is sufficient to establish the convention of counting as true any implication with false hypothesis and true conclusion.

So, in three of the four possible cases of the truth table, an implication is counted as true. Unless an implication is to be regarded as a tautology, the remaining case is false, and this of course is the case where the hypothesis is true and the conclusion is false. This completes the determination of the truth table of an implication $p \Rightarrow q$.

ON THE CONTINUITY OF MONOTONIC FUNCTIONS

A. F. BEARDON, University of Maryland

The following result appears in many well-known texts (e.g. [1] p. 78).

THEOREM 1. *Let f be a real-valued, continuous, strictly increasing function on $[a, b]$. Then f^{-1} is also a continuous, strictly increasing function.*

The topological importance of this result seems to overshadow the following elementary, but more general, result.

THEOREM 2. *Let f be a real valued, strictly increasing function defined on a set D of real numbers. Then f^{-1} is a continuous, strictly increasing function if D is an open, closed, or connected set.*

Proof. Clearly f^{-1} is strictly increasing. Moreover it is easily proved that if x_0 is an interior point of D then f^{-1} is continuous at $f(x_0)$ and thus the result is true if D is open.

If D is connected then it must necessarily be an interval and if c is an endpoint of D which belongs to D it is evident that f^{-1} is continuous at $f(c)$.

Finally, suppose that D is closed. We can extend f to a function f^* whose domain includes all open intervals (a, b) where $a, b \in D$ and $(a, b) \cap D = \varnothing$ by the equation

$$f^*(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \qquad x \in (a, b),$$

and the extended function f^* is strictly increasing on its domain of definition D^*. The assumption that D^* is disconnected leads immediately to a contradiction and hence $(f^*)^{-1}$ is continuous on $f^*(D^*)$. Thus the restriction of $(f^*)^{-1}$ to $f(D)$, namely f^{-1}, is continuous on $f(D)$ and the proof is complete.

The following result is a restatement of Theorem 2.

THEOREM 3. *Let f be a real-valued, strictly increasing function defined on a set D of real numbers. Then f is continuous on D if $f(D)$ is an open, closed, or connected set.*

Finally, we have the following result.

THEOREM 4. *Let f be a real-valued, strictly increasing function defined on a set D of real numbers which is open, closed or connected. Then f maps relatively open subsets of D onto relatively open subsets of $f(D)$.*

Proof. From Theorem 2 we can deduce that f^{-1} is continuous on $f(D)$. Equivalently, ([1] p. 80), $(f^{-1})^{-1}=f$ maps relatively open subsets of D onto relatively open subsets of $f(D)$ and this is the required result.

Reference

1. T. M. Apostol, Mathematical Analysis, Addison-Wesley, Reading, Mass., 1957.

ANOTHER PROOF OF ASCOLI'S THEOREM

ADAM KLEPPNER, University of Maryland

One of the forms of Ascoli's Theorem asserts that a subset A of $C(X)$, where X is a locally compact Hausdorff space, is relatively compact for the topology of uniform convergence on compact sets if and only if A is equicontinuous and, for each $x\in X$, the set $A(x)=\{f(x): f\in A\}$ is bounded. Although the necessity of these conditions is easily established, most proofs of their sufficiency rely on a diagonal argument of Tychonoff's Theorem, or perhaps something more complicated. We shall give here a short proof which avoids these procedures and depends instead on the following fact.

LEMMA. *Let A be a family of bounded functions on the topological space X, suppose that X is a finite union of subsets, $X=\bigcup_1^n N_i$, and for each i let $A_i=A\,|\,N_i=\{f\,|\,N_i: f\in A\}$. Provide A and each A_i with the sup norm. If for some $\epsilon>0$ each A_i contains a finite $(\epsilon/2)$-dense set, then A contains a finite ϵ-dense set.*

To prove this, we provide $\prod_1^n A_i$ with the metric $d((f_i),(g_i))=\max_i \|f_i-g_i\|_i$, where $\|\ \|_i$ is the sup norm on N_i. If D_i is a finite $(\epsilon/2)$-dense set in A_i, then $\prod_1^n D_i$ is a finite $(\epsilon/2)$-dense set in $\prod_1^n A_i$. Thus every subset of $\prod_1^n A_i$ contains a finite set which is at least ϵ-dense. The natural embedding of A into $\prod_1^n A_i$, $f\to(f\,|\,N_1,\cdots,f\,|\,N_n)$, is an isometry so that A contains a finite ϵ-dense set.

To prove the theorem it must be shown that for each compact $K\subset X$ and each $\epsilon>0$, A contains a finite set which is ϵ-dense for the pseudometric of uniform convergence on K. Thus we may restrict everything to K and assume that X is compact. Because A is equicontinuous there exists for each $\epsilon>0$ and each $x\in X$ a compact neighborhood N_x of x such that $|f(y)-f(x)|<\epsilon/6$ for all $y\in N_x$ and $f\in A$. A finite number of these neighborhoods, $N_i=N_{x_i}$, $1\leqq i\leqq n$, cover X. Each

of the sets $A(x_i)$ is bounded; hence there exists for each i a finite number of functions $f_{ij} \in A$ such that the set $\{f_{ij}(x_i)\}$ is $\epsilon/6$-dense in $A(x_i)$. For each i and each $f \in A$ there exists f_{ij} such that $|f(x_i) - f_{ij}(x_i)| < \epsilon/6$. For all $y \in N_i$ we have

$$|f(y) - f_{ij}(y)| \leqq |f(y) - f(x_i)| + |f(x_i) - f_{ij}(x_i)| + |f_{ij}(x_i) - f_{ij}(y)| < \epsilon/2.$$

Thus $A_i = A \mid N_i$ contains a finite set which is $(\epsilon/2)$-dense for the sup norm on A_i, and by the lemma A contains a finite ϵ-dense set.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

Send manuscripts to R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457

SUMMER INSTITUTES FOR COLLEGE TEACHERS

LEONARD GILLMAN, Chairman, CUPM Panel on College Teacher Preparation
DON THOMSEN, Chairman, MAA Committee on Institutes

Among various types of programs for continuing the education of college mathematics teachers, one which has natural logistic advantages is the summer institute. Through its Committee on Institutes, the MAA has been directly involved in conducting several institutes for experienced college professors. Moreover, a study of existing summer programs for college teachers has been an item of high priority on the agenda of CUPM's newly formed Panel on College Teacher Preparation. This Panel has conferred with a number of persons who have directed two or more such programs, and is now planning further study which may lead to some specific recommendations for summer institute courses.

Institutes held in recent years have pursued various objectives:

1. One kind of institute is directed at training teachers of precollege teachers and is particularly concerned with curricular reforms and experiments in the teaching of precollege mathematics.

2. Another provides fundamental instruction for college teachers not prepared for their current or forthcoming teaching assignments. At this kind of institute the subject matter is often undergraduate mathematics presented from a higher standpoint.

3. For teachers who have adequate general background but need special study to introduce a new course, there have been institutes devoted to specific topics in mathematics such as linear algebra, topology, probability and statistics, or applied mathematics.

4. There have been several institutes devoted to recent research with the objective of stimulating active interest in research and encouraging the development of seminars and other scholarly activities at the home institutions of the participants.

Other types of institutes would also serve important purposes. For instance, college teachers could be brought together to collaborate in the preparation of instructional materials, or in the reorganization of courses and curricula to take into account recent trends in mathematics education.

The range of possible programs within each type of institute is wide, and any particular program may combine features of several types. The purposes above are all extremely worth while, and the Panel is convinced that a greatly expanded effort in the direction of summer institutes of all types is urgently needed. In the programs for which we have data, applications have outnumbered acceptances by more than six to one. Even if summer institutes are to reach only ten percent of all college mathematics teachers each summer, the number of institutes will need to be approximately tripled.

Accordingly, the CUPM Panel on College Teacher Preparation and the Association's Committee on Institutes strongly urge that many more institutions submit proposals for summer institutes for college teachers, that supporting agencies extend their support of such institutes to the level just described, and that directors of the institutes publicize their programs as widely as possible to eligible college teachers.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

COLLABORATING EDITORS: J. BARLAZ, Rutgers—The State University; L. CARLITZ, Duke University; HASKELL COHEN, Louisiana State University; H. EVES, University of Maine; M. S. KLAMKIN, Ford Scientific Laboratory; R. C. LYNDON, University of Michigan; and A. WILANSKY, Lehigh University.

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P.O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before July 31, 1967. ***Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.***

E 1965. *Proposed by Hyman Gabai and Steven Szabo, University of Illinois and UICSM*

Given any triangle ABC, let P and Q be the midpoints of sides AB and AC, respectively. Let G and G' be points on the perpendicular bisector of AB such

that $GG'=AC$, P is the midpoint of GG', and segment GP contains no points of the interior of triangle ABC. Let H and H' be points on the perpendicular bisector of AC such that $HH'=AB$, Q is the midpoint of HH', and segment QH contains no points of the interior of triangle ABC. Prove (a) $GH^2+G'H'^2=AB^2+AC^2$ and (b) $G'H^2+GH'^2=BC^2$.

E 1966. *Proposed by M. S. Klamkin, Ford Scientific Laboratory*

Show how to construct a regular tetrahedron if the vertices lie on four given parallel planes.

E 1967. *Proposed by W. L. Blanchard, University of Rochester*

Prove that for any positive integers a, b the set of all $ma+nb$ (m, n positive integers) includes all multiples of (a, b) larger than $[a, b]$. Show also that this is the best possible statement. Note: (a, b) is the greatest common divisor and $[a, b]$ is the least common multiple.

E 1968. *Proposed by Erwin Just, Bronx Community College, N. Y.*

Prove that, for any polynomial $p(x)$ and any integer k, there exist k consecutive values of x such that $p(x)$ is composite.

E 1969. *Proposed by J. M. Quoniam, Saint-Etienne, France*

The centers of the internal squares constructed on the sides BC, CA, AB of a triangle are collinear if $\cot\omega=2$, where ω is the Brocard angle of ABC.

E 1970. *Proposed by R. E. Dowds, State University College of New York, Fredonia*

Define $f(x)=\sin(1/x)$, $x\neq 0$, and $f(0)=0$. Is the function f a derivative? In other words, does there exist a function F such that $F'(x)=f(x)$ for all x, including $x=0$?

E 1971. *Proposed by R. A. Whiteman, Illinois Institute of Technology*

Let $g(x)=a_mx^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0$, where the coefficients are real numbers and $a_m>0$. Define a polynomial $G(x)$ by $G(x)=[g(x)]^2-g'(x)$. Prove that, if $g(x)$ has m distinct real roots, then

(a) for m equal to an odd positive integer, the polynomial $G(x)$ has $m+1$ real roots and $m-1$ complex roots, and

(b) for m equal to an even positive integer, $G(x)$ has m real roots and m complex roots.

E 1972. *Proposed by E. P. Starke, Plainfield, N. J.*

There are n consecutive positive integers whose squares can be separated into two sets having the same sum, if and only if there exist two positive integers r, s, $r>s$, such that $n=r^2-s^2$.

E 1973. *Proposed by H. P. Bieri, Spiez, Switzerland*

Consider triangle ABC and its circumcircle. Let A_1, A_2; B_1, B_2; C_1, C_2 be the trisection points of arcs BC, CA, AB of the circumcircle, arranged so that B, A_1, A_2, C, B_1, B_2, A, C_1, C_2, B appear in this cyclic order on the circumference. Prove that A_1B_2, B_1C_2, C_1A_2 determine the side lines of a triangle homothetic to the Morley triangle of ABC.

E 1974. *Proposed by Richard Harter, Brookline, Mass.*

Let the n variables $x_1, \cdots, x_n$ satisfy a relationship $f(x_1, \cdots, x_n)=0$. Define the nth order determinant A as follows: $A_{ii}=1$, $A_{ij}=\partial x_i/\partial x_j$, $i\neq j$. Prove $A=-(n-2)2^{n-1}$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Notice by the Editors. The lost correspondence regarding Solutions of Elementary Problems E 1835 through E 1894 has at last been located and will be processed as rapidly as possible. The editors are very grateful to contributors who responded to our SOS in the February issue (p. 199) and supplied duplicates of their earlier solutions. However, no further duplicates are now needed.

Integral Triangles with Given Perimeter

E 1825 [1965, 1020]. *Proposed by A. K. Austin, The University, Sheffield, England*

Find the number of noncongruent triangles with integral sides and perimeter n.

Solution by Sidney Spital and Yu Chang, California State Polytechnic College, Pomona. We shorten the notation for two useful quantities: $h=[(n+2)/3]$ and $k=[(n+1)/2]$, where the brackets denote the greatest integer function. From the perimeter length n we attempt to select (nondegenerate) triangles of side lengths a, b, c. Two conditions are needed to assure noncongruence and satisfaction of the triangle inequality: (1) ordering the sides, $c\geqq b\geqq a$ and (2) limiting the largest side c to $n/3\leqq c<n/2$. We begin the enumeration of all such triangles with integral sides by choosing c from $\{h, h+1, \cdots, k-1\}$. Since any selection leaves $(n-c)$ of perimeter remaining, the intermediate side b may be chosen in N_c ways from

$$\{[(n-c+1)/2], [(n-c+1)/2]+1, \cdots, c\}.$$

Hence $N_c=c-[(n-c+1)/2]+1$ which easily converts to

$$N_c=\tfrac{1}{4}(6c-2n+3)+\tfrac{1}{4}(-1)^{n-c}.$$

The desired number of triangles is therefore given by

$$\sum_{c=h}^{k-1} N_c = \sum_{c=h}^{k-1} \tfrac{1}{4}\,(6c - 2n + 3) + \sum_{c=h}^{k-1} \tfrac{1}{4}(-1)^{n-c}.$$

The first sum is the sum of an arithmetic progression and equals $\frac{1}{4}(k-h) \cdot (3k+3h-2n)$. The second sum adds terms which alternate between $\pm\frac{1}{4}$; thus in no case does its absolute value exceed $\frac{1}{4}$. Hence

$$f(n) = \left\{\tfrac{1}{4}(k - h)(3k + 3h - 2n)\right\},$$

where $\{\ \}$ denotes the nearest integer and $f(n)$ is the required number of triangles.

Also solved by Irving Adler, Walter Bluger, J. M. Brooke, L. Carlitz, H. K. Crowder, Jorge Dou (Spain), R. B. Eggleton (Australia), J. M. Elkin, A. R. Erskine, N. J. Fine, D. M. Hancasky, D. C. B. Marsh, Liselotte Miller, Peter Nyikos, Brian O'Haire, D. A. Penner, Walter Penney, L. A. Ringenberg, Simon Vatriquant (Belgium), and the proposer. Several solvers gave the correct result but without any indication of the method of deduction.

Vatriquant obtained the result by using the recurrence relations $f(2m+4)=f(2m+1)$, $f(2m)=f(2m-12)+m-3$. Fine used $f(2n)=f(n)+f(n-1)+f(n-2)+f(n-3)$ together with $f(2m+4)=f(2m+1)$. Miller found the problem and complete solution as no. 30 in Yaglom and Yaglom, *Challenging Mathematical Problems with Elementary Solutions*.

The final result is given in many forms. Dou writes

$$f(n) = \left[\frac{n^2 + 3n + 21 - (-1)^n 3n}{48}\right].$$

Eggleton puts $n=12p+2q$, for $0 \leqq q < 6$ and $p \geqq 0$. Then

$$f(12p + 2q) = 3p^2 + pq + \left[\frac{q^2 - q + 5}{10}\right];$$

$f(n)$ for odd n is obtained from $f(2m+1)=f(2m+4)$. Bluger defines p and q so that $\frac{1}{2}n=6p+q$ if n is even, $\frac{1}{2}(n+3)=6p+q$ if n is odd, where q is so chosen that $|q| \leqq 3$. Then for $q=\pm 3$, $f(n) = 3p^2+pq+1$; in all other cases, $f(n)=3p^2+pq$.

Quadrilateral with Rational Sides and Diagonals

E 1826 [1965, 1020]. *Proposed by A. K. Austin, The University, Sheffield, England*

Does there exist a quadrilateral $ABCD$ such that the sides and diagonals have rational length, but AO, BO, CO, DO are all irrational? O is the intersection of the diagonals.

I. *Solution by D. A. Penner, University of Maryland.* Set up the quadrilateral with vertices (0, 0), $(r, 0)$, (x_1, y_1), (x_2, y_2) respectively. Since all sides and diagonals have rational length, the squares of the lengths are rational, and it is easy to show that x_1, x_2, r, y_1y_2 and y_1^2 are all rational, as well as $\sqrt{(x_1^2+y_1^2)}$. The intersection of the diagonals is found to be

$$\left(\frac{rx_1y_2}{ry_1 + x_1y_2 - x_2y_1}, \frac{ry_1y_2}{ry_1 + x_1y_2 - x_2y_1}\right).$$

The distance of this intersection from the vertex (0, 0) is

$$\frac{ry_2}{ry_1 + x_1y_2 - x_2y_1}\sqrt{(x_1^2 + y_1^2)} = \frac{(y_1y_2)r}{y_1^2r + (y_1y_2)x_1 - y_1^2x_2}\sqrt{(x_1^2 + y_1^2)}.$$

But every term is rational, so AO is rational. Thus a quadrilateral of the requested type is impossible.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* No. If the sides and diagonals of $ABCD$ are rational, then from the Law of Cosines, the cosine of the angle formed by any pair of these lines is rational; hence the sines of the angles have rational squares. From the Law of Sines, OA, OB, OC, OD have rational squares, also $OA+OC$ and $OB+OD$ are rational. For positive values this implies that the segments of the diagonals must also be rational.

Also solved by R. B. Eggleton (Australia), and Michael Goldberg.

Goldberg refers to Dickson, *History of the Theory of Numbers*, vol. 2, p. 217, where there is given the proof due to E. E. Kummer (J. Reine Angew. Math., 37 (1948) 1–20.)

Simultaneous Linear Equations

E 1827 [1965, 1020]. *Proposed by Joseph Arkin, Suffern, N. Y.*

Given an integer $n \geqq 2$ and the system of $n-1$ equations:

$$0 = a_nx_n + \binom{n}{1}x_{n-1} + \binom{n}{2}x_{n-2} + \binom{n}{3}x_{n-3} + \cdots + \binom{n}{n-2}x_2 + 1,$$

$$0 = a_{n-1}x_n + \binom{n-1}{1}x_{n-1} + \binom{n-1}{2}x_{n-2} + \binom{n-1}{3}x_{n-3} + \cdots + \binom{n-1}{n-2}x_2 + 1,$$

$$0 = a_{n-2}x_n + \binom{n-2}{1}x_{n-1} + \binom{n-2}{2}x_{n-2} + \binom{n-2}{3}x_{n-3} + \cdots + 0 + 1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 = a_3x_n + \binom{3}{1}x_{n-1} + \binom{3}{2}x_{n-2} + 0 + \cdots + 0 + 1,$$

$$0 = a_2x_n + \binom{2}{1}x_{n-1} + 0 + 0 + \cdots + 0 + 1,$$

in $n-1$ quantities $x_2, x_3, \cdots, x_n$. Determine the value of the one quantity x_n in terms of the a_j.

Solution by Uri J. Schild, Bar Ilan University, Ramat Gan, Israel. Introducing $m=n-k$ in the well-known formula $\sum_{i=0}^{m-1}\binom{n}{i}B_i=0$ and multiplying each term by $\binom{n}{k}$, we obtain

$$(*) \qquad \binom{n}{k} + \sum_{i=1}^{n-k-1} \binom{n-i}{k}\binom{n}{i} B_i = 0,$$

where the B_i are Bernoulli numbers. Consider the determinant

$$D(c_2, c_3, \cdots, c_n) = \begin{vmatrix} c_n & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-3} & \binom{n}{n-2} \\ c_{n-1} & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-3} & \binom{n-1}{n-2} \\ c_{n-2} & \binom{n-2}{1} & \binom{n-2}{2} & \cdots & \binom{n-2}{n-3} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_3 & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 \\ c_2 & \binom{2}{1} & 0 & \cdots & 0 & 0 \end{vmatrix}.$$

Multiplying row i by $\binom{n}{i-1}B_{i-1}$ and adding to row one, for all $i=2, 3, \cdots, n-1$, we see by (*) that all elements in row one except the first become 0. The nonzero element in row one multiplied by its cofactor C_{11} yields the value of D. Here $C_{11} = (-1)^{n-1}(n-1)!$. By Cramer's rule we therefore obtain for x_n:

$$x_n = \frac{D(-1, -1, \cdots, -1)}{D(a_2, a_3, \cdots, a_n)} = \frac{-\sum_{i=0}^{n-2}\binom{n}{i}B_i}{\sum_{i=0}^{n-2} a_{n-i}\binom{n}{i}B_i} = \frac{nB_{n-1}}{\sum_{i=0}^{n-2} a_{n-i}\binom{n}{i}B_i}.$$

Also solved by Sidney Spital, and the proposer.

Differential Equation

E 1828 [1965, 1021]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Form the differential equation for which the general solution is

$$y = \sum_{k=1}^{n} A_k e^{a_k x},$$

where the constants A_k and a_k are arbitrary.

I. *Editorial Comment.* In the original statement, the problem was misprinted to read $\sum_{k=1}^{\infty} A_k e^{a_k x}$, for which no solution is possible with arbitrary a_k.

David Cohoon of Purdue University offered a solution of the problem in this form provided the a_k are assumed given—the solution being possible only if $|a_k| \to \infty$ as $k \to \infty$:

$$\left\{ \prod_{k=1}^{\infty} \left(1 - \frac{D_x}{a_k}\right) E_k(x, D_x) \right\} y(x) = 0,$$

where

$$E_k(x, D_x) = \exp\left(\frac{D_x}{a_k} + \frac{1}{2}\frac{D_x^2}{a_k^2} + \cdots + \frac{1}{k+1}\frac{D_x^{nk}}{a_k^{nk}}\right)$$

as in the Weierstrass factor theorem.

II. *Solution by J. Barlaz, Rutgers—The State University.* We must eliminate a_k, A_k from the expressions for $y, y', \cdots, y^{(2n)}$. We have

$$y^{(i)} = \sum_{k=1}^{n} a_k^i A_k e^{a_k x} \qquad i = 0, 1, \cdots, n.$$

From the $n+1$ equations we eliminate $A_k e^{a_k x}$, $k = 0, 1, \cdots, n$ and obtain

$$\begin{vmatrix} y & 1 & 1 & \cdots & 1 \\ y' & -a_1 & -a_2 & \cdots & -a_n \\ y'' & (-a_1)^2 & (-a_2)^2 & \cdots & (-a_n)^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y^{(n)} & (-a_1)^n & (-a_2)^n & \cdots & (-a_n)^n \end{vmatrix} = 0.$$

This is linear of the nth order, of the form

$$(*) \qquad c_0 y + c_1 y' + c_2 y'' + \cdots + c_n y^{(n)} = 0,$$

where $c_0, \cdots, c_n$ are arbitrary constants with a single dependency relation. Differentiate (*) n times to eliminate the $n+1$ constants. The final result is

$$\begin{vmatrix} y & y' & \cdots & y^{(n)} \\ y' & y'' & \cdots & y^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ y^{(n)} & y^{(n+1)} & \cdots & y^{(2n)} \end{vmatrix} = 0.$$

Similar solution contributed by D. C. B. Marsh.

A Polynomial Identity

E 1829 [1965, 1021] *Proposed by M. J. Pascual and S. D. Beck, Watervliet Arsenal, N. Y.*

If m is a positive integer and x is arbitrary, establish the following identity:

$$\sum_{k=m}^{2m-1}\binom{2m-1}{k}x^k(1-x)^{2m-1-k}=\sum_{k=m}^{2m-1}\binom{k-1}{m-1}x^m(1-x)^{k-m}.$$

Solution by Leonard Carlitz, Duke University. With change of index after dividing through by $x^m(1-x)^{m-1}$, the stated equality can be written in the form

$$\sum_{k=0}^{m-1}\binom{2m-1}{k+m}\left(\frac{x}{1-x}\right)^k=\sum_{k=0}^{m-1}\binom{k+m-1}{m-1}(1-x)^{k-m+1}.$$

For $y=x/(1-x)$, $1+y=1/(1-x)$, this becomes

$$(*)\qquad \sum_{k=0}^{m-1}\binom{2m-1}{k+m}y^k=\sum_{k=0}^{m-1}\binom{k+m-1}{m-1}(1+y)^{m-k-1}.$$

The right member of (*) is equal to

$$\sum_{k=0}^{m-1}\binom{k+m-1}{m-1}\sum_{j=0}^{m-k-1}\binom{m-k-1}{j}y^j$$

$$=\sum_{j=0}^{m-1}y^j\sum_{k=0}^{m-j-1}\binom{k+m-1}{m-1}\binom{m-k-1}{j}.$$

Since it is known that

$$\sum_{k=0}^{n}\binom{a+k}{a}\binom{b+n-k}{b}=\binom{a+b+n+1}{n},$$

we have

$$\sum_{k=0}^{m-j-1}\binom{k+m-1}{m-1}\binom{m-k-1}{j}=\binom{2m-1}{m-j-1}=\binom{2m-1}{m+j}$$

from which (*) follows at once.

Also solved by J. D. Austin, L. J. Burton, M. A. Bershad, Carey M. Jensen, D. C. B. Marsh, Sidney Spital, K. L. Yocom, and the proposer.

The proposer gives a solution using probability theory. Yocom shows

$$f_{m+1}(x)-f_m(x)=x^m(1-x)^m\binom{2m-1}{m}(2x-1)=g_{m+1}(x)-g_m(x),$$

where $f_m(x)$, $g_m(x)$ are the left, right members of the given identity.

Semigroup Which Is Both Left and Right Regular

E 1830 [1965, 1021]. *Proposed by Michael Gemignani, University of Notre Dame*

Let $\{S,\cdot\}$ be a semigroup such that, given any $a\in S$, x and y can be found in S such that $(xa)a=a$, and $a(ay)=a$. Prove (1) Every element $a\in S$ has at least one two-sided identity, i.e. given a, $\exists e_a$ in S with $e_aa=ae_a=a$; and (2) each

$a \in S$ has at least one two-sided inverse, say b_a, such that $b_a a = a b_a = e_a$, e_a being a two-sided identity for a.

Solution by D. F. Dawson, North Texas State University. Let $a \in S$ and $xaa = a = aay$, S being associative. Let $e_a = xa = xaay = ay$. Then $e_a a = xaa = a = aay = ae_a$. Let $b_a = xxa = xay = ayy$. Then $b_a a = xxaa = xa = e_a = ay = aayy = ab_a$.

Also solved by G. A. Alberts, José Asseo, T. A. Atchison, D. A. Blaeuer, W. A. Bodden, Mary Beth Bridgham, J. L. Brown, Jr., P. L. Chabot, Jon Cole, A. J. D'Aristotle, M. J. DeLeon, R. B. Eggleton (Australia), E. W. Ewing, R. W. Feldman, Neal Felsinger, J. E. Homer, Jr., G. A. Johanson, Erwin Just, U. S. Kahlon (India), H. P. Konen, R. N. Leggett, Jr., C. W. Leininger, Edward Lester, Marijo LeVan, Steve Ligh, C. C. Lindner. J. L. Linnstaedter, L. B. Loyd, W. A. McWorter, D. C. B. Marsh, J. J. Martinez, Jose Mason & C. J. Maxson, Helen Merkel, G. L. Musser, D. E. Myers, D. E. Nixon, F. D. Parker, S. N. Rao & A. M. Vaidya (India), Henry Ricardo, M. D. Richek, Azriel Rosenfeld, M.S.R.K. Sastry, Carol Ann Schiller, P. S. Schnare, A. L. Selman & R. B. McNeil, Ronald Shubert, D. L. Silverman, Al Somayajulu, J. F. Standish, D. R. Stark, R. L. Taylor, Kenneth Tolo, P. R. Winter, and the proposer.

For other properties of such semigroups, Leininger refers to Clifford & Preston, *The Algebraic Theory of Semigroups*, vol. 1, pp. 121–123.

Polynomial over an Embedded Field

E 1831 [1965, 1021]. *Proposed by William Becker, New York University*

Let F and G be fields with G (properly) embedded in F. Let $P(x)$ be a polynomial with coefficients in G such that $x \in G$ implies $P(x) \in G$, and $x \in (F-G)$ implies $P(x) \in (F-G)$. Must $P(x)$ be linear? (Compare E 1642 [1964, 914] and E 1696 [1965, 550].)

I. *Solution by E. W. Ewing, Western Kentucky State College.* No. Let $G = \{0, 1\}$ be the field of integers modulo 2. Now x^2+x+1 is irreducible over G. Let $F = \{0, 1, \alpha, \alpha+1\}$ be the splitting field for x^2+x+1 where $\alpha^2+\alpha+1=0$ or $\alpha^2 = -\alpha-1 = \alpha+1$. Let $P(x) = x^2+1$. It is easy to see that $P(\alpha) = \alpha$ and $P(\alpha+1) = \alpha+1$, and so, $P(x)$ satisfies the conditions of the problem but $P(x)$ is not linear.

II. *Solution by J. T. Lloyd, Lehigh University.* If G denotes the field of algebraic numbers and F denotes the complexes, then any polynomial $P(x)$ over G has the property that $x \in G$ implies $P(x) \in G$ and $x \in (F-G)$ implies $P(x) \in (F-G)$.

Also solved by L. J. Burton, Erwin Just, Marijo LeVan, C. C. Lindner, D. C. B. Marsh, Azriel Rosenfeld, M. S. R. K. Sastry, S. Smith, Jr., Preston Stein, J. E. Wilkins, Jr., and K. L. Yocom.

A Ladder and Wall Problem

E 1832 [1965, 1021]. *Proposed by Alan Sutcliffe, Congleton, Cheshire, England*

There is a well-known problem in which a ladder rests against a wall and just touches a cube having one face in contact with the wall. The lengths of the ladder and an edge of the cube are given, and it is required to find the distance

of the ends of the ladder from the base of the wall. When are all four quantities integers?

Solution by M. S. Demos, Drexel Institute of Technology. Let z be the length of the ladder, e the edge of the cube, and x and y the distances of the ends of the ladder from the base of the wall. Then $x^2+y^2=z^2$ and (from similar triangles) $xy=e(x+y)$ are necessary and sufficient conditions for the given figure. For integral solutions, suppose $(x, y)=d$, $x=ad$, $y=bd$, $(a, b)=1$. Then $dab=e(a+b)$, and $(ab, a+b)=1$. Therefore $d=k(a+b)$ where k is any integer, and it follows that $e=kab$, $x=ak(a+b)$, $y=bk(a+b)$, $z=ck(a+b)$, where a, b, c are any primitive solution of $a^2+b^2=c^2$ $[a=u^2-v^2, b=2uv, c=u^2+v^2, u>v, (u, v)=1, 2\mid uv]$.

Also solved by R. H. Anglin, J. W. Baldwin, W. Bluger, Michael Goldberg. D. M. Hancasky, J. A. H. Hunter, Sam Kravitz, L. I. Lowell, D. C. B. Marsh, Liselotte Miller, Norman Miller, C. S. Ogilvy, J. S. Vigder, and the proposer.

Only Hunter and Kravitz answered the first question of the problem. Using the notation of the above solution we have

$$2x, 2y = e + w \pm \sqrt{(w + e)(w - 3e)},$$

where

$$w = \sqrt{(e^2 + z^2)} = (x^2 + xy + y^2)/(x + y).$$

An Identity in Binomial Coefficients

E 1833 [1965, 1022]. *Proposed by Alan Sutcliffe, Congleton, Cheshire, England*

If $m=\min(r, s)$ and $M=\max(r, s)$, prove that

$$\sum_{r=0}^{N}\sum_{s=0}^{N}\frac{(N+m)!}{r!s!(N-M)!}=\sum_{r=0}^{N}\frac{(N+r+1)!}{r!(r+1)!(N-r)!}.$$

Solution by David Zeitlin, Minneapolis, Minn. Rewrite the proposed identity as

$$\sum_{r=0}^{N}\sum_{s=0}^{N}\binom{N}{M}\binom{N+m}{m}=\sum_{r=0}^{N}\binom{N+1}{r+1}\binom{N+r+1}{r}.$$

We will use several well-known and easily proved relations on binomial coefficients:

$$\binom{N}{r}+\binom{N}{r+1}=\binom{N+1}{r+1}, \quad \sum_{s=0}^{r}\binom{N+s}{s}=\binom{N+r+1}{r},$$

$$\sum_{r=0}^{N}\binom{N+r}{r}\sum_{s=r+1}^{N}\binom{N}{s}=\sum_{r=0}^{N}\binom{N}{r+1}\sum_{s=0}^{r}\binom{N+s}{s}$$

$$=\sum_{r=0}^{N}\binom{N}{r+1}\binom{N+r+1}{r}.$$

Using the above and replacing m, M by their values, the desired result is proved

in the following steps:

$$\sum_{r=0}^{N}\binom{N}{r}\sum_{s=0}^{r}\binom{N+s}{s}+\sum_{r=0}^{N}\binom{N+r}{r}\sum_{s=r+1}^{N}\binom{N}{s}$$
$$=\sum_{r=0}^{N}\binom{N}{r}\binom{N+r+1}{r}+\sum_{r=0}^{N}\binom{N}{r+1}\binom{N+r+1}{r}$$
$$=\sum_{r=0}^{N}\binom{N+1}{r+1}\binom{N+r+1}{r}.$$

Also solved by L. Carlitz, P. A. Scheinok, and the proposer.

An Instance of a Ramanujan Result

E 1834 [1965, 1022]. *Proposed by Douglas Lind, University of Virginia*

Prove that

$$\pi=\sqrt{\{r+r'+2\sqrt{[2r+r'+3\sqrt{(3r+r'+\cdots)}]}\}},$$

where $r=\pi-3$ and $r'=r^2+3r+1$.

Solution by J. L. Standig, Alexandria, Va. With $r=\pi-3$ and $r+r'=\pi^2-2\pi-2$, we note that

$$\pi+n+1\equiv\frac{(\pi+n)^2-[r(n+1)+r']}{n+2}$$

for $n=0, 1, \cdots$, so that

$$\pi+n=\sqrt{\{r(n+1)+r'+(n+2)(\pi+n+1)\}}. \tag{*}$$

Now, for $n=0$, $\pi=\sqrt{\{r+r'+2(\pi+1)\}}$. In this, replace $\pi+1$ by its value from (*) with $n=1$ to get

$$\pi=\sqrt{\{r+r'+2\sqrt{[2r+r'+3(\pi+2)]}\}}.$$

We now replace $\pi+2$ using (*) and repeat as often as we please, to obtain the desired result.

Also solved by R. O. Davies & A. Weinmann (England), P. G. Kirmser, and the proposer.

Editorial Note. The proposer refers to Ramanujan, *Collected Papers*, Cambridge, 1927, p. 323:

$$n(n+k)=n\sqrt{\{(k-2)n+k''+(n+1)\sqrt{[(k-2)(n+1)+k''}}$$
$$+(n+2)\sqrt{\{(k-2)(n+2)+k''+\cdots\}]\}},$$

where $k''=k^2-k-1$, and this is the proposed relation with $k=\pi-1$, $n=1$, $r=k-2$. Ramanujan, however, gives only formal justification (similar to that provided in the above solution).

Davies and Weinmann offer the following proof that the successive replacements for $\pi+1$, $\pi+2$, etc., do indeed give a convergent sequence whose limit is π. Let

$$T_n=\sqrt{\{r+r'+2\sqrt{[2r+r'+3\sqrt{\{\cdots+(n-1)\sqrt{[(n-1)r+r'+n\sqrt{(nr+r'+n+1)}]\}}]\}}}.$$

We have from Standig's solution:

$$\pi = \sqrt{\{r + r' + 2\sqrt{[2r + r' + 3\sqrt{\{ \cdots + (n-1)\sqrt{[(n-1)r + r' + n\sqrt{\{nr + r' + (n+1)(\pi + n)\}}]}\}}]}}.$$

Therefore $T_n < \pi$ for all n. On the other hand,

$$\begin{aligned}(n+\pi)^{2^{-n}} T_n &= (n+\pi)^{2^{-n}} \sqrt{\{r + r' + 2\sqrt{[\cdots + (n-1)\sqrt{\{(n-1)r + r + n\sqrt{(nr + r' + n + 1)}\}}]}\}} \\ &= \sqrt{\{(n+\pi)^{2^{-n+1}}(r + r') + 2\sqrt{[(n+\pi)^{2^{-n+2}}(2r + r') + 3\sqrt{\{\cdots + (n-1)\sqrt{[(n+\pi)^{2^{-1}}((n-1)r + r') + n\sqrt{(n+\pi)(nr + r' + n + 1)}]}\}}]}\}} \\ &\geqq \sqrt{\{r + r' + 2\sqrt{[\cdots (n-1)\sqrt{\{(n-1)r + r' + n\sqrt{[nr + r' + (n+1)(n+\pi)]}\}}]}\}} = \pi.\end{aligned}$$

Thus $T_n \geqq \pi/(n+\pi)^{2^{-n}} \to \pi$ as $n \to \infty$. Thus $T_n \to \pi$ as required.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before September 30, 1967. ***Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.***

5470. *Proposed by Edwin Hewitt, University of Washington*

Let $(X, \mathfrak{A}, \mu)$ be a measure space such that if $A \in \mathfrak{A}$ and $\mu(A) = \infty$, then there is a set $B \in \mathfrak{A}$ such that $B \subset A$ and $0 < \mu(B) < \infty$. Let p be a real number such that $p > 1$ and let f be a complex-valued $\mathfrak{A}$-measurable function on X such that $f \notin \mathfrak{L}_p(X, \mathfrak{A}, \mu)$. Let $p' = p/(p-1)$. Prove that the set $\{\phi \in \mathfrak{L}_{p'}(X, \mathfrak{A}, \mu) : f\bar{\phi} \in \mathfrak{L}_1, \int_X f\bar{\phi}\, d\mu = 0\}$ is dense in $\mathfrak{L}_{p'}(X, \mathfrak{A}, \mu)$.

5471. *Proposed by W. A. McWorter, University of British Columbia*

Let G be a group and let σ be a nontrivial automorphism of G, such that for each $x \in G$, $\sigma(x) = x$ or $\sigma(x) = x^{-1}$. Prove that G is solvable.

5472. *Proposed by James Duemmel, University of Montana*

Let $J_n(x)$ be the Bessel function of the first kind of order n, $n \geqq 0$. Let $J_n'(\lambda) = 0$, $\lambda > 0$. Prove that $\lambda > n$.

5473. *Proposed by A. A. Goldstein and J. V. Ryff, University of Washington*

Let $F(r) = r^2 + 1$. Let $[,]$ represent an inner product, with $\|x\|^2 = [x, x]$. Prove that for any real number r, and any x, y with $\|x\| \geqq \|y\|$,

$$\|x\|^{2r} + \|y\|^{2r} - 2\|x\|^r\|y\|^r \frac{[x, y]}{\|x\| \cdot \|y\|} \leqq \begin{cases} \|x\|^{2r-2}\|x - y\|^2 F(r), & r \geqq 1, \\ \|y\|^{2r-2}\|x - y\|^2 F(r), & r < 1. \end{cases}$$

5474. *Proposed by Benjamin Volk, Far Rockaway, N. Y.*

Given an integer x and a sequence $\{\lambda_n\}$, $\lambda_0=1$, $\lambda_k\neq 0$. Define the $\{\lambda_n\}$ kth power of x, $x^{(k)}_{\{\lambda_n\}}=x^{(k)}$ by

$$0^{(k)}=\begin{cases}1, & k=0,\\ 0, & k>0,\end{cases}\qquad \frac{(x+1)^{(k)}}{\lambda_0\lambda_1\cdots\lambda_k}=\sum_{j=0}^{k}\frac{x^{(k-j)}}{\lambda_0\lambda_1\cdots\lambda_{k-j}}\cdot\frac{1^{(j)}}{\lambda_0\lambda_1\cdots\lambda_j}.$$

Then prove

$$\frac{(x+y)^{(k)}}{\lambda_0\lambda_1\cdots\lambda_k}=\sum_{j=0}^{k}\frac{x^{(k-j)}}{\lambda_0\lambda_1\cdots\lambda_{k-j}}\cdot\frac{y^{(j)}}{\lambda_0\lambda_1\cdots\lambda_j}.$$

This generalizes a result in Gloria Olive, *Generalized Powers*, this MONTHLY, 72 (1965) 619–627.

5475. *Proposed by Louis Comtet, Viroflay, France*

Let s be an integer >0, E an at most countable set, and $\mathfrak{V}$ a system of subsets of E which has the property that each member of E is a member of at most s sets of $\mathfrak{V}$. What can be said about Card $\mathfrak{V}$?

5476. *Proposed by W. A. McWorter, University of British Columbia*

Let $I=\{1,\cdots,n\}$. Let T be a subset of $I\times I$. Define M_T to be the set of all $n\times n$ matrices $A=(a_{ij})$ over a field F such that (i,j) in $I\times I\setminus T$ implies $a_{ij}=0$ in F. Characterize all T such that M_T is a subalgebra of the complete $n\times n$ matrix algebra over F.

5477. *Proposed by D. J. Newman, Yeshiva University*

Let $|\alpha|=1$. Show that there is a nontrivial bounded variation function, $f(x)$, on $[0,1]$ which satisfies $\alpha f(x)=f(x/2)+f((x+1)/2)$.

5478. *Proposed by Benjamin Volk, Far Rockaway, N. Y.*

Let $F(z)=\sum_{n=0}^{\infty}a_nz^n$ be analytic for $|z|<1$, and let $\{F(x_n)\}_{n=0}^{\infty}$ form a convergent sequence for some sequence of nonnegative reals $\{x_n\}$ converging to 1. If $\{a_n\}$ lies in some central sector of the closed unit disk of angle less than π, prove that $\sum_{n=0}^{\infty}|a_n|<\infty$.

5479. *Proposed by Fred Richman and John Thomas, New Mexico State University*

Let N be an odd integer. Can a rectangle be dissected into N nonoverlapping triangles, all having the same area?

SOLUTIONS OF ADVANCED PROBLEMS

The Coefficients of $(\sum c_ix^i)^n$

5325 [1965, 913; 1966, 901]. *Proposed by D. J. Newman, Yeshiva University, and W. E. Weissblum, AVCO Research*

Suppose $0 \leqq c_i \leqq 1$ and $\sum c_i = 2$. Prove that all the coefficients of $(\sum c_i x^i)^n$ are $\leqq \binom{n}{q}$, where $q = [n/2]$.

Editorial Note. Solution I, as given, is invalid since $2\binom{n}{q} \geqq \binom{n+1}{q}$. The solver, and the editors, offer apologies.

The First Zero of $\int_1^\infty \cos xt\, d\mu(x)$

5367 [1966, 206]. *Proposed by D. J. Newman, Yeshiva University*

Let $d\mu(x)$ be any finite positive measure and set $\phi(t) = \int_1^\infty \cos xt d\mu(x)$. Prove that $\phi(t)$ must have a zero in $[0, \pi]$. Prove also that π cannot be replaced by any smaller number.

Solution by the proposer. This problem arises in working with "Band Pass" signals. The first part of the problem follows easily from the evaluation of $I = \int_0^\pi \sin t\phi(t)dt$. Reversing integrations, it follows that

$$I = \int_1^\infty \frac{1 + \cos \pi x}{1 - x^2} d\mu(x) \leqq 0.$$

Now $\phi(0) \geqq 0$, and so $\phi(t)$ must vanish in $[0, \pi]$.

To obtain the second part of the problem consider the measure function defined by placing masses $1, \rho, \rho^2, \cdots$ at the points $1, 3+\epsilon, 5+2\epsilon, \cdots$, respectively $(0 < \rho < 1, \epsilon > 0)$. Then

$$\phi(t) = \sum_{k=0}^{\infty} \rho^k \cos t[k(2 + \epsilon) + 1] = \frac{\cos t - \rho \cos(1 + \epsilon)t}{1 + \rho^2 - 2\rho \cos(2 + \epsilon)t}.$$

With ρ chosen close to 1, the first positive zero of the numerator of $\phi(t)$ is pushed close to $2\pi/(2+\epsilon)$ which can be taken arbitrarily close to π.

Lattice Points within Curves of Constant Width

5368 [1966, 206]. *Proposed by Joseph Hammer, University of Sidney, Australia*

A curve of constant width greater than $\pi/(\pi - \sqrt{3})$ contains a lattice point.

Solution by G. D. Chakerian, University of California at Davis. If K is a curve with constant width b, perimeter L, and area A, then $L = \pi b$ (Barbier's theorem), and $A \geqq (\pi - \sqrt{3})b^2/2$ (Blaschke-Lebesgue theorem). Thus, with $b = \pi/(\pi - \sqrt{3})$, we obtain $L/2A \leqq 1$, so by a theorem of E. A. Bender (this MONTHLY, 69 (1962) 742–744), K contains a lattice point.

However, one can do better than this, since any set of constant width b contains a circle of radius $(1 - 1/\sqrt{3})b$ (H. G. Eggleston, *Convexity*, Cambridge 1958, p. 125), and any circle of diameter $> \sqrt{2}$ contains a lattice point. Thus K contains a lattice point if $b \geqq (\sqrt{6} + 3\sqrt{2})/4$. The same kind of argument shows that a body of constant width $b \geqq \{(2n+2)\sqrt{n} + n\sqrt{(2n+2)}\}/2(n+2)$ in Euclidean n-space contains a lattice point. (Note that the constant is asymptotic

to $\sqrt{n}(1+1/\sqrt{2})$.) We offer the question as to what is the smallest value of b for each dimension.

Also solved by G. R. Lyndaker, Andy Vince, and the proposer.

Determinant Type Functions of Infinite Matrices

5369 [1966, 206]. *Proposed by Albert Wilansky, Lehigh University*

Let M be the set of infinite matrices of real numbers each of which is row finite and column finite (i.e., each row and column has finitely many nonzero entries).

Let $f: M \to$ Reals satisfy $f(I)=1$, $f(AB)=f(BA)$.

Show that $\{A: f(A)=1\} \not\subset G$, where G is the set of invertible members of M. (The result is false for finite matrices of fixed order.)

Solution by James Washenberger, Iowa State University. Let the infinite matrices A and B be defined by $A=(\delta_{i+1,j})$ and $B=(\delta_{i,j+1})$ where $\delta_{n,m}$ is the Kronecker delta and $i, j=1, 2, \cdots$. Then $AB=I$ but $BA \neq I$. In fact, since the first column of BA consists entirely of zeros, BA is not a member of G. Let $C=BA$, then $f(C)=f(BA)=f(AB)=f(I)=1$. Thus C is a member of $\{A: f(A)=1\}$ but C is not in G.

Also solved by Roy O. Davies (England), J. T. Rosenbaum, W. R. Scott, G. N. Wollan, and the proposer. Davies observes that BA can never be inverted if $AB=I \neq BA$. Clearly, therefore, the situation in this problem is peculiar to infinite matrices.

Nilpotent Rings

5371 [1966, 311]. *Proposed by W. A. McWorter, University of British Columbia*

Let R be a ring finitely generated by $x_1, \cdots, x_n$. Suppose that every x_i is nilpotent in R and R satisfies the identity $xyx+yxy=0$ for every x and y which are products of the x_i (x and y monomials in the x_i). Prove that R is nilpotent. (R nilpotent means that for some m every product of m elements vanishes.)

Solution by W. G. Leavitt, University of Nebraska. For any j

$$x_j^2 x_j^2 = -x_j^2 x_j x_j^2 = -x_j^5.$$

Thus $x_j^4 = \pm x_j^m$ for any $m \geqq 4$, so, since x_j is nilpotent, $x_j^4=0$. Similarly (for any i, j) $w=x_i(x_jx_ix_j)x_i=-x_jwx_j$. Thus $w=x_j^4wx_j^4=0$ and so $(x_ix_j)^3=0$.

Now, for any i, j, k we have

$$u = x_ix_jx_kx_i = -x_j(x_kx_ix_jx_k) = x_jx_ix_jx_kx_ix_j = x_jux_j.$$

Thus, again, $u=0$. Similarly, for any i, j, k, r,

$$v = x_ix_jx_kx_rx_i = -x_jx_k(x_rx_ix_jx_kx_r) = x_jx_kx_ix_jx_kx_rx_ix_jx_k = x_jx_kvx_jx_k.$$

Thus, since $(x_jx_k)^3=0$, we have $v=0$.

Proceeding similarly we find that $x_iux_i=0$ in all cases in which u is a product of two or more symbols. But in every product of $3n+1$ or more symbols some

symbol, say x_i, must appear at least 4 times. The product must then contain x_i^4 or it must contain x_iux_i where u has at least two symbols. Thus every product of $3n+1$ or more symbols is zero.

Also solved by Anders Bager (Denmark), K. E. Eldridge, Erwin Just, Ka Menehune, F. C. Stewart, G. N. Pandya, and the proposer.

Linear Factors in a Polynomial

5372 [1966, 311]. *Proposed by L. Carlitz, Duke University*

Let p be a prime >2. Show that the polynomial $P(x)=x^{p+1}-\frac{1}{2}x^2-2c$ has linear factors (mod p) if and only if c is a quadratic residue (mod p). Moreover, when linear factors occur each has multiplicity two.

I. *Solution by E. S. Langford, U. S. Naval Postgraduate School.* a is a zero of P iff $a^{p+1}-\frac{1}{2}a^2-2c\equiv\frac{1}{2}a^2-2c\equiv 0$; that is, a is a zero of P iff $a^2\equiv 4c$, which is obviously true iff c is a quadratic residue (mod p).

If $c=0$, then $P(x)=x^2Q(x)$, where $Q(x)=x^{p-1}-\frac{1}{2}$. Q is obviously irreducible, by Fermat's theorem.

Suppose then that $c\not\equiv 0$ is a quadratic residue. Write $c=(a/2)^2$. Then $x=+a$ and $x=-a$ will be the only zeros of P. It can be verified that

$$P(x) = (x-a)^2(x+a)^2Q(x),$$

where $Q(x)=x^{p-3}+2a^2x^{p-5}+3a^4x^{p-7}+\cdots+\frac{1}{2}(p-1)a^{p-3}$. But

$$Q(a) \equiv Q(-a) \equiv (1+2+\cdots+\tfrac{1}{2}(p-1))a^{p-3} \equiv \frac{(p-1)(p+1)a^{p-3}}{8}$$

$$\equiv -\frac{a^{p-3}}{8}.$$

This cannot be zero, so Q is irreducible.

II. *Solution by J. B. Muskat, University of Pittsburgh.* For $P(x)$ we find $P'(x)=(p+1)x^p-x\equiv 0$ and $P''(x)\equiv -1$ for all x, showing that linear factors occur with exact exponent 2.

Also solved by Anders Bager (Denmark), J. L. Brenner, M. S. Demos, H. M. Edgar, M. A. Ettrick, M. G. Greening (Australia), John B. Kelly, Kenneth Kramer, W. A. McWorter, H. F. Mattson, Jr., Ka Menehune, Charles Wells, K. S. Williams, (England), and the proposer.

Wells shows that the result is still valid if p is replaced by p^n and the polynomial is taken over GF(p^n) with c a square in GF(p^n).

Almost every contributor noticed the misprinted sign of the term $2c$ in the original statement [1966, 311]. The result when $P(0)=2c$ does not hold unless $p\equiv 1$ (mod 4).

Irreducible Quadratic Factors over GF(p^n)

5373 [1966, 312]. *Proposed by L. Carlitz, Duke University*

Let p be a prime >2.

(a) Show that $x^{p+1}+ax+b$, where $a\not\equiv 0 \pmod{p}$, has no irreducible quadratic factor (mod p).

(b) Show that $x^{p+1}+ax^2+b$, where $ab\not\equiv 0 \pmod{p}$, has an irreducible quadratic factor (mod p) if and only if $(1-a)b$ is a quadratic nonresidue (mod p).

Solution by Charles Wells, Western Reserve University. Let $q=p^n$, p an odd prime, n a positive integer. We consider the polynomials $f(x)=x^{q+1}+ax+b$ ($a, b\in \mathrm{GF}(q)$, $a\neq 0$) and $g(x)=x^{q+1}+ax^2+b$ ($a, b\in \mathrm{GF}(q)$, $ab\neq 0$).

(a) If f has an irreducible quadratic factor, then it has a root $r\in \mathrm{GF}(q^2)$ which has the property that $r^{q^2}=r$ but $r^q\neq r$. But then $r^{q+1}=-ar-b$, and, raising this to the qth power, $r^{q+1}=-ar^q-b$ since $a^q=a$, $b^q=b$. Then $ar+b=ar^q+b$, $r=r^q$, a contradiction.

(b) If g has an irreducible quadratic factor, then it has a root r as in (a). Then, as in (a), it follows that $ar^{2q}+b=ar^2+b$, so that $r^{2q}=r^2$. Since $r^q\neq r$, this means $r^q=-r$. Hence $-r^2=r^{q+1}$, and since $g(r)=0$, $-r^2+ar^2+b=0$. But this means $r^2=b(1-a)^{-1}$. Thus $b(1-a)^{-1}$, and therefore $b(1-a)$, cannot be a square in $\mathrm{GF}(q)$ since $r\notin \mathrm{GF}(q)$.

Conversely, if $b(1-a)$ is not a square in $\mathrm{GF}(q)$, then we may write $b(1-a)^{-1}=r^2$, $r\in \mathrm{GF}(q^2)$, and $r^q\neq r$. But $r^2\in \mathrm{GF}(q)$, so $r^{2q}=r^2$. Then $(r^{q+1})^2=r^{2q}\cdot r^2=r^4$, so $r^{q+1}=\pm r^2$. Since $r^q\neq r$, $r^{q+1}=-r^2$. Then $g(r)=-r^2+ar^2+b=-b(1-a)^{-1}+ab(1-a)^{-1}+b=-b+b=0$, thus yielding the irreducible quadratic factor x^2-r^2.

Also solved by M. A. Ettrick, M. G. Greening (Australia), Kenneth Kramer, H. F. Mattson, Jr., W. A. McWorter, Ka Menehune, and the proposer.

Sequences almost Linearly Related

5374 [1966, 312]. *Proposed by Benjamin Volk, Far Rockaway, N. Y.*

Under what conditions for $\{a_n\}$, $\{b_n\}$ may we find nontrivial A, B such that $\lim\{Aa_n+Bb_n\}$ exists?

Solution by Robert Goldstein, The Northern Polytechnic, London, England. We first note the trivial cases:

1. If $A=B=0$, then for any pair of sequences $\{a_n\}$, $\{b_n\}$, the sequence $\{Aa_n+Bb_n\}$ converges to zero.

2. If one of A, B may be zero, then $\{Aa_n+Bb_n\}$ converges if one of $\{a_n\}$, $\{b_n\}$ converges (with suitable choice of A, B).

In what follows we require $A\neq 0$, $B\neq 0$. We then prove that the necessary and sufficient condition for the existence of nontrivial A, B such that $\{Aa_n+Bb_n\}$ is convergent, is that there exist P, Q ($P\neq 0$) such that $b_n=Pa_n+Q+\epsilon_n$, where ϵ_n denotes a sequence which tends to zero as $n\to\infty$.

(i) The condition is necessary. Given that $\{Aa_n+Bb_n\}$ converges, we have $Aa_n+Bb_n=c+o(1)$, so $b_n=(-A/B)a_n+(c/B)+o(1)$. Hence $P=-A/B$, $Q=c/B$.

(ii) The condition is sufficient. Let $b_n = Pa_n + Q + o(1)$. Choose $A = -P$, $B = 1$, then $Aa_n + Bb_n = Q + o(1)$ which converges to Q as $n \to \infty$.

It is easily seen that (a) α is a limit point of $\{a_n\}$ if and only if $\beta = P\alpha + Q$ is a limit point of $\{b_n\}$; and (b) $b_n/a_n \sim P$ as $n \to \infty$ through T, where T is an increasing subsequence of the natural numbers such that $a_n \to \infty$, $b_n \to \infty$ as $n \to \infty$ through T.

These results may be used to obtain necessary conditions on $\{a_n\}$ and $\{b_n\}$ to fulfill the requirements of the problem. Thus, (a) yields the result that the linear function $y = Px + Q$ sets up a one-one correspondence between the set of limit points of $\{a_n\}$ and that of $\{b_n\}$. Moreover, $\{a_n\}$ is bounded if and only if $\{b_n\}$ is bounded.

For bounded sequences with a finite number of limit points, we have: Let $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ be the limit points of the bounded sequence $\{a_n\}$ and $\beta_1 < \beta_2 < \cdots < \beta_k$ be the limit points of the bounded sequence $\{b_n\}$. Then the necessary condition for the existence of nontrivial A, B such that $\{Aa_n + Bb_n\}$ converges is that one of the systems of linear equations in P and Q given by

$$\beta_i = P\alpha_i + Q \qquad (i = 1, 2, \cdots, k) \tag{1}$$

$$\beta_{k-i+1} = P\alpha_i + Q \qquad (i = 1, 2, \cdots, k) \tag{2}$$

is consistent.

If $k = 1$, i.e. $\{a_n\}$ and $\{b_n\}$ are both convergent, these conditions become vacuous. Indeed, in this case, for any A, B, $\{Aa_n + Bb_n\}$ converges.

The Sequence $a_{n+1} = a_n(1 - a_n^2)$

5375 [1966, 312]. *Proposed by Benjamin Volk, Far Rockaway, N. Y.*

Given a sequence of real numbers $\{a_n\}$ satisfying $0 < a_1 < 1$, $a_{n+1} = a_n(1 - a_n^2)$. Clearly $a_n \downarrow 0$. For which α is it true that $a_n = O(n^{-\alpha})$? How finely may we estimate the convergence of $\{a_n\}$?

I. *Solution by Sidney Spital and Yu Chang, California State Polytechnic College, Pomona.* In L. M. Milne-Thompson, *Calculus of Finite Differences*, 1960, p. 342, we find the given difference equation treated via Haldene's method. There it is found that

$$a_n^2 = \frac{1}{2n}\left[1 + 3a_n^2 \log a_n + O(a_n^2)\right] \qquad (n \nearrow \infty,\ a_n \searrow 0) \tag{1}$$

which shows that we may write

$$a_n = 1/\sqrt{2n(1 + \epsilon_n)}, \qquad \text{where } \epsilon_n \to 0. \tag{2}$$

Substitution of (2) into (1) gives, after slight rearrangement,

$$2n = 2n(1 + \epsilon_n) - \tfrac{3}{2}(\log n - \log(1 + \epsilon_n)) + O(1).$$

Invoking $\log(1+\epsilon_n)=O(1)$, we find $\epsilon_n=3\log n/4n+O(1/n)$. Hence, after binomial expansion of $1/\sqrt{(1+\epsilon_n)}$,

$$a_n=\frac{1}{\sqrt{2n}}-\frac{3}{8\sqrt{2}}\cdot\frac{\log n}{n^{3/2}}+O\left(\frac{1}{n^{3/2}}\right).$$

II. *Solution by I. N. Baker, Imperial College, London, England.* In Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. de France, 47 (1919) 161–271, Section 11, the general case

$$z_{n+1}=z_n-az_n^{p+1}+\cdots, \qquad a>0,$$

is discussed, and it is shown that for z_1 in a suitable region (e.g., $0<|z|<\delta$, $|\arg z|<\pi/p-\epsilon$) one has $z_n=\{npa+O(n^{1-1/p})\}^{-1/p}$. Thus in our example,

$$a_n=\{2n+O(n^{1/2})\}^{-1/2}$$

for $0<a_1<\delta$ in the first instance, and in fact in $0<a_1<1-\epsilon$ for any ϵ, since $a_n\downarrow 0$.

Also solved by Leopold Flatto, Robert Goldstein (England), Amram Meir, and George Piranian.

A Problem in Convergence of Infinite Series

5376 [1966, 312]. *Proposed by Jürgen Elstrodt, University of Münster, West Germany*

Denote by $\log_p x$ the p-fold iterated natural logarithm of x ($\log_1 x=\log x$, $\log_{p+1}x=\log(\log_p x)$). Abel has shown that the infinite series

$$\sum_{n=N}^{\infty}\frac{1}{n\log n\cdots\log_{p-1}n(\log_p n)^{\alpha}}$$

(N sufficiently great, p a fixed natural number) converges if $\alpha>1$ and diverges if $\alpha\leqq 1$. Now let p increase with n, define $p(n)$ for $n\geqq 3$ by $1<\log_{p(n)}n\leqq e$, and prove that the infinite series

$$\sum_{n=3}^{\infty}\frac{1}{n\log n\cdots\log_{p(n)}n} \tag{1}$$

diverges. Prove also that the infinite series

$$\sum_{n=9}^{\infty}\frac{1}{n\log n\cdots\log_{q(n)-1}n(\log_{q(n)}n)^{\alpha}} \tag{2}$$

with $q(n)=[\frac{1}{2}p(n)]$ converges if $\alpha>1$ and diverges if $\alpha\leqq 1$. (For real x we denote by $[x]$ the unique integer satisfying $[x]\leqq x<[x]+1$.)

Solution by Addison Fischer, Student, Washington Irving High School, Clarksburg, W. Va. First of all, let us introduce the function $\epsilon(p)$ defined in this manner:

$\epsilon(0)=1$; $\epsilon(p+1)=e^{\epsilon(p)}$. Since $\log_p \epsilon(p)=1$ and $\log_p \epsilon(p+1)=e$, then for $\epsilon(p)<n \leqq \epsilon(p+1)$ we have $p(n)=p$. Similarly, for $\epsilon(2p)<n\leqq\epsilon(2p+2)$, $q(n)=p$.

The series (1) converges if and only if the sequence of partial sums

$$\sum_{n=3}^{[\epsilon(k+1)]} \frac{1}{n \log n \cdots \log_{p(n)} n}$$

does. This partial sum is equivalent to

$$\sum_{i=1}^{k} \sum_{n=[\epsilon(i)]+1}^{[\epsilon(i+1)]} \frac{1}{n \log n \cdots \log_i n} > \sum_{i=1}^{k} \int_{[\epsilon(i)]+1}^{\epsilon(i+1)} \frac{dx}{x \log x \cdots \log_i x}$$

$$= \sum_{i=1}^{k} \{\log_{i+1} x\}_{[\epsilon(i)]+1}^{\epsilon(i+1)} > \sum_{i=1}^{k} (1 - \log 2);$$

where the last step is a result of the inequality

$$\log_{i+1}([\epsilon(i)]+1) = \log_{i+1}([e^{\epsilon(i-1)}]+1) < \log_{i+1}(2e^{\epsilon(i-1)})$$

$$= \log_i (\log 2 + \epsilon(i-1)) < \log_i (1+\epsilon(i-1)) \leqq \cdots \leqq \log_b (1+\epsilon(b-1))$$

$$\leqq \cdots \leqq \log 2, \qquad i \geqq b \geqq 1.$$

Hence, clearly, the series diverges.

In case $\alpha \leqq 1$, the series (2) is not less than

$$\sum_{n=9}^{\infty} \frac{1}{n \log n \cdots \log_{q(n)-1} n \cdot \log_{q(n)} n} > \sum_{n=9}^{\infty} \frac{1}{n \log n \cdots \log_{q(n)} n \cdots \log_{p(n)} n}$$

which diverges because of (1).

In determining the convergence of this series for $\alpha>1$ we need only consider

$$\sum_{n=16}^{\infty} \frac{1}{n \log n \cdots \log_{q(n)-1} n (\log_{q(n)} n)^{\alpha}}$$

$$= \sum_{n=1}^{\infty} \sum_{m=[\epsilon(2n)]+1}^{[\epsilon(2n+2)]} \frac{1}{m \log m \cdots \log_{n-1} m (\log_n m)^{\alpha}}$$

$$< \sum_{n=1}^{\infty} \int_{[\epsilon(2n)]}^{\epsilon(2n+2)} \frac{dx}{x \log x \cdots (\log_n x)^{\alpha}} = -\frac{1}{\alpha-1} \sum_{n=1}^{\infty} \left\{ \frac{1}{(\log_n x)^{\alpha-1}} \right\}_{[\epsilon(2n)]}^{\epsilon(2n+2)}$$

$$< \frac{1}{\alpha-1} \sum_{n=1}^{\infty} \frac{1}{(\log_n [\epsilon(2n)])^{\alpha-1}} < \frac{1}{\alpha-1} \sum_{n=1}^{\infty} \frac{1}{(\log_n \epsilon(2n-1))^{\alpha-1}}$$

$$= \frac{1}{\alpha-1} \sum_{n=0}^{\infty} \frac{1}{(\epsilon(n))^{\alpha-1}} < \frac{1}{\alpha-1} \sum_{n=0}^{\infty} \frac{1}{(e^{\alpha-1})^n};$$

since $\epsilon(n) \geqq e^n$ (equality iff $n=0$ or 1). This last series is convergent, hence our proof is complete.

Also solved by Robert Goldstein (England), Sidney Spital, Dieter Wolke (Germany), and the proposer.

Wolke noted that the method used to prove the divergence of (1) may also be used to prove the divergence of

$$\sum_{n=n_0}^{\infty} \frac{1}{n \log n \cdots \log_{p(n)-k-1} n \cdot (\log_{p(n)-k} n)^a}$$

in which k is any fixed positive integer and a is real. He raises the question as to the required magnitude of $f(n)$ to secure convergence properties for

$$\sum_{n=n_0}^{\infty} \frac{1}{n \log n \cdots \log_{p(n)-f(n)-1} n \cdot (\log_{p(n)-f(n)} n)^a}$$

analogous to the results for (2), and he observes that the methods used for (2) allow $f(n)=[\log_k p(n)]$ for any fixed positive integer k.

On Commutative Rings

5377 [1966, 312]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College*

In a ring R each element x satisfies the equation $x=x^{n+1}$ for some integer n. Prove that $x^n y=yx^n$ for each x and y contained in R.

I. *Solution by Sheila M. Kaye, McGill University.* We must assume $n\equiv n(x)>0$.

(a) $x^{n(x)}$ is idempotent, since $(x^{n(x)})^2=x^{n(x)+1}x^{n(x)-1}=x^{n(x)}$.

(b) Let $ab=0$. Then $ba=(ba)^{n(ba)+1}=b(ab)^{n(ba)}a=0$.

(c) Let x, $y\in R$ and let $n=n(x)$. Then from (a), $yx^n=yx^{2n}$, whence $(y-yx^n)x^n=0$. Also from (b), $x^n(y-yx^n)=0$, so that $x^ny=x^nyx^n$.

(d) Similarly, from $x^ny=x^{2n}y$, we deduce $yx^n=x^nyx^n$, which completes the proof.

II. *Solution by Harsh Pittie, Princeton University.* Actually much more is known. The condition $x^{m(x)}=x$, $m(x)>1$ implies that R is commutative (see N. Jacobson, *Structure Theories for algebraic algebras of bounded degree*, Annals of Math., 46 (1945), pp. 695 ff). The given condition may be replaced by $x^{m(x)}-x$ is in the center of R (I. N. Herstein, *A generalization of a theorem of Jacobson*, Amer. J. of Math., 73 (1951), pp. 756 ff). See also Ayoub and Ayoub, *On the commutativity of rings*, this MONTHLY, 71 (1964) 267–271.

Also solved by Anders Bager (Denmark), Joseph Beckely, Robert Breusch, K. E. Eldridge, C. G. Fain, Neil Grabois, Charles Green, M. G. Greening (Australia), T. W. Hargrave, G. A. Heuer, R. J. Hursey, Jr., M. S. Kaplan, B. G. Klein, S. Lajos (Hungary), Joanne Larson, Steve Ligh, Chin-Yeang Lin, C. C. Lindner, Charles McCracken, W. A. McWorter, Ka Menehune, Stanton Philipp, Bob Prielipp, A. R. Prince (England), William Scott, K. N. Sigmon, F. C. Stewart, Charles Wells, P. R. Winter, Qazi Zameeruddin (India), and the proposers.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Correspondence about reviews is welcomed.

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (Teacher Training), L (Library purchase), 15 (junior level)—18 (second graduate year).

Computers and Numerical Analysis.

Six Figure Logarithmic Tables. By C. Attwood. Pergamon, L. I. City, N. Y., 1966. ix+298 pp. $4.75 (paper). Logs, cologs, antilogs. Eight historical facsimilies.

Six Figure Trigonometrical Tables. By C. Attwood. Pergamon, L. I. City, 1966. ix+164 pp. $3.45 (paper). Degrees and minutes, degrees and hundredths. Five attractive plates from Ptolemy, Briggs, Rheticus, Theodosius.

The Programming Language LISP: Its Operation and Applications. Edited by Edmund C. Berkeley and Daniel G. Bobrow. M.I.T., Cambridge, Mass., 1966. ix+382 pp. $5.00 (paper). " . . . Remarkable . . . language . . . not only . . . governs . . . a computer, but . . . with great flexibility and power for expressing processes in mathematics, logic, and symbol manipulation in general." S, P.

Elementary Numerical Analysis. By S. D. Conte. McGraw-Hill, New York, 1965. x+278 pp. $7.95. " . . . geared to . . . students of engineering and science . . . no attempt . . . to give rigorous proofs of all theorems . . . heuristic methods are introduced when insight is thereby enhanced . . . assumed that a computer is available . . . and that a large number of the algorithms . . . will be programmed . . . assumed that the student already knows how to write programs . . . The language used . . . is FORTRAN IV, and the computer . . . is the IBM 7090 . . . " T (15).

Threshold Logic: A Synthesis Approach. By Michael L. Dertouzos. M.I.T., Cambridge, Mass., 1965. xii+256 pp. $6.00. This book is concerned with the logical capabilities of threshold elements and with their application to the design of digital systems. Most of the results are from the author's doctoral thesis submitted to the Department of Electrical Engineering of M.I.T. in 1964. P.

Introduction to Algol Programming. By Torgil Ekman and Carl-Erik Froberg. Studentlitteratur, Lund, Sweden, and Oxford Univ. Press, 1965. 123 pp. "Ever since the advent of the computers, programming has been a necessary condition for automatic treatment of problems. Through the creation of a number of problem oriented programming languages, it became possible also for non-specialists to perform programming. Among all these programming languages, ALGOL holds a special position because of its international status, and still more because of its strict and logical structure. These circumstances make ALGOL extremely well suited for educational purposes, and especially in Europe it has obtained a strong position at most universities and technical institutes. This book has been written primarily in order to give the students at higher educational institutes a modern text-book in ALGOL." T (13–16), S, P, L.

Ten Place Tables of the Jacobian Elliptic Functions I. By Henry E. Fettis, and James C. Caslin. ARL 65-180, Office of Aerospace Research, USAF, Wright-Patterson AFB, Ohio, 1965. iv+562 pp.

Analyse numérique linéaire. By Noel Gastinel. Hermann, Paris, 1966. xii+364 pp. 60 F. xii+364 Matrix calculations. S, P.

Numerical Solutions of Nonlinear Differential Equations. Proceedings of an Advanced Symposium Conducted by the Mathematics Research Center, U. S. Army, at the University of Wisconsin, Madison, May 9–11, 1966. Edited by Donald Greenspan, Wiley, New York, 1966. x+347 pp. $7.75. "One of the major disappointments in the development of modern mathematics is that it has provided the applied scientist with no general *analytical* methods for solving non-linear ordinary and non-linear partial differential equations. This situation has been brightened considerably, however, with the development of the modern digital computer and with the simultaneous revitalization and growth of the study of *numerical* methods, and the primary objective of the symposium from which these proceedings emerged was to present the most current research in this direction." P.

Brief Numerical Methods. By Wendell E. Grove. Prentice-Hall, Englewood Cliffs, N.J., 1966. xi+117 pp. $6.75. Developed out of a three hour a week, twelve week course taught at the General Motors Institute by the author who is Associate Professor of Mathematics there. T (14–15).

Fundamentals of Digital Machine Computing. By Guenther Hintze. Springer, NewYork, 1966. ix+225 pp. $6.40. " . . . topics include the basic circuitry, logical organization and programming concepts common to all large modern computers. . . . This book was photo-offset by Linofil equipment activated by punched paper tape produced from an RCA 301 computer by means of programmed instructions created by Rocappi, Inc.," T, S, P.

Threshold Logic. By Sze-Tsen Hu. University of California Press, Berkeley, Cal., 1966. xiv+338 pp. $8.00. Threshold devices and related switching functions.

Introduction to Computing. By T. E. Hull. Prentice Hall, Englewood Cliffs, N. J., 1966. xi+212 pp. $7.50. Algorithms, stored-programs, basic programs, compilers and monitors, FORTRAN (80 pp.), planning and debugging, numerical methods, non-numerical applications, simulation, automata, languages. T (Introductory), S, P, L.

Tables of Lagrange Interpolation Coefficients. By K. A. Karpov. Translated by D. E. Brown. Pergamon Press, L. I. City, N. Y., 1965. viii+74 pp. $5.75. L.

The Method of Averaging Functional Corrections: Theory and Applications. By A. Yu. Luchka. Academic Press, New York, 1965. xvii+136 pp. $3.45 (paper). Approximating algorithms for integral, differential, integro-differential, and other equations. P.

Handbook for Computing Elementary Functions. By L. A. Lyusternik, O. A. Chervonenkis, and A. R. Yanpol'skii. Translated by G. J. Tee. Pergamon, L. I. City, N. Y. 1965. xiii+251 pp. $10.00. P, L.

Ten-Decimal Tables of the Logarithms of Complex Numbers and for the Transformation from Cartesian to Polar Coordinates, tables of the functions ln x, arctan x, $\frac{1}{2}$ ln $(1+X^2)$, $\sqrt{1+X^2}$. Edited by L. A. Lyusternik. Translated by D. E. Brown. Pergamon, L. I. City N. Y., 1965. ix+110 pp. $7.50.

An Introduction to Computer Programming. By Henry Mullish. Gordon and Breach, New York, 1966. xi+244 pp. $5.00 (paper) $14.50 (cloth). " . . . the essentials of programming for large scale electronic computers . . . to help create a substantial understanding of the techniques employed in solving many of today's common tasks . . . " FORTRAN used throughout. Coding sheets included. T (11–14), S (15–17), P.

A Fortran IV Primer. By E. I. Organick. Addison-Wesley, Reading, Mass., 1966. viii +263 pp. $4.95 (paper). T (12–18).

Numerical Mathematical Analysis. 6th ed. By James B. Scarborough. Johns Hopkins Press, Baltimore, 1966. xxi+608 pp. $8.50. This old standby, first published in 1930 and in successive editions in 1950, 1955, 1958, and 1962 is now enlarged slightly by "the addition of the trapezoid rule, an additional method of finding the roots of numerical equations, and formulas and computing schedules for the numerical solution of simultaneous differential equations of the second order by the Runge-Kutta method". S, P, L (if you don't have a recent edition).

Introduction to Numerical Analysis. By Gerard P. Weeg and Georgia B. Reed. Blaisdell, Waltham, Mass., 1966. vii+184 pp. $7.50. Errors, real roots of equations, polynomial approximations and finite differences, integration, ordinary differential equations, linear equations and matrices, least squares, Gaussian quadrature. T (14–15, one term course). S.

Theoretical Numerical Analysis. By Burton Wendroff. Academic Press, New York, 1966. xi+239 pp. $10.95. " . . . as a legitimate branch of mathematics . . . narrow, often deep path through . . . interpolation, approximation, numerical solution of ordinary and partial differential equations . . . systems of equations . . . a few computer problems . . . " T (17), P. (Reviewed in *American Scientist*, December 1966, 436A.)

Texts on Linear Algebra.

Matrices and Linear Transformations. By Charles Cullen. Addison-Wesley, Reading, Mass., 1966. 272 pp. $8.95. One or two term course for "science, engineering, and mathematics students . . . matrix theory point of view rather than . . . abstract approach . . . " T (14–15).

Elementary Matrix Theory. By Howard Eves. Allyn and Bacon, Boston, 1966. xvi+325 pp. $9.50 " . . . one-semester introduction . . . for the general student . . . the elementary, or concrete, approach is employed and no previous course in abstract algebra is required. Matrices are studied over the field of complex numbers . . . occasional references to . . . real numbers . . . and optional . . . extension to . . . abstract rings and fields . . . " T (14–15), TT.

Introduction to Matrices and Linear Transformations. By Daniel T. Finkbeiner, II. 2nd ed. Freeman, San Francisco, 1966. xi+297 pp. $7.75. Changes from first edition included earlier introduction of the dual space, use of cyclic subspaces in proofs of the Hamilton-Cayley theorem and the Jordan canonical form, more complete treatment of metric concepts, expanded discussion of combinatorial equivalence and applications to linear programming, and a new appendix on different matrix notations. T (14–15).

Linear Algebra. By Serge Lang. Addison-Wesley, Reading, Mass., 1966. x+294 pp. $8.95. For a semester or year course. Vectors in euclidean space, vector spaces, matrices as linear mappings, determinants, bilinear maps, polynomials and matrices, triangulation, spectral theorem, primary decomposition, multilinear products (tensors), groups, rings, convexity. T (14–15).

Matrices and Transformations. By Anthony J. Pettofrezzo. Prentice-Hall, Englewood Cliffs, N. J. 1966. x+133 pp. $3.95. Elementary definitions and consequences, linear transform atoms, eigenvalues and eigenvectors for training high school teachers. T (13–14).

Vector Spaces of Finite Dimension. By G. C. Shephard. Oliver & Boyd, London, 1966. viii+200 pp. $3.25 (from Wiley, the U. S. distributor). Linear algebra without ma-

trices and determinants followed by interpretation in matrix terms. Assumes the most elementary parts of abstract algebra, together with the properties of matrices and determinants. T (15–16), S.

Logic, Foundations, Philosophy.

The Undecidable: Basic papers on undecidable propositions, unsolvable problems and computable functions. Edited by Martin Davis. Raven Press, Hewlett, N. Y., 1965. 440 pp. $9.75. This anthology consists of papers by Gödel, Church, Turing, Rosser, Kleene and Post, most of which have been available heretofore only in the periodicals in which they originally appeared. A few, such as Gödel's Princeton Lectures, have not been published previously or appear here for the first time in English translation. P. L.

Symbolic Logic and the Real Number System. By A. H. Lightstone. Harper & Row, New York, 1965. ix+225 pp. $7.50. T (15), S, TT.

A Theory of Sets. By Anthony P. Morse. Academic Press, New York, 1965. xxxi+130 pp. $7.95. An original "formal unified treatment of logic and set theory." T (18–19), S, P, L.

Elementary Logic. By Willard Van Orman Quine. Harvard Univ., Cambridge, Mass., 1966. xii+129 pp. $3.50. Much revised from its first (1941) edition. T (15).

Selected Logic Papers. By W. V. Quine. Random, New York, 1966. x+250 pp. $6.95. Twenty-three technical papers on logic (1934–1960), including one historical and one survey paper. S (16–19), P, L.

The Ways of Paradox and other essays. By W. V. Quine. Random House, New York, 1966. x+258 pp. $6.95. Twenty-one papers (1934–1964) on logic, foundations, history and philosophy of mathematics, complementing his previous collections *From a Logical Point of View* and *Word and Object.* S (15–19), P, L.

Non-Standard Analysis. By A. Robinson. North Holland, Amsterdam, 1966. xi+293 pp. $10.00. Survey of the theory since its origin (1960) in the idea "that the concepts and methods of contemporary mathematical logic are capable of providing suitable framework for the development of the Differential and Integral Calculus by means of infinitely small and infinitely large numbers." Final chapter is on the history of calculus. P.

Number Systems of Analysis. By G. Cuthbert Webber. Addison-Wesley, Reading, Mass., 1966. x+213 pp. $7.95. From the natural through complex numbers from modified Peano via Cauchy sequences. T (14–15), S.

Probability and Statistics.

Ergodic Theory and Information. By Patrick Billingsley. Wiley, New York, 1965. xiii +193 pp. $8.50. First of a Series of Tracts on Probability and Mathematical Statistics "intended to make new mathematical developments generally accessible while the pulse of life still beats in them. . . . This book grew out of a survey . . . designed to follow a brief course in measure-theoretic probability . . . not for the experts, but for those to whom the subject is new." S, P.

An Introduction to Probability Theory and Its Applications. Vol. II. By William Feller. Wiley, New York, 1966. xviii+626 pp. $12.00. Similar to Vol. I. but "harder mathematics" including continuous distributions, measure, higher dimensional spaces, limit theorems, Markov processes, renewal theory, Laplace transform methods, etc. T (16–17), S, P, L.

Theory of Probability. By Bernard Harris (Univ. of Wisconsin). Addison-Wesley, Reading, Massachusetts, 1966. ix+294 pp. $9.75. Probability based on the Kolmogorov axioms, random variables and distributions, moments, random experiments, limit theorems. " . . . extensive use of characteristic functions and the Riemann Stieltjes integrals, so that the discrete and continuous cases may be treated simultaneously." T (15–16).

A First Course in Stochastic Processes. By Samuel Karlin. Academic Press, New York, 1966. xi+502 pp. $11.75. Assumes something like Feller's Volume I. T (16), S, P.

Introduction to Measure and Probability. By J. F. C. Kingman and S. J. Taylor. Cambridge Univ. Press, 1966. $12.50. Finite measures in general spaces (including Lebesgue and Lebesgue-Stieltjes) in chapters 1–6, introductory functional analysis in chapters 7–8, followed by probability, random variables, characteristic functions, independence, joint distributions, and stochastic processes in chapters 10–15. T (16–17), S, P.

Classical and Contagious Discrete Distributions. Proceedings of the International Symposium held at McGill University, Montreal, Canada, August 15–August 20, 1963. Edited by Ganapati P. Patil. Pergamon Press, L. I. City, 1965. xiv+552 pp. $21.00. P, L.

The Linear Hypothesis: A General Theory. By. G. A. Seber. Edited by M. G. Kendall. Hafner, New York, 1966. vii+115 pp. $4.15. Linear hypothesis theory (formerly "under the guise of regressional analysis, analysis of variance, and experimental design") in a vector space framework. S, P, L.

The Structure of Lie groups. By G. Hochschild. Holden-Day, San Francisco, London, Amsterdam, 1965. x+230 pp. $11.75.

This excellent introduction is not written for the innocent. The exposition is at the level of the well trained advanced graduate student or young Ph.D. who wishes to inform himself quickly of a vital area. For the nonexpert (e.g., the reviewer) the way is eased by the first 75 pages, the main theme appearing only in Chapter VII. Each chapter is well organized almost as an independent unit, and each is of just the right length, long enough to be informative and short enough to avoid discouraging the reader, and ends with two to six nontrivial problems. One wishes that the author would share his considerable expository talents (in, perhaps, a 350 page book) with a somewhat less knowledgeable audience.

F. HAIMO, Washington Univ., St. Louis

Differential Periodic Maps. By. P. E. Conner and E. E. Floyd. Academic Press, New York, 1966. 148 pp. $6.50.

This book represents almost all original research. The expository material present is extremely brief and condensed. Indeed, it is little more than a method of presenting notation. Hence, the reader is expected to have a substantial experience with the topics discussed in the book. These are by chapter headings: The bordism groups; Computation of the bordism groups; The G-bordism groups; Differentiable involutions; Differentiable actions of $(Z_2)^k$; Differentiable involutions and bundles; The structure of $\Omega_*(Z_p)$, p an odd prime; Fixed points of maps of odd prime period; Actions of finite abelian groups of odd prime power order.

LOUIS AUSLANDER, City Univ. of New York

Positive Solutions of Operator Equations. By. M. A. Krasnoselskii. Translated from the Russian by Richard E. Flaherty and edited by Leo F. Boron. Noordhoff, Groningen, 1964. 381 pp. $13.75.

"This book," according to its annotation, "is devoted to a systematic study of an important aspect of nonlinear functional analysis." The aspect referred to is the positivity of many standard nonlinear operators with respect to a cone in the space of definition. (A cone in a real Banach space E is a closed subset such that if $u,v \in K$ then $\alpha u+\omega v \in K$ for all $\alpha, \omega \geqq 0$ and if $u \neq 0$ is in E then either u or $-u$ is not in K.) The first two chapters discuss the geometry of Banach spaces with cones and linear operators which are positive with respect to the cone. (An operator is called positive if it maps K into K.) The next three chapters are devoted to theory of non-linear operators. They include Fréchet differentiation, fixed point theorems, eigenvectors, concave and convex operators, and successive approximation methods. The last chapter deals with applications of the previous theory to such topics as non-linear integral equations, non-linear elliptic boundary value problems, periodic solutions of ordinary differential equations and the Monge-Ampère equations.

Due to the inordinately large number of definitions, the reviewer found the book very tedious reading. There is a summary at the back of the book, however, which should allow a reader to open the book to, for example, the last chapter and read it with only a modicum of difficulty. The reviewer cannot see this book being used for a text in either an undergraduate or a graduate course, though it would make rather interesting material for a graduate or post-graduate seminar.

A. R. BRODSKY, University of California, Los Angeles

Elementary Abstract Algebra. By E. M. Patterson and D. E. Rutherford. Wiley, New York, 1965. viii+211 pp. $3.50.

The strength of R. Dedekind's abstractions in dealing with classical problems of algebra guided and inspired E. Noether to create a new method of expounding algebraic insights centered on concepts and their relations from which the solutions of solvable problems are derived. The first edition of "Moderne Algebra" by van der Waerden established the success of this method of exposition inasmuch as it reflected a much broader attitude towards algebraic relationships than the problem-centered approach of Dedekind's predecessors. In fact today, about forty years later, "modern algebra" has become "algebra" and a book on "elementary abstract algebra" providing a "thorough, but not too hurried, introduction to some of the basic ideas of abstract algebra" has become desirable and may indeed "be suitable for senior school pupils who wish to have a preview of the algebra they are likely to encounter at University." It follows that the choice of topics and the sequence in which they are expounded is determined exclusively by their usefulness towards learning and understanding abstract algebra, that only scant mention is given to the problems of number theory and geometry which led eventually to the evolution of concepts like binary operations (I), semi-groups, groups, permutations, isomorphisms, homomorphism (II), rings, integral domains, fields, matrices, characteristics, ideals and factorrings (III), polynomial and Euclidean rings, algebraic extensions of a field, algebraically closed fields (IV), vector spaces, unitary spaces, normed vector spaces, linear algebras and Lie algebras (V). It means also that there are no references whatever to the history of the subject—frequent references to other books in the same series take their place. On almost every page the text is remarkable for the care given to exposition and motivation of the concepts gradually brought into view. Marvels of careful exposition are, e.g., §23 on matrices and §34 on polynomials in a single indeterminate. Each chapter contains a few well-selected exercises the solutions of which are given at the end of the book.

HANS ZASSENHAUS, The Ohio State University

Introductory Numerical Analysis of Elliptic Boundary Value Problems. By Donald Greenspan. Harper and Row, New York, 1965. 164 pp. $7.00.

As the title implies this monograph introduces the reader to constructive solutions for elliptic problems. The opening chapter constitutes a dozen pages of prerequisite analytic concepts.

Through carefully prepared stages the author swiftly moves from an analysis of the standard two dimensional Laplace equation (Chapter 2) to multi-dimensional linear elliptic problems (Chapter 6). Along the way the reader is initiated to the mode of attack: a motivational development of appropriate difference equations followed by a convergence proof. (A ten page appendix analyzes some iterative methods for actually solving both linear and nonlinear systems.)

The final chapter (7) is titled Nonlinear Problems. By its very nature the material here quickly reaches research frontiers. Various open problems are presented. Chapter 7 in particular concludes with a carefully referenced detailed discussion of four "Research Problems."

Algorithms are clearly labeled (called "Methods"); theorems and lemmas on the whole are concisely stated, a rather unusual feat in numerical analysis; numerical examples are presented in a form which one can assimilate readily without becoming overwhelmed by a surplus of digits. Motivation is provided by associating the various differential equations with physical phenomena. The author continually teaches that problems need be well posed. There is an abundance of exercises and a very extensive bibliography.

It is unfortunate that the author chose to omit any analysis of rate of convergence or of error propagation—at either the discretization or iteration phases. Otherwise, this excellent text is admirably suited for use with either advanced undergraduates or beginning graduates. Though it will likely appeal primarily to mathematics majors, it should certainly be recommended for suitable serious students of the sciences. The price (about $4\frac{1}{4}$ cents per page) seems almost double what would be reasonable.

M. A. FELDSTEIN, Brown University

Einleitung in die Algebra und die Theorie der algebraische Gleichungen. By F. Nevanlinna. Birkhäuser Verlag, Basel and Stuttgart, 1965. 218 pages.

This book is intended as a clear, leisurely, well-motivated introduction to so-called "modern abstract algebra." The method chosen is a development of the machinery for a treatment of the theory of algebraic equations, both by the classical approach and by Galois theory. It is intended as a self-contained text for a short, introductory course (no number-theoretic or algebraic preparation necessary), or as a book for self-study, preparing the reader for "broader, more difficult" books, like van der Waerden's *Algebra*. The reviewer would rate it as an unqualified success in these aims, and something more.

The early chapters of the book give an excellent, fairly standard introduction to number theory, groups, integral domains and fields. There is more than enough here for the usual "first course in algebra" given to undergraduates in this country. (Quadratic reciprocity, perfect fields and Euclidean rings are even treated here.) These are followed by chapters on algebraic equations and algebraic field extensions. Here the theorem of the primitive element is proved, abelian extensions and ruler and compass constructions are discussed; then the equations of degree two, three and four are treated. Then, Abel's Theorem on the unsolvability by radicals of the general equation of degree n is proved by "classical" means. It is only after this that Galois theory is developed, in the last chapter, climaxing in a second proof of Abel's Theorem. The quality of exposition is very high throughout the book. The exercises (131 of them) are excellent, providing examples, drill, and extensive excursions. The paper and typography are good, and the reviewer was unable to find any errors—mathematical or typographical—in his reading of the book.

Were the language of the book English, this would be a top candidate for use in a first

course in algebra at almost any level. As it is, Professor Nevanlinna has provided a beautiful treatment of material which virtually no students in standard algebra courses get to see (the classical approach to Abel's Theorem), and there is enough material of interest throughout the book to recommend it to almost anyone who wants to read some algebra.

FREDERICK HOFFMAN, Institute for Defense Analyses, Princeton, N. J.

Commutative Normed Rings. By I. Gelfand, D. Raikov, and G. Shilov. Chelsea, Bronx, N. Y., 1964. 306 pp. $6.50.

The original (1960) and the German translation (1964) were reviewed in Math. Reviews 23, A 1242 and 29, 6327. Chapter VIII appeared as an appendix in the original, and Chapter IX is new. Neither appeared in the German translation. The material of Chapters V, VII, IX exists elsewhere only in the periodical literature. There is perpetuated (p. 83) what were once doubted and are now known to be false [E. Kallin, Proc. Nat. Acad. Sci., USA, 119 (1963), 821–824; MR 27, 2878]: the original statement and "proof" that a *locally* analytic function on a commutative normed ring is globally analytic. Section 13, where the error occurs, is now of principally historic interest and the closest relevant and valid material may be found in the following: R. Arens, Pacific J. Math. 11 (1961), 405–429; MR 25, 4373.

B. R. GELBAUM, University of California, Irvine

Algebraic Numbers. By Serge Lang. Addison-Wesley, Reading, Mass., 1964. xi+163 pp. $7.75.

This volume is a most welcome addition to the literature of the subject. The interplay of algebra and analysis, together with the author's lively style and erudition, combine to produce a book which should excite and stimulate its readers. Lang was a student of Artin, and we are fortunate to find in him an able expositor of the wonderful tradition in number theory associated with his teacher.

The first half of the book presupposes something like volume I of van der Waerden's algebra, the latter half, complex variables, elementary point set topology and a little distribution theory. The treatment strikes a balance between the global methods of earlier books such as Hecke's and the emphasis on the local theory to be found in some newer books such as that of Weiss. Beginning with a nicely arranged sketch of classical ideal theory in Dedekind domains, the author deals with algebraic integers, the different and discriminant, cyclotomic fields, parallelotopes, ideles and adeles, functional equation of ζ-functions, density of primes, the Brauer-Siegel theorem, and Weil's approach to explicit formulas.

BASIL GORDON, University of California at Los Angeles

Enumerability, Decidability, Computability. (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Vol. 127) by H. Hermes. Translated by G. T. Herman, and O. Plassman. Academic Press, New York, and Springer, Berlin, 1965. 245 pp. $9.75.

This is an introduction to the theory of recursive functions from a classical (non-intuitionistic) point of view. (Neither "Intuitionism" nor "Brouwer" nor "Heyting" appears in the index!). As in Martin Davis' book, *Computability and Unsolvability*, the framework is established by the discussion of Turing machines; μ-recursiveness also plays a large role. Applications of the theory include undecidability of the predicate calculus and incompleteness of arithmetic. The text is both unusually elementary and unusually clear. There are relatively easy problems at the end of every few paragraphs. For an undergraduate course in mathematics (Foundations . . .), or an undergraduate or graduate course in philosophy (Logic . . .), this competently written little book may be just the thing.

H. E. KYBURG, JR., University of Rochester

Notes on Logic. By Roger C. Lyndon. Van Nostrand, Princeton, N. J., 1966. vi+97 pp. $2.50.

This set of thirty-four lectures provides a compact and formal, yet surprisingly readable, introduction to mathematical logic. The treatment focuses attention on the important problems and goals of the subject, suppresses details of some philosophical interpretations, and ignores technical skills. There are no exercises.

The initial lectures treat, in addition to the program undertaken in logic, the grammar and semantics of sentential and predicate languages. The discussion of interpretations and structures is noteworthy. The important questions of decidability, axiomatizability, adequacy, and categoricity are raised and discussed in the context of the compactness and Skolem-Löwenheim theorems. Lyndon treats topics all too frequently excluded from a first course in logic: Gentzen's system of natural deduction in various formulations, proof of the Herbrand-Gentzen theorem, and Craig's theorem. The book ends with a concise but accurate discussion of the theorems of Tarski, Gödel, and Church on expressibility, incompleteness, and decidability, respectively.

The notes are worth the attention of any student of logic who has had an introductory course. They could serve as a difficult introduction to the subject for an exceptionally mature student.

D. L. KREIDER, Dartmouth College

Topological Vector Spaces. By Helmut H. Schaefer. Macmillan, New York, 1966. ix+294 pp. $10.95.

The book under review "is intended to be a systematic text" on topological vector spaces. For nearly a decade after the appearance in 1953 of Bourbaki's "Espaces Vectoriels Topologiques," no text-book on this topic appeared in English. The author's preliminary versions of the matter in this book were shaping up around 1958. In 1960, Köthe's "Topologische lineare Räume" appeared, and in 1963 Kelley, Namioka, *et al*, published what is perhaps the first English text-book on this topic. Schaefer's book preserves to a large extent the spirit of the Bourbaki texts, and includes also spectral properties of positive operators and a brief introduction to tensor products and nuclear spaces.

Brief summary of the contents: Chapter I: topological vector spaces and the related basic material such as metrization, completion and complexification. Chapter II: locally convex spaces. Chapter III: the space of linear mappings and the topologies on it. Chapter IV: fairly exhaustive account of the duality theory. Chapter V: ordered topological vector spaces. Each chapter is followed by numerous exercises which often supplement the text.

The book can be highly recommended for general reading and as a largely self-contained text-book.

M. S. RAMANUJAN, University of Michigan

Iterative Methods for the Solution of Equations. By J. F. Traub. Prentice-Hall, Englewood Cliffs, N. J., 1964. 18+310 pp. $12.95.

The author is concerned with the theory of iteration algorithms for the numerical determination of roots of equations and systems of equations. He is primarily concerned with the determination of a real root α of a real single-valued function $f(x)$ which has a certain number of continuous derivatives in the neighborhood of α. He assumes that the multiplicity of the root is an integer, that the root is isolated, and that a close approximation to the root is available. He is not concerned with the determination of initial values or with convergence criteria for the iteration algorithms. The theory is local, rather than global.

Four classes of iteration functions (IF) are considered: "one-point IF" (such as the Newton-Raphson method), "one-point IF with memory" (such as the secant method),

"multipoint IF" (similar to Runge-Kutta) and "multipoint IF with memory." Most of the IF are derived by the use of hyperosculatory interpolation, described in detail in Appendix A. Both inverse and direct interpolation are used.

The book is a scholarly piece of work with a considerable amount of new material and an extensive bibliography. It should become a basic reference for anyone desiring to do theoretical work in the field. On the other hand, it is not, and surely was not intended to be, a practical guide for writing a general-purpose computer program for automatically handling a large class of cases.

D. M. YOUNG, JR., University of Texas

Elements of the Theory of Nonlinear Oscillations. By N. V. Butenin. Blaisdell, Waltham, Mass. 1965. 226 pp. $7.50.

From the author's preface: "The author of this book has sought to give the reader at least an introduction to the theory of nonlinear vibrations, to acquaint him with a few of the methods of studying nonlinear systems, and to apply these methods to the solution of concrete problems." He succeeds, but at the cost of leaving the reader with the impression that the theory of nonlinear differential equations is quite primitive, and that the construction of approximations to the integral curves by graphical methods is still an important task. The publisher states that the book is "well suited for use as an advanced text in engineering schools." In my opinion, this claim should end with the phrase "in underdeveloped countries where the engineer will not have access to computers or the need of theory." On the other hand, the book contains many extremely interesting examples which are discussed in great detail, e.g., the dependence of the motion of a clock on its means of suspension (Kelvin's problem) and the self-excited oscillations of an autopilot.

The book is well designed and easy to read. In general, the translation (the book was first published in Leningrad in 1962) is careful, but stiff. There are the usual minor errors and omissions which can give pleasure to a reviewer but are otherwise unimportant. Once (page 188) after an "It is obvious that . . . " sentence, the editor felt obligated to insert a parenthetical remark "This is not entirely obvious; the reader should make certain that he understands the situation." I agree; it wasn't obvious.

W. T. KYNER, University of Southern California

Theoretical Numerical Analysis. By Burton Wendroff. Academic Press, New York, London, 1966. xi+239 pp. $10.95. (Telegr. Rev. page 340.)

This textbook is suitable for students at the graduate level. Occasionally some quite advanced concepts are introduced, but it is remarkable that the author does present proofs to some very deep theorems without requiring advanced mathematical knowledge. He follows what I consider the sound practice of selecting a limited number of algorithms in each area without attempting to be comprehensive. Many of the algorithms and proofs presented were until recently only available in the research literature. On the other hand the treatment of some areas is very sketchy (e.g., multistep methods and stability in the chapter on ordinary differential equations receive scant attention). Also the chapter on partial differential equations omits almost entirely the consideration of elliptic difference equations, but makes up for this by giving an excellent treatment of stability for initial value problems. Considering the size of the book the omission of some important topics was inevitable.

There seems to be developing in this country at least two types of numerical analysts: the mathematician type and the computer scientist type. This book is definitely intended for the former. The influence of the computer is somewhat superficial, occurring only in the selection of algorithms and in a few computer based exercises. There is little discussion of such computer science related aspects of algorithms as rates of convergence, round-off error propagation, storage problems, and computational efficiency.

S. D. CONTE, Purdue University

Estimation Theory. By Ralph Deutsch. Prentice-Hall, Englewood Cliffs, N. J., 1965. xiv+269 pp. $10.95. (Telegr. Rev., January 1967.)

Deutsch includes under the umbrella of *estimation theory* the ideas of least squares, polynomial approximation, smoothing and prediction of time series, as well as the topic usually called statistical estimation theory.

To understand these ideas one of course needs to know some mathematics and statistics. In writing a book of this sort, then, an author can either assume that his readers come prepared with the necessary background, or he can provide it for them. Deutsch has chosen the latter alternative, and has also included some historical notes on his subject, numerous exercises, and an extensive bibliography (!). This is an ambitious undertaking for a book of 269 pages. Nonetheless, I think Deutsch has largely succeeded in giving a good introductory account of the topic of estimation theory, as well as some of the mathematics, statistics, and numerical analysis necessary for the theory and for implementing the techniques.

The breadth of coverage and the brevity of the book dictate that the penetration into some topics is not too deep. Further, readers interested in mathematical rigor, i.e., careful attention to details in definitions and proofs, will have to look elsewhere. Similarly, readers seeking to implement techniques will find Deutsch's discussions helpful in getting an overview of the difficulties involved, but will have to seek particulars elsewhere. The extensive bibliography will help in these regards.

On balance, this is a worthwhile book. It could serve well as a course text provided the teacher is prepared to supplement, correct, and make the presentation rigorous.

E. M. SCHEUER, The RAND Corporation

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. C. James, Harvey Mudd College, represented the Association at the 75th Anniversary Convocation and Anniversary Conference of the California Institute of Technology from October 24 to October 27, 1966.

Mrs. Artha J. Snyder, Angelo State College, represented the Association at the inauguration of President G. E. Murray of Texas Technological College on November 1, 1966.

Professor W. L. Waltmann, Wartburg College, represented the Association at the inauguration of President P. F. Sharp of Drake University on October 28, 1966.

California Institute of Technology: Associate Professors R. A. Dean and F. B. Fuller have been promoted to Professors; Assistant Professor D. E. Knuth has been promoted to Associate Professor.

University of California at Santa Barbara: Dr. H. B. Keynes, Wesleyan University, has been appointed Assistant Professor; Assistant Professor D. A. Sprecher, Syracuse University, has been appointed Associate Professor; Dr. J. M. Zelmanowitz, University of Wisconsin, has been appointed Assistant Professor; Assistant Professor T. K. Boehme has been promoted to Associate Professor.

Cleveland State University: Associate Professor J. M. Egar, University of Akron, has been appointed Associate Professor; Assistant Professor R. L. Pruitt, Capital University, has been appointed Assistant Professor.

University of Connecticut: Dr. L. J. DeLuca, Syracuse University, has been appointed Assistant Professor; Associate Professor G. N. Raney has been promoted to Professor and has accepted the position of Acting Head of the Department of Mathematics.

Eastern Montana College: Mr. D. F. Furness, Church College of Hawaii, has been appointed Assistant Professor; Dr. W. A. Stannard, Montana State University, has been appointed Associate Professor.

Illinois Institute of Technology: Mr. Seymour Kass has been promoted to Assistant Professor; Associate Professor J. J. Mehlberg has been promoted to Professor.

Ithaca College: Mrs. Shirley O. Hockett, Cornell University, and Mr. Necdet Ucoluk, Clarkson College of Technology, have been appointed Assistant Professors.

Kansas State University: Dr. R. S. Sabharwal, Washington State College, has been appointed Assistant Professor; Assistant Professor F. B. Sloat has been promoted to Associate Professor.

Lock Haven State College: Mr. M. J. Brady, Waynesboro High School, and Mr. D. R. Wagner, Monclair High School, have been appointed Assistant Professors.

University of Maryland: Dr. J. J. Benedetto, The Institute for Fluid Dynamics and Applied Mathematics, Dr. P. J. Gowen, University of Virginia, and Dr. D. I. Schneider, Johns Hopkins University, have been appointed Assistant Professors; Associate Professor Carol R. Karp has been promoted to Professor.

Miami University: Dr. Syed Asadulla, University of Florida, Dr. R. M. Bullock, University of Cincinnati, and Dr. S. E. Payne, Florida State University, have been appointed Assistant Professors; Associate Professor S. E. Bohn has been promoted to Professor; Assistant Professors R. G. Laatsch and S. H. Tung have been promoted to Associate Professors.

Mississippi State University: Dr. W. M. Causey, University of Kansas, has been appointed Assistant Professor; Dr. F. Virginia Rohde, University of Chattanooga, has been appointed Professor; Assistant Professor C. T. Scarborough, Jr., Wayne State University, has been appointed Associate Professor.

University of New Hampshire: Associate Professor H. F. Bechtell, Bucknell University, has been appointed Associate Professor; Assistant Professors R. H. Balomenos, W. E. Bonnice, and D. M. Burton have been promoted to Associate Professors; Associate Professor E. H. Batho has been promoted to Professor.

University of New Mexico: Assistant Professor H. L. Bentley, Rensselaer Polytechnic Institute, has been appointed Assistant Professor; Professor G. M. Wing, University of Colorado, has been appointed Professor; Assistant Professor M. F. Janowitz has joined the faculty of Western Michigan University as an Associate Professor; Assistant Professors Heinz Renggli and Arthur Steger have been promoted to Associate Professors.

North Carolina State University: Dr. J. R. Kolb, University of Maryland, has been appointed Assistant Professor; Dr. J. A. Marlin has been promoted to Assistant Professor; Assistant Professor Kwangil Koh has been promoted to Associate Professor.

Professor W. A. Lafferty, Northwest Missouri State College, died on October 9, 1966. He was a member of the Association for twenty-seven years.

Professor H. P. Pettit, Marquette University, died in August, 1966. He was a member of the Association for forty-six years.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE MISSOURI SECTION

The spring meeting of the Missouri Section of the MAA was held at the University of Missouri at Rolla on April 30, 1966, Professor Charles Hatfield, Chairman of the Section, presiding. Sixty-four persons were present, of whom forty-six were members.

At the business meeting, the following officers were elected for 1966–1967: Chairman, Professor Dale Woods, Northeast Missouri State Teachers College; Vice-Chairman, Professor F. W. Wilke, University of St. Louis at St. Louis; Secretary-Treasurer, Professor E. C. Pringle, Northeast Missouri State Teachers College.

Professor John Andrews reported on the high school testing program and Professor Paul Burcham on the University of Missouri (Columbia) testing program for entering freshmen.

During the morning session, Professor Guido Weiss of Washington University, St. Louis, gave the main address on "The CUPM General Curriculum in Mathematics for Colleges."

A program of new films was scheduled for the afternoon session. The following papers were presented during the morning session:

1. *A projective method for linear equations*, by R. F. Keller, University of Missouri, Columbia

2. *Solution of a difference equation by means of a contour integral*, by Gary Walls, Northeast Missouri State Teachers College.

3. *Some vector families in E^3 with integral components, integral length and useful orthogonality properties*, by J. F. Gray, Society of Mary, Kirkwood.

4. *On arbitrary large postulate sets for the propositional calculus*, by John Bridges, Southwest Missouri State College, Springfield.

C. A. JOHNSON, *Secretary-Treasurer*

NOVEMBER MEETING OF THE INDIANA SECTION

On Saturday, November 5, 1966, the Indiana Section of the MAA met at Purdue University. Approximately 150 persons attended, including 62 members of the Association who registered.

The group was welcomed by Professor F. H. Haas, Dean of the School of Science, and the sessions were presided over by Professors G. R. MacLane and R. E. Zink.

At the business meeting, it was decided unanimously that the Section would recognize the top two competitors from Indiana on the Putnam Examination by awarding them a year's membership in the Association and announcing their achievement to the membership. Professor Paul Mielke reported on a team competition held among eight small colleges in the state last year and announced plans to repeat the undertaking this year in view of its success.

The program for the meeting was as follows:

1. *Blood flow and asymptotic uniqueness of a boundary value problem*, by H. M. Lieberstein, Indiana University.

2. *A reduction of Diophantine problems*, by A. Adler, Indiana University.

3. *Mathematics and space science*, by H. Pollard, Purdue University.

4. *Convergence of Fourier series*, by C. Goffman, Purdue University.

5. *An instability result for a linear differential-difference equation with constant coefficients*, by H. Bailey, Rose Polytechnic Institute.

6. *Ordinary differential equations in Banach space and functional equations of the retarded type*, by Z. Vorel, Rose Polytechnic Institute.

7. *Pointwise periodicity and Boolean algebras*, by F. B. Wright, Tulane University (Guest Speaker).

GEORGE PEDRICK, *Secretary*

NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The twelfth annual meeting of the Northeastern Section of the MAA was held at Trinity College, Hartford, Connecticut, on November 26, 1966. The registered attendance was 151, including 131 members of the Association. Chairman Hartley Rogers, of the Massachusetts Institute of Technology, presided at the morning session and Vice-Chairman Robin Robinson, of Dartmouth College, presided at the afternoon session.

At the business meeting, the following officers were elected for the coming year: Chairman, Robin Robinson, Dartmouth College; Vice-Chairman, Guilford Spencer III, Williams College; Secretary-Treasurer, George Best, Phillips Academy.

The morning program was devoted to the following talks:

1. *Equivalences of differentiable structures*, by J. R. Munkres, Massachusetts Institute of Technology.

2. *Markov chains and potential theory*, by L. J. Snell, Dartmouth College.

Following the afternoon business meeting, the program continued with a panel discussion of *The CUPM report on the general curriculum in mathematics for colleges.* The panel was moderated by A. B. Willcox, Amherst College. Panelists were: V. O. McBrien, College of the Holy Cross; E. E. Moise, Harvard University; R. A. Rosenbaum, Wesleyan University; N. J. Schoonmaker, University of Vermont.

G. W. BEST, *Secretary-Treasurer*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The forty-first annual meeting of the Philadelphia Section of the MAA was held at Villanova University, Villanova, Pennsylvania, on November 19, 1966. The Chairman, Professor Russell Remage Jr., University of Delaware, presided at the meeting. The meeting was attended by 172 persons including 137 members of the Association.

At the business meeting the following officers were elected: Chairman: Professor Emil Amelotti, Villanova University; Member of the Executive Committee, Professor R. J. Kohlmeyer, Albright College.

The following papers were presented:

1. *Some aspects of homological algebra-background and recent developments*, by J. C. Moore, Princeton University.

2. *Components of mathematical systems*, by P. C. Hammer, The Pennsylvania State University.

3. *A brief trip through the affine plane*, by Samuel Gulden, Lehigh University.

4. *The CUPM general curriculum in mathematics for colleges*, by D. W. Western, Franklin and Marshall College; J. A. Meier, Millersville State College; David Rosen, Swarthmore College.

5. Film: *Let us teach Guessing*, with George Polya.

V. V. LATSHAW, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Forty-eighth Summer Meeting, University of Toronto, Toronto, Ontario, Canada, August 28–30, 1967.

Fifty-first Annual Meeting, San Francisco, California, January 25–27, 1968.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1967.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1967.

INDIANA, Wabash College, Crawfordsville, May 13, 1967.

IOWA, Drake University, Des Moines, April 21, 1967.

KANSAS, Fort Hays State College, Hays, April 22, 1967.

KENTUCKY, Murray State University, Murray, April 1, 1967.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, April 22, 1967.

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, St. John's University, Collegeville, May 6, 1967.

MISSOURI, Northeast Missouri State Teachers College, Kirksville, April 29, 1967.

NEBRASKA, University of South Dakota, Vermillion, May 6, 1967.

NEW JERSEY

NORTHEASTERN, Mt. Allison University, Sackville, New Brunswick, June 23–24, 1967.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 22, 1967.

OKLAHOMA-ARKANSAS

PACIFIC NORTHWEST, University of Montana, Missoula, June 16–17, 1967.

PHILADELPHIA, University of Delaware, Newark, Delaware, November 18, 1967.

ROCKY MOUNTAIN, Western State College of Colorado, Gunnison, May 12–13, 1967.

SOUTHEASTERN

SOUTHERN CALIFORNIA

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TEXAS, Austin College, Sherman, April 14–15, 1967.

UPPER NEW YORK STATE, State University College, Plattsburgh, May 20, 1967.

WISCONSIN, St. Norbert College, DePere, May 6, 1967.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, New York, N. Y., December 26–31, 1967.

AMERICAN MATHEMATICAL SOCIETY, Toronto, Ontario, Canada, Aug. 29–Sept. 1, 1967.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Michigan State University, June 19–23, 1967.

ASSOCIATION FOR COMPUTING MACHINERY, Sheraton-Park, Washington, D. C., August 29–31, 1967.

ASSOCIATION FOR SYMBOLIC LOGIC, Chicago, May 4–5, 1967.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 23–25, 1967.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Convention Center, Las Vegas, Nevada, April 18–21, 1967.

OPERATIONS RESEARCH SOCIETY OF AMERICA, New York Hilton Hotel, May 31–June 2, 1967.

PI MU EPSILON, Toronto, Ontario, Canada, August 29–30, 1967.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D. C., June 12–15, 1967 (Symposiums on applied probability and fluid dynamics).



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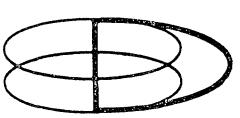
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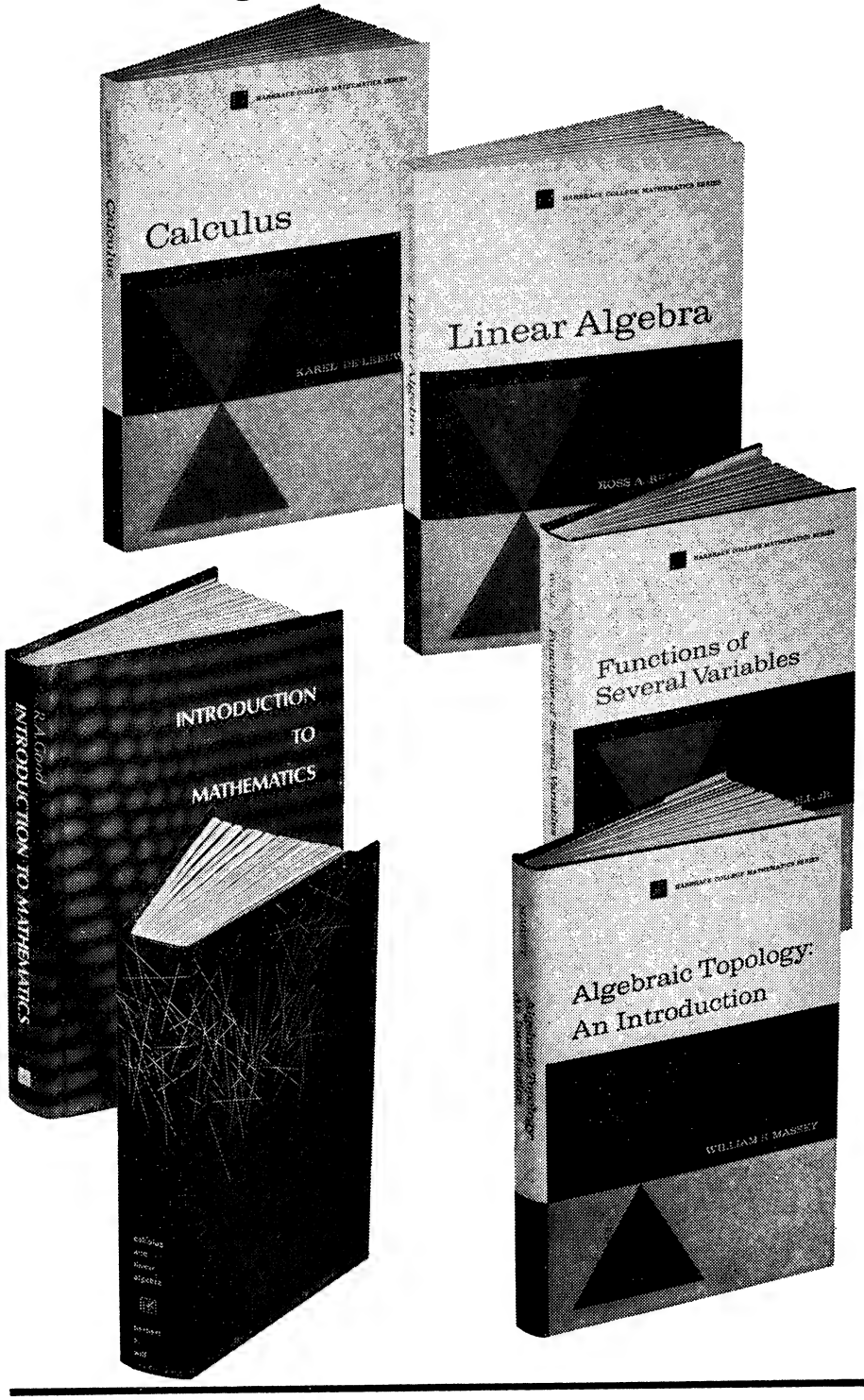
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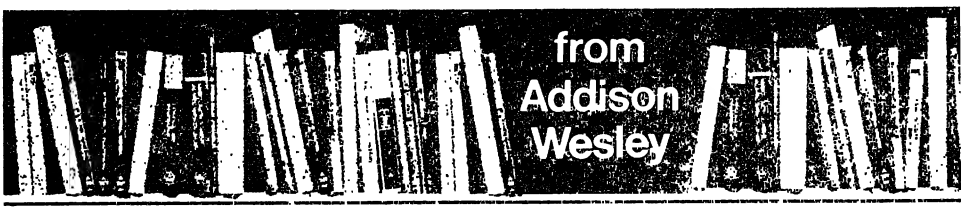
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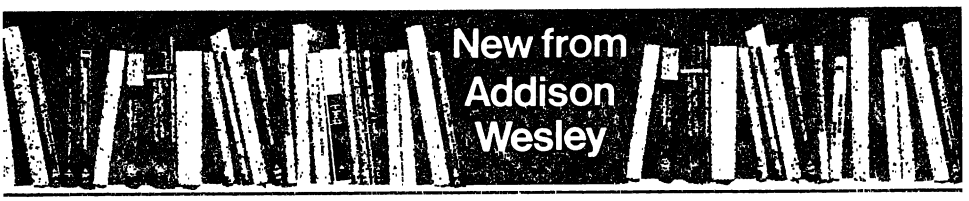
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APRIL 1967

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $6.00. For nonmembers the subscription price is $10.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

LATTICE POINT COVERINGS BY PLANE FIGURES

IVAN NIVEN, University of Oregon, and H. S. ZUCKERMAN, University of Washington

1. Let S be a bounded set of points in the Euclidean plane with a unit distance defined. If a rectangular coordinate system is imposed a certain number of points of S are lattice points, i.e., points with integral coordinates. Let $m(S)$ be the minimum number of lattice points, and $M(S)$ the maximum number, for all possible choices of the coordinate system. For example if S is a unit disk, i.e., a circle of diameter 1 with interior, then $m(S)=0$ and $M(S)=2$. These definitions of $m(S)$ and $M(S)$ could be given in terms of a fixed rectangular coordinate system, with the set S being rotated and translated freely in the plane. This alternative formulation is useful in proofs.

If a set S satisfies $m(S)\geqq 1$, we say that S has the *lattice point covering property* because S covers at least one lattice point no matter how the coordinate system is imposed. This paper gives necessary and sufficient conditions for the lattice point covering property in the cases of three types of simple plane figures, with boundary and interior included. The three types are the ellipse, the triangle and the parallelogram, treated in Sections 2, 3, 4, respectively.

For the circle plus interior, denoted by C say, it is clear that C has the lattice point covering property if and only if $r\geqq\sqrt{2}/2$, where r is the radius of the circle. Another relatively easy case is that of the rectangle, say of dimensions a by b with $a\leqq b$. Necessary and sufficient conditions that a rectangle have the lattice point covering property are that $a\geqq 1$ and $b\geqq\sqrt{2}$; (see [1]).

2. The Ellipse.

THEOREM 1. *An ellipse with axes of lengths $2a$ and $2b$ has the lattice point covering property if and only if $4a^2b^2\geqq a^2+b^2$.*

Alternative statement: *an ellipse has the lattice point covering property if and only if the point $(\frac{1}{2},\frac{1}{2})$ is inside or on the ellipse when a rectangular coordinate system is imposed with origin at the center of the ellipse and coordinate axes coinciding with the axes of the ellipse.*

Proof. The equivalence of the two statements follows at once from the equation $x^2/a^2+y^2/b^2=1$ for the ellipse with the coordinate system imposed.

Assume first that $4a^2b^2\geqq a^2+b^2$. Consider a lattice of points in the uv plane: $u=0$, $v=0$; $u=1$, $v=0$; $u=0$, $v=1$; etc. Impose this lattice of points on the xy plane where the ellipse E has the equation $x^2/a^2+y^2/b^2=1$. Let m denote the slope (relative to the xy coordinate system) of the u-axis. We can suppose that $-1<m\leqq 1$ if we interchange u and v when appropriate.

The lines of lattice points lie on $v=n$ relative to the uv coordinate system, where n is an integer. For an appropriate choice of n the line of lattice points $v=n$ will have equation

$$y = mx + k \text{ with } -\tfrac{1}{2}\sqrt{1+m^2} < k \leqq \tfrac{1}{2}\sqrt{1+m^2}$$

because $y=mx+k_1$ and $y=mx+k_2$ are a unit distance apart if $|k_1-k_2|=\sqrt{1+m^2}$.

Now consider the possible intersections of $y=mx+k$ with the ellipse E. When y is eliminated the equations give

$$x = \{-mka^2 \pm ab(b^2 + m^2a^2 - k^2)^{1/2}\}/(b^2 + m^2a^2). \tag{1}$$

The condition $4a^2b^2 \geqq a^2+b^2$ implies

$$a^{-2} + b^{-2} \leqq 4, \qquad a^{-2} < 4, \qquad b^{-2} < 4,$$

$$k^2 < \frac{1}{4}(1 + m^2) = \frac{1}{4} + \frac{m^2}{4} < b^2 + m^2a^2.$$

Hence we see from (1) that there are two points of intersection, say (x_1, mx_1+k) and (x_2, mx_2+k). We prove that the distance between these points is at least one, and it follows that the ellipse covers a lattice point. Denoting this distance by d we see that

$$d^2 = (1 + m^2)(x_1 - x_2)^2 = 4a^2b^2(1 + m^2)(b^2 + m^2a^2 - k^2)/(b^2 + m^2a^2)^2.$$

Using $k^2 \leqq (1+m^2)/4$ and direct calculation we get

$$(d^2 - 1)(b^2 + m^2a^2)^2 \geqq (m^4a^2 + b^2)(4a^2b^2 - a^2 - b^2) + 4m^2a^2b^2(a^2 + b^2 - 1).$$

The right member is not negative because $4a^2b^2 \geqq a^2+b^2$ and because this together with $a^2+b^2 \geqq 2ab$, $a>0$, $b>0$ imply

$$4a^2b^2 \geqq 2ab, \qquad 2ab \geqq 1, \qquad a^2 + b^2 \geqq 1.$$

Hence $d \geqq 1$ is established.

Conversely, if $4a^2b^2 < a^2+b^2$ then the point $x=\frac{1}{2}$, $y=\frac{1}{2}$ is outside the ellipse $x^2/a^2+y^2/b^2=1$. Then set the lattice of points down on the ellipse with $u=0$, $v=0$ at $x=-\frac{1}{2}$, $y=-\frac{1}{2}$ and with $u=1$, $v=0$ at $x=\frac{1}{2}$, $y=-\frac{1}{2}$. Then the ellipse contains no lattice point (u, v).

3. The triangle.

THEOREM 2. *A triangle with sides of lengths a, b, c, with $a \leqq b \leqq c$, has the lattice point covering property if and only if $2\Delta(c-1) \geqq c^2$ where Δ is the area of the triangle.*

It is convenient to break the proof into a number of steps. We let $A_1A_2A_3$ be an arbitrary triangle, a_i the length of the side opposite A_i, h_i the length of the altitude through A_i, α_i the size of the angle at A_i, and Δ the area. Throughout the letters i, j, k will denote the three distinct integers 1, 2, 3 in some order.

We define three numbers w_i as follows. If $h_i<1$ then $w_i=0$. If $h_i \geqq 1$ then w_i is the length of the segment P_iQ_i where P_i lies on A_iA_j, Q_i on A_iA_k, P_iQ_i is parallel to A_jA_k, and P_iQ_i is at a unit distance from A_jA_k.

LEMMA 3. *If $w_i \geqq 1$ for all $i=1, 2, 3$ then $A_1A_2A_3$ has the lattice point covering property.*

Proof. Let $A_1A_2A_3$ be set on the plane. Through each vertex A_i construct the four half lines emanating from A_i, two parallel (but in opposite directions) to

each coordinate axis. It is not hard to see that at least one of these twelve half lines will intersect the side of the triangle opposite the corresponding vertex. We can suppose the vertex is A_i and let K denote the intersection with A_jA_k. Lines parallel to A_iK through P_i and Q_i will intersect A_jA_k. Call these intersections

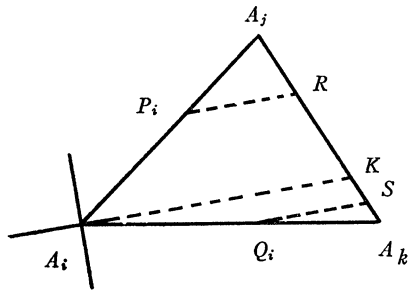


FIG. 1

R and S. Then RSQ_iP_i is a parallelogram contained in the triangle $A_1A_2A_3$. Since $w_i \geqq 1$, a line of lattice points parallel to A_iK will intersect P_iQ_i. Furthermore, this line will have on it a lattice point in RSQ_iP_i because $P_iR \geqq 1$.

LEMMA 4. (a) *If* $w_i > 0$ *then* $w_i - 1 = \{2\Delta - (a_i + h_i)\}/h_i$.
(b) *If* $w_i \geqq 1$ *and* $a_j \leqq a_i$ *then* $w_j \geqq 1$.

Proof. (a) We have $h_i > 1$. By a comparison of similar triangles, using $a_ih_i = 2\Delta$, we see that

$$\frac{h_i - 1}{h_i} = \frac{w_i}{a_i}, \qquad w_i = \frac{2\Delta - a_i}{h_i}, \qquad w_i - 1 = \frac{2\Delta - a_i - h_i}{h_i}.$$

(b) Since $w_i \geqq 1$ we have $h_i > 1$, and also we have $a_jh_j = 2\Delta = a_ih_i$, $h_j \geqq h_i$, and therefore $w_j > 0$. Furthermore

$$a_i - a_j \geqq (a_i - a_j)\sin\alpha_k, \quad a_i + a_j\sin\alpha_k \geqq a_j + a_i\sin\alpha_k, \quad a_i + h_i \geqq a_j + h_j.$$

Using these results and part (a) we find

$$w_j - 1 = \frac{2\Delta - (a_j + h_j)}{h_j} \geqq \frac{2\Delta - (a_i + h_i)}{h_j} = \frac{h_i}{h_j}(w_i - 1) \geqq 0.$$

LEMMA 5. *If* $w_i < 1$ *for some* $i = 1, 2, 3$, *then* $A_1A_2A_3$ *does not have the lattice point covering property.*

Proof. Let $m = 1, 2$, or 3 be such that $a_m = \max(a_1, a_2, a_3)$. Using Lemma 4(b) we see that $w_m < 1$. We also have $\alpha_m = \max(\alpha_1, \alpha_2, \alpha_3)$, and hence $\alpha_n < \pi/2$ if $n \neq m$. If $h_m < 1$ let T be the point A_m. If $h_m \geqq 1$ let T be the midpoint of P_mQ_m. Place the triangle on the plane with T at $(\frac{1}{2}, 1+\epsilon)$ and with the side opposite A_m on the line $y = \epsilon$. For ϵ sufficiently small but positive, the triangle $A_1A_2A_3$ covers no lattice point.

LEMMA 6. *If* $a_m=\max(a_1, a_2, a_3)$ *then the condition* $w_i\geqq 1$ *for all* $i=1, 2, 3$ *is equivalent to the condition* $2\Delta(a_m-1)\geqq a_m^2$.

Proof. Lemma 4(b) shows that $w_i\geqq 1$ for all $i=1, 2, 3$ is equivalent to the condition $w_m\geqq 1$. If $w_m\geqq 1$ then using Lemma 4(a) we obtain

$$2\Delta(a_m - 1) - a_m^2 = 2\Delta a_m - a_m h_m - a_m^2 = a_m(2\Delta - h_m - a_m)$$
$$= a_m h_m(w_m - 1) \geqq 0.$$

Conversely if $2\Delta(a_m-1)\geqq a_m^2$ then

$$a_m^2(h_m - 1) > a_m^2 h_m - a_m^2 - a_m h_m = 2\Delta(a_m - 1) - a_m^2 \geqq 0.$$

Therefore $h_m>1$, hence $w_m>0$, and by Lemma 4(a)

$$w_m - 1 = \frac{2\Delta - (a_m + h_m)}{h_m} = \frac{2\Delta a_m - a_m^2 - a_m h_m}{a_m h_m} = \frac{2\Delta(a_m - 1) - a_m^2}{2\Delta} \geqq 0.$$

These lemmas together imply Theorem 2.

4. The parallelogram. Let $ABCD$ be a parallelogram whose smaller interior angle γ, say at A, satisfies $0<\gamma<\pi/2$; thus $ABCD$ is not a rectangle. Let a be the distance between the parallel lines AB and CD, and b the distance between AD and BC.

LEMMA 7. *If* $a<1$ *or* $b<1$ *the parallelogram does not have the lattice point covering property.*

Proof. The parallelogram can be fitted between, for example, the lines $y=0$ and $y=1$.

LEMMA 8. *Assume* $a\geqq 1$ *and* $b\geqq 1$. *Let a rectangular coordinate system be imposed on* $ABCD$, *and define the angle* θ *thus: consider the four half lines emanating from* A, *parallel to the axes; let* AK *be that one of the four half lines such that* K *is on the same side of the line* AB *as are* C *and* D, *and such that the angle* BAK, *denoted by* θ, *satisfies* $0<\theta\leqq\pi/2$. *If* $\theta\geqq\gamma$, *i.e. if* AK *does not intersect the interior of the parallelogram, then there is a lattice point inside or on* $ABCD$.

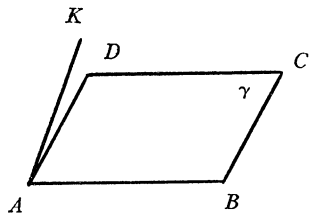


FIG. 2

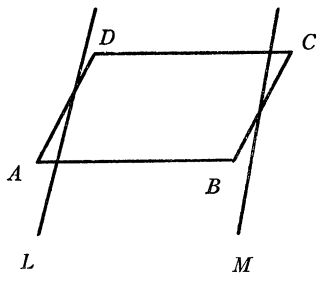


FIG. 3

Proof. Suppose there is no such lattice point. The lattice lines which are parallel to AK intersect the line AB extended at infinitely many points, any two adjacent points being at a distance $\csc\theta$. However, the distance between the parallel lines AD and BC being b, we see that

$$\text{length } AB = b\cdot\csc\gamma \geqq b\csc\theta \geqq \csc\theta.$$

Hence some lattice line parallel to AK intersects the line segment AB. Let L be the rightmost such line, i.e. the one whose intersection with the segment AB is closest to B. Let M be the next such lattice line to the right, so that M does not intersect the segment AB. Now L cannot intersect the segment DC since the distance between AB and DC is $a\geqq 1$, and we are presuming that there is no lattice point inside or on $ABCD$. Furthermore L cannot intersect the segment BC since $\theta\geqq\gamma$, and L coincident with BC is not possible. Hence L intersects AD. For convenience label the regions of the plane outside the parallelogram, bounded by the extensions of the sides, by (1), (2), $\cdots$, (8) as shown in Fig. 4.

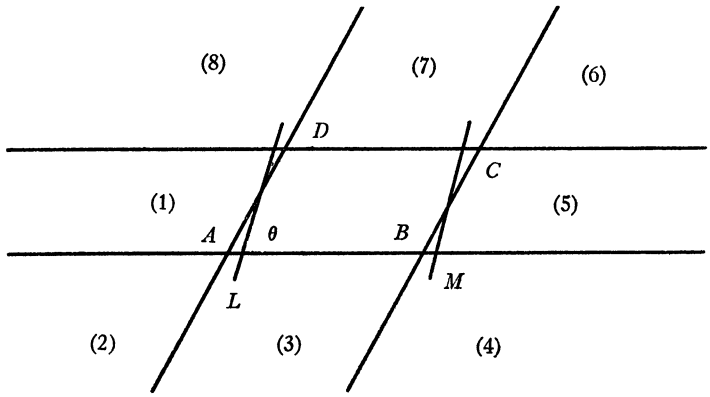


FIG. 4

No point of the parallelogram belongs to any of these regions (1), $\cdots$, (8). We assign the boundaries between regions in any fashion with one exception: the extension of CD beyond the point D belongs to region (1). Since $a\geqq 1$ there is a lattice point on L in region (1). Let P be such a point closest to the line CD. Let Q be the lattice point on M such that the distance $PQ=1$.

In which region is the point Q located? It cannot be in (7) or (8) since $\theta\leqq\pi/2$. It cannot be in (4), (5) or (6) since $b\geqq 1$. It cannot be in (2) or (3); in fact no point on M lies in (2) or (3) because $\theta\geqq\gamma$ and the line M intersects the line AB to the right of B. Hence Q lies in region (1). But this means that the line M has points in both regions (1) and (5), from which it follows that a segment of M of length at least 1 lies in $ABCD$. Thus we have a contradiction and Lemma 8 is proved.

LEMMA 9. *Let $ABCD$ be a parallelogram with $a\geqq 1$ and $b\geqq 1$ that does not have the lattice point covering property. Let a coordinate system be imposed on $ABCD$ so*

that no lattice point is inside or on the parallelogram. Then

$$a < \sqrt{2}\cdot\cos\left(\frac{\pi}{4} - \theta\right) \leqq \sqrt{2}, \qquad b < \sqrt{2}\cdot\cos\left(\frac{\pi}{4} - \gamma + \theta\right) \leqq \sqrt{2},$$

where θ is defined as in Lemma 8.

Proof. Note that $0<\theta<\gamma$ by Lemma 8. Slide the parallelogram down on the lines AD and BC extended until a lattice point appears on the line segment AB. With the parallelogram in this position there is at least one lattice point P on the segment AB but no lattice points in the part of the parallelogram off this segment. Let Q be the lattice point such that $PQ=1$ and PQ makes an angle θ with PB. Let S be the lattice point such that $PS=1$, $QS=\sqrt{2}$, and the turn from the direction PQ to PS is counterclockwise. Let R be the lattice point diagonally opposite P in the unit square $PQRS$. We prove that Q lies to the right

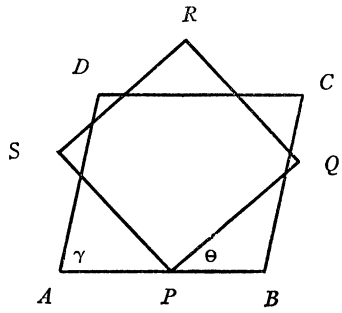


FIG. 5

of BC, i.e. on that side of BC away from AD; that S lies to the left of AD, i.e. on that side of AD away from BC; and that R lies above DC, i.e. on that side of DC away from AB.

First since $0<\theta<\gamma$ we see that Q lies above the line AB and to the right of AD. Also $a>1$ so Q cannot lie above CD. Hence Q lies to the right of BC. Likewise S is above AB, below CD, and to the left of BC. Hence S is to the left of AD. Now $b\geqq 1$ implies that R lies between the lines AD and BC since $SR=RQ=1$. But R does not lie below AB, and so R lies above CD.

Now impose a new coordinate system with the origin at P, the positive end of the x axis along PB away from A, and the positive end of the y axis from P towards CD. Then P is (0, 0) and R is $(\cos\theta-\sin\theta, \cos\theta+\sin\theta)$. But R lies above CD, so $\cos\theta+\sin\theta>a$ and this establishes the first part of the lemma.

Next impose a new xy coordinate system, discarding the preceding one, with origin at Q, x axis parallel to BC and with positive end in the direction B to C, and with positive end of the y axis from Q towards AD. Then Q is (0, 0) and S is

$$\left(\cos\left(\frac{\pi}{2}-\gamma+\theta\right)-\sin\left(\frac{\pi}{2}-\gamma+\theta\right),\right.$$
$$\left.\cos\left(\frac{\pi}{2}-\gamma+\theta\right)+\sin\left(\frac{\pi}{2}-\gamma+\theta\right)\right).$$

But S is to the left of AD and Q to the right of BC, and so

$$\cos\left(\frac{\pi}{2}-\gamma+\theta\right)+\sin\left(\frac{\pi}{2}-\gamma+\theta\right)>b.$$

This implies the second inequality of Lemma 9, and thus the proof is complete. The following result follows at once.

LEMMA 10. *If $a\geqq 1$ and $b\geqq\sqrt{2}$ (or $a\geqq\sqrt{2}$ and $b\geqq 1$) then $ABCD$ has the lattice point covering property.*

LEMMA 11. *Assume that $1\leqq a<\sqrt{2}$ and $1\leqq b<\sqrt{2}$. Define $\alpha=\arccos a/\sqrt{2}$ and $\beta=\arccos b/\sqrt{2}$, so that $0<\alpha\leqq\pi/4$ and $0<\beta\leqq\pi/4$. If $ABCD$ does not have the lattice point covering property then $\alpha+\beta+\gamma>\pi/2$.*

Proof. By Lemma 9 we see that

$$\cos\left(\frac{\pi}{4}-\gamma+\theta\right)>\cos\beta \quad\text{and}\quad \cos\left(\frac{\pi}{4}-\theta\right)>\cos\alpha,$$

$$\left|\frac{\pi}{4}-\gamma+\theta\right|<\beta \quad\text{and}\quad \left|\frac{\pi}{4}-\theta\right|<\alpha,$$

$$-\beta<\frac{\pi}{4}-\gamma+\theta<\beta \quad\text{and}\quad -\alpha<\frac{\pi}{4}-\theta<\alpha,$$

$$\frac{\pi}{4}-\alpha<\theta<\gamma+\beta-\frac{\pi}{4}, \qquad \alpha+\beta+\gamma>\frac{\pi}{2}.$$

THEOREM 12. *Let a and b be the distances between the pairs of opposite sides, say with $a\leqq b$, of a parallelogram $ABCD$ with an interior angle $\gamma<\pi/2$. The parallelogram has the lattice point covering property if and only if $a\geqq 1$ and one of the following conditions holds:*

(i) $b\geqq\sqrt{2}$;

(ii) $b<\sqrt{2}$ *and* $\alpha+\beta+\gamma\leqq\pi/2$, *where* $\alpha=\arccos a/\sqrt{2}$ *and* $\beta=\arccos b/\sqrt{2}$.

Proof. For convenience we use the contrapositive form of the statement: a parallelogram does not have the lattice point covering property if and only if $a<1$, or $1\leqq a\leqq b<\sqrt{2}$ with $\alpha+\beta+\gamma>\pi/2$. These conditions are necessary because of Lemmas 10 and 11.

That the condition $a<1$ is sufficient follows from Lemma 7. So we assume

$1 \leqq a \leqq b < \sqrt{2}$ and $\alpha+\beta+\gamma > \pi/2$ and prove that the parallelogram does not have the lattice point covering property. The second inequality along with $\alpha > 0$, $\beta > 0$, and $\gamma < \pi/2$ implies

$$\frac{\pi}{4} - \alpha < \frac{\pi}{4} + \alpha, \qquad \frac{\pi}{4} - \alpha < \beta + \gamma - \frac{\pi}{4},$$

$$\gamma - \beta - \frac{\pi}{4} < \frac{\pi}{4} + \alpha, \qquad \gamma - \beta - \frac{\pi}{4} < \beta + \gamma - \frac{\pi}{4}.$$

Hence we can choose a real number θ satisfying

$$\max\left(\frac{\pi}{4} - \alpha, \gamma - \beta - \frac{\pi}{4}\right) < \theta < \min\left(\frac{\pi}{4} + \alpha, \beta + \gamma - \frac{\pi}{4}\right).$$

This real number θ also satisfies

$$0 < \theta < \gamma, \qquad \left|\frac{\pi}{4} - \theta\right| < \alpha, \qquad \left|\frac{\pi}{4} - \gamma + \theta\right| < \beta. \tag{2}$$

To establish these we note first that since $\alpha \leqq \pi/4$, $\beta \leqq \pi/4$,

$$0 \leqq \frac{\pi}{4} - \alpha < \theta < \beta + \gamma - \frac{\pi}{4} \leqq \gamma.$$

The second and third inequalities (2) follow directly from

$$\frac{\pi}{4} - \alpha < \theta < \frac{\pi}{4} + \alpha \quad \text{and} \quad \gamma - \beta - \frac{\pi}{4} < \theta < \beta + \gamma - \frac{\pi}{4}.$$

Now we start with a unit square $RSTU$ of lattice points and construct a

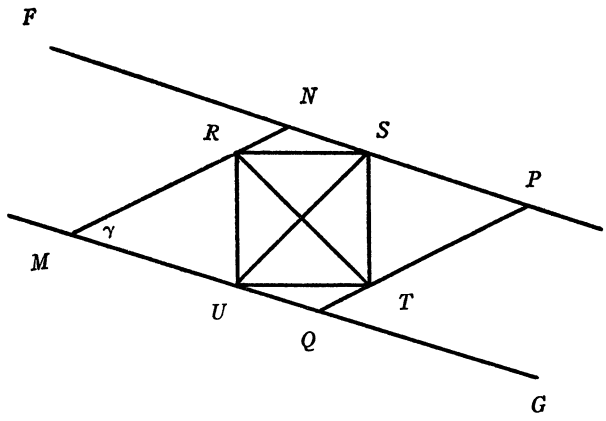


FIG. 6

parallelogram $MNPQ$ as shown in Fig. 6. Here FS and GU are parallel lines such that angle FSR=angle $GUT=\theta$. Also angle $RMU=\gamma$, N is the intersection of FS and MR, P is on FS, Q is on GU, and PQ passes through T. It can be seen that the parallelogram $MNPQ$ covers no lattice points other than R, S, T, U because N and Q lie between the lines UR and ST extended, and M and P lie between RS and TU extended. We see that

$$\text{angle } RTQ = \frac{\pi}{4} + \gamma - \theta, \quad \text{angle } USN = \theta + \frac{\pi}{4}.$$

Hence the distance between MN and PQ is, by use of (2),

$$RT \sin\left(\frac{\pi}{4} + \gamma - \theta\right) = \sqrt{2} \cos\left(\frac{\pi}{4} - \gamma + \theta\right) > \sqrt{2} \cos \beta = b.$$

Also the distance between NP and MQ is

$$US \sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2} \cos\left(\frac{\pi}{4} - \theta\right) > \sqrt{2} \cos \alpha = a.$$

But the parallelogram $ABCD$ described in the theorem has altitudes a and b and angle γ, and thus $ABCD$ can be inscribed inside $MNPQ$, with no boundary point of $MNPQ$ lying inside or on $ABCD$. This establishes that $ABCD$ does not have the lattice point covering property.

THEOREM 13. *Let a and b be the distances between the pairs of opposite sides, say with $a \leqq b$, of a parallelogram or rectangle with an interior angle $\gamma \leqq \pi/2$. The figure has the lattice point covering property if and only if $a \geqq 1$ and one of the following conditions holds:*

(i) $b \geqq \sqrt{2}$;

(ii) $b < \sqrt{2}$ *and* $2 \cos \gamma \geqq a\sqrt{2-b^2} + b\sqrt{2-a^2}$.

Proof. This differs from Theorem 12 in the formulation of condition (ii) and in the extension to $\gamma = \pi/2$. Now a rectangle of dimensions a by b with $a \leqq b$ has the lattice point covering property if and only if $a \geqq 1$ and $b \geqq \sqrt{2}$. (See [1].) This is what both Theorems 12 and 13 reduce to in case $\gamma = \pi/2$, because condition (ii) is satisfied in neither theorem. For $\gamma < \pi/2$ it is easy to establish the equivalence of condition (ii) in the two theorems. For $\alpha+\beta+\gamma \leqq \pi/2$ implies

$$\alpha + \beta \leqq \frac{\pi}{2} - \gamma, \quad \sin(\alpha + \beta) \leqq \sin\left(\frac{\pi}{2} - \gamma\right) = \cos \gamma,$$

$$\frac{\sqrt{2 - a^2}}{\sqrt{2}} \cdot \frac{b}{\sqrt{2}} + \frac{a}{\sqrt{2}} \cdot \frac{\sqrt{2 - b^2}}{\sqrt{2}} \leqq \cos \gamma,$$

and this gives condition (ii) of Theorem 13; also the steps of this argument are reversible.

Theorem 13 can be formulated in nongeometric language in the following

way. Consider the inequalities

$$(3)\qquad \begin{aligned} a_1 &\leqq c_{11}x + c_{12}y \leqq a_1 + a, \\ b_1 &\leqq c_{21}x + c_{22}y \leqq b_1 + b, \end{aligned}$$

and the equalities

$$(4)\qquad c_{11}^2 + c_{12}^2 = 1, \qquad c_{21}^2 + c_{22}^2 = 1, \qquad |c_{11}c_{21} + c_{12}c_{22}| = c.$$

THEOREM 14. *Let a, b and c be real numbers satisfying the conditions $a>0$, $b>0$, $a\leqq b$, $0\leqq c\leqq 1$. The inequalities* (3) *have a solution in integers x, y for every set of real numbers a_1, b_1, c_{11}, c_{12}, c_{21}, c_{22} satisfying* (4) *if and only if $a\geqq 1$ and one of the following conditions holds*:

(i) $b\geqq\sqrt{2}$;

(ii) $b<\sqrt{2}$ *and* $2c\geqq a\sqrt{2-b^2}+b\sqrt{2-a^2}$.

Supported in part by N.S.F. Contracts GP 3801 (I. Niven) and GP 3788 (H. S. Zuckerman).

Added in proof: The proof of Lemma 3 is incorrect. A correction will be forthcoming.

Reference

1. Ivan Niven, Diophantine Approximations, Interscience, New York, 1963, p. 54.

PRODUCTS OF POINTS—SOME SIMPLE ALGEBRAS AND THEIR IDENTITIES

TREVOR EVANS, Emory University

Introduction. The collection of algebras we introduce here were discovered during a study of groupoids satisfying the identity

$$(xy)(yz) = y \quad \text{for all } x, y, z.$$

The original question was whether there existed a nontrivial groupoid satisfying this identity. Rather surprisingly, it turns out that if a finite groupoid satisfies this identity, then it must contain a square number of elements. From this, it is not too hard to find a model of the identity. If S is any nonempty set, then the multiplication

$$(a, b)\cdot(c, d) = (b, c)$$

defined on the set of all ordered pairs of elements of S does indeed satisfy $xy\cdot yz = y$, identically.

This suggests that we look at other multiplications of the same type. On a set S, we can always introduce two different associative multiplications by defining a product ab to be a, for each a, b in S or by defining ab to be b, for each a, b in S. There are sixteen analogues of these multiplications on the set of all

ordered pairs of elements of S, each of which is described by a rule of the form

$$(a, b)\cdot(c, d) = (\ ,\)$$

where the right-hand side $(\ ,\)$ is some ordered pair selected from a, b, c, d. Because of the interpretation of each of these products as a geometric construction on the graph of the cartesian product $S\times S$, we call them *products of points*.

Although the product $(a, b)\cdot(c, d)=(b, c)$ is probably the most interesting, several of the other multiplications yield identities worth studying. For example, the product

$$(a, b)\cdot(c, d) = (a, d)$$

is a special case of what is usually called a rectangular band [2]. This product satisfies the identities

$$(xy)z = x(yz), \qquad xyz = xz, \qquad x^2 = x \quad \text{for all } x, y, z.$$

That is, the product defines an idempotent semigroup satisfying the further identity $xyz=xz$. Conversely, any idempotent semigroup satisfying the identity $xyz=xz$ is a rectangular band.

As another example, consider the product

$$(a, b)\cdot(c, d) = (a, c).$$

Since $(a, x)\cdot(a, y)=(a, a)$ for all x, y in S, this groupoid is the union of disjoint constant semigroups $K_a=\{(a, x): x\in S\}$. Now this multiplication satisfies the identities

$$(xy)z = xz, \qquad x(yz) = xy \quad \text{for all } x, y, z,$$

and we show that any groupoid satisfying these identities is, in fact, a union of disjoint constant semigroups.

Our aim, in general, is to study the varieties of groupoids defined by the identities which these 'products of points' satisfy. By a *variety* of algebras, in general, we mean a class of algebras which can be described axiomatically by a set of identities. For example, the classes of groups, rings, lattices, loops, semigroups, etc. are all varieties. By a *variety of groupoids*, we mean the class of all groupoids satisfying some given set of identities. Not all types of algebras of interest form varieties (for example, fields do not) but varieties of algebras are on the one hand sufficiently general to be interesting to the universal algebraist and, on the other hand, sufficiently special to provide nontrivial theorems. The final section of the paper is a brief discussion of some of the problems in universal algebra which are illustrated by the algebras and identities we have studied.

1. The identity $xy\cdot yz=y$. The two products defined on the cartesian product $S\times S$

$$(a, b)(c, d) = (b, c)$$
$$(a, b)\circ(c, d) = (d, a)$$

are very closely related. In fact, $x \circ y = yx$. Thus, it will suffice to look at only one of them and we choose the first.

We digress briefly to note that not only can we express the ($\circ$) operation above in terms of this product—we can obtain all products of points in terms of this fundamental product. We will not give the complete list. Here are some examples. Let $x=(a, b)$, $y=(c, d)$. Then,

$$(xx)x = (a, a), \qquad x(xx) = (b, b), \qquad (xy)x = (c, a),$$
$$(yx)y = (a, c), \qquad x(yy) = (b, d), \qquad y(xx) = (d, b).$$

It is also true, but not quite so easy to see, that the $(a, b)(c, d)=(b, c)$ product (and the ($\circ$) operation) are the only products of points which can be used in this way to construct all such products. We cannot construct the pair (b, c) from (a, b), (c, d) and the ($*$) operation $(a, b) * (c, d)=(a, d)$, for example, since any repeated product using the ($*$) operation must always yield a pair having a or c as its first component, and b or d as its second.

Our main concern in the study of the $(a, b) \cdot (c, d)=(b, c)$ product, however, is not the above property but an interesting identity this product satisfies. For any (a, b), (c, d), (e, f) in $G=S\times S$, we have

$$\{(a, b)(c, d)\}\{(c, d), (e, f)\} = (b, c)(d, e)$$
$$= (c, d)$$

and hence this product satisfies

$$(xy)(yz) = y \quad \text{for all } x, y, z \text{ in } G.$$

Clearly, if S is finite and contains n elements, then this groupoid G contains n^2 elements. The converse of this statement is also true. Let G be a finite groupoid satisfying the identity $xy \cdot yz = y$, for all x, y, z in G. Let $x=tu$, $y=uv$ in this identity. Then $(tu \cdot uv)(uv \cdot z)=uv$ for all t, u, v, z in G. That is

$$u(uv \cdot z) = uv, \quad \text{for all } u, v, z \text{ in } G.$$

Similarly, we obtain the identity

$$(x \cdot uv)v = uv, \quad \text{for all } x, u, v \text{ in } G.$$

For each a in G, we define, as usual, the *left* and *right multiplications* L_a, R_a by

$$xL_a = ax, \qquad xR_a = xa, \quad \text{for all } x \text{ in } G.$$

In terms of these mappings, the above identities may be written

$$L_uR_zL_u = L_u, \qquad R_vL_xR_v = R_v, \quad \text{for all } x, u, v, z \text{ in } G.$$

We will write $r(P)$ for the number of elements of the range of a mapping $P: G \to G$. From $L_uR_zL_u = L_u$, it follows that $r(L_u) \leqq r(R_z)$ and from $R_vL_xR_v = R_v$, we obtain $r(R_v) \leqq r(L_x)$. Hence, $r(L_x) = r(R_y)$, for all x, y in G. Let n denote the number of elements in the range of each left and right multiplication.

Now let a be any element in G and let b be any element in the range of R_a. That is, $ua=b$, for some u in G. Since $(x\cdot ua)a=ua$, for all x in G, we obtain

$$(xb)a = b \quad \text{for all } x \text{ in } G.$$

But xb takes on n different values as x ranges over G. Hence, R_a maps at least n different elements on b.

Let y be any element mapped onto b by R_a. That is, $ya=b$. Now, if $x\in G$, $xy\cdot ya=(xy)b$. But $xy\cdot ya=y$. Hence, $y=(xy)b$. That is, y is in the range of R_b. Since the range of R_b contains n elements, there are at most n elements mapped onto b by R_a.

It follows that R_a maps exactly n elements onto each element in its range. Since R_a has n elements in its range, G contains n^2 elements.

THEOREM. *A finite model of the identity $xy\cdot yz=y$, for all x, y, z contains n^2 elements, for some positive integer n. Conversely, for each positive integer n, there is a model of the identity containing n^2 elements.*

The groupoid on $S\times S$ with the multiplication $(a, b)\cdot(c, d)=(b, c)$ satisfies many more identities then merely $xy\cdot yz=y$. It is easy to find others such as

$$(x\cdot x^2)(y^2\cdot y) = xy, \qquad (xy)^2(xy) = x\cdot x^2,$$
$$(x\cdot x^2)^2 = x\cdot x^2, \qquad (x\cdot y^2)(z\cdot w^2) = yz^2.$$

In fact, we can find enough identities to characterize completely groupoids of this type.

THEOREM. *Let G be a groupoid satisfying the identities*
(i) $xy\cdot yz=y$ *for all* x, y, z
(ii) $(x\cdot x^2)(y^2\cdot y)=xy$ *for all* x, y
(iii) $(xy)^2(xy)=x\cdot x^2$ *for all* x, y.
Then G is isomorphic to the groupoid on S^2 (where S is the set of idempotents of G) with multiplication $(a, b)\cdot(c, d)=(b, c)$.

Proof. From (i) $x^2\cdot x^2=x$ and hence, putting $y=x^2$ in (ii), we obtain

$$(x\cdot x^2)^2 = x\cdot x^2 \quad \text{for all } x \text{ in } G.$$

Hence, for each x, $x\cdot x^2$ is an idempotent in G. Let S be the set of all such idempotents. Since $(xy)^2(xy)=x\cdot x^2$, we see that any element in S may also be written as $v^2\cdot v$ for some v in G. Hence, any element in $S\times S$ may be written as $(u\cdot u^2, v^2\cdot v)$ for some u, v in G.

Now $u\cdot u^2=(uv)^2(uv)$. Furthermore, $v^2\cdot v=(uv\cdot vw)^2(uv\cdot vw)$ by identity (i) and $(uv\cdot vw)^2\cdot(uv\cdot vw)=(uv)(uv)^2$ by identity (iii). Hence, $(u\cdot u^2, v^2\cdot v)=((uv)^2(uv), (uv)(uv)^2)$ and so any element in $S\times S$ may be written as $(x^2\cdot x, x\cdot x^2)$ for some x in G.

Let us now introduce on $S\times S$, the multiplication $(a, b)\cdot(c, d)=(b, c)$. We will show the correspondence

$$(x^2\cdot x, x\cdot x^2)\leftrightarrow x$$

is an isomorphism between this groupoid on $S\times S$ and the groupoid G.

The correspondence is certainly one-one since $x=y$ implies $(x^2\cdot x,\ x\cdot x^2)=(y^2\cdot y,\ y\cdot y^2)$ and $(x^2\cdot x,\ x\cdot x^2)=(y^2\cdot y,\ y\cdot y^2)$ implies $(x^2\cdot x)(x\cdot x^2)=(y^2\cdot y)(y\cdot y^2)$ which, in turn, by identity (i) implies $x=y$.

To see that it is an isomorphism, consider the product

$$(x^2\cdot x,\ x\cdot x^2)(y^2\cdot y,\ y\cdot y^2)\ =\ (x\cdot x^2,\ y^2\cdot y)$$

in the groupoid on $S\times S$. Now

$$(x\cdot x^2,\ y^2\cdot y)\ =\ ((xy)^2(xy),\ (xy)(xy)^2)$$

by identity (iii) and by our earlier result $v^2\cdot v=(uv)(uv)^2$. But $((xy)^2(xy),\ (xy)(xy)^2)$ corresponds to xy in G under the one-one mapping described above. Hence, the mapping is an isomorphism.

REMARKS. The identities listed in the preceding theorem are not necessarily the simplest which will characterize this particular product of points; it may be that one other identity, in addition to $xy\cdot yz=y$ is sufficient. The identity $xy\cdot yz=y$ by itself is not sufficient since it can be shown that the free groupoid on one generator satisfying this identity is infinite.

The multiplication $(a,\ b)\cdot(c,\ d)=(d,\ a)$ which we mentioned at the beginning of this section also satisfies the identity $(xy)(yz)=y$ and hence a completely similar analysis may be carried out for this product.

2. Rectangular bands. We remarked earlier that on a nonempty set S we can always construct two semigroups

(i) *the left-zero semigroup on S*, with multiplication given by $ab=a$, for all $a,\ b$ in S,

(ii) *the right-zero semigroup on S*, with multiplication given by $ab=b$, for all $a,\ b$ in S.

The groupoid on $S\times S$ with multiplication

$$(a,\ b)\cdot(c,\ d)\ =\ (a,\ d)$$

is simply the direct product of the left and right-zero semigroups on S and similarly, the multiplication $(a,\ b)(c,\ d)=(c,\ b)$ is the direct product of the right and left-zero semigroups on S. Hence, both of these products on $S\times S$ are associative and, in fact, satisfy the further identities

$$x^2\ =\ x,\qquad xyz\ =\ xz\quad \text{for all } x,\ y,\ z.$$

(We omit parentheses in products such as xyz, from now on, since the multiplication is associative.)

By a *rectangular band*, we mean a semigroup having as elements all ordered pairs in $S\times T$, where S, T are any sets, and with a product defined by $(s_1,\ t_1)\cdot(s_2,\ t_2)=(s_1,\ t_2)$. The product defined above on $S\times S$ is the special case $S=T$. Furthermore, the identities $x^2=x$, $xyz=xz$ are also satisfied by any rectangular band. We now prove the converse of this.

THEOREM. *A semigroup satisfying the identities* $xyz=xz$ *for all* x, y, z; $x^2=x$ *for all* x, *is a rectangular band.*

Proof. Let B be a semigroup satisfying these identities. We first construct two homomorphic images of B which are left and right-zero semigroups, respectively.

Consider the relation λ on B defined by

$$x \lambda y \text{ if and only if } xt = yt, \quad \text{for all } t.$$

In terms of left multiplications, this condition may be written as $L_x=L_y$. Clearly, λ is an equivalence relation on B. In addition if $x \lambda y$ and $z \lambda w$, then $(xz)t=xt=yt=(yw)t$ for all t in B. Hence, $xz \lambda yw$. That is, λ is a congruence relation on B. Since $(xy)t=xt$ for all x, y, t in B, $xy \lambda x$. That is, if C_x denotes the congruence class containing x, then $C_{xy}=C_x$. Hence, in the quotient semigroup B_λ, $C_xC_y=C_x$ and so B_λ is a left-zero semigroup.

Similarly, we define a congruence relation on B by $x \rho y$ if and only if $tx=ty$ for all t. The corresponding quotient algebra B_ρ is a right-zero semigroup.

Consider the direct product of B_λ and B_ρ. If C_x denotes the λ-congruence class of B containing x and D_x denotes the ρ-congruence class of B containing x, then the mapping

$$\alpha: x \to (C_x, D_x)$$

is a homomorphism of B into $B_\lambda \times B_\rho$. In fact, it is onto $B_\lambda \times B_\rho$, since $(C_x, D_y) = (C_{xy}, D_{xy})$. It is also one-one since if $(C_x, D_x)=(C_y, D_y)$, then $xt=yt$, $tx=ty$ for all t. Hence, $x=xx=yx=yy=y$.

This completes the proof. We have shown that B is isomorphic to the rectangular band on the cartesian product of the set of all C_x's and the set of all D_x's.

There is clearly a very close connection between the product discussed in this section and the product $(a, b)\cdot(c, d)=(b, c)$ of the preceding section. An abstract setting for this connection is provided by the following theorem:

Let E *be an idempotent semigroup satisfying the further identity* $xyz=xz$, *for all* x, y, z, *and let* $\theta: E \to E$ *be an anti-automorphism of* E *of period two. Then the operation* $(\circ)$ *defined by* $x \circ y=(xy)\theta$ *satisfies the identity*

$$(x \circ y) \circ (y \circ z) = y.$$

The proof is purely computational. We remark that this result implies that a finite idempotent semigroup satisfying $xyz=xz$ for all x, y, z and having an anti-automorphism of order two necessarily contains a square number of elements.

3. Unions of constant semigroups. The product in $S \times S$ given by

$$(a, b)\cdot(c, d) = (a, c)$$

satisfies the identities

$$xy\cdot z = xz, \qquad x\cdot yz = xy \quad \text{for all } x, y, z.$$

The subset K_a of all pairs in $S \times S$ of the form (a, x), a fixed, x any element in S, is closed under this multiplication and in fact, K_a is a constant semigroup since $(a, x) \cdot (a, y) = (a, a)$. (By a *constant semigroup*, we mean a set and a multiplication satisfying $xy = zw$ for all x, y, z, w, i.e. all products are the same. It is easy to verify that such a multiplication is associative.)

Furthermore, if K_a is the constant semigroup consisting of all pairs (a, x) and if K_b is the constant semigroup consisting of all pairs (b, x), then all products of an element in K_a and in K_b (in that order) are equal and lie in K_a.

We will now show that the two identities $xy \cdot z = xz$ and $x \cdot yz = xy$ characterize such unions of constant semigroups. Let G be a groupoid satisfying these identities. We begin by decomposing G by the equivalence relation

$$x \equiv y \text{ if and only if } xt = yt \quad \text{for all } t \text{ in } G$$

i.e. $x \equiv y$ if and only if $L_x = L_y$. If $x \equiv y$, then $t \cdot xt = t \cdot yt$ and so, by the identity $x \cdot yz = xy$, we have

$$tx = ty \quad \text{for all } t \text{ in } G.$$

It follows that this is a congruence relation on G since $x \equiv y$, $z \equiv w$ implies $xz = xw = yw$. The resulting quotient groupoid is a left zero semigroup since $xy \equiv x$.

Let C_u be the congruence class containing u and let x, y be any elements in C_u. Now $xy \cdot t = xt$ for all t and hence $xy \equiv x$. That is, $xy \in C_u$ and so C_u is a subgroupoid of G. Furthermore, if $x, y, z, w \in C_u$ then $xy = xw = zw$. Hence, C_u is a constant semigroup.

We now know that G is the union of disjoint constant semigroups and that these are congruence classes for a congruence relation yielding a left-zero semigroup as quotient algebra. The product of an element in C_u and one in C_v (in that order) belongs to C_u. Furthermore, for any x, z in C_u and y, w in C_v, $xy = zw$ since $xy = zy$ and $zy = zw$. Hence, the product of any element in C_u and any element in C_v is some fixed element in C_u.

This is all the information we can obtain from the identities $xy \cdot z = xz$ and $z \cdot yz = xy$ since if we take disjoint sets C_α, (α in some indexing set I), and pick, for each pair C_α, C_β, a fixed element $c_{\alpha\beta}$ in C_α, then the product on $\bigcup_{\alpha \in I} C_\alpha$ defined by $xy = c_{\alpha\beta}$ if $x \in C_\alpha$ $y \in C_\beta$ satisfies the identities $xy \cdot z = xz$, $x \cdot yz = xy$.

THEOREM. *A groupoid satisfies the identities $xy \cdot z = xz$, $x \cdot yz = xy$ if and only if it is the union of disjoint constant semigroups C_α where all products xy, with $x \in C_\alpha$, $y \in C_\beta$ are equal and belong to C_α.*

The multiplication on S^2,

$$(a, b) \cdot (c, d) = (d, b)$$

also satisfies the identities $xy \cdot z = xz$ and $x \cdot yz = xy$. In this case, the constant subsemigroups are the sets $\{(x, b) : x \in S\}$.

Two other 'products of points' yield to similar treatment. The multiplications

$$(a, b)\cdot(c, d) = (c, a) \qquad (a, b)\cdot(c, d) = (b, d)$$

each satisfies the identities $xy\cdot z = yz$, $x\cdot yz = xz$, for all x, y, z. If G is a groupoid satisfying these identities then the relation $x \equiv y$ if $R_x = R_y$ defines a congruence relation and the congruence classes are constant subsemigroups of G. In fact, we obtain a characterization of G completely analogous to the description in the preceding theorem with *left-zero semigroup* replaced by *right-zero semigroup*.

4. The other products. There are eight other products of points still to consider. The products

$$(a, b)\cdot(c, d) = (a, b), \qquad (a, b)\cdot(c, d) = (c, d)$$

yield left-zero and right-zero semigroups and there is nothing more to be said about them. The closely related products

$$(a, b)\cdot(c, d) = (b, a), \qquad (a, b)\cdot(c, d) = (d, c)$$

are more interesting. We will look at the first one only since the second may be treated similarly. The product $(a, b)\cdot(c, d) = (b, a)$ satisfies the identities

$$xy\cdot z = x, \qquad xy = xz \quad \text{for all } x, y, z.$$

Let G be a groupoid satisfying these identities. From the second, we see that $R_y = R_z$ and so all right multiplications of G are the same. From the first identity, $xR_yR_z = x$ and so the square of each right multiplication of G is the identity mapping. This is all that can be said about G since any groupoid which has all right multiplications equal and of order two necessarily satisfies the identities $xy\cdot z = x$, $xy = xz$. Using these properties of the right multiplications of G, it is a simple matter to construct a table for a groupoid satisfying these identities. One with three elements is shown in Fig. 1.

	a	b	c
a	a	a	a
b	c	c	c
c	b	b	b

FIG. 1.

The remaining four products are:

$$(a, b)\cdot(c, d) = (a, a), \qquad (a, b)\cdot(c, d) = (b, b)$$
$$(a, b)\cdot(c, d) = (c, c), \qquad (a, b)\cdot(c, d) = (d, d).$$

The first two satisfy the identity $xy = (xz)w$ and the second pair satisfies $xy = z(wy)$. We will consider only the identity $xy = (xz)w$. This implies the identities

$$xy = xz, \qquad xy\cdot y = xy, \qquad xy\cdot z = x\cdot yz \quad \text{for all } x, y, z.$$

Hence a groupoid G satisfying $xy=(xz)w$ is associative, all of its right multiplications are equal, $R_y=R_z$, and furthermore, are idempotent, $R_y^2=R_y$. The first two identities $xy=xz$, $xy\cdot y=xy$ are, in fact, equivalent to the single identity $xy=xz\cdot w$ and imply the associative law. These properties of the right multiplications again make it easy to construct examples of groupoids satisfying $xy=(xz)w$. One with three elements is shown in Fig. 2.

	a	b	c
a	b	b	b
b	b	b	b
c	c	c	c

FIG. 2

5. Some problems in universal algebra. We conclude by mentioning some examples of problems in universal algebra which are illustrated by the algebras we have studied above. We recall that a variety of groupoids is the class of all groupoids satisfying some given set of identities. By Birkhoff's Theorem, a class of groupoids V is a variety if and only if V is closed under taking subalgebras, homomorphic images and (unrestricted) direct products.

It seems very difficult to decide whether a variety of groupoids defined by a finite set of identities (or even by just one identity) actually contains nontrivial algebras. If $u(x, y, \ldots)=v(x, y, \ldots)$ is a groupoid identity and V is the variety of all groupoids satisfying it, then we see that (i) if both sides $u(x, y, \ldots)$, $v(x, y, \ldots)$ are products of variables then V contains constant semigroups (ii) if one side, say $v(x, y, \ldots)$ is simply one variable y, and the other side $u(x, y, \ldots)$ either begins or ends with y, then V contains either left- or right-zero semigroups. It is for this reason that we first studied the identity $xy\cdot yz=y$. It is the simplest identity for which it is not fairly easy to decide whether there is a groupoid satisfying it which contains more than one element. The similar-looking identities

$$x(y\cdot yz)=y, \qquad x(yy\cdot z)=y, \qquad (xy\cdot y)z=y, \qquad (x\cdot yy)z=y$$

have only trivial models.

The identity $xy\cdot yz=y$ is interesting for another reason. Let us call the set of orders of the finite algebras in a variety, the *spectrum* of the variety. What can be said about the spectrum of a variety? It is closed under multiplication since the direct product of two groupoids in a variety lies in the variety. Any set of positive integers generated multiplicatively by some finite set of primes is the spectrum of a variety of groupoids given by a single identity. For, let $p_1, p_2, \cdots, p_t$ be distinct primes and let $n=p_1p_2\cdots p_t$. The variety of abelian groups satisfying $x^n=1$ can be described as a variety of groupoids satisfying a single identity [4] and the groupoids in this variety have orders $p_1^{i_1}p_2^{i_2}\cdots p_t^{i_t}$; $i_s\geqq 0$. As another example, the variety of idempotent totally symmetric quasigroups may be defined as a variety of groupoids satisfying $x(xy)=y$, $xy=yx$,

$x^2=x$ and the spectrum of this variety consists of all positive integers of the form $6n+1$ or $6n+3$, [1]. Our main result in Section 1 may be stated as: the spectrum of the groupoid variety defined by $xy\cdot yz=y$ is the set of all squares, $\{1, 4, 9, 16, \cdots\}$. The author has generalized this result [3] and obtained, for each positive integer n, a variety of groupoids having as spectrum all nth powers. For example, the variety defined by the identities $\{(x\cdot yz)(wz)\}(zt)=z$, $(x\cdot yz)w=(x\cdot yz)t$, for all x, y, z, w, t, has $\{1^3, 2^3, 3^3, \cdots\}$ as its spectrum.

A set of groupoid identities I is said to be *equationally complete* if it is maximal, that is, if for any groupoid identity $u=v$, either $u=v$ is deducible from I or the addition of $u=v$ to I yields a set of identities $I\cup\{u=v\}$ which has no nontrivial models. A variety is said to be equationally complete if its set of identities are equationally complete. Although it is known that there are uncountably many different equationally complete groupoid varieties [5], the problem of finding explicit sets of identities which are equationally complete is not easy. It would take us too far afield to prove it but we note that the variety of Section 1 defined by

$$xy\cdot yz = y, \qquad (xy)^2(xy) = xy, \qquad (x\cdot x^2)(x\cdot x^2) = x\cdot x^2$$

is equationally complete. None of the other varieties described are equationally complete. In fact, in each case, either the identity $xy=x$ or $xy=y$ may be added to the identities.

The problem of deciding whether a variety of groupoids is nontrivial, that is, contains nontrivial groupoids, is equivalent to the problem of deciding whether the identity $x=y$ follows from the defining identities of the variable. This is a special case of the problem of deciding, for an arbitrary identity, whether or not it holds in a particular variety. This question is easy to answer for all the varieties we study except for the variety defined by $xy\cdot yz=y$. For example, an identity is a consequence of the identities $(xy)z=xz$, $x(yz)=xy$ if and only if it is of the form $(x\cdots)\cdot(y\cdots)=(x\cdots)\cdot(y\cdots)$, where $x\cdots$ is any groupoid expression having x as the first letter. To decide when an identity is a consequence of $xy\cdot yz=y$ requires more sophisticated techniques and an analysis of the structure of a free groupoid satisfying this identity. The test may be described as follows. Let $w_1(x, y, z, \cdots)$, $w_2(x, y, z, \cdots)$ be groupoid 'words' in the variables $x, y, z, \cdots$. Let the identities $xy\cdot yz=y$, $x(xy\cdot z)=xy$, $(x\cdot yz)z=yz$ be used to shorten w_1 and w_2, in the sense that any occurrence of a left-hand side of one of these three identities in w_1 or w_2 be replaced by the corresponding right-hand side. Eventually, we obtain from w_1 and w_2 new expressions $w_1^*(x, y, z, \cdots)$, $w_2^*(x, y, z, \cdots)$ which cannot be shortened further in this manner. Then $w_1=w_2$ is an identity which is a consequence of $xy\cdot yz=y$ if and only if w_1^* and w_2^* are identical.

Added in Proof: After the abstract of [3] appeared, D. Knuth sent me a description of a number of different ways to approach the study of the identity $(xy)(yz)=y$ and he obtained a simpler set of identities than that given here to characterize the product, $(a, b)\cdot(c, d)=(b, c)$. A former student of mine, Marshall Saade, has found two identities which characterize the algebras of n-

tples described in [3] and has studied products of n-tples which correspond to the products of ordered pairs in this paper.

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THE STELLATED FORMS OF THE SIXTEEN-CELL

B. L. CHILTON, State University of New York at Buffalo

Introduction. Let II be a regular or semi-regular convex polyhedron. A *stellated* form II′ of II is defined to be a polyhedron satisfying the following five restrictions:

(i) The faces of II′ must lie in the planes of the faces of II.

(ii) If F_1 and F_2 are congruent faces of II and are related by some symmetry operation of II, then the faces F_1' and F_2' of II′ which lie in the planes of F_1 and F_2, respectively, must likewise be congruent.

(iii) A face F' of II′ must possess at least the rotational symmetry of the face F of II in whose plane F' lies.

(iv) The parts included in any plane must all be "accessible" in the completed solid (i.e., on the "outside").

(v) Excluded from consideration shall be cases where the parts can be divided into two sets, each giving a solid with as much symmetry as the whole figure.

The above restrictions are based on conditions devised by J. C. P. Miller for the stellated forms of the icosahedron, $\{3, 5\}$.

The stellated forms of the five regular polyhedra have been exhaustively enumerated. The following table condenses the information. (Note that we

Name of convex polyhedron	*Number of stellated forms*	*Date(s) of discovery of nontrivial forms*	*Discoverer(s)*
Tetrahedron $\{3, 3\}$	1	—	—
Cube $\{4, 3\}$	1	—	—
Octahedron $\{3, 4\}$	2	1619	Kepler
Dodecahedron $\{5, 3\}$	4	1619, 1809	Kepler, Poinsot
Icosahedron $\{3, 5\}$	59	1900, 1938	Brückner, Coxeter *et al.*

consider a polyhedron to be a stellated form of itself; we distinguish it as "the trivial stellated form." It will be seen that two of the regular polyhedra possess no nontrivial stellated forms.)

In addition, the stellated forms of certain semiregular polyhedra have been investigated. D. Luke has worked with the rhombic dodecahedron. E. Hess and M. Brückner began the task of stellating the rhombic triacontahedron, and J. D. Ede has recently published a paper in which he has depicted a "complete face" of this polyhedron and in which he describes a number of its stellations.

The techniques developed by Coxeter and others to enumerate the stellated forms of the regular and semiregular polyhedra can be adapted for use on the four-dimensional regular polytopes (and conceivably on polytopes in Euclidean space of any number of dimensions). The purpose of the present article is to apply these techniques to the 16-cell, $\{3, 3, 4\}$. This has been chosen because it is the "simplest" four-dimensional regular polytope which possesses nontrivial stellated forms. (The simplex $\{3, 3, 3\}$ and the hypercube $\{4, 3, 3\}$ have "dihedral" angles $\leqq \pi/2$, which immediately rules out the possibility of nontrivial forms in these cases.) Miller's five conditions can easily be altered for our purposes, by replacing "face" by "cell" and "plane" by "hyperplane."

Stellating the 16-cell. The work is carried out by embedding the 16-cell in a Cartesian coordinate system, and finding the analytic representations of the planes, lines, and points of intersection of sets chosen from among the sixteen hyperplanes of its cells. The points of intersection of four or more hyperplanes are found to comprise four sets, classified as to arrangement and distance from the origin. These points are the vertices of the various stellated forms of the 16-

TABLE I. THE FOUR SETS OF VERTICES

Set	*Typical member*	*Number of vertices*	*Distance from origin*	*Convex polytope formed by set*
1	(2, 0, 0, 0)	8	2	$\{3, 3, 4\}$
2	(1, 1, 1, 1)	16	2	$\{4, 3, 3\}$
3	(2, 2, 2, 0)	32	$2\sqrt{3}$	$t_1\gamma_4$
4	(4, 2, 2, 2)	64	$2\sqrt{7}$	Variant of $t_{0,1}\gamma_4$

cell. Complete information is given in Table I. Note that for each set of points, we have given the coordinates of a "typical" or "representative" point; all the points of the set can be found by listing all ordered quadruples obtainable from the given one by the operations of permutation and change of sign.

The vertices of Set 1 are the vertices of the original 16-cell, and because of the particular coordinates chosen, the edge is $2\sqrt{2}$. The convex hull formed by Set 2 is an 8-cell (hypercube) of edge 2. The vertices of Set 3 are the midpoints of the edges of a hypercube, and as such, form a cell-regular polytope consisting of 8 cuboctahedra and 16 tetrahedra, all of edge $2\sqrt{2}$. In the notation of [2, pp. 146, 158], this figure is $\left\{\begin{smallmatrix}3,3\\4\end{smallmatrix}\right\}$ or $\left\{\begin{smallmatrix}4\\3,3\end{smallmatrix}\right\}$. Set 4 gives a polytope that can best be

described as an "expanded" hypercube in the following sense: Each of the 8 cubic cells of a hypercube of edge 4 is translated bodily away from the center by such a distance that the faces originally in contact are separated by a square prism of height $2\sqrt{2}$ (so that there are 24 such prisms altogether). The vertices are distributed among 16 regular tetrahedra of edge $2\sqrt{2}$. This polytope, while not cell-regular, is a variant of the cell-regular polytope $t_{0,3}\gamma_4$.

Table II is intended as a supplement to the later tables. We consider the line of intersection of three or more cell-hyperplanes of the 16-cell. On this line will be found a number of points of Table I. These can be linearly ordered. What we refer to as a "1-region" in Table II is the segment of such a line between two

TABLE II. THE SEVEN TYPES OF 1-REGION

Type	1 1	1 2	1 3	1 4	2 3	3 3	3 4
Length	$2\sqrt{2}$	2	$2\sqrt{2}$	4	2	$2\sqrt{2}$	$2\sqrt{2}$

TABLE III. THE EIGHT TYPES OF 2-REGION

Type	*Description*
1 1 1	Equilateral triangle
1 1 2	Isosceles right triangle, right angle at 2
1 1 3	Equilateral triangle
1 2 3	Isosceles right triangle, right angle at 2
1 3 3	Equilateral triangle
1 3 4	Isosceles right triangle, right angle at 3
2 3 4	Isosceles right triangle, right angle at 2
3 3 4	Equilateral triangle

consecutive points. We classify the 1-regions according to their end points: e.g., "1 2" refers to a 1-region with one end from Set 1 and the other from Set 2.

The plane of intersection of two or more cell-hyperplanes is covered by a network of lines of intersection. We wish to consider the minimal finite regions into which the plane is cut by this network (minimal in the sense that if P_1 and P_2 are distinct points in the interior of the same region, then the segment $\overline{P_1P_2}$ will not intersect a line of the network). These regions are the "2-regions" of Table III.

In Table IV, we do the same to a cell-hyperplane: we consider its decomposition by its intersections with the other cell-hyperplanes not parallel to it, and classify the minimal finite regions into seven categories.

Table V concerns itself with the division of E^4 by the cell-hyperplanes. The regions can be classified into five "layers," of which the innermost is the 16-cell itself.

The layers **b**, **c**, and **d** completely surround the origin; thus, for instance, in

TABLE IV. THE SEVEN TYPES OF 3-REGION

Type	*Description*	*Composition: Type of face*	*Number*
1 1 1 1	Regular tetrahedron	1 1 1	4
1 1 1 2	Tetrahedron	1 1 1	1
		1 1 2	3
1 1 2 3	Tetrahedron	1 1 2	1
		1 1 3	1
		1 2 3	2
1 1 3 3	Regular tetrahedron	1 1 3	2
		1 3 3	2
1 2 3 3	Tetrahedron	1 2 3	2
		1 3 3	1
		2 3 3	1
2 3 3 3 4	Triangular dipyramid	2 3 3	3
		3 3 4	3
1 3 3 4	Tetrahedron	1 3 3	1
		1 3 4	2
		3 3 4	1

TABLE V. THE FIVE LAYERS

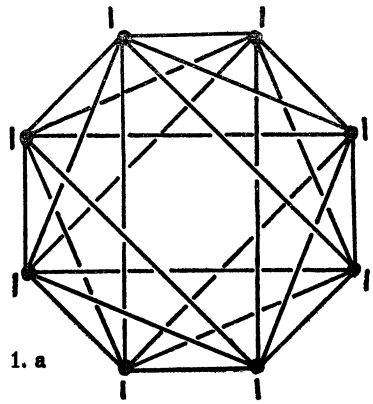
Layer	*Component of layer*	*Structure of component*	*Number of components in layer*
a	16-cell	16 regular tetrahedra 1 1 1 1	1
b	Simplex	1 tetrahedron 1 1 1 1	16
		4 tetrahedra 1 1 1 2	
c	Simplex	2 tetrahedra 1 1 1 2	32
		6 tetrahedra 1 1 2 3	
d	Simplex	2 tetrahedra 1 1 2 3	96
		1 tetrahedron 1 1 3 3	
		2 tetrahedra 1 2 3 3	
e		3 tetrahedra 1 2 3 3	64
		1 dipyramid 2 3 3 3 4	
		3 tetrahedra 1 3 3 4	

the solid obtained by uniting the 16-cell with layer **b**, no part of the 16-cell "remains visible." The layer **b** by itself is not a valid stellated form, by Miller's condition (iv). Thus, if layer **b** is to be present, layer **a** must also be present. Likewise, if layer **c** or **d** is present, then all layers interior to it must be present. Our procedure for naming the stellated forms thus obtained appears in Table VI.

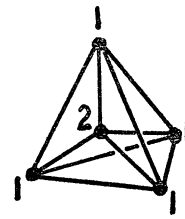
Layer **e** is unique in that it is "center-exposing": there are "holes" in it through which the origin can be reached. Thus, layer **e** by itself constitutes a

TABLE VI. THE SIX STELLATED FORMS

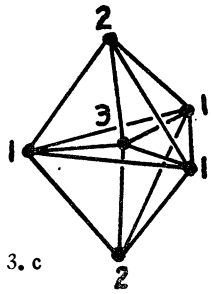
Name	*Composition*
A	**a**
B	**a**∪**b**
C	**a**∪**b**∪**c**
D	**a**∪**b**∪**c**∪**d**
e	**e**
De	**a**∪**b**∪**c**∪**d**∪**e**



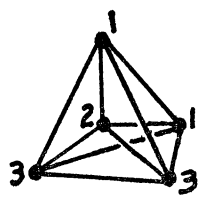
1. a



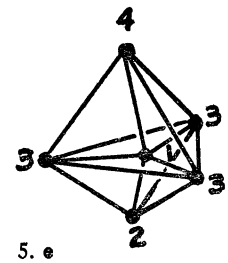
2. b



3. c



4. d



5. e

PLATE I. Components of layers.

valid stellated form; and by combining it with **D**, we obtain still another valid form.

The stellated form **A** is the 16-cell itself. **B** is the compound of two hypercubes, listed by Coxeter [2, p. 150] as $[2\gamma_4]\beta_4$. The vertices of this compound

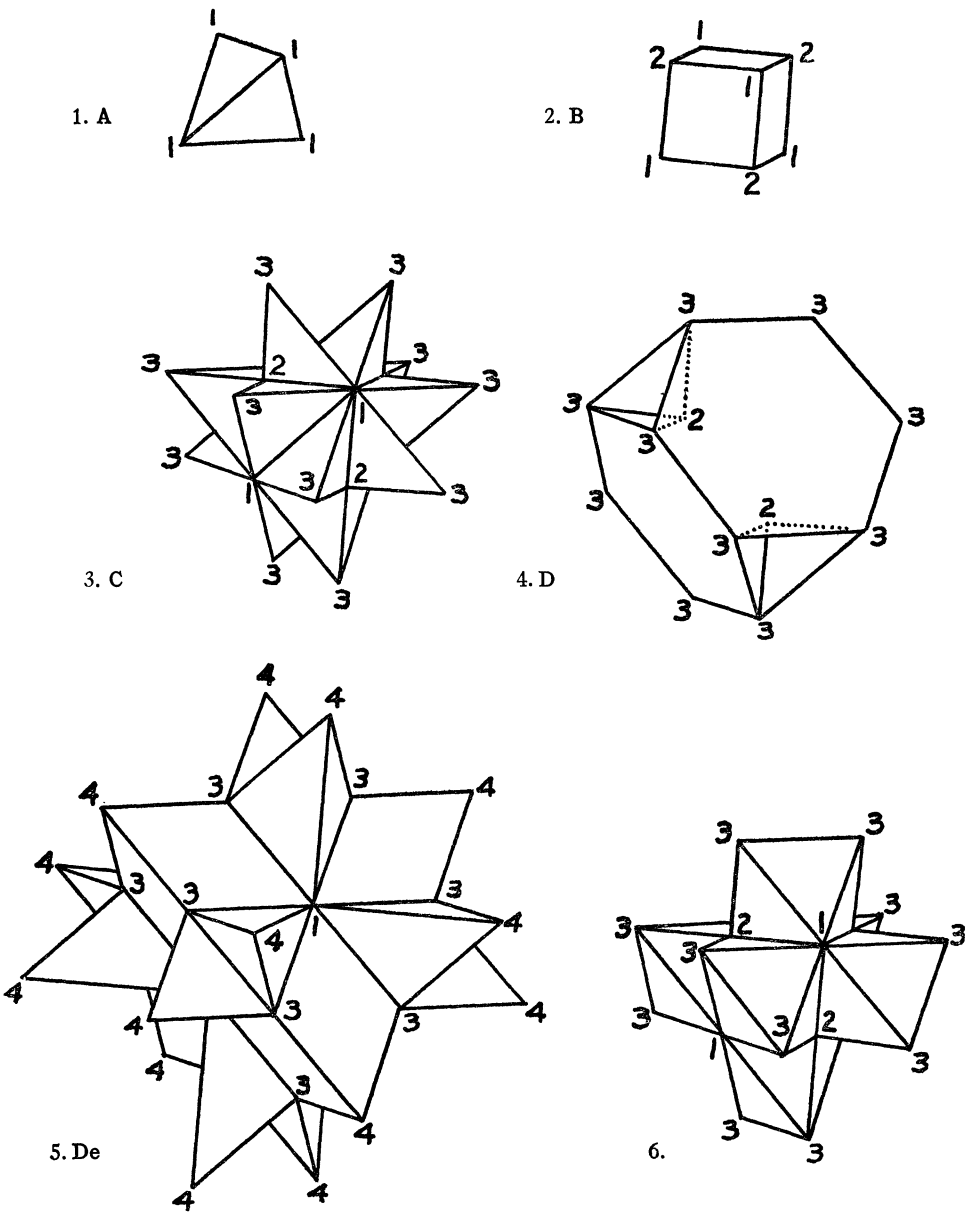


PLATE II. Cells of stellated forms.

(the vertices of our Set 1 and Set 2) are the vertices of a regular 24-cell, $\{3, 4, 3\}$, but 8 of these are shared by two hypercubes whereas the other 16 belong to only one hypercube. The "visible" portion of **B** consists of 64 tetrahedra 1 1 1 2. **C** is formed by inserting into the indentations between pairs of these the 32 figures **c**. This gives a polytope whose visible portions are 192 tetrahedra 1 1 2 3. Into the indentations between pairs of these will fit the 96 simplexes **d**, giving the polytope **D**. Two adjacent cells of **D** which fit together at one of the hexagonal faces are related by a turn through $\pi/3$ radians, so that the triangular depressions are not adjacent. The hexagonal faces are the equatorial hexagons of the cuboctahedra of the semi-regular polytope $\left\{\begin{smallmatrix}3,3\\4\end{smallmatrix}\right\}$.

Placed on **D**, the figures **e** leave the cells 1 1 3 3 of **D** exposed; hence the figures **e** by themselves leave the center of symmetry exposed.

Plate I illustrates the components of the five layers **a**, **b**, **c**, **d**, and **e**. The vertices are numbered according to the sets to which they belong. Plate II illustrates the cells of the stellated forms. The form **e** has "hollow" cells, each one like the solid of Figure II-5 with the solid of Figure II-6 removed.

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ON THE THEORY OF PRE-p-RINGS

H. J. WEINERT, Teachers Training College, Potsdam, Germany

Let p be a prime. Generalizing the concept of p-rings, A. Abian and W. A. McWorter [1] call an associative and commutative ring R with characteristic p a pre-p-ring if $xy^p = x^p y$ for every x and y in R. They prove the following theorem: *Every pre-p-ring R is a direct sum $R = B \oplus N$ of a p-ring B and a nil ring N, where even $x^{p+2} = 0$ for every $x \in N$.* Of course, N is a pre-p-ring, too. It was proved in [1] that N is the radical of R and hence N is uniquely determined by R. Moreover, it is not difficult to show that B is also uniquely determined by R.

A simple calculation shows the following converse of this theorem: *The direct sum $R = B \oplus N$ of a p-ring B and a pre-p-nil-ring N is a pre-p-ring.* Since the structure of p-rings is known, there remains to investigate only the pre-p-nil-rings, which is the purpose of this paper.

First we determine the pre-p-nil-rings as certain algebras over the prime field F_p of characteristic p (Theorem 1). Many pre-p-nil-rings, among them those which may be generated by one element, arise as semigroup rings or as semi-

group rings with a factor system over F_p (cf. Theorem 2 and Theorem 3), but this is not true for all pre-p-nil-rings. In the second part we deal with subdirect decompositions of pre-p-nil-rings. We recall that every p-ring is a subdirect sum and in the finite case even a direct sum of prime fields F_p (cf. [4], [5]); in particular, every subdirectly irreducible p-ring is isomorphic to F_p. Just so every pre-p-nil-ring is a subdirect sum of subdirectly irreducible pre-p-nil-rings. But here we have for every characteristic p even an infinite number of subdirectly irreducible pre-p-nil-rings (likewise of finite as of infinite order), which are not isomorphic (Theorem 6). Another contrast to the situation with p-rings is the existence of finite pre-p-nil-rings, which are subdirectly reducible although they have no decomposition into a direct sum. Finally, for the subdirect irreducibility of a pre-p-nil-ring N it is necessary (Theorem 4) and in the finite case necessary and sufficient (Theorem 5), that the annihilator of N be a principal ideal different from (0).

1. By an algebra over F_p we mean in the following a ring R (for our purposes always associative and commutative), which is also a vector space over the prime field F_p of characteristic p. Hence every element of R may be written uniquely in the form

$$\sum_{i\in I} a_i w_i \quad a_i \in F_p, \text{ only a finite number } a_i \neq 0 \tag{1}$$

by the help of a basis $\{w_i\} \subseteq R$, where I is an index system of an arbitrary cardinal number.

THEOREM 1. *Let R be an algebra over F_p with a basis $\{w_i\}$ having the following properties:*

(a) *$w_i w_i^p = w_i^p w_j$ for every w_i and w_j.*

(b) *For every w_i there is a (minimal) natural number $t = t(w_i) \geqq 2$ with $w_i^t = 0$.*

Then R is a pre-p-nil-ring and, conversely, every pre-p-nil-ring N is an algebra of this kind.

We note that (a) and (b) imply $t(w_i) \leqq p+2$ for every w_i (cf. the proof of Lemma 1 in [1]), whereas conversely (a) is trivial if (b) always holds with $t(w_i) \leqq p$.

Proof. By well-known rules for characteristic p we have for arbitrary elements (1) of the algebra R

$$\left(\sum a_i w_i\right)^{p^2} = \sum a_i w_i^{p^2} = 0,$$

$$\left(\sum a_i w_i\right)\left(\sum b_j w_j\right)^p = \sum a_i b_j w_i w_j^p = \sum a_i b_j w_i^p w_j = \left(\sum a_i w_i\right)^p \left(\sum b_j w_j\right).$$

Hence R is a pre-p-nil-ring. Conversely, every pre-p-nil-ring N obviously is a unitary F_p-module and therefore a vector space over F_p with a basis $\{w_i\} \subseteq N$. Now, (a) and (b) are true even for all elements of N.

With the aid of this theorem many pre-p-nil-rings may easily be given. For example, one starts with a commutative semigroup $H=\{0, w_i\}$, $i\in I$ with zero 0, which satisfies (a) and (b), and takes $\{w_i\}=H\setminus\{0\}$ with the multiplication determined by H as basis of an algebra R over F_p. Such an algebra R is what we call here the *semigroup ring of H over F_p*. (More precisely, in a first step one has to take also $0\in H$ as a basis element of an algebra R^* over F_p, where $0\neq 0^*$ for the zero 0^* of R^*. The considered ring R then arises by "identifying" of these zeros, i.e. as residue class ring $R=R^*/(0-0^*)$ of R^* modulo the ideal $(0-0^*)$; cf. e.g. [6], Section 66.) If we choose

$$H = \{w, w^2, \cdots, w^{t-1}, w^t = w^{t+1} = \cdots = 0\}, \quad 2 \leqq t \leqq p+2, \tag{2}$$

i.e. the cyclic semigroup of order t and period 1 (cf. [2]), (a) and (b) hold and we have

THEOREM 2. *Every pre-p-nil-ring N, which is generated by one element, is a semigroup ring R of a cyclic semigroup H of period 1 and order t $(2\leqq t\leqq p+2)$ over F_p and conversely. Hence for a fixed characteristic p there exist exactly $p+1$ pre-p-nil-rings (up to isomorphisms), generated by one element.*

Proof. From the text above, it is clear that R is a pre-p-nil-ring, generated by the element w. Conversely, let w be a generating element of a pre-p-nil-ring N. Then all the powers of w form a subsemigroup (2) of N, and for the semigroup ring R of H over F_p we have $R\subseteq N$, from which by assumption $R=N$.

Let again $H=\{0, w_i\}$ be a (commutative) semigroup with zero 0. We say that R is a *semigroup ring of H over F_p with a factor system* $\{c_{w_i,w_j}\}$, if R is an algebra over F_p with the basis $\{w_i\}$, for which multiplication is defined by

$$w_i \circ w_j = c_{w_i,w_j} w_i w_j, \qquad c_{w_i,w_j} \in F_p. \tag{3}$$

(Here we have also identified the zeros as explained in the paragraph preceding Theorem 2. Hence the values of the factors c_{w_i,w_j} for $w_iw_j=0$ are irrelevant and may be chosen arbitrarily.)

It is known that R is associative if and only if (cf.e.g. [6])

$$c_{w_i,w_j} c_{w_iw_j,w_k} = c_{w_i,w_jw_k} c_{w_j,w_k} \quad \text{for } w_iw_jw_k \neq 0,$$

whereas commutativity is preserved if and only if

$$c_{w_i,w_j} = c_{w_j,w_i} \quad \text{for } w_iw_j \neq 0.$$

The special case $c_{w_i,w_j}=1\in F_p$ for every w_i, w_j corresponds to the semigroup ring in the above sense.

THEOREM 3. *Let $H=\{0, w_i\}$ be a commutative semigroup with zero 0, which satisfies* (a) *and* (b). *Then every semigroup ring R of H over F_p with a factor system $\{c_{w_i,w_j}\}$ is a pre-p-nil-ring, if $c_{w_i,w_i}\neq 0$ for $w_i^2\neq 0$.*

(This condition is superfluous if $t(w_i)\leqq p$ for every $w_i\in H$. On the other hand, it means no loss of generality, because all semigroup rings with factor sys-

tems (yet even all monomial algebras, cf. [7], Section 4) are included, if one only considers factor systems with $c_{w_i,w_j}=0 \Leftrightarrow w_iw_j=0$.)

Proof. From $w_i^t=0$ it follows immediately that

$$\underbrace{w_i \circ \cdots \circ w_i}_{t} = c_{w_i,w_i}^{t-1} w_i^t = 0.$$

Moreover, the property (a) of the multiplication $\cdot$ of the w_i carries over to the multiplication $\circ$ according to

$$w_i \circ \underbrace{w_j \circ \cdots \circ w_j}_{p} = c_{w_i,w_j} c_{w_j,w_j}^{p-1} w_i w_j^p = c_{w_i,w_j} c_{w_i,w_i}^{p-1} w_i^p w_j = \underbrace{w_i \circ \cdots \circ w_i}_{p} \circ w_j.$$

This is not trivial only in the case $w_iw_j^p=w_i^pw_j\neq 0$ and $c_{w_i,w_j}\neq 0$. But then, by assumption, $c_{w_i,w_i}\neq 0$ and $c_{w_j,w_j}\neq 0$; hence the $(p-1)$-th powers of these factors are equal to $1\in F_p$. Therefore R is a pre-p-nil-ring by Theorem 1.

We note at once that not all pre-p-nil-rings, not even those which are finite or subdirectly irreducible (cf. Section 2), arise as semigroup rings with a factor system according to Theorem 3 (hence also not as monomial algebras, cf. [7]). We restrict ourselves to a simple counter-example for characteristic $p=2$ and consider the algebra $N=R$ over F_2 with the basis $\{u, u^2, v, v^2, q\}$ and the multiplication table

	u	u^2	v	v^2	q
u	u^2	0	u^2+v^2	q	0
u^2	0	0	q	0	0
v	u^2+v^2	q	v^2	0	0
v^2	q	0	0	0	0
q	0	0	0	0	0

It is easy to see that by Theorem 1 R is a pre-2-nil-ring. But it is impossible to choose a basis $\{w_1, \cdots, w_5\}\subseteq R$ in such a way that for every w_i, w_j the product w_iw_j is a multiple of one of the elements of this basis (i.e. here of course equal to zero or to the basis element itself, since we have F_2 as operator domain). In order to prove this assertion, we observe first that in each basis there must be two elements w_1 and w_2 such that the terms u or $u+v$ occur in w_1 and v or $u+v$ occur in w_2 (and of course not $u+v$ in both). Then we get from

$$w_1^2,\ w_2^2,\ w_1w_2,\ w_1w_2^2 = w_1^2w_2$$

up to the order of succession the four linearly dependent elements u^2 or u^2+q, v^2 or v^2+q, u^2+v^2 or u^2+v^2+q, q, which can not be obtained in the basis $\{w_i\}$.

2. The considerations of this section concerning the decomposition of pre-p-nil-rings as subdirect sums (for the concepts and properties of subdirect sums, used in the following, cf. e.g. [4]) are based on the following

PROPOSITION. *Every pre-p-nil-ring N is isomorphic to a subdirect sum of subdirectly irreducible pre-p-nil-rings N_i, and conversely every subdirect sum of pre-p-nil-rings is again a pre-p-nil-ring.*

The first part is an immediate consequence of the fact that a homomorphic image of a pre-p-nil-ring is also one. For the second, it suffices to show that the direct sum of pre-p-nil-rings N_i is also a pre-p-nil-ring, which follows essentially from $x_i^{p+2}=0$ for every x_i of each pre-p-nil-ring N_i.

Therefore our main interest is the study of subdirectly irreducible pre-p-nil-rings. As already mentioned in the introduction, the situation is here much more complicated than for p-rings, but, some information is given in what follows, where the annihilator Q of N, i.e. the ideal of all elements $q\in N$ with $qx=0$ for every $x\in N$, is of some importance.

THEOREM 4. *Every subdirectly irreducible pre-p-nil-ring N has an annihilator $Q\neq(0)$, which is a principal ideal of N and hence it consists of exactly p elements $q, 2q, \cdots, pq=0$.*

This assertion is even true for nilpotent rings with characteristic p and is a special case of Theorem 2 of [3].

We give first an example of a pre-p-nil-ring without an annihilator ($\neq 0$), which is then subdirectly reducible by Theorem 4. Let H be the commutative semigroup with zero 0, generated by the (countable) infinite set of elements $u_1, u_2, \cdots$, with the defining relations

$$u_i^2 = 0 \quad \text{for every } i.$$

Then every nonzero element of H has a unique representation

$$u_{k_1}u_{k_2}\cdots u_{k_m} \quad \text{with} \quad k_1 < k_2 < \cdots < k_m. \tag{4}$$

Now we regard the semigroup ring N of H over F_p (with arbitrary characteristic p), which is according to section 1 a pre-p-nil-ring. This ring N has only the zero as annihilator, hence it is a ring of the desired kind. Indeed, every element $x\neq 0$ of N is a linear combination of a finite number of elements (4), and we have $xu_j\neq 0$ for every j greater than all the indices k_m of the factors u_{k_m} which occur in the terms of x.

By the help of this example, we see that the converse of Theorem 4 is false. For this purpose we take the direct sum $N\oplus Q$ of N with a zero-ring Q of p elements. Then $N\oplus Q$ is a pre-p-nil-ring with the principal ideal $Q\neq(0)$ as annihilator, though it is subdirectly reducible. But we have

THEOREM 5. *A finite pre-p-nil-ring N always has an annihilator $Q\neq(0)$ and it is subdirectly irreducible if and only if Q is a principal ideal.*

Proof. For the first assertion we take an arbitrary element $x_1\neq 0$ of N. Either we have $x_1x_2\neq 0$ for every $x_2\in N$, or there is an element $x_2\in N$ with $x_1x_2\neq 0$. In the same manner we regard $(x_1x_2)x_3$ and so on. Since N is finite and every

element of N is nilpotent, we obtain in this way an annihilator element $q = x_1x_2 \cdot \cdot \cdot x_n \neq 0$. Hence $Q \neq (0)$. Moreover, it follows from this consideration, that every ideal $A \neq (0)$ of N contains at least one annihilator element $q \neq 0$.

Now let Q be a principal ideal. Then we have $Q = (q)$ for every nonzero $q \in Q$. Therefore the intersection of all ideals $A \neq (0)$ of N contains Q and hence N is subdirectly irreducible. If the latter is true, then Q is a principal ideal by Theorem 4.

COROLLARY. *Every pre-p-nil-ring N, generated by one element w, is subdirectly irreducible.*

Proof. According to Theorem 2, N is the semigroup ring of the semigroup $\{w, w^2, \cdot \cdot \cdot, w^{t(w)-1}\}$ and the annihilator of N is the principal ideal $Q = (w^{t(w)-1})$.

We note in this connection, that subrings of such a pre-p-nil-ring N throughout may be subdirectly reducible. For example, suppose that $t(w) \geqq 4$, and let U be the subring of N which is generated by the elements $w^2, \cdot \cdot \cdot, w^{t(w)-1}$. Then $w^{t(w)-2} \neq 0$ and $w^{t(w)-1} \neq 0$ are elements of the annihilator Q of U, which is therefore not a principal ideal. Hence U is subdirectly reducible by Theorem 5. Contrasting the situation with finite p-rings (cf. the introduction), U is not directly reducible if $t(w) \geqq 6$. In order to prove this, let us assume that $U = U_1 \oplus U_2$. Then at least in one of the U_i, say in U_1, there must occur an element $x = cw^2 + \cdot \cdot \cdot$ with $cw^2 \neq 0$, perhaps among other terms. Suitable multiplications with w^2 and w^3 show that $cw^{t(w)-1} \in U_1$, then $cw^{t(w)-2} \in U_1$ and therefore $Q \subseteq U_1$. As we must have $Q \cap U_2 \neq (0)$ (cf. the proof of Theorem 5), this contradicts $U_1 \cap U_2 = (0)$.

Moreover, besides the pre-p-nil-rings generated by one element, there are many others, which are likewise subdirectly irreducible (for example, the ring presented at the end of Section 1 with the annihilator $Q = (q)$, or those subrings of the pre-p-nil-ring N without annihilator constructed after Theorem 4, which are generated by the elements $u_1, \cdot \cdot \cdot, u_k$). In this respect we have even the following

THEOREM 6. *For every characteristic p there is an infinite number of (not isomorphic) subdirectly irreducible pre-p-nil-rings which have* (i) *a finite,* (ii) *an infinite order.*

Proof. In order to construct such rings we take an index system I of an arbitrary cardinal number. Let H be the commutative semigroup with zero 0, which is generated by the elements u_i $(i \in I)$, v_i $(i \in I)$ and q with the defining relations

$$\left.\begin{aligned} u_iu_j &= 0 \\ v_iv_j &= 0 \end{aligned}\right\} \text{ for every } i \in I, j \in I;$$

$$u_iv_j = 0 \quad \text{for every } i \in I, j \in I \text{ with } i \neq j;$$

$$u_iv_i = q \quad \text{for every } i \in I.$$

From these it follows that $qu_i = qv_i = q^2 = 0$. Then H consists exactly of the ele-

ments $0, u_i, v_i, q$, and the semigroup ring N of H over F_p is a pre-p-nil-ring according to section 1. Moreover, $Q=(q)$ is just the annihilator of N, and we have $q\in A$ for every ideal $A\neq(0)$ of N. Hence N is subdirectly irreducible. If the index systems I and I' have different cardinal numbers, the same is true for the corresponding pre-p-nil-rings N and N', from which our theorem follows.

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HYPERSPHERES ASSOCIATED WITH AN n-SIMPLEX

H. E. VAUGHAN and HYMAN GABAI, University of Illinois

1. Introduction. Suppose that A_0 and A_1 are two points of an oriented line. For any point P of this line, let V, V_0, and V_1 be the directed distances from A_0 to A_1, from P to A_1, and from A_0 to P, respectively. Then $V_0+V_1=V$. It is well known that if $\boldsymbol{a}_0$, $\boldsymbol{a}_1$, and $\boldsymbol{p}$ are the position vectors of A_0, A_1, and P, respectively, with respect to any origin O, then

$$\boldsymbol{p}=\boldsymbol{a}_0\frac{V_0}{V}+\boldsymbol{a}_1\frac{V_1}{V}. \tag{1}$$

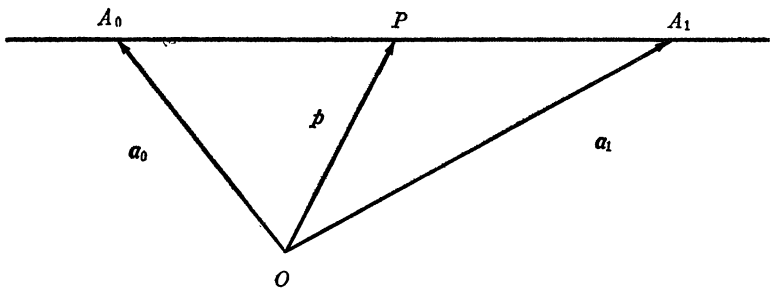


FIG. 1

In another notation which we shall sometimes use:

$$P - O = (A_0 - O)\frac{V_0}{V} + (A_1 - O)\frac{V_1}{V}.$$

(This notation is introduced in [1]. Translations of Euclidean space are elements of a vector space of the same dimension. Using the notation in question, $Y-X$ is the translation which maps X onto Y, and the image of a point Z under this mapping is $Z+(Y-X)$.)

In this paper we generalize this result and obtain information concerning some hyperspheres associated with an n-simplex.

As an indication of how (1) can be generalized, note that the numbers V, V_0, and V_1 can be described as the sensed length-measures of the oriented 1-simplexes A_0A_1, PA_1, and A_0P. (Since P may be A_0 or A_1, the second or third of these simplexes may be degenerate.) In order to speak of sensed length-measures (or equivalently, of directed distances) the line containing A_0 and A_1 must be oriented. But, since such measures enter into (1) only as ratios, which orientation one chooses is immaterial as far as (1) is concerned. It will be convenient, however, to assume that the orientation of the line is that determined by the oriented simplex A_0A_1 (equivalently, that the chosen positive direction is from A_0 to A_1). With this choice, $V>0$.

2. The position vector of a point in n-space. Now, suppose that $A_0 \ldots A_n$ is an oriented nondegenerate n-simplex σ. Without loss of generality we assume that the n-space containing σ is oriented so that the sensed (n-dimensional) measure V of σ is positive. Let P be any point of this n-space and, for $0 \leqq i \leqq n$, let σ_i be the oriented (possibly degenerate) n-simplex obtained from σ by replacing A_i by P. More precisely, σ_i is the image of σ under the linear mapping which, for $j \neq i$, leaves A_j fixed and which maps A_i onto P. Let V_i be the sensed measure of σ_i. Then $\sum_{i=0}^{n} V_i = V$. (See, for example, [2], page 234.) Let $\boldsymbol{a}_0, \cdots, \boldsymbol{a}_n$, and $\boldsymbol{p}$ be position vectors of $A_0, \cdots, A_n$ and P respectively with respect to any origin O. Then,

THEOREM 1. $\boldsymbol{p} = \sum_{j=0}^{n} \boldsymbol{a}_j (V_j/V)$.

Proof. For $1 \leqq j \leqq n$, let $\boldsymbol{b}_j = A_j - A_0 = \boldsymbol{a}_j - \boldsymbol{a}_0$, and let $\boldsymbol{r} = P - A_0 = \boldsymbol{p} - \boldsymbol{a}_0$. Since P belongs to the space spanned by $(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ there are numbers $x_1, \cdots, x_n$ such that $\boldsymbol{r} = \sum_{j=1}^{n} \boldsymbol{b}_j x_j$. The linear transformation previously described which maps σ on σ_i has, with respect to the basis $(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$, the matrix which differs from the identity only in having $(x_1, \cdots, x_n)$ as its ith row. Hence, the determinant of this transformation is x_i. Consequently, $V_i = Vx_i$ (see, for example, [2], page 300) and

$$\boldsymbol{p} - \boldsymbol{a}_0 = \boldsymbol{r} = \sum_{j=1}^{n} (\boldsymbol{a}_j - \boldsymbol{a}_0)\frac{V_j}{V}.$$

The theorem follows at once on noting that $1 - \sum_{j=1}^{n} (V_j/V) = V_0/V$. (That

$V_i = Vx_i$ can be seen without using determinants. Let s_i be the $(n-1)$-measure of the face of σ opposite A_i and let h_i be the altitude of σ from A_i. Then ([3], p. 123), $V = h_i s_i/n$. The altitude of s_i from P is $h_i|x_i|$, and x_i is positive or negative according as P and A_i are on the same or opposite sides of the hyperplane containing the face of σ opposite A_i. It follows that $V_i = (h_i x_i)s_i/n$. Hence, $V_i = Vx_i$.)

Note that Theorem 1 provides an interpretation of the barycentric coordinates of P with respect to $A, \cdots, A_n$. In this form, these coordinates are obtained in [3] and, for the 2-dimensional case, in [4]. It still seems appropriate to include the proof given here since it makes use of the ideas of linear algebra which are basic for the rest of this paper.

It is interesting to consider some 2-dimensional applications of Theorem 1. If P is the centroid of $\triangle A_0A_1A_2$, it is well known that $\triangle PA_1A_2$, $\triangle A_0PA_2$, and

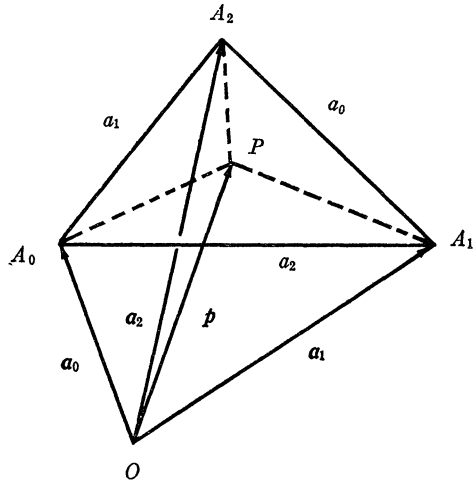


FIG. 2

$\triangle A_0A_1P$ (that is, σ_0, σ_1, and σ_2) all have the same area, and this is one-third the area of $\triangle A_0A_1A_2$. Consequently, $V_0/V = V_1/V = V_2/V = \frac{1}{3}$ and, by Theorem 1, we have the familiar formula:

$$p = \frac{a_0 + a_1 + a_2}{3}$$

for the position vector of the centroid.

If P is the incenter of $\triangle A_0A_1A_2$ then the triangles σ_0, σ_1, and σ_2 have the same altitude. Hence, their areas are proportional to the measures of the sides A_1A_2, A_2A_0, and A_0A_1. Taking these measures to be a_0, a_1, and a_2, we see from Theorem 1 that the position vector of the incenter is given by:

$$p = \frac{a_0a_0 + a_1a_1 + a_2a_2}{a_0 + a_1 + a_2}.$$

If P is the circumcenter of $\triangle A_0A_1A_2$ it is not hard to show that the sensed area-measure of σ_i is $(a_iR \cos \angle A_i)/2$, where R is the radius of the circumcircle. So, by Theorem 1, the position vector of the circumcenter is given by:

$$\boldsymbol{p} = \frac{\boldsymbol{a}_0(a_0 \cos \angle A_0) + \boldsymbol{a}_1(a_1 \cos \angle A_1) + \boldsymbol{a}_2(a_2 \cos \angle A_2)}{a_0 \cos \angle A_0 + a_1 \cos \angle A_1 + a_2 \cos \angle A_2}.$$

Returning for a moment to the proof of Theorem 1, we note that the numbers $x_1, \cdots, x_n$ are determined uniquely by P (and the points $A_0, \cdots, A_n$), and that, conversely, an arbitrary choice of these numbers determines a unique point P. Hence:

COROLLARY 1. *If* $\boldsymbol{p} = \sum_{j=0}^{n} \boldsymbol{a}_j y_j$ *and* $\sum_{j=0}^{n} y_j = 1$ *then, for* $0 \leqq j \leqq n$, $y_j = V_j/V$.

COROLLARY 2. *If* $\sum_{j=0}^{n} y_j = 1$ *then there is a point* P *such that, for* $0 \leqq j \leqq n$, $V_j = Vy_j$.

For example (reversing an earlier argument), it follows from Corollary 1, and the well-known formula for the position vector of the centroid P of an n-simplex σ, that each of the $n+1$ n-simplexes which has for its vertices P and the vertices of a face of σ has for its measure $V/(n+1)$, where V is the volume of σ. As a further consequence, the distance of P from a face of σ is $nV/(n+1)a_i$, where a_i is the measure of the face.

3. Hyperspheres tangent to $n+1$ hyperplanes. Theorem 1 and Corollary 1 give us a way of finding the hyperspheres each of which is tangent to the $n+1$ hyperplanes which contain the faces of an n-simplex. (In the case $n=2$, these are the inscribed circle and the three escribed circles of a triangle.) For $0 \leqq j \leqq n$, let π_j be the hyperplane which contains the face of σ opposite A_j, and let a_j be the measure of this face. Then,

COROLLARY 3. *Let* P *be the center of a hypersphere which, for* $0 \leqq j \leqq n$, *is tangent to* π_j, *and let* ϵ_j *be* 1 *or* -1 *according as* P *and* A_j *are on the same or opposite sides of* π_j. *Then* $\sum_{j=0}^{n} \epsilon_j a_j > 0$ *and the position vector of* P *is given by*:

$$\boldsymbol{p} = \sum_{j=0}^{n} \boldsymbol{a}_j(\epsilon_j a_j) \Big/ \sum_{j=0}^{n} \epsilon_j a_j. \tag{2}$$

Conversely, for any choice of the ϵ_j*'s such that each is* 1 *or* -1 *and* $\sum_{j=0}^{n} \epsilon_j a_j > 0$, *the point whose position vector is given by* (2) *is equidistant from all of the* π_j's.

Proof: Suppose that h is the radius of the hypersphere in question. Then, for $0 \leqq j \leqq n$, $V_j = (h_j \epsilon_j a_j)/n$, and $V = (1/n)h \sum_{j=0}^{n} \epsilon_j a_j > 0$. Hence, (2) follows from Theorem 1, and the remainder of the corollary follows from Corollary 1. Note that the radius h of the hypersphere is given by:

$$h = nV \Big/ \sum_{j=0}^{n} \epsilon_j a_j.$$

As an application of Corollary 3, the position vectors of the three escribed circles of a triangle of side measures a_0, a_1, and a_2, are (in terms of the position vectors of its vertices)

$$\frac{-a_0a_0+a_1a_1+a_2a_2}{-a_0+a_1+a_2},\qquad \frac{a_0a_0-a_1a_1+a_2a_2}{a_0-a_1+a_2},\quad\text{and}\quad \frac{a_0a_0+a_1a_1-a_2a_2}{a_0+a_1-a_2}.$$

As another application, we see that each n-simplex has an inscribed hypersphere the position vector of whose center is

$$\sum_{j=1}^{n} a_ja_j \Big/ \sum_{j=1}^{n} a_j,$$

when $a_0, \cdots, a_n$ are the position vectors of its vertices and $a_0, \cdots, a_n$ are the measures of the faces opposite these vertices. It follows that the incenter and the centroid of a simplex coincide if and only if all faces of the simplex have the same measure. (In particular, the incenter and centroid of a triangle coincide if and only if the triangle is equilateral.)

Since the measure of any face of an n-simplex is less than the sum of the measures of the other faces, we can choose in Corollary 3, any one of the ϵ_j's to be -1 and the remainder to be 1. So, in addition to an inscribed hypersphere, each n-simplex has $n+1$ "escribed" hyperspheres—that is, hyperspheres each of which is tangent to just one face of the simplex as well as being tangent to the planes which contain the remaining faces. It may (and, for $n>3$, it will) be possible to choose two or more of the numbers ϵ_j of Corollary 3 to be -1. For each such choice there will be a hypersphere which is tangent to each of the planes π_j but is not tangent to any face of the simplex. The case $n=3$—that of a tetrahedron—is interesting, partly because it is comparatively easy to analyze. In this case, in addition to the inscribed sphere and the four escribed spheres, there may be tangent spheres whose centers belong to one or another of the six wedge-shaped regions each of which has an edge of the tetrahedron on its boundary. Such a sphere is determined by choosing two of the numbers $\epsilon_0, \cdots, \epsilon_3$ to be -1 and two to be 1. It follows that a wedge will contain the center of a tangent sphere if and only if the sum of the area-measures of the two faces which intersect in the edge which is on the boundary of the wedge is less than the sum of the areas of the other two faces. (So, the wedges corresponding to two opposite edges cannot both contain centers of tangent spheres; and none of the wedges in the complement of a tetrahedron whose faces all have the same area can contain the center of a tangent sphere.) For a more complete analysis, suppose that the area-measures of the faces of a tetrahedron are such that $a_0 \leqq a_1 \leqq a_2 \leqq a_3$. In this case there will be a sphere corresponding to the choice

(a) $\epsilon_0=\epsilon_1=-1$ and $\epsilon_2=\epsilon_3=1$ if and only if $a_0+a_1<a_2+a_3$,

one corresponding to the choice

(b) $\epsilon_0=\epsilon_2=-1$ and $\epsilon_1=\epsilon_3=1$ if and only if $a_0+a_2<a_1+a_3$,

and one corresponding to one of the choices

(c) $\epsilon_0=\epsilon_3=-1$ and $\epsilon_1=\epsilon_2=1$, or $\epsilon_0=\epsilon_3=1$ and $\epsilon_1=\epsilon_2=-1$ if and only if $a_0+a_3\neq a_1+a_2$.

Notice that if $a_0+a_2<a_1+a_3$ then $a_0+a_1<a_2+a_3$, and that if $a_0+a_3\neq a_1+a_2$ then $a_0+a_2<a_1+a_3$. Hence, for a given tetrahedron, there may be only the inscribed and the four escribed spheres, or there may be, in addition the sphere obtained by the choice (a), or the two spheres obtained by the choices (a) and (b), or the three spheres obtained from the choices (a), (b), and (c). (These results are obtained in another way in [5], page 87.)

Considerations, similar to the preceding, show that there are at least 12 hyperspheres each of which is tangent to all of the five hyperplanes containing the faces of a nondegenerate 4-simplex. As is easily seen, this minimum is attained in the case of a 4-simplex whose faces have measures proportional to 1, 1, 1, 1, and 2. As follows from Corollary 3, there can be at most 16 hyperspheres tangent to all five hyperplanes. This maximum is attained in the case of a 4-simplex all of whose faces have the same measure.

In the general case it follows from Corollary 3 that there are at most 2^n hyperspheres each of which is tangent to all of the $n+1$ hyperplanes containing the faces of a nondegenerate n-simplex. For $n>1$ this maximum number is attained in the case of an n-simplex all but one of whose faces have the same measure, the remaining face having a measure $n-1$ times as great as the others. (For even n, the maximum is also attained by n-simplexes all of whose faces have the same measure.) The minimum number of such hyperspheres seems more difficult to determine. For n odd, consideration of simplexes all of whose faces have the same measure shows that the minimum is at most

$$2^n-\binom{n+1}{(n+1)/2}\Big/2.$$

For n even (and at least 4), considerations of simplexes all but one of whose faces have the same measure, that of the remaining face being twice as great, shows that the minimum is at most

$$2^n-\binom{n}{(n+2)/2}.$$

The use of these formulas together with some hack-work shows that, for $n=5$, the minimum is 22, and that, for $n=6$, the minimum is at least 36 and at most 49.

4. The circumscribed hypersphere. We have given earlier a formula for the position vector of the center of the circumcircle of a triangle (and have suggested a derivation of it based on Theorem 1). We shall now use a different approach to find the position vector of the center of the hypersphere circumscribed about an n-simplex (see the corollary to Theorem 2, below) and apply Theorem 1 to interpret the result. To begin with, we obtain an equation for the hypersphere in question. As previously, $A_0, \cdots, A_n$ is a nondegenerate n-simplex σ and, as in the proof of Theorem 1, for $1\leqq j\leqq n$, $\boldsymbol{b}_j=A_j-A_0$.

THEOREM 2. *Let S be the hypersphere which contains the vertices of σ. Then, a point X belongs to S if and only if*

$$X - A_0 = \sum_{j=1}^{n} b_j x_j, \tag{3}$$

where the numbers $x_1, \cdots, x_n$ satisfy the equation

$$\left(\sum_{j=1}^{n} b_j x_j \right)^2 = \sum_{j=1}^{n} b_j^2 x_j. \tag{4}$$

Moreover, the radius R of the hypersphere S is given by

$$R^2 = \sum_{j=1}^{n} b_j^2 \frac{V_j}{2V}, \tag{5}$$

where the volumes V_j are the volumes described in the beginning of section 2.

Proof. Let P be the center of S and let $\mathbf{r} = P - A_0$. Since, for $1 \leqq j \leqq n$, the radius of S which is perpendicular to the chord A_0A_j bisects this chord, it follows that

$$\mathbf{b}_j \cdot \mathbf{r} = b_j^2/2 \qquad (1 \leqq j \leqq n). \tag{6}$$

Now, $X \in S$ if and only if $\|X-P\| = \|P-A_0\|$. Since $X-P = (X-A_0) + (A_0-P) = (X-A_0) - \mathbf{r}$ it follows that $X \in S$ if and only if

$$\|(X - A_0) - \mathbf{r}\| = \|\mathbf{r}\|. \tag{7}$$

By (3), (7) is equivalent to:

$$\left(\sum_{j=1}^{n} b_j x_j - \mathbf{r} \right)^2 = \mathbf{r}^2.$$

Since, by (6),

$$\left(\sum_{j=1}^{n} b_j x_j - \mathbf{r} \right)^2 = \left(\sum_{j=1}^{n} b_j x_j \right)^2 - \sum_{j=1}^{n} b_j^2 x_j + \mathbf{r}^2,$$

(7) is equivalent to (4).

(Notice that in case $n=2$ and $\mathbf{b}_1$ and $\mathbf{b}_2$ are orthogonal, (4) reduces to the polar equation: $r^2 = b_1 r \cos\theta + b_2 r \sin\theta$ of the circle which passes through the pole A_0 of a polar coordinate system and the points A_1 and A_2 with polar coordinates (b_1, θ) and $(b_2, \pi/2)$, respectively; $(b_i = \|\mathbf{b}_i\|)$.)

The position vector $\mathbf{r}$, with respect to A_0, of the center P of S can be found in terms of the basis $(\mathbf{b}_1, \cdots, \mathbf{b}_n)$ by solving the equations (6). Assuming that $\mathbf{r} = \sum_{j=1}^{n} \mathbf{b}_j y_j$, we find, from (6), that

$$\sum_{j=1}^{n} (\mathbf{b}_i \cdot \mathbf{b}_j) y_j = b_i^2/2 \qquad (1 \leqq i \leqq n).$$

So, by Cramer's rule, $y_j = G_j/G$, $(1 \leqq j \leqq n)$ where G is the Gram determinant of $(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ and G_j is obtained from G by replacing the entry in its ith row and jth column (j fixed, $1 \leqq i \leqq n$) by $b_i^2/2$. Hence,

$$\boldsymbol{r} = \sum_{j=1}^{n} \boldsymbol{b}_j \frac{G_j}{G} \cdot \tag{8}$$

As in the proof of Theorem 1, if, with respect to some arbitrary origin O, the position vectors of $A_0, \cdots, A_n$, and P are $\boldsymbol{a}_0, \cdots, \boldsymbol{a}_n$, and $\boldsymbol{p}$, respectively, then $\boldsymbol{r} = \boldsymbol{p} - \boldsymbol{a}_0$ and, for $1 \leqq j \leqq n$, $\boldsymbol{b}_j = \boldsymbol{a}_j - \boldsymbol{a}_0$. Substituting in (8) we obtain

$$\boldsymbol{p} = \sum_{j=0}^{n} \boldsymbol{a}_j \frac{G_j}{G},$$

where, for $1 \leqq j \leqq n$, G_j is as described above and $G_0 = G - \sum_{j=1}^{n} G_j$.

Since $\sum_{j=0}^{n} G_j/G = 1$ it follows from Corollary 1 to Theorem 1 that, for $0 \leqq j \leqq n$, $G_j/G = V_j/V$. Since it is a property of the Gram determinant G that

$$V = \frac{1}{n!}\sqrt{G},$$

(see, for example, [6], page 170) it follows that $V_j = (1/n!)(G_j/\sqrt{G})$, $(0 \leqq j \leqq n)$.

The radius R of the hypersphere S is easily computed in terms of the lengths of the sides of σ issuing from A_0 and the sensed measures of σ and σ_j, $1 \leqq j \leqq n$. To do so, note that, since $P = A_0 + \boldsymbol{r}$, $A_0 + \boldsymbol{r} 2 \in S$. Since

$$\boldsymbol{r}^2 = \sum_{j=1}^{n} \boldsymbol{b}_j \frac{2G_j}{G}$$

it follows that

$$R^2 = \|\boldsymbol{r}\|^2 = \frac{1}{4}\left(\sum_{j=1}^{n} \boldsymbol{b}_j \frac{2V_j}{V}\right)^2$$

$$= \frac{1}{4}\sum_{j=1}^{n} b_j^2 \frac{2V_j}{V} = \sum_{j=1}^{n} b_j^2 \frac{V_j}{2V} \cdot$$

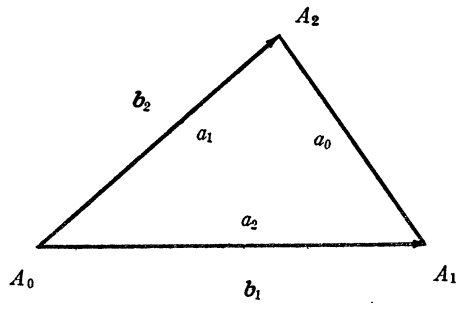


FIG. 3

This proves (5).

Finally, let us consider the 2-dimensional case. In this case

$$G = \begin{vmatrix} b_1^2 & b_1 \cdot b_2 \\ b_1 \cdot b_2 & b_2^2 \end{vmatrix}, \quad \text{and} \quad G_1 = \frac{1}{2} \begin{vmatrix} b_1^2 & b_1 \cdot b_2 \\ b_2^2 & b_2^2 \end{vmatrix}.$$

Since $b_1^2 = a_2^2$, $b_2^2 = a_1^2$, and $b_1 \cdot b_2 = a_1 a_2 \cos \angle A_0$ it follows that

$$G = a_1^2 a_2^2 (1 - \cos^2 \angle A_0) = a_1^2 a_2^2 \sin^2 \angle A_0$$

and

$$2G_1 = a_1^2 (a_2^2 - a_2 a_1 \cos \angle A_0) = a_1^2 a_2 (a_2 - a_1 \cos \angle A_0) = a_1^2 a_2 a_0 \cos \angle A_1.$$

Hence,

$$V_1 = \frac{G_1}{2\sqrt{G}} = \frac{a_1 a_0 \cos \angle A_1}{4 \sin \angle A_0} = \frac{a_1 R \cos \angle A_1}{2} \left(= \frac{a_1^2 \cot \angle A_1}{4} \right).$$

By symmetry, similar formulas hold for V_2 and V_0. (As noted previously, these formulas can also be obtained by more elementary methods.) They yield formulas for the distance between the circumcenter and the sides and, by summation, formulas for the area of the triangle.

This work was supported by National Science Foundation Grant G23554.

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SOME RESULTS IN VECTOR POTENTIAL THEORY

J. M. DOYLE, University of Illinois, Chicago

1. Introduction. Traditionally, the development of methods of formal solution for the displacement equilibrium equations of classical elasticity has paralleled work done in connection with the solution of Laplace's equation. This is perhaps to be expected inasmuch as this set of equations represents a generalization of second order, elliptic differential operators to vector functions. Beginning

with the work of Betti [1] in 1872 and continuing for the next forty years or so, the Italian geometers produced an extensive volume of literature dealing with the adaptation of potential theory methods to the theory of elasticity. In 1895, Lauricella [2] noting that solutions to the field equations can be constructed which are analogous to potentials of single and double layer distributions, demonstrated that the discontinuities encountered by the functions or their associated stress vectors are similar to those of harmonic functions or their normal derivatives in ordinary potential theory.

It is the purpose of this paper to outline a modern, concise and hopefully, a clearer treatment of the work just mentioned and at the same time to extend the ideas to the time-independent equations of classical elastokinetics. Throughout, the results from ordinary potential theory, as set forth by O. D. Kellogg [3], are utilized whenever applicable. Where such results cannot be utilized directly, slight modifications of the procedures employed by Kellogg can be used to carry out the various limit processes. For the sake of brevity and in order not to bore the reader with a great deal of computational detail, only end results have been included here. Solutions are constructed by integration over volume or surface of the inner product of a vector density and a singular dyadic which is, except at the singular point, a solution to the basic differential equation. These singular dyadics are akin to the function e^{ikr}/r or its normal derivative as singularities for the Helmholtz equation. Following the previous work, two different surface potentials are defined; one which is similar to a single layer and the other resembling a double layer. In each case the function and its derivatives are examined as the point at which they are evaluated approaches the surface along a normal. Not unexpectedly, the behavior of the function and the associated stress vector are the same as for their counterparts in the scalar case. That is, the vector itself is continuous across the surface for the single layer type while its stress vector is not and so on.

Aside from their significance from the theoretical point of view, it is possible that the results established here can be utilized in formulating certain boundary value problems in elastic body motion. This, of course, includes static problems as a special case.

II. Preliminary Information.

1. *Notation and Definitions.* Throughout this paper, the conventional vector and dyadic notation is employed. Vectors are shown as typed letters with a bar above and dyadics are symbolized by typed letters with a double bar above. The dot and cross placed between two vectors indicate scalar or vector multiplication respectively. The symbol ∇ represents the standard del operator. Operations on dyadics with this operator follow the convention of Weatherburn [4].

Regular regions and regular surfaces are defined according to Kellogg [3]. In general D represents a regular open region, B its bounding surface, and $D+B$ the closed region.

2. *Basic Equations and Identities.* The basic differential equation with which we are here concerned is

$$\bar{\bar{C}}\cdot\bar{u} + k^2\bar{u} = (b^2 - a^2)\nabla\nabla\cdot\bar{u} + a^2\nabla^2\bar{u} + k^2\bar{u} = 0, \tag{2.1}$$

where a and b represent the propagation speeds of shear and dilatational waves. This equation is understood to hold for all points in a region D. Two common types of boundary conditions which are of interest are: (1) specification of the vector $\bar{u}$ at each point of the surface B, and (2) specification of the surface stress vector at points of B. The surface stress vector is given by

$$\bar{T}(\bar{u}) = \rho\left[2a^2\frac{d\bar{u}}{dn} + (b^2 - 2a^2)(\nabla\cdot\bar{u})\bar{n} + a^2\bar{n}\times\nabla\times\bar{u}\right], \tag{2.2}$$

where $\bar{n}$ is the unit outward normal on B and ρ is the mass density of the material.

For two vectors $\bar{u}$ and $\bar{v}$ which, along with their first derivatives, are continuous in $D+B$, the following reciprocity relation can be shown to be true:

$$\int_D \rho[\bar{u}\cdot(\bar{\bar{C}}\cdot\bar{v}) - \bar{v}\cdot(\bar{\bar{C}}\cdot\bar{u})]d\tau = \int_B [\bar{u}\cdot\bar{T}(\bar{v}) - \bar{v}\cdot\bar{T}(\bar{u})]d\sigma. \tag{2.3}$$

III. Potential of Volume Distribution. In this section a vector function is constructed which is analogous to the potential of a volume distribution. The vector is generated by integration of the product of a piecewise continuous vector density distribution $\bar{\lambda}$ and a singular dyadic $\bar{\bar{U}}$:

$$\bar{u}(P) = \int_D \bar{\lambda}(Q)\cdot\bar{\bar{U}}_s(P, Q)d\tau_Q, \tag{3.1}$$

where P: (x, y, z), Q: (ξ, η, ζ), $r^2 = (\xi-x)^2+(\eta-y)^2+(\zeta-z)^2$,

$$\bar{\bar{U}}_s(P, Q) = \frac{1}{4\pi\rho k^2}\nabla\nabla\left(\frac{e^{\alpha r(P,Q)} - e^{\beta r(P,Q)}}{r(P, Q)}\right) + \frac{e^{\alpha r(P,Q)}}{4\pi\rho a^2 r(P, Q)}\bar{\bar{I}},$$

$$\alpha = \frac{ik}{a}; \quad \beta = \frac{ik}{b}; \quad \text{and} \quad i^2 = -1.$$

It is observed that the integrand in (3.1) is singular and therefore the integral must be checked for convergence. In addition if information about the various derivatives of $\bar{u}$ is required, further investigation is necessary. Before proceeding with this task, we shall look carefully into the behavior of $\bar{\bar{U}}_s$ in the vicinity of $r=0$. After the various exponentials in their appropriate series are expanded, $\bar{\bar{U}}_s$ takes the form:

$$\bar{\bar{U}}_s = \frac{b^2 - a^2}{8\pi\rho a^2 b^2}\left(\frac{1}{r}\bar{e}_r\bar{e}_r\right) + \frac{1}{4\pi\rho a^2}\left(1 - \frac{b^2 - a^2}{2b^2}\right)\frac{1}{r}\bar{\bar{I}} + 0(\alpha, \beta). \tag{3.2}$$

In this expression, $\bar{e}_r$ is a unit vector directed from P to Q. We remark that the

expression for $\overline{\overline{U}}_s$ given in (3.2) is, with the exception of the final order term, the Kelvin solution for displacement due to a point load in an infinite medium. Since the order term disappears when k vanishes, all results obtained in this paper are directly applicable even if the undifferentiated term of (2.1) is nonexistent.

Inasmuch as the second term on the right of (3.2) is merely the singularity employed in ordinary potential theory it need not be treated further here. Also, since we need be concerned only with the various quantities in the immediate vicinity of $r=0$, the final term on the right side of (3.2) will not be necessary. Let $\bar{u}$ be a sum of two functions $\bar{u}_1$ and $\bar{u}_2$ where:

$$
\begin{aligned}
\bar{u}_1(P) &= \frac{b^2 - a^2}{8\pi\rho a^2 b^2} \int_D \frac{\bar{\lambda}(Q)}{r(P,Q)} \cdot \bar{e}_r \bar{e}_r d\tau_Q, \\
\bar{u}_2(P) &= \frac{1}{4\pi\rho a^2}\left(1 - \frac{b^2 - a^2}{2b^2}\right)\overline{\overline{I}} \cdot \int_D \frac{\bar{\lambda}(Q)}{r(P,Q)}\, d\tau_Q.
\end{aligned} \tag{3.3}
$$

Then $\bar{u}_2$ and its first derivatives are continuous throughout all of space. Proofs of this are numerous, the treatment of the entire subject given by O. D. Kellogg [3] being quite easy to follow.

Each term of the integral expression for $\bar{u}_1(P)$ is seen to be of the order r^{-1}; hence the integral is convergent and the function exists at all points. Furthermore, first derivatives of the function with respect to the coordinates at P will give rise to terms of order r^{-2} in the integrand which will again yield a convergent integral; in turn this means that the first derivatives exist at all points. Continuity of the function and its first order derivatives is also readily established.

IV. Potential of Surface Distribution.

1. *Definition.* Just as in the previous section, we may extend the ideas of potential theory to vector surface distributions. We define a vector distribution $\bar{\mu}(Q)$ which satisfies a uniform Hölder condition on a regular closed surface B, and construct a vector function $\bar{u}$:

$$
\bar{u}(P) = \int_B \bar{\mu}(Q) \cdot \overline{\overline{U}}_s(P, Q) d\sigma_Q. \tag{4.1}
$$

Call D_i the volume enclosed by B and D_e all space external to B. For all points P in D_i or D_e, $\bar{u}(P)$ satisfies (2.1). Clearly, the function defined by (4.1) exists and is continuous for all points in space, except possibly points on the surface B. However, since the singularity in the integrand is of order r^{-1}, it is at once obvious that $\bar{u}(P)$ exists and is continuous for points in B as well.

2. *Tangential Derivatives.* The behavior of the various partial derivatives of $\bar{u}(P)$, defined by (4.1), is best studied by establishing a set of cartesian axes at a point p of B such that the z-axis is normal to the surface and the xy plane is tangent to the surface.

As with the function itself, the existence and continuity of its partial deriva-

tives at interior points of D_i or D_e presents no problem. We are concerned, however, with the behavior of these quantities at points P which approach p along the normal to the surface at p.

As previously, we write the integral, without the terms of $\bar{\bar{U}}_s$ which are constant or of positive powers of r

$$(4.2)\quad \bar{u}(P) = \int_B \frac{\bar{\mu}(Q)}{r(P,Q)} \cdot \left[\frac{b^2 - a^2}{8\pi\rho a^2 b^2} \bar{e}_r \bar{e}_r + \frac{1}{4\pi\rho a^2} \left(1 - \frac{b^2 - a^2}{2b^2} \right) \bar{\bar{I}} \right] d\sigma_Q,$$

and again focus attention on the term involving $\bar{e}_r\bar{e}_r$. With the coordinate system oriented as mentioned, select P on the z axis. The object is to examine the limit of the various partial derivatives as z approaches zero. For example, consider the derivative with respect to x. The members of the dyadic $\bar{e}_r\bar{e}_r$ involving $\bar{\imath}\bar{k}$ and $\bar{k}\bar{\imath}$ are of the form

$$(4.3)\quad \bar{I}_1 = (\bar{\imath}\bar{k} + \bar{k}\bar{\imath}) \cdot \left[3 \int_B \bar{\mu}(Q) \frac{\xi^2(\zeta - z)}{r^5(P,Q)} d\sigma_Q - \int_B \bar{\mu}(Q) \frac{\zeta - z}{r^3(P,Q)} d\sigma_Q \right].$$

The limit of the second integral is given in [3]. It is

$$(4.4)\quad \lim_{z \to 0} - \int_B \bar{\mu}(Q) \frac{\zeta - z}{r^3(P,Q)} d\sigma_Q = 2\pi\bar{\mu}(p) - \int_B \bar{\mu}(Q) \frac{\zeta - z}{r^3(P,Q)} d\sigma_Q.$$

To calculate the limiting value of the other integral, the technique utilized by Kellogg may be repeated with only minor modifications.

After carrying out the necessary steps one obtains

$$(4.5)\quad \lim_{z \to 0} 3 \int_B \bar{\mu}(Q) \frac{\xi^2(\zeta - z)}{r^5(P,Q)} d\sigma_Q = -2\pi\bar{\mu}(p) + 3 \int_B \bar{\mu}(Q) \frac{\xi^2 \zeta}{r^5(P,Q)} d\sigma_Q;$$

we may now write the desired result

$$(4.6)\quad \lim_{z \to 0} \bar{I}_1 = (\bar{\imath}\bar{k} + \bar{k}\bar{\imath}) \cdot \int_B \bar{\mu}(Q) \left[\frac{\partial}{\partial x} \left(\frac{\xi(\zeta - z)}{r^3} \right) \right]_{z=0} d\sigma_Q.$$

We note that the convergence of the integral in (4.6) is readily established if one recalls that

$$|\zeta| \leqq A(\xi^2 + \eta^2), \qquad |\xi| \leqq r.$$

Utilizing the same type of analysis, it may be shown that the limits of the various other terms in the integral are of the same form as that just obtained. The combination of all these results yields

$$(4.7)\quad \left. \frac{\partial \bar{u}}{\partial x} \right|_p = \int_B \bar{\mu}(Q) \cdot \left[\frac{\partial \bar{\bar{U}}_s}{\partial x} \right]_{z=0} d\sigma.$$

It follows at once that

$$\left.\frac{\partial \bar{u}}{\partial y}\right|_p = \int_B \bar{\mu}(Q)\cdot\left[\frac{\partial \bar{\bar{U}}_s}{\partial y}\right]_{z=0} d\sigma. \tag{4.8}$$

These two expressions represent derivatives of the function $\bar{u}$ in tangential directions. The derivative in any tangential direction is a linear combination of these two. In summary, the tangential derivatives, when defined for points on B as their limiting values as the surface is approached along normals, exist and are continuous throughout space.

3. *Normal Derivatives.* The study of normal derivatives is no different in character than the work just completed. Using the set of axes employed previously, the normal derivatives are simply the derivatives with respect to z.

As the details are quite similar to the previous work, only the results will be given here. The plus and minus signs in the denominator of the derivative symbol indicate that the surface is approached through positive or negative values of z respectively. A slight refinement is given by replacing z by n_p which indicates, naturally, the normal to the surface at point p.

$$\frac{\partial \bar{u}}{\partial n_p^{\pm}} = \frac{b^2-a^2}{8\pi a^2 b^2 \rho}\bar{\mu}(p)\cdot(\mp 2\pi\bar{\bar{I}} \pm 4\pi\bar{k}\bar{k}) + \frac{1}{4\pi a^2\rho}\left(1-\frac{b^2-a^2}{2b^2}\right)\bar{\mu}(p)\cdot(\mp 2\pi\bar{\bar{I}})$$
$$+ \int \bar{\mu}(Q)\cdot\left.\frac{\partial \bar{\bar{U}}_s}{\partial n_p}\right|_{z=0} d\sigma.$$

The results just presented are, as were the others, uniform as to location of p by virtue of the uniform Hölder condition imposed on $\bar{\mu}(Q)$. For this reason the vector $\bar{u}$ and its derivatives are continuously differentiable in either D_i or D_e (cf. [3], Theorem VII, p. 165).

4. *Stress Vector of $\bar{u}$.* It is of interest to look at the limiting value of the stress vector as points on B are approached along normals. Inasmuch as the stress vector is a combination of first derivatives, the desired limit can be obtained by a judicious use of the results already obtained. For convenience, the axes used for computation will have their origin at point p of B and are oriented as described before. In these considerations, the normal will be taken as positive if directed outward from the region D, and the limit will be approached through negative values of z.

The first term of the stress vector, (2.2) is the normal derivative of $\bar{u}$, and therefore its behavior can be examined by using the results in (4.9).

The remaining two terms of the stress vector expression contain the dilatation and the rotation vector of $\bar{u}$. Both of these quantities behave as ordinary potentials for small values of r and therefore their limits are known. When the individual results are summed, we obtain

$$\bar{T}^{-}(\bar{u}) = 1/2\bar{\mu}(p) + \int_B \bar{\mu}(Q)\cdot\bar{\bar{T}}[\bar{\bar{U}}_s]_p d\sigma, \tag{4.10}$$

in which

$$(4.11)\quad \bar{\bar{T}}[\bar{\bar{U}}_s]_p = \rho\left[2a^2\frac{d\bar{\bar{U}}_s}{dn_p} + (b^2 - 2a^2)\bar{n}_p(\nabla\cdot\bar{\bar{U}}_s)_p + a^2\bar{n}_p\times(\nabla\times\bar{\bar{U}}_s)_p\right]$$

and the minus sign with the left member indicates that the approach was made from the negative or inward side of B. The sign on the leading term on the right would, of course, be reversed had the approach been from the opposite side. The result expressed in (4.10) represents the natural extension of a similar result applying to normal derivatives in the ordinary potential theory.

V. Potential of "Double Layer." Here the analog of a double layer in ordinary potential theory is considered. The kernel for such a distribution is the normal derivative of $1/r$. The corresponding kernel in our case is of the form of the dyadic kernel in (4.11) except that the derivatives are taken with respect to the roving point Q and evaluated there. As in the ordinary potential theory, we shall assume that the density distribution, $\bar{\nu}(Q)$ and its first and second derivatives are continuous. The function, which satisfies (2.1) for points not on B, is written

$$(5.1)\quad \bar{u}(P) = \int \bar{\nu}(Q)\cdot\bar{\bar{T}}_Q[\bar{\bar{U}}_s(P, Q)]d\sigma_Q.$$

As may be demonstrated, this vector is discontinuous across the surface B. To show this one recognizes that all derivatives of functions of the radius r, with respect to coordinates at Q are just the negative of the corresponding derivatives with respect to the coordinate of the fixed point P. In addition, if the coordinate system is set, as before, with the origin at p and the xy plane tangent to B, the various components of the integrand are similar to the terms arising out of the calculation of the stress vector appropriate to the vector obtained in the last section. They differ in that they have a different sign, and the density is multiplied by a direction cosine. Application of reasoning similar to that leading to the result given in (4.10) leads one to:

$$(5.2)\quad \bar{u}^- = -1/2\bar{\nu}(p) + \int \bar{\nu}(Q)\cdot\bar{\bar{T}}_Q[\bar{\bar{U}}_s(p, Q)]d\sigma_Q.$$

It is understood that the first term on the right changes in sign if the surface is approached from the opposite direction. Again, since the condition is satisfied uniformly on B, the function $\bar{u}$ is continuous in any closed region where B is only approached from one side.

We turn now to the question of continuity of first derivatives and change our mode of attack somewhat. As always, points not on B cause no great concern, and so once more our attention is directed to points p on the bounding surface.

Utilizing the reciprocity relation one may express $\bar{\nu}$ in the following way

$$\bar{v}(P) = -\int_{D^*} (\bar{\bar{C}}\cdot\bar{v})\cdot\bar{\bar{U}}_s d\tau + \int_{B^*} \bar{T}(\bar{v})\cdot\bar{\bar{U}}_s d\sigma - \int_{B^*} \bar{v}(Q)\cdot\bar{\bar{T}}[\bar{\bar{U}}_s]d\sigma,$$

where D^* and B^* are a regular region and its bounding surface. If then, D^* and B^* are constructed in the same way as explained in ([3], p. 170), the continuity of first derivatives is readily shown in a neighborhood of any point of B. Therefore the field $\bar{u}$ as defined by (5.1) is, along with its first derivatives, continuous in a closed region D bounded by B provided B is approached from one side only.

Acknowledgment: The author wishes to express his sincere appreciation to Professor M. Stippes whose suggestion of the feasibility of this work gave impetus to its pursuit.

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MATHEMATICAL NOTES

TYCHONOFF'S THEOREM FOR THE SPACE OF MULTIFUNCTIONS

Y.-F. LIN, University of South Florida

1. Introduction. The classical theorem of Tychonoff states that *the cartesian product of a collection of compact topological spaces is compact relative to the product topology* [3]. "This theorem is," according to J. L. Kelley [2], "unquestionably the most useful theorem on compactness and it is probably the most important single theorem of general topology." We propose to carry this theorem to a theorem for the space of multivalued functions.

2. Space of multifunctions. Let X and Y be two topological spaces, then $F \subset X \times Y$ is said to be a *multifunction* on X to Y if F is a correspondence between elements in X and nonvoid subsets in Y. That is, to be precise, for each x in X

$$\pi_2((\{x\} \times Y) \cap F) \neq \square,$$

where π_2 is the second projection of $X \times Y$ to Y, and $\square$ denotes the empty set. We shall write $F\colon X \Rightarrow Y$ for a multifunction on X to Y and $F(x)$ for the set $\pi_2((\{x\} \times Y) \cap F)$. If $\{X_\lambda | \lambda \in \Lambda\}$ is a family of topological spaces, discrete or otherwise, then the *cartesian M-product* $P_M\{X_\lambda | \lambda \in \Lambda\}$ of the family of spaces $\{X_\lambda | \lambda \in \Lambda\}$ is the set of all multifunctions $F\colon \Lambda \Rightarrow \cup\{X_\lambda | \lambda \in \Lambda\}$ such that $F(\lambda) \subset X_\lambda$. Since each (single valued) function is also a multifunction, the cartesian product $P\{X_\lambda | \lambda \in \Lambda\}$ is a subset of the cartesian M-product $P_M\{X_\lambda | \lambda \in \Lambda\}$. We shall topologize $P_M\{X_\lambda | \lambda \in \Lambda\}$ in a manner analogous to the product topol-

ogy for $P\{X_\lambda|\lambda\in\Lambda\}$, and in such a way that the product topology will be its subtopology (relativized topology). Recall that the product topology for $P\{X_\lambda|\lambda\in\Lambda\}$ is obtained by taking as a subbasis the collection of all sets $\{f\in P\{X_\lambda|\lambda\in\Lambda\}|f(\lambda)\in U_\lambda\}$ for all λ in Λ and for all open sets U_λ in X_λ. The statement "$f(\lambda)\in U_\lambda$" has two meanings if the function f is replaced by a multifunction F; precisely, the statements "$F(\lambda)\subset U_\lambda$" and "$F(\lambda)\cap U_\lambda\neq\square$" both mean "$f(\lambda)\in U_\lambda$" if F is a single valued function f. Therefore we make the following definition.

DEFINITION. *Let $\{X_\lambda|\lambda\in\Lambda\}$ be a collection of topological spaces. By the M-product topology for $P_M\{X_\lambda|\lambda\in\Lambda\}$ is meant the topology defined by taking the totality of sets*

$$\{F\in P_M\{X_\lambda\,|\,\lambda\in\Lambda\}\,|\,F(\lambda)\subset U_\lambda\},\tag{1}$$

$$\{F\in P_M\{X_\lambda\,|\,\lambda\in\Lambda\}\,|\,F(\mu)\cap V_\mu\neq\square\},\tag{2}$$

as subbasic open sets, where U_λ and V_μ are any open sets in X_λ and X_μ respectively, and λ, u are any members of Λ.

Following A. D. Wallace [4], we shall write simply $[U_\lambda]$ and $]V_\mu[$ respectively for the sets (1) and (2) above.

THEOREM. *If $\{X_\lambda|\lambda\in\Lambda\}$ is a collection of compact spaces, then the cartesian M-product $P_M\{X_\lambda|\lambda\in\Lambda\}$ is compact relative to the M-product topology.*

Proof. By a lemma of Alexander [1], to show that $P_M\{X_\lambda|\lambda\in\Lambda\}$ is compact it suffices to show every covering by subbasic open sets has a finite subcovering. Let $\mathbf{S}$ be the totality of $[U_\lambda]$ and $]V_\mu[$ where U_λ and V_μ are open in X_λ and X_μ respectively. Suppose, to get a contradiction, that the space $P_M\{X_\lambda|\lambda\in\Lambda\}$ is not compact. Then there exists a subcollection $\mathbf{C}$ of $\mathbf{S}$ such that $\mathbf{C}$ covers $P_M\{X_\lambda|\lambda\in\Lambda\}$ but every finite subcollection of $\mathbf{C}$ fails to cover $P_M\{X_\lambda|\lambda\in\Lambda\}$. For each λ in Λ, let $\mathbf{U}_\lambda$ ($\mathbf{V}_\lambda$) be the collection of all open sets U_λ (V_λ) in X_λ such that the sets $[U_\lambda]$ ($]V_\lambda[$) are in $\mathbf{C}$. Then no finite subcollection of $\mathbf{V}_\lambda$ covers X_λ and hence by compactness of X_λ every $X_\lambda-\cup\{V_\lambda|\,V_\lambda\in\mathbf{V}_\lambda\}$ is not empty. Let $F\colon\Lambda\Rightarrow\cup\{X_\lambda|\lambda\in\Lambda\}$ be defined by

$$F(\lambda)=X_\lambda-\cup\{V_\lambda|V_\lambda\in\mathbf{V}_\lambda\}$$

for each $\lambda\in\Lambda$. Then $F(\lambda)$ is not in $\mathbf{V}_\lambda$ for all λ in Λ; also, $F(\lambda)$ cannot be in $\mathbf{U}_\lambda$ for all λ in Λ. Thus, F is not in any member of $\mathbf{C}$ and, the fact that $\mathbf{C}$ is a covering of $P_M\{X_\lambda|\lambda\in\Lambda\}$ is contradicted.

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NOTE ON ITERATION OF CONCAVE FUNCTIONS

M. KUCZMA AND A. SMAJDOR, Katowice, Poland

Let $f(x)$ be a function which is continuous, strictly increasing and concave downward in an interval (a, b), $-\infty \leqq a < b \leqq \infty$, and moreover $a < f(x) < x$ in (a, b). The condition of concavity implies that at every point of (a, b) both the lateral derivatives of $f(x)$ exist and are decreasing functions. In the sequel $f'(x)$ will denote the right-sided derivative of $f(x)$.

Let $f^n(x)$ denote the nth iterate of $f(x)$ defined by the recurrence $f^{n+1}(x) = f[f^n(x)]$, $n = 0, 1, 2, \cdots$, where $f^0(x)$ is the identity function. It follows from the properties of $f(x)$ that for every n the function $f^n(x)$ is defined, continuous, and strictly increasing in (a, b). Moreover, $a < f^{n+1}(x) < f^n(x) < x$ in (a, b), and $\lim_{n\to\infty} f^n(x) = a$ for every $x \in (a, b)$.

The purpose of the present note is to prove the following result, of importance in the theory of functional equations and in the theory of iterations (cf. [1], [2]).

THEOREM. *Let $f(x)$ have the properties described above. If*

$$\lim_{x\to a+0} f'(x) = 1, \tag{1}$$

then

$$\lim_{n\to\infty} \frac{f^{n+1}(x) - f^n(x)}{f^{n+1}(y) - f^n(y)} = 1 \tag{2}$$

for every $x, y \in (a, b)$.

Proof. Without loss of generality we may assume that $y < x$. (For $y = x$, (2) is evident and the case $y > x$ is reduced to that considered by taking the reciprocals.) Then there exists a positive integer k such that

$$f^k(x) < y < x. \tag{3}$$

Since f is concave, if $t_1 < t_0$, $t_1, t_0 \in (a, b)$ then

$$f'(t_0) \leqq \frac{f(t_1) - f(t_0)}{t_1 - t_0} \leqq f'(t_1). \tag{4}$$

Taking $t_0 = f^n(x)$, $t_1 = f^{n+1}(x)$ in (4), we obtain

$$f'[f^n(x)] \leqq \frac{f^{n+2}(x) - f^{n+1}(x)}{f^{n+1}(x) - f^n(x)} \leqq f'[f^{n+1}(x)],$$

whence by (1)

$$\lim_{n\to\infty} \frac{f^{n+2}(x) - f^{n+1}(x)}{f^{n+1}(x) - f^n(x)} = 1$$

and

$$\lim_{n\to\infty}\frac{f^{n+k+1}(x)-f^{n+k}(x)}{f^{n+1}(x)-f^{n}(x)}=\lim_{n\to\infty}\prod_{i=0}^{k-1}\frac{f^{n+i+2}(x)-f^{n+i+1}(x)}{f^{n+i+1}(x)-f^{n+i}(x)}=1. \tag{5}$$

Since $f'(x)$ is decreasing, it follows from (1) that $f'(x)\leqq 1$ in (a, b). Consequently the function $f(x)-x$ is decreasing in (a, b), i.e.

$$f(u)-u\leqq f(v)-v, \quad \text{whenever } v<u, \quad u, v\in(a, b). \tag{6}$$

Both the expressions in (6) are negative; hence

$$\frac{f(u)-u}{f(v)-v}\geqq 1, \quad \text{whenever } v<u, \quad u, v\in(a, b). \tag{7}$$

Setting $u=f^n(x)$ and $v=f^n(y)$ in (7) we get

$$\frac{f^{n+1}(x)-f^{n}(x)}{f^{n+1}(y)-f^{n}(y)}\geqq 1. \tag{8}$$

On the other hand, by (3), $f^{k+n}(x)<f^n(y)$ whence, by (6), $f^{n+1}(y)-f^n(y)\leqq f^{k+n+1}(x)-f^{k+n}(x)$ and

$$\frac{f^{n+1}(x)-f^{n}(x)}{f^{k+n+1}(x)-f^{k+n}(x)}\geqq\frac{f^{n+1}(x)-f^{n}(x)}{f^{n+1}(y)-f^{n}(x)}. \tag{9}$$

We now obtain (2) from (8), (9) and (5).

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DIAGONAL EQUIVALENCE TO MATRICES WITH PRESCRIBED ROW AND COLUMN SUMS

RICHARD SINKHORN, University of Houston

The author shows in [1] that corresponding to each positive square matrix A there is a unique doubly stochastic matrix of the form D_1AD_2 where D_1 and D_2 are diagonal matrices with positive main diagonals. This doubly stochastic matrix can be obtained as a limit of the iterative process of alternately normalizing the rows and columns of A. It is the intent of this paper to generalize this result in the following way.

THEOREM. *Let $r_1, \cdots, r_m$, $c_1, \cdots, c_n$ be fixed positive numbers. Then, corresponding to each positive $m\times n$ matrix A, there is a unique matrix of the form D_1AD_2 with row sums $\mu r_1, \cdots, \mu r_m$ and column sums $c_1, \cdots, c_n$ where $\mu=\sum_j c_j/\sum_i r_i$. D_1 and D_2 are respectively $m\times m$ and $n\times n$ diagonal matrices*

with positive diagonals and are themselves unique up to a scalar multiple.

The iterative process of alternately scaling the rows and columns of A to have row and column sums respectively r_i and c_j can be used to find D_1AD_2. The subsequence from the iteration in which column sums are scaled converges to D_1AD_2 while the subsequence in which the row sums are scaled converges to $(1/\mu)D_1AD_2$. In particular, if $\sum_i r_i = \sum_j c_j$ then the entire iteration converges to D_1AD_2.

The uniqueness part of the theorem is immediate. Suppose that D_1AD_2 and $D_1'AD_2'$ satisfy the requirements of the theorem where

$$D_1 = \operatorname{diag}(x_1, \cdots, x_m), \qquad D_1' = \operatorname{diag}(x_1', \cdots, x_m'),$$
$$D_2 = \operatorname{diag}(y_1, \cdots, y_n), \quad \text{and} \quad D_2' = \operatorname{diag}(y_1', \cdots, y_n').$$

Then, if $p_i = x_i'/x_i$ $(i=1, \cdots, m)$, and $q_j = y_j'/y_j$ $(j=1, \cdots, n)$,

$$\begin{aligned}\operatorname{Min} p_i &= p_{i_1} = \mu r_{i_1}\Big(\sum_j x_{i_1}a_{i_1j}q_jy_j\Big)^{-1} \\ &\geqq \mu r_{i_1}(\operatorname{Max} q_j)^{-1}\Big(\sum_j x_{i_1}a_{i_1j}y_j\Big)^{-1} = (\operatorname{Max} q_j)^{-1},\end{aligned}$$

and, similarly, $\operatorname{Max} q_j \leqq (\operatorname{Min} p_i)^{-1}$.

These inequalities are compatible only if each inequality is an equality. But equality in the former case forces $q_j = \operatorname{Max} q_j = q$, a positive constant for $j=1, \cdots, n$; in the latter case it forces $p_i = \operatorname{Min} p_i = p$, a positive constant for $i=1, \cdots, m$. Either equality then shows that $q = p^{-1}$. Thus $D_1' = pD_1$, $D_2' = p^{-1}D_2$ and $D_1AD_2 = D_1'AD_2'$.

The existence part of the theorem follows at once from the following lemma. The proof of the lemma justifies the statements regarding the iteration.

LEMMA. *Let $V \subseteq R^m \times R^n$ denote the set of pairs of vectors (x, y) with positive components such that $\sum_i x_i a_{ij} y_j = c_j$ $(j=1, \cdots, n)$, and $\|x\| = \operatorname{Max}|x_i| \leqq \delta^{1/2}a^{-1/2}$, $\|y\| = \operatorname{Max}|y_j| \leqq \delta^{1/2}a^{-1/2}$ where $\delta = \operatorname{Max}(r_i, c_j)$ and $a = \operatorname{Min} a_{ij}$, and let*

$$\phi(x, y) = \operatorname*{Max}_i r_i^{-1} \sum_j x_i a_{ij} y_j - \operatorname*{Min}_i r_i^{-1} \sum_j x_i a_{ij} y_j$$

have domain of definition V. Then $\phi(\hat{x}, \hat{y}) = 0$ for some $(\hat{x}, \hat{y}) \in V$.

Proof. First note that V is nonvoid since it contains (x^0, y^0) with components

$$x_{i0} = \delta^{1/2}a^{-1/2}, \quad y_{j0} = c_j\delta^{-1/2}a^{1/2}\Big(\sum_i a_{ij}\Big)^{-1}.$$

Construct a sequence $(x^k, y^k) \in V$ as follows beginning with (x^0, y^0): Let $\alpha_{ik} = r_i^{-1}\sum_j x_{ik}a_{ij}y_{jk}$ and $\beta_{jk} = c_j^{-1}\sum_i \alpha_{ik}^{-1}x_{ik}a_{ij}y_{jk}$ and use

$$x_{i,k+1} = \delta^{1/2}a^{-1/2}M_k^{-1}\alpha_{ik}^{-1}x_{ik}$$

$$y_{j,k+1} = \delta^{-1/2}a^{1/2}M_k\beta_{jk}^{-1}y_{jk}$$

as the components of (x^{k+1}, y^{k+1}) where $M_k = \operatorname{Max}_i \alpha_{ik}^{-1}x_{ik}$.

Since for any j

$$\beta_{jk}^{-1}y_{jk} = (\sum_i \alpha_{ik}^{-1}x_{ik}a_{ij})^{-1}c_j \leqq (\alpha_{ik}^{-1}x_{ik}a_{ij})^{-1}c_j \leqq (\alpha_{ik}^{-1}x_{ik})^{-1}a^{-1}\delta$$

for all i, it is true in particular that $\beta_{jk}^{-1}y_{jk} \leqq \mathrm{Min}_i(\alpha_{ik}^{-1}x_{ik})^{-1}a^{-1}\delta = M_k^{-1}a^{-1}\delta$ for all j. It readily follows that each $(x^k, y^k) \in V$.

Denoting Min α_i and Max α_i by α_{i_1} and α_{i_2} and Min β_j and Max β_j by β_{j_1} and β_{j_2} respectively, we see that

$$\begin{aligned}
\phi(x^{k+1}, y^{k+1}) &= \alpha_{i_2,k+1} - \alpha_{i_1,k+1} \\
&= r_{i_2}^{-1}\sum_j x_{i_2,k+1}a_{i_2j}y_{j,k+1} - r_{i_1}^{-1}\sum_j x_{i_1,k+1}a_{i_1j}y_{j,k+1} \\
&= r_{i_2}^{-1}\sum_j \alpha_{i_2k}^{-1}x_{i_2k}a_{i_2j}\beta_{jk}^{-1}y_{jk} - r_{i_1}^{-1}\sum_j \alpha_{i_1k}^{-1}x_{i_1,k}a_{i_1j}\beta_{jk}^{-1}y_{jk} \\
&= r_{i_2}^{-1}\alpha_{i_2k}^{-1}x_{i_2k}a_{i_2j_2}\beta_{j_2k}^{-1}y_{j_2k} + r_{i_2}^{-1}\sum_{j\neq j_2} \alpha_{i_2k}^{-1}x_{i_2k}a_{i_2j}\beta_{jk}^{-1}y_{jk} \\
&\quad - r_{i_1}^{-1}\alpha_{i_1k}^{-1}x_{i_1k}a_{i_1j_1}\beta_{j_1k}^{-1}y_{j_1k} - r_{i_1}^{-1}\sum_{j\neq j_1} \alpha_{i_1k}^{-1}x_{i_1k}a_{i_1j}\beta_{jk}^{-1}y_{jk} \\
&\leqq r_{i_2}^{-1}\alpha_{i_2k}^{-1}x_{i_2k}a_{i_2j_2}\beta_{j_2k}^{-1}y_{j_2k} + r_{i_2}^{-1}\beta_{j_1k}^{-1}\sum_{j\neq j_2} \alpha_{i_2k}^{-1}x_{i_2k}a_{i_2j}y_{jk} \\
&\quad - r_{i_1}^{-1}\alpha_{i_1k}^{-1}x_{i_1k}a_{i_1j_1}\beta_{j_1k}^{-1}y_{j_1k} - r_{i_1}^{-1}\beta_{j_2k}^{-1}\sum_{j\neq j_1} \alpha_{i_1k}^{-1}x_{i_1k}a_{i_1j}y_{jk} \\
&= \beta_{j_2k}^{-1}(\alpha_{i_2k}^{-1}x_{i_2k}a_{i_2j_2}r_{i_2}^{-1}y_{j_2k}) + \beta_{j_1k}^{-1}(1 - \alpha_{i_2k}^{-1}x_{i_2k}a_{i_2j_2}r_{i_2}^{-1}y_{j_2k}) \\
&\quad - \beta_{j_1k}^{-1}(\alpha_{i_1k}^{-1}x_{i_1k}a_{i_1j_1}r_{i_1}^{-1}y_{j_1k}) - \beta_{j_2k}^{-1}(1 - \alpha_{i_1k}^{-1}x_{i_1k}a_{i_1j_1}r_{i_1}^{-1}y_{j_1k}) \\
&= (\beta_{j_1k}^{-1} - \beta_{j_2k}^{-1})(1 - \alpha_{i_2k}^{-1}x_{i_2k}a_{i_2j_2}r_{i_2}^{-1}y_{j_2k} - \alpha_{i_1k}^{-1}x_{i_1k}a_{i_1j_1}r_{i_1}^{-1}y_{j_1k}) \\
&\leqq (\beta_{j_1k}^{-1} - \beta_{j_2k}^{-1})(1 - 2d_k),
\end{aligned}$$

where $d_k = \mathrm{Min}_{i,j}\, \alpha_{ik}^{-1}x_{ik}a_{ij}r_i^{-1}y_{jk}$.

But

$$\alpha_{ik}^{-1}x_{ik} = (\sum_j a_{ij}y_{jk})^{-1}r_i \geqq (\sum_j a_{ij})^{-1}r_i a^{1/2}\delta^{-1/2} \geqq \rho R^{-1}a^{1/2}\delta^{-1/2}$$

and

$$y_{jk} = (\sum_i a_{ij}x_{ik})^{-1}c_j \geqq (\sum_i a_{ij})^{-1}c_j a^{1/2}\delta^{-1/2} \geqq \rho R^{-1}a^{1/2}\delta^{-1/2},$$

where $\rho = \mathrm{Min}(r_i, c_j)$ and $R = \sum_{i,j} a_{ij}$. Thus, for all k

$$d_k \geqq \rho R^{-1}a^{1/2}\delta^{-1/2}a\delta^{-1}\rho R^{-1}a^{1/2}\delta^{-1/2} = \rho^2 R^{-2}a^2\delta^{-2}.$$

Furthermore, it is clear that $\alpha_{i_2k}^{-1} \leqq \beta_{jk} \leqq \alpha_{i_1k}^{-1}$ for all j and k. Hence

$$\begin{aligned}
\phi(x^{k+1}, y^{k+1}) &\leqq (\alpha_{i_2k} - \alpha_{i_1k})(1 - 2\rho^2 R^{-2}a^2\delta^{-2}) \\
&= (1 - 2\rho^2 R^{-2}a^2\delta^{-2})\phi(x^k, y^k),
\end{aligned}$$

and it follows that $\phi(x^k, y^k) \to 0$.

Since $\|x^k\| \leqq \delta^{1/2}a^{-1/2}$ and $\|y^k\| \leqq \delta^{1/2}a^{-1/2}$ for all k, the sequence $\{(x^k, y^k)\}$ has a limit point $(\hat{x}, \hat{y}) \in R^m \times R^n$. The continuity of ϕ shows in particular that

$$\phi(\hat{x}, \hat{y}) = \lim_{k \to \infty} \phi(x^k, y^k) = 0.$$

It is seen that $\beta_{j_2k}^{-1} \leqq \alpha_{i,k+1} \leqq \beta_{i_1k}^{-1}$ for all i and k. Thus $\alpha_{i_1k} \leqq \alpha_{i_1,k+1} \leqq \alpha_{i_2,k+1} \leqq \alpha_{i_2k}$ and, since $\alpha_{ik}^{-1}x_{ik} \geqq \rho R^{-1}a^{1/2}\delta^{-1/2}$, $x_{ik} \geqq \alpha_{ik}\rho R^{-1}a^{1/2}\delta^{-1/2} \geqq \alpha_{i_10}\rho R^{-1}a^{1/2}\delta^{-1/2}$ $(i = 1, \cdots, m;\ k = 0, 1, \cdots)$. This along with $y_{jk} \geqq \rho R^{-1}a^{1/2}\delta^{-1/2}$ $(j = 1, \cdots, n;\ k = 0, 1, \cdots)$, shows that $\hat{x}$ and $\hat{y}$ have positive coordinates.

Then, since $\|x^k\| \leqq \delta^{1/2}a^{-1/2}$ and $\|y^k\| \leqq \delta^{1/2}a^{-1/2}$ for all k, and $\sum_i x_{ik}a_{ij}y_{jk} = c_j$ for all j and k, $\|\hat{x}\| \leqq \delta^{1/2}a^{-1/2}$, $\|\hat{y}\| \leqq \delta^{1/2}a^{-1/2}$, and $\sum_i \hat{x}_i a_{ij}\hat{y}_j = c_j$ for all j. Thus $(\hat{x}, \hat{y}) \in V$ and the lemma is proved.

The proof of the theorem may now be completed. Using the $(\hat{x}, \hat{y})$ of the lemma, set $D_1 = \operatorname{diag}(\hat{x}_1, \cdots, \hat{x}_m)$ and $D_2 = \operatorname{diag}(\hat{y}_1, \cdots, \hat{y}_n)$. Then $\sum_i \hat{x}_i a_{ij}\hat{y}_j = c_j$ $(j = 1, \cdots, n)$, and, since $\phi(\hat{x}, \hat{y}) = 0$, $\sum_j \hat{x}_i a_{ij}\hat{y}_j = Kr_i$ $(i = 1, \cdots, m)$, where K does not depend upon i. But

$$K \sum_i r_i = \sum_i \sum_j \hat{x}_i a_{ij}\hat{y}_j = \sum_j c_j = \mu \sum_i r_i;$$

hence $K = \mu$.

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A NOTE ON EXTREME POINTS

A. K. Chaudhuri and E. Tarafdar, Regional Engineering College, Durgapur 9, India

Let $F: X \to Y$ be a mapping of a topological space X into another topological space Y. A point $x \in X$ will be said to be an extreme point of X with respect to the mapping $F: X \to Y$, if there exists an open neighborhood $N(x)$ of x such that $F(x) \in \text{Bdry } F(N(x))$, where Bdry $A = A - A^0$, A^0 standing for the interior of the set A. The set of all extreme points of X with respect to the mapping $F: X \to Y$ will be denoted by $E_F(X, Y)$ and the mapping F will be said to be extreme at every point $x \in E_F(X, Y)$ and the value of the mapping $F(x)$, $x \in E_F(X, Y)$ will be called an extreme value of the mapping $F: X \to Y$.

Lemma 1. *If x is an extreme point of X with respect to the mapping $F: X \to Y$, and $N(x)$ is an open neighborhood of x such that $F(x) \in$ Bdry $F(N(x))$, then if $N'(x)$ is any other open subneighborhood of x such that $N'(x) \subset N(x)$, then $F(x)$ will also belong to* Bdry $F(N'(x))$.

Proof. We have by hypothesis $F(x) \in \text{Bdry } F(N(x))$. Let $N'(x)$ be an open subneighborhood of x such that $N'(x) \subset N(x)$. If possible, let $F(x) \in F(N'(x))^0$.

Since $N'(x) \subset N(x)$, therefore $F(N'(x)) \subset F(N(x))$. Hence if $F(x) \in F(N'(x)^0$, then $F(x)$ must also belong to $F(N(x))^0$ which is a contradiction. Hence the lemma is proved.

THEOREM. *If $F: X \to Y$ is a continuous mapping of a locally connected space X satisfying the second axiom of countability into the real line Y, then the set of the extreme values of the mapping F must be countable at most.*

Proof. Let x be any point $\in E_F(X, Y)$. Then there exists an open neighborhood $N(x)$ of x such that $F(x) \in \text{Bdry } F(N(x))$. Since X is locally connected, there exists a connected open neighborhood $U(x)$ of x such that $U(x) \subset N(x)$. Then according to the Lemma 1, $F(x) \in \text{Bdry } F(U(x))$. Let us consider the family Ω of open connected sets such that for any arbitrarily chosen $x \in E_F(X, Y)$ it will be possible to find at least one $U \in \Omega$ such that $x \in U$ and $F(x) \in \text{Bdry } F(U)$. Since X satisfies the second axiom of countability, there is a countable open base $\{O_\alpha\}$ of X. Hence it is possible to find an $O \in \{O_\alpha\}$ such that $x \in O \subset U$. Hence as x runs over $E_F(X, Y)$ the corresponding set O runs over a subfamily of $\{O_\alpha\}$. Since the sets O are countable, the corresponding sets $U \in \Omega$ are also countable. Let $\{U_\alpha\}$ be the countable family of sets U so obtained.

For any $x \in E_F(X, Y)$ it will be possible, obviously, to select one $U_\alpha \in \{U_\alpha\}$ such that $x \in U_\alpha$ and $F(x) \in \text{Bdry } F(U_\alpha(x))$, by construction of the family $\{U_\alpha\}$. Since F is continuous and U_α is connected for each α, $F(U_\alpha)$ is also connected for each α and it will have at most two boundary points. Also $\{F(U_\alpha)\}$ is countable family of connected sets. Hence $\bigcup_\alpha \text{Bdry } F(U_\alpha)$ is countable at most. Again let $y \in F(E_F(X, Y))$. Therefore $F^{-1}(y) \cap E_F(X, Y) \neq \phi$. Let x_1 be a point $\in F^{-1}(y) \cap E_F(X, Y)$. Then there exists a $U \in \{U_\alpha\}$ such that $x_1 \in U$ and $F(x_1) \in \text{Bdry } F(U)$. Hence $y = F(x_1) \in \text{Bdry } F(U)$. Therefore $F(E_F(X, Y)) \subset \bigcup_\alpha \text{Bdry } F(U_\alpha)$ which is at most countable, proving the theorem.

IMBEDDING AN ARBITRARY RING IN A NONTRIVIAL NEAR-RING

JAMES R. CLAY, University of Arizona

Let R be a ring and let M be a module over R as defined in [1] or [2]. Let $N = R \times M$. For (r_1, m_1), $(r_2, m_2) \in N$, define $+$ by

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2) \tag{1}$$

and define $*$ by

$$(r_1, m_1) * (r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1). \tag{2}$$

It is immediate that $(N, +)$ is a group and that $(N, *)$ is a semigroup.

Let $(r_3, m_3) \in N$ also. Then

$$\begin{aligned}[(r_1, m_1) + (r_2, m_2)] * (r_3, m_3) &= (r_1 + r_2, m_1 + m_2) * (r_3, m_3) \\ &= ((r_1 + r_2)r_3, (r_1 + r_2)m_3 + m_1 + m_2)\end{aligned} \tag{3}$$

and

$$
\begin{aligned}
&(r_1, m_1) * (r_3, m_3) + (r_2, m_2) * (r_3, m_3) \\
&\qquad = (r_1r_3, r_1m_3 + m_1) + (r_2r_3, r_2m_3 + m_2) \\
&\qquad = (r_1r_3 + r_2r_3, r_1m_3 + m_1 + r_2m_3 + m_2) \\
&\qquad = ((r_1 + r_2)r_3, (r_1 + r_2)m_3 + m_1 + m_2).
\end{aligned} \tag{4}
$$

From (3) and (4) we see that $*$ is right distributive over $+$.

It is easy to see that $*$ is not left distributive over $+$. Let (r, g) be an arbitrary element of N and suppose g' is not the zero element of M. Then

$$(0, g') * [(r, g) + (r, g)] = (0, g') * (r + r, g + g) = (0, g') \tag{5}$$

and

$$(0, g') * (r, g) + (0, g') * (r, g) = (0, g') + (0, g') = (0, g' + g'). \tag{6}$$

From (5) and (6) we see that $*$ cannot be left distributive over $+$.

It follows that $(N, +, *)$ is a near-ring but not a ring. It remains to show that R can be imbedded in N. Define $\phi: R \to R$ by $\phi(r) = (r, 0)$. It is easy to see that $\phi(r_1+r_2) = \phi(r_1)+\phi(r_2)$ and that $\phi(r_1r_2) = \phi(r_1) * \phi(r_2)$ for $r_1, r_2 \in R$. Certainly ϕ is one-to-one onto $N' = \{(r_1,0): r \in R\}$, and $(N', +, *)$ is a ring. Hence R and N' are isomorphic and ϕ is an imbedding map.

Similarly, the map $\pi: M \to N$ defined by $\pi(m) = (0, m)$ imbeds M into the group $(N, +)$.

If R has an identity element 1 and if $1m = m$ for every $m \in M$, then $(1, 0)$ is an identity for the near-ring N.

Let R^+ denote the additive group of R. Then R^+ is a module over R and R can be imbedded into the near-ring $R \times R^+$ as above.

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ON THE THEOREM ON THE PRIMITIVE ELEMENT

J. Sonn and H. Zassenhaus, Ohio State University

In 1947 Artin remarked [1] that the known proofs of the theorem on the existence of a primitive element, in a finite separable extension of an arbitrary ground field, were uniformly marred by being divided into two cases of finite and infinite fields. The purpose of this note is to present a unified proof of the theorem.

Theorem. (Theorem on the Primitive Element) *Let E be a finite separable extension of a field F of degree $n > 1$. Then $E = F(\theta)$ for some θ in E.*

Proof. Let D be a minimal splitting field of E over F. Then there are exactly n distinct isomorphisms of E into D, say $s_1, s_2, \cdots, s_n$ where s_1 is the identity isomorphism. The $n-1$ subfields defined by the equations $s_i(a)=a$, $i=2, \cdots, n$, are, by the degree theorem, of degree at most $n/2$ over F. It remains to prove that there is an element θ in E lying outside of all these $n-1$ subfields; for then θ is of degree n over F and hence generates E.

Let $u_1, \cdots, u_n$ be a basis of E over F, and consider the set S of the 2^n elements $\sum_{i=1}^{n} e_i u_i$, $e_i=0, 1$. If it can be shown that each of the $n-1$ subfields contains at most $2^{n/2}$ elements of S, then the inequality $(n-1)2^{n/2}<2^n$ for $n>1$ completes the proof. That this is indeed the case is an immediate consequence of the following lemma.

LEMMA 1. *Let L be a linear space of dimension n over a field F, with basis $u_1, \cdots, u_n$. Then any subspace W of dimension m contains at most 2^m elements of the set $S=\left\{ \sum_{i=1}^{n} e_i u_i \middle| e_i=0, 1 \right\}$.*

Proof. Let $w_1, \cdots, w_m$ be a basis of W, and let $s \in S \cap W$. Then $s=\sum_{i=1}^{m} c_i w_i (c_i \in F)$ and, if $w_i=\sum_{j=1}^{n} w_{ij} u_j (w_{ij} \in F)$, there results a system of linear equations $\sum_{i=1}^{m} c_i w_{ij}=s_j$ $(j=1, \cdots, n)$, where $s=\sum_{j=1}^{n} s_j u_j$. Since this system has rank m, the unique solution $c_1, \cdots, c_m$ is completely determined by m of the coordinates s_j of s. As s varies in $S \cap W$, each of these m coordinates can be assigned at most two values (0 or 1) giving at most 2^m possible solutions $c_1, \cdots, c_m$.

When is the maximum value 2^m attained?

To answer this question the use of the language of affine geometry is appropriate.

DEFINITION. *A d-dimensional parallelotope with base point P_0 and edge vectors $u_1, u_2, \cdots, u_d$ is defined as the subset π_d of the n-dimensional affine space A_n over F that is obtained by producing the set $S_d=\left\{ \sum_{i=1}^{d} e_i u_i \middle| e_i=0, 1 \right\}$ of 2^d linear combinations of d independent n-vectors $u_1, u_2, \cdots, u_d$ over F from the point P_0 of the A_n. The members of π_d are called the vertices of π_d.*

Every vertex of π_d can serve as base point. If the characteristic of F is not 2, then for a fixed base point, only the $d!$ permutation s of a given system of edge vectors leads to other systems of edge vectors. If the characteristic of F is 2 then any of the $(2^d-1)(2^d-2) \cdots (2^d-2^{d-1})$ systems of d linearly independent vectors of S_d will be systems of edge vectors, no matter which vertex will be chosen as base point. We note that in the latter case S_d is a d-dimensional linear space over the prime field.

LEMMA 2. *If a d-dimensional linear manifold W of the A_n has 2^d points in common with an n-dimensional parallelotope π_n, then the intersection $W \cap \pi_n$ is a d-dimensional parallelotope.*

For any point P_0 of $W \cap \pi_n$ as basepoint, there is a system of edge vectors $u_1, \cdots, u_n$ of π_n such that the edge vectors of $W \cap \pi_n$ are of the form

(1) $u_{\alpha_{i-1}+1} + \cdots + u_{\alpha_i}$ $(1 \leqq i \leqq d;\ 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_d \leqq n)$.

The converse is obvious.

Proof. If the characteristic of F is 2 apply the remark preceding the lemma. If $d=0$ then Lemma 2 is trivial.

Let char $F \neq 2$. Let $d>0$, $n>0$. Apply induction over $d+n$. Let W be a $(d+1)$-dimensional linear manifold having at least 2^{d+1} points in common with π_{n+1} one of which be P_0. Let $u_1, u_2, \cdots, u_{n+1}$ be a system of edge vectors of π_{n+1} from P_0. Let P_1 be the vertex obtained upon producing u_{n+1} from P_0. Let L_0, L_1 be the linear manifolds that are generated by producing the linear combinations of $u_1, \cdots, u_n$ from P_0, P_1 respectively. It follows that the intersections $L_0 \cap \pi_{n+1}$, $L_1 \cap \pi_{n+1}$ are disjoint n-dimensional parallelotopes, at least one of which has 2^d points in common with W. Say there are 2^d points in common to $L_0 \cap \pi_{n+1}$ and W. If W is contained in L_0 then the statement of Lemma 2 follows by application of the inductive hypothesis to L_0, $L_0 \cap \pi_{n+1}$, W. If W is not contained in L_0, then the intersection $W \cap L_0$ is a d-dimensional linear manifold. By the inductive hypothesis, the intersection $W \cap L_0 \cap \pi_n$ is a d-dimensional parallelotope with base point P_0 and edge vectors (1) (after suitable renumbering of $u_1, \cdots, u_n$). It follows that $W \cap L_1$ is a d-dimensional linear manifold which has 2^d points in common with $L_1 \cap \pi_{n+1}$. By the inductive hypothesis, $L_1 \cap \pi_{n+1}$ is a d-dimensional parallelotope. There is a vertex P_1' of $W \cap L_1 \cap \pi_{n+1}$ such that the vector from P_0 to P_1' is of the form: $u_{\beta_1} + u_{\beta_2} + \cdots + u_{\beta_s} + u_{n+1}$ $(0 < \beta_1 < \beta_2 < \cdots < \beta_s < n+1)$ with minimal value of s. The vertices of $W \cap L_1 \cap \pi_{n+1}$ are obtained by producing certain nontrivial linear combinations of the vectors (1) from P_1'. Because of the minimality of s none of the d index sets $(1 \leqq i \leqq d)$ is contained in the index set $\{\beta_1, \beta_2, \cdots, \beta_s\}$. The coefficients of the linear combinations are 0 or 1. Since there are actually 2^d such linear combinations, all combinations of 0's and 1's do occur. Hence there is no overlap between any index set $\{\alpha_{i-1}+1, \cdots, \alpha_i\}$ and $\{\beta_1, \cdots, \beta_s\}$. Thus the lemma follows. (Renumber $u_1, \cdots, u_{n+1}$ again if necessary!)

There is an extension of the theorem of the primitive element to any extension E' of a field of prime characteristic p that is obtained by a separable extension E of finite degree n followed by one purely inseparable adjunction of degree $p^v > 1$. It is again claimed that E' can be obtained by the adjunction of just one element to F.

We prove the extension by means of Lemma 1 as follows: There are precisely n distinct isomorphisms of E' into a minimal splitting field of E' over F, say $s_1, \cdots, s_n$. The $n-1$ subfields E_i' of E which are defined by the equations: $s_i(a) = s_1(a)$ $(1 < i \leqq n)$ are of degree at most $np^v/2$ over F. Furthermore there is precisely one extension E_1' of E of degree p^{v-1} contained in E'. An element θ of E' which is not contained in any one of the fields $E_1', \cdots, E_n'$ must be a primitive element. Indeed, θ^{p^v} is an element of E which does not belong to any subfield $E_i = E \cap E_i'$ $(1 < i \leqq n)$ and hence it generates E over F. But since θ

is not contained in E_1' it follows that $E(\theta)=E'$ and thus $E'=F(\theta)$.

Let $u_1, u_2, \cdots, u_{np^v}$ be a basis of E over F; consider the set S of the 2^{np^v} elements $\sum_{i=1}^{np^v} e_i u_i$, $e_i=0,1$. From Lemma 1 it follows that S intersects E_i in at most $2^{np^v/2}$ elements and since $n2^{np^v/2}<2^{np^v}$ it follows that S contains a primitive element.

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DEGENERATE LOCI AND PROPER CYLINDRICAL SURFACES

D. C. B. MARSH, Colorado School of Mines

1. In a previous paper [1; Theorem 9] the author gave a necessary and sufficient condition for the existence of a "curve-locus" for a nonhomogeneous quadratic equation in n variables. Not considered was the equally important problem of determining all such loci. This note serves to resolve that question and in so doing provides an alternate proof of a recent theorem by Underwood [2; Theorem 6].

The reader may wish to refer to these definitions:

A *linear plotting rule* (L.P.R.) for $f(1, x_1, x_2, \cdots, x_n)=0$ is a pair of equations of the form

$$y_i = c_{i0} + \sum_{j=1}^{n} c_{ij}x_j, \quad i = 1, 2,$$

where $(c_{11}, c_{12}, \cdots, c_{1n})$ and $(c_{21}, c_{22}, \cdots, c_{2n})$ are linearly independent and, for each subscript j, c_{1j} and c_{2j} are not both zero. Alternatively, an L.P.R. is given by $y=Cx$ where $x=(1, x_1, x_2, \cdots, x_n)'$, $y=(1, y_1, y_2)'$, and

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ c_{10} & c_{11} & c_{12} & \cdots & c_{1n} \\ c_{20} & c_{21} & c_{22} & \cdots & c_{2n} \end{bmatrix}$$

is of rank 3 and has no zero column. (Underwood places no restriction on the rank of C, but the author finds it both advantageous and reasonable to do so.)

Under these conditions one may eliminate two of the x_j, say x_1 and x_2, from $f(1, x_1, x_2, \cdots, x_n)$ to obtain $g(1, y_1, y_2, x_3, \cdots, x_n)$ which is then regarded as a function of y_1 and y_2 alone.

The *locus* of $g(1, y_1, y_2, x_3, \cdots, x_n)=0$ is the totality of points (y_1, y_2) obtained by the L.P.R. from all sets $\{x_j\}$ which satisfy $f(1, x_1, x_2, \cdots, x_n)=0$. *A* locus is a *curve-locus* under an L.P.R. if all x_j occur vacuously in g; that is, if $g(1, y_1, y_2, x_3, \cdots, x_n)=g(1, y_1, y_2)$.

Since a nondegenerate conic section in the y_1, y_2-plane can represent the

A CONGRUENCE FOR $\tau(p^{4m})$

J. M. GANDHI, University of Alberta, Edmonton

1. Ramanujan's τ function is defined by

$$x\prod_{r=1}^{\infty}(1-x^r)^{24}=\sum_{n=1}^{\infty}\tau(n)x^n. \tag{1.1}$$

Gandhi [1] and Lehmer in an unpublished paper, made the conjecture that if p is a prime and θ be determined by $p=2^\theta n-1$, n being odd, then

$$\tau(p)\equiv 0 \pmod{2^\theta}, \qquad \tau(p)\not\equiv 0 \pmod{2^{\theta+1}}. \tag{1.2}$$

This conjecture was verified to be correct by Gandhi [1] for values of $\theta=1, 2, \cdots, 9$.

On the basis of (1.2), Gandhi [1] proved the following congruences

$$\tau(p^{4m+1})\equiv 0 \pmod{2^\theta}, \qquad \tau(p^{4m+1})\not\equiv 0 \pmod{2^{\theta+1}} \tag{1.3}$$

$$\tau(p^{4m+2})-1\equiv 0 \pmod{2^\theta}; \qquad \tau(p^{4m+2})-1\not\equiv 0 \pmod{2^{\theta+1}}. \tag{1.4}$$

$$\tau(p^{2^km-1})\equiv 0 \pmod{2^{\theta+k-1}}; \qquad \tau(p^{2^km-1})\not\equiv 0 \pmod{2^{\theta+k}}. \tag{1.5}$$

These congruences enable us to find the maximum power of 2 that will divide $\tau(p^{2m+1})$ and also the maximum power of 2 that will divide $\tau(p^{4m+2})-1$. In this note we prove that for $k>1$,

$$\begin{aligned}&\tau(p^{2^kR})-1\equiv 0 \pmod{2^{k+2\theta-2}},\\ &\tau(p^{2^kR})-1\not\equiv 0 \pmod{2^{k+2\theta-1}},\end{aligned} \tag{1.6}$$

and this gives us the maximum power of 2 that will divide $\tau(p^{4m})-1$. Since

$$\tau(mk)=\tau(m)\tau(k), \quad (m,k)=1, \tag{1.7}$$

using the above congruences we are now in a position to tell the maximum power of 2 that divides $\tau(n)$ when n is odd but not an odd square, while when n is an odd square, then also we know the maximum power of 2 which will divide $\tau(n)-1$.

2. To prove (1.6) we use the following identification given by Gandhi [1]

$$\tau(p^m)=\sum_{j=0}^{[m/2]}(-1)^j\binom{m-j}{m-2j}p^{11j}\tau^{m-2j}(p). \tag{2.1}$$

Let us consider $m=2^kR$, $k>1$. Changing the summation index in (2.1) to $r=2^{k-1}R-j$, we have

$$\tau(p^{2^kR})-1=\sum_{r=1}^{2^{k-1}R}(-1)^r\binom{2^{k-1}R+r}{2r}p^{11(2^{k-1}R-r)}\tau^{2r}(p). \tag{2.2}$$

Now consider

$$(2.3) \qquad \binom{2^{k-1}R + r}{2r} = \frac{(2^{k-1}R + r) \cdots (2^{k-1}R) - (2^{k-1}R - r + 1)}{(2r)!}.$$

Excluding the term $2^{k-1}R$, there are $(2r-1)$ factors of which r are odd factors and $(r-1)$ even factors, hence the numerator contains a factor $2^{k-1+r-1}$. By elementary methods it can be proved that the highest power of 2 which can divide $(2r)!$ is 2^{2r-1}. This proves that 2^{k-1-r} divides the left hand side of (2.3). Assuming (1.2), $\tau(p) = 2^\theta u$, u being odd. For each $r \geqq 0$ the corresponding term in (2.2) is divisible by

$$2^{k-1-r}\, 2^{\theta(2r)} = 2^{k-1+(2\theta-1)r}.$$

The term for $r=1$ in (2.2) is an odd multiple of $2^{k+2\theta-2}$, hence the congruence (1.6) follows.

(Note that Congruence (1.5) of paper [1] is misprinted and should read as (1.4) given in this paper.)

Acknowledgment. My thanks are due to Professor H. Gupta for his kind encouragement. The author is on leave from the University of Rajasthan, Jaipur, India.

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A CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION

M. S. SRIVASTAVA, Princeton University

1. Introduction. Characterization of the exponential distribution has recently been considered by many authors (Ferguson [1], and Tanis [2]) and several theorems based on properties of some suitable statistics have been given. The purpose of this note is to add one more to the existing set of theorems to characterize the exponential distribution whose probability density function is given by

$$(1) \qquad f(x) = (1/\sigma)e^{-(x-\theta)/\sigma}, \qquad x > \theta,\ \sigma > 0.$$

2. A characterization of the exponential distribution. The following theorem characterizes the exponential distribution.

THEOREM. *Let F be an absolutely continuous distribution function of the random variable X with $F(\theta) = 0$, $\theta > 0$ and with probability density function $f(x)$ (hereafter, pdf). Let $X_1 < X_2 < \cdots < X_n$ denote the order statistics of a random sample of size n from this distribution. Then, in order that the statistics $X_{m+1} - X_m$ and X_m for fixed m, $1 \leqq m < n$, be independent, it is necessary and sufficient that the random variable X has the exponential distribution.*

Proof. Necessity: Let $X_m = U$ and $X_{m+1} = V$. Then the pdf. of U is

$$(n!/(m-1)!(n-m)!)(F(u))^{m-1}(1-F(u))^{n-m}f(u),$$

and the joint pdf. of U and V is

$$(n!/(m-1)!(n-m-1)!)(F(u))^{m-1}(1-F(v))^{n-m-1}f(u)f(v).$$

Consequently, the conditional pdf. of $V \mid U=u$ is

$$(n-m)[(1-F(v))^{n-m-1}/(1-F(u))^{n-m}]f(v). \tag{2}$$

It is our desire to show that if $V-U$ and U are independent, then X has the exponential distribution. Because of the independence of $V-U$ and U, $E(V-U) = E(V-U \mid U=u)$ and is free of u. Thus

$$\begin{aligned} E(V-U) &= E(V-U \mid U=u) \\ &= (n-m)\int_u^\infty (v-u)[(1-F(v))^{n-m-1}/(1-F(u))^{n-m}]f(v)dv, \end{aligned} \tag{3}$$

and is free of u. Differentiating (3) with respect to u, we have with probability one

$$0 = -\int_u^\infty [(1-F(v))^{n-m-1}/(1-F(u))^{n-m}]f(v)dv + \frac{(n-m)f(u)}{1-F(u)}E(V-U).$$

Since $E(V-U)$ is independent of u and the first expression on the right is constant, we have

$$f(u)/(1-F(u)) = \beta, \tag{4}$$

where β is a constant different from zero. Consequently,

$$1-F(u) = e^{-\beta u+d}, \tag{5}$$

and

$$f(u) = \beta e^{-\beta u+d}, \tag{6}$$

where d is a constant of integration. But, since f is a probability density function over the range θ to ∞, it follows that $d=\beta\theta$, and $\beta>0$. Hence,

$$f(x) = \beta e^{-\beta(x-\theta)}, \qquad x>\theta, \beta>0. \tag{7}$$

Sufficiency. Now, suppose that the distribution of X is given by (7). Then the joint density of U and V is

$$= \beta^2(n!/(m-1)!(n-m-1)![1-e^{-\beta(u-\theta)}]^{m-1}e^{-(n-m)\beta(v-\theta)-\beta(u-\theta)}. \tag{8}$$

Consequently, the joint density of $Z=V-U$ and $W=U$ is

$$= \beta^2(n!/(m-1)!(n-m-1)!)[1-e^{-\beta(w-\theta)}]^{m-1}e^{-(n-m+1)\beta(w-\theta)}e^{-(n-m)\beta z}, \tag{9}$$

$$z>0, w>\theta.$$

Hence, Z and W are independent. The density of Z is given by

$$(n - m)\beta e^{-(n-m)\beta z}, \qquad z > 0,$$

and the density of W is given by

$$\beta(n!/(m-1)!(n-m)!)\left[1 - e^{-\beta(w-\theta)}\right]^{m-1} e^{-(n-m+1)\beta(w-\theta)}, \qquad w > \theta.$$

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A CHARACTERISTIC PROPERTY OF THE ELLIPSOID

ROLF SCHNEIDER, University of Frankfurt, Germany

Stamm [3] proved (what had been suggested by Blaschke [1], p. 232) that an ovaloid in three-space must be a sphere if it has the following property: The surface area of the zone lying between two parallel planes which cut the surface is always proportional to the distance between the planes. Here we shall give a similar characterization of the ellipsoid by a certain volume property.

Let B be a centrally symmetric convex body with interior points in three-dimensional Euclidean space, and let z be its centre. Let $T(u)$ be the supporting plane of B with unit normal vector u such that u points into that half-space which does not contain B. Let $h>0$ be a real number not greater than the distance of z from $T(u)$, and denote by $C(u, h)$ the solid "cap" which one of the planes parallel to $T(u)$ and at distance h from it cuts from B. Let $V(u, h)$ be the volume of the convex hull of $C(u, h)$ and z. For a solid sphere of radius r we have $V(u, h) = \frac{2}{3}\cdot\pi r^2 h$, and hence for a solid ellipsoid $V(u, h)$ is, for fixed u, proportional to h, where the proportionality factor depends on u. We want to invert this assertion, but we are only able to do this under the additional assumption that the boundary ∂B of B is sufficiently smooth:

THEOREM. *Let the centrally symmetric convex body B have its boundary of class C^2 and of positive Gaussian curvature. If $V(u, h)$ is always proportional to h, where the proportionality factor may depend on u, then B is bounded by an ellipsoid.*

Proof. We shall make use of the so-called "affine support function," known from the affine differential geometry of surfaces (see Blaschke [2]):

Let x be a variable point on the surface ∂B, where local parameters u^1, u^2 have been introduced. (We denote a point and its position vector with respect to some coordinate system by the same symbol.) Put

$$x_i = \frac{\partial x}{\partial u^i} \quad \text{and} \quad b_{ik} = |\, x_1, x_2, x_{ik}\,|, \quad 1 \leqq i, k \leqq 2,$$

where $|\, x_1, x_2, x_{ik}\,|$ denotes the determinant of the vectors x_1, x_2, x_{ik}. The determinant $\det(b_{ik})$ of the matrix with elements b_{ik} is positive since the curvature

of ∂B is positive. The function

$$\phi(x) = [\det(b_{ik})]^{-1/4} \left| x_1, x_2, x - z \right| \tag{1}$$

(which we may choose positive) is called the "affine support function" of ∂B with respect to z, since it resembles in some sense the ordinary support function $p(x)$, which is defined as the distance of z from the tangent plane of ∂B at x.

In the following we write $V(x, h)$ instead of $V(u, h)$, where x is the unique point of ∂B where $u = u(x)$ is assumed as outer unit normal vector (this is possible since ∂B is regular and of positive curvature). Let $\overline{V}(x, h)$ be the volume of the cap $C(u(x), h)$. Now, by a formula of Blaschke ([2], p. 128):

$$\phi(x)^2 = \frac{9}{4\pi} \lim_{h \to 0} \frac{[V(x, h) - \overline{V}(x, h)]^2}{\overline{V}(x, h)} \tag{2}$$

(which shows that $\phi(x)$ is invariant against unimodular affine transformations of space which leave z fixed).

Let $K(x)$ be the curvature of ∂B at x. Another formula of Blaschke [1], p. 120, says that

$$K(x) = \lim_{h \to 0} \left(\frac{\pi h^2}{\overline{V}(x, h)} \right)^2 . \tag{3}$$

Since we have assumed that

$$V(x, h) = k(x)h \tag{4}$$

with some function $k(x)$, (3) leads to

$$\lim_{h \to 0} \frac{[V(x, h) - \overline{V}(x, h)]^2}{\overline{V}(x, h)} = k(x)^2 \lim_{h \to 0} \frac{h^2}{\overline{V}(x, h)} = \frac{k(x)^2}{\pi} K(x)^{1/2}. \tag{5}$$

If in (4) we choose $h = p(x)$, then by the symmetry of B we get

$$V/2 = k(x)p(x), \tag{6}$$

where V is the volume of B. Furthermore, it is not difficult to deduce from (1) a representation of ϕ in terms of p and K: Writing $g_{ik} = x_i \cdot x_k$, we have

$$p(x) = [\det(g_{ik})]^{-1/2} \left| x_1, x_2, x - z \right|,$$
$$K(x) = [\det(g_{ik})]^{-1} \det(\bar{b}_{ik}),$$

where $\bar{b}_{ik} = [\det(g_{ik})]^{-1/2} b_{ik}$. This leads to

$$\phi(x) = p(x)K(x)^{-1/4}. \tag{7}$$

Now, combining (2), (5), (6), and (7), we arrive at

$$p(x) = (3V/4\pi)^{1/2} K(x)^{1/4}. \tag{8}$$

Let dA denote the element of surface area of ∂B at x. Integrating over the whole boundary of B, we have the well-known formula

$$\int p(x)dA = 3V.$$

Applying (8), we get

$$\left(\int K(x)^{1/4}dA\right)^2 = 12\pi V. \tag{9}$$

The integral $\Omega=\int K(x)^{1/4}dA$ (which can be shown to be invariant against unimodular affine transformations) is known as the "affine surface area" of ∂B. For this functional Blaschke ([2], Section 73) has proved an "isoperimetric inequality of affine differential geometry," namely

$$\Omega^2 \leqq 12\pi V.$$

Here equality characterizes ellipsoids; hence, according to (9), our theorem is proved.

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PARTIALLY ORDERED FIELDS

RALPH DEMARR, University of Washington

It is well known that the real number system is a totally ordered field which is complete, in the sense that every nonincreasing sequence of nonnegative numbers has a greatest lower bound. It is of some interest to know if there are partially ordered fields which are complete in this same sense. In this note we will show that the real number field is the only partially ordered field having this completeness property. For details on partially ordered sets and algebraic systems the reader may consult Birkhoff [1], Fuchs [2], or McShane [3].

By a partially ordered field F we mean a field (not necessarily commutative) in the usual algebraic sense which is also a partially ordered set subject to the following conditions (x, y, z denote arbitrary elements of F under the specified restrictions in each condition):

(a) if $x\leqq y$, then $x+z\leqq y+z$;
(b) if $0\leqq x$ and $0\leqq y$; then $0\leqq xy$;
(c) $0<1$, where 1 is the multiplicative identity;
(d) if $0<x$, then $0<x^{-1}$;
(e) for any x there exist $y\geqq 0$ and $z\geqq 0$ such that $x=y-z$.

This is equivalent to saying that F is directed. The reader should specifically

note that we do not assume that F is lattice ordered; see [2, pp. 138–140] for a detailed discussion on lattice ordered fields. In addition to the above conditions which define a partially ordered field we shall require a weak form of order completeness called Dedekind σ-completeness; see [3, pp. 9–11]:

(f) If $\{x_n\}$ is a sequence of elements from F such that $x_1 \geqq x_2 \geqq \cdots \geqq 0$, then $\inf\{x_n\}$ exists. If $y = \inf\{x_n\}$, then this means that $y \leqq x_n$ for all n and if $z \leqq x_n$ for all n, then $z \leqq y$. Using the fact that F is a partially ordered abelian group under addition, condition (f) can be used to show that if $\{x_n\}$ is a sequence of elements from F such that $x_1 \leqq x_2 \leqq \cdots \leqq y$, then $\sup\{x_n\}$ exists. We can in fact define $\sup\{x_n\} = y - \inf\{y - x_n\}$. A partially ordered field which satisfies condition (f) above will be called a Dedekind σ-complete field. We will now prove the main theorem.

THEOREM. *Every Dedekind σ-complete field (not necessarily commutative) is order isomorphic to the real number field.*

Proof. We will first show that every Dedekind σ-complete field F is totally ordered; i.e., if $x \in F$, then $x \leqq 0$ or $0 \leqq x$. We do this by first showing that if $u \in F$ and $u \geqq 0$, then $u \leqq 1$ or $1 \leqq u$. Thus, if $u \geqq 0$, define $x_n = 1 + u + \cdots + u^n$ for all $n = 1, 2, 3, \cdots$. It is clear that $1 \leqq x_1 \leqq x_2 \leqq \cdots$; hence, $1 \geqq x_1^{-1} \geqq x_2^{-1} \geqq \cdots \geqq 0$. Therefore, $w = \inf\{x_n^{-1}\}$ exists and, of course, $w \geqq 0$. We now consider two cases.

CASE I. $w = 0$. From the fact that $(1-u)x_n = 1 - u^{n+1} \leqq 1$ we see that $1 - u \leqq x_n^{-1}$ for all n; hence, $1 - u \leqq w = 0$, which means that $1 \leqq u$.

CASE II. $w \neq 0$. This means that $x_n \leqq w^{-1}$ for all n. Hence, $v = \sup\{x_n\}$ exists. If we put $y_n = v - x_n$, then $u^{n+1} \leqq y_n$ and $\inf\{y_n\} = 0$. Therefore, $(1-u)x_n = 1 - u^{n+1} \geqq 1 - y_n$ from which it follows that $u - 1 \leqq y_n x_n^{-1} - x_n^{-1} \leqq y_n$ for all n. Hence, $u - 1 \leqq \inf\{y_n\} = 0$, which means that $u \leqq 1$.

These two cases exhaust all possibilities. Hence, if $u \geqq 0$, then $u \leqq 1$ or $1 \leqq u$. Now if we take any $x \in F$, then there exist y, $z \in F$ such that $y \geqq 0$ and $z \geqq 0$ and $x = y - z$. If $z = 0$, then $x = y \geqq 0$. If $z \neq 0$, then $yz^{-1} \geqq 0$ and, therefore, $yz^{-1} \leqq 1$ or $1 \leqq yz^{-1}$. If $yz^{-1} \leqq 1$, then $y \leqq z$ and, hence, $x = y - z \leqq 0$. If $1 \leqq yz^{-1}$, then $z \leqq y$ and, hence, $x = y - z \geqq 0$. In any event, $x \leqq 0$ or $0 \leqq x$.

Using the fact that F is Dedekind σ-complete, we may show that it has the following property: if x, $y \in F$ and $0 \leqq nx \leqq y$ for all $n = 1, 2, 3, \cdots$, then $x = 0$; for the proof see [2, p. 90]. From this we can establish that F is Archimedean (see [2], p. 12). Hence, we can use Theorem 1 on p. 126 of [2] to assert that F is order isomorphic to a subfield of the real number field. Since F contains a subfield which is order isomorphic to the rationals and since every real number is the infimum of a nonincreasing sequence of rationals, we can use the fact that F is Dedekind σ-complete to assert that F is order isomorphic to the real number field.

We will now show by examples that the above theorem could not be proved without explicitly assuming axioms (c), (d), (e), and (f). Axioms (a) and (b) are

quite natural and, therefore, we will not question them. In the first two examples F will denote the field of rational numbers and $\leqq_o$ will denote the usual ordering of the real numbers.

Example I. Axiom (c) is a consequence of axioms (b) and (d). Therefore, we must eliminate (c) and (d) simultaneously. Let $P=\{0\}\cup\{x: 2\leqq_o x\}$. Define $x\leqq y$ to mean that $y-x\in P$. It is easily verified that axioms (a), (b), (e), and (f) hold. F is not totally ordered.

Example II. Let $P=\{0\}\cup\{x: 1\leqq_o x\}$. Define $x\leqq y$ to mean that $y-x\in P$. Axioms (a), (b), (c), (e), and (f) hold. F is not totally ordered.

Example III. Let F be the field of complex numbers and define $x\leqq y$ to mean that $y-x$ is real and $0\leqq_o y-x$. Axioms (a), (b), (c), (d), and (f) hold. F is not totally ordered.

Example IV. Let F be the collection of all real-valued rational functions defined on the real line. It is clear that F is a field. We now partially order F as follows: if $x, y\in F$, then $x\leqq y$ if and only if $x(t)\leqq_o y(t)$ for all real numbers t for which both functions are defined. If we define $x_n(t)=(1+t^2+\cdots+t^{2n})^{-1}$, then $x_1\geqq x_2\geqq\cdots\geqq 0$, but $\inf\{x_n\}$ does not exist. Thus, F is a partially ordered field, but it is not Dedekind σ-complete. It is clear that F is not totally ordered and, hence, cannot be order isomorphic to the real number field.

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THE BEHAVIOR OF SOLUTIONS OF THE EQUATIONS $x''\pm xx'^2\pm x^3=0$

W. R. Utz, University of Missouri

The differential equation

$$x'' + xx'^2 + q^2x^3 = 0, \qquad x' = dx/dt,$$

is satisfied by $x=\pm\sin qt$ and by $x=\pm\cos qt$. The fact that these periodic and oscillatory functions satisfy the equation suggests that all solutions may have these properties. This is established in Theorem 1. One part of the proof of this theorem is similar to the proof of a theorem of E. McHarg [1] but neither theorem implies the other.

Theorem 1. *In the equation*

$$x'' + xx'^2 + xf(x) = 0 \tag{1}$$

let $f(x)$ be an even integrable function, $f(x)>0$ for $x=0$,

$$F(x) = \int_0^x uf(u)du \to \infty \quad as\ x\to\infty.$$

Assume that solutions of (1) *are unique. If* $x=x(t)$ *is a solution of* (1) *valid for all* t, *then* $x(t)$ *is oscillatory* (*if* $x\not\equiv 0$) *and periodic.*

Proof. Let $x(t)$ satisfy (1) for all t. There is no loss in generality in taking $x(0)\geqq 0$ since $-x(t)$ also satisfies (1). For this reason, if $x(0)=0$, we may assume $x'(0)>0$.

First consider the case $x_0=x(0)\geqq 0$, $v_0=x'(0)>0$ (the remaining case being $x_0=x(0)>0$, $v_0=x'(0)\leqq 0$).

In the (x, v)-plane, $x=x(t)$, $v=v(t)$ satisfy

$$\frac{dv}{dx} = -\frac{x}{v}(v^2+f(x)) \tag{2}$$

wherein $v=x'$. Let $A(x, v)=F(x)+v^2/2$. Since $F(x)\to\infty$ as $x\to\infty$ it follows that for each real $c>0$, $A(x, v)=c$ is a simple closed curve symmetric with respect to both the x-axis and v-axis in the (x, y)-plane. Moreover, since $-x(t)$ satisfies (1) if $x(t)$ satisfies (1), any integral curve of (2) is symmetric about the v-axis and it is closed if it cuts the positive v-axis and also cuts the negative v-axis.

Consider $P(x_0, v_0)$ in the (x, v)-plane and let C be the corresponding integral curve of (2) through P. So long as $x>0$, $v>0$, C has negative slope hence cannot turn toward the only singular point, (0, 0). On the other hand,

$$dA/dt = -xv^3 < 0$$

on C hence C lies within the curve $A(x, v)=F(x_0)+v_0^2/2$. According to the Poincaré-Bendixson Theorem, C cuts the x-axis at some point $x=a$, $v=0$ with $0<a<\alpha$ where $F(\alpha)=F(x_0)+v_0^2/2$.

Suppose that this occurs for $t=t_1>0$. That is, $x(t_1)=a>0$, $x'(t_1)=0$, $x''(t_1)=-af(a)<0$. We now require that for some $t_2>t_1$, $x(t_2)=0$ and $v(t)=x'(t)<0$ for $t_1<t\leqq t_2$. It is clear that, other conditions satisfied, $v(t_2)=x'(t_2)<0$ because $dA/dt>0$ in quadrant IV hence C can not turn toward the singularity (0, 0).

Now, considering $x=x(t)$ in the (t, x)-plane we have $x(t_1)>0$ and, by continuity, $x(t)>0$ for $t>t_1$ unless $x(t)$ has the desired zero beyond t_1. It is clear that if there is no $t_2>t_1$ for which $x(t_2)=0$, then for all $t>t_1$, $x'(t)=v(t)<0$ since $dv/dt>0$ on C until $x=0$ and since $x''(t)<0$ until $x=0$. The combination $x(t)>0$, $x'(t)<0$, $x''(t)<0$ for all real $t>t_1$ is impossible as is seen by an application of the mean-value theorem [2]. Thus there must be a $t=t_2$ for which $x(t_2)=0$ and C meets the negative v-axis.

If $x_0=0$, then this case of the proof is finished. If the initial conditions were $x_0>0$ (and, of course, $v_0>0$) then C must be continued through the quadrants III and then II to meet the positive v-axis. In quadrant III this is done as in quadrant I by the Poincaré-Bendixson Theorem. The continuation in quadrant II is done as in quadrant IV to secure the intersection on the positive v-axis. Continuing to P is automatic since the integral curves are symmetric with respect to the v-axis.

Finally, if one has initial conditions $x_0>0$, $v_0\leqq 0$ one follows C across the

negative v-axis and through quadrants III and II, as before, to secure an intersection on the positive v-axis. This completes the proof.

The proof given for Theorem 1 may be used to yield the same theorem for the equation

$$x'' + h(x)g(x') + h(x)f(x) = 0$$

if $g(x')$ is an even function, $g(x')>0$ for $x'>0$, $h(x)$ is an odd integrable function, $h(x)>0$ for $x>0$, and $f(x)$ is as above. One now takes

$$A(x, v) = \int_0^x h(u)f(u)du + v^2/2.$$

The solutions of the equation

$$x'' - xx'^2 - x^3 = 0, \qquad x' = dx/dt,$$

are eventually monotone as is clear from the following theorem.

THEOREM 2. *In the equation*

$$x'' = xF(x, x') \tag{3}$$

assume that $F(x, x')>0$ *for* $x^2+x'^2\neq 0$ *and that solutions of the equation are unique. If* $x=x(t)$, $x(t)\not\equiv 0$, *is a solution of* (3) *valid for all* $t\geqq t_0$, *then* $x(t)$ *is eventually monotone. If* $x(t)$ *has a zero, then* $x(t)\to\infty$ *or* $x(t)\to-\infty$.

Proof. Suppose that for some $t_1\geqq t_0$, $x(t_1)=0$. Since $x(t)\not\equiv 0$ and solutions are unique, $x'(t_1)\neq 0$. If $x'(t_1)>0$, it is clear that $x'(t)>0$ for all $t\geqq t_1$ since $x''(t)>0$ so long as $x>0$. Thus, for all $t>t_1$, $x(t)>0$, $x'(t)>0$ and $x''(t)>0$. Hence $x(t)\to\infty$ unless $x(t)\to c<\infty$. The latter case is impossible by the mean value theorem [2].

If $x'(t_1)<0$, then $x'(t)<0$ for all $t\geqq t_1$ since $x''(t)<0$ so long as $x(t)<0$. Thus, for all $t>t_1$, $x(t)<0$, $x'(t)<0$ and $x''(t)<0$. Hence $x(t)\to-\infty$ unless $x(t)\to c>-\infty$. The latter case is again impossible.

Suppose that $x(t)=0$ for $t\geqq t_0$. If $x(t_0)>0$, then $x'(t_0)>0$ implies $x'(t)>0$ for all $t>t_0$ since $x''(t)>0$ and $x(t)\to\infty$ as $t\to\infty$. If $x'(t_0)<0$, then either $x'(t)\leqq 0$ for all $t\geqq t_0$ or, in case $x'(t)$ changes sign, $x'(t)>0$ for all large t and $x(t)\to\infty$ as $t\to\infty$. At most one change of sign is possible for $x'(t)$ for $t>t_0$ since $x''(t)>0$ for all $t\geqq t_0$. In either case, $x(t)$ is eventually monotone.

Similarly, if $x(t_0)<0$, then $x''(t_0)<0$. If $x'(t_0)\leqq 0$, then $x(t)\to-\infty$ as $t\to\infty$. If $x'(t_0)>0$, then either $x'(t_0)\geqq 0$ for all $t\geqq t_0$ or, again, $x'(t)$ may have one zero after which $x'(t)<0$ and $x(t)\to-\infty$. In all cases $x(t)$ is eventually monotone and the theorem is proved.

The behavior of solutions of

$$x'' - xx'^2 + q^2x^3 = 0, \quad \text{and} \tag{4}$$

$$x'' + xx'^2 - q^2x^3 = 0 \tag{5}$$

is not as easily established. As a clue to what may happen we notice that (4) is satisfied by $x=\pm\sinh qt$ and (5) is satisfied by $x=\pm\cosh qt$.

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NEAR-HOMOGENEITY ON 2-MANIFOLDS

H. C. WISER, Washington State University

The concept of continuous near-homogeneity was introduced in [5]. In this note, this and other related types of homogeneity are considered as applied to proper subcontinua of a 2-manifold. A space X is said to be *near-homogeneous* if for each $x\in X$ and each open set U in X there is a homeomorphism h of X onto itself such that $h(x)\in U$. If for each x and U there is an isotopy H_t of X onto itself such that $H_1(x)\in U$, the space is said to be *continuously near-homogeneous*. In the following, $H(X)$ will be the isotopy path of X under the isotopy H_t. A space X is *locally homogeneous* if for each two points x and y of X there are neighborhoods U_x and U_y and a homeomorphism h on U_x such that $h(U_x)=U_y$ and $h(x)=y$.

LEMMA 1. *If H_t is an isotopy of a simple triod T which moves the branch point of T, then $H(T)$ contains a simple closed curve.*

Proof. Since $H(T)$ is the continuous image of a locally connected set it must be locally connected. Suppose $H(T)$ contains no simple closed curve, i.e., $H(T)$ is a dendrite. Let b be the branch point of T, and A the unique arc from $H_0(b)$ to $H_1(b)$. There are uncountably many values of t in $[0, 1]$ such that the corresponding points $H_t(b)$ are disjoint points of A, branch points of $H_t(T)$, and thus branch points of the dendrite—a contradiction.

THEOREM 1. *No continuously near-homogeneous proper subcontinuum X of a 2-manifold M contains a simple triod.*

Proof. Suppose X contains a triod T. Let $p\in X-T$ (since T is clearly not continuously near-homogeneous) and D a disk on M containing p and not intersecting T. There is an isotopy h of X onto itself which carries the branch point of T into D and thus an isotopy H_t and a triod T_1 in $h(T)$ such that $H(T_1)$ lies entirely in D and $H_0(b)\neq H_1(b)$ where b is the branch point of T_1. By Lemma 1 there is a simple closed curve J in $D\cap X$ and since $X\neq J$ there is an isotopy g_t of J such that $g(J)\subset D$ and $g_1(J)\neq J$. As in Theorem 10 of [5] such an isotopy would sweep through some disk in D and we are led to the contradiction that X is a 2-manifold without boundary.

COROLLARY 1. *A continuously near-homogeneous arcwise connected nondegenerate proper subcontinuum of a 2-manifold must be a simple closed curve.*

Proof. This is an immediate consequence of Theorem 1 above and Theorem 9 of [7].

THEOREM 2. *If X is a continuously near-homogeneous proper subcontinuum of a 2-manifold which contains a simple closed curve J, then X is a simple closed curve.*

Proof. Suppose $p \in X - J$ and U is an open set containing p such that $U \cap J = 0$. Let H be an isotopy on X such that $H_1(J) \cap U \neq 0$, then $H(J)$ must be a continuum filled with the simple closed curves $\{H_t(J)\}$ for $0 \leqq t \leqq 1$. Any two of these simple closed curves must be identical or fail to intersect since X contains no triod by Theorem 1. By Theorem 7 of [7], $H(J)$ must be an annulus or Möbius strip which contradicts Theorem 1.

COROLLARY 2. *If X is a locally connected continuously near-homogeneous nondegenerate proper subcontinuum of a 2-manifold, then X is a simple closed curve.*

Proof. By Corollary 2 of [5], X must contain a simple closed curve and the result follows from Theorem 2.

REMARK. In [1] R. H. Bing gives an example of a planar continuously near-homogeneous continuum which is not a simple closed curve. This continuum is also locally homogeneous.

THEOREM 3. *If a nondegenerate proper subcontinuum X of a 2-manifold is aposyndetic and continuously near-homogeneous then X is a simple closed curve.*

Proof. In Theorem 10 of [7] the analogous result for homogeneous continua is established. The properties of X in the proof of that theorem which depend on homogeneity are (1) X contains no simple triod, (2) if X contains a simple closed curve then it is a simple closed curve, and (3) X contains no separating point. For continuously near-homogeneous continua, these three properties are established by Theorem 1 and Theorem 2 above, and by Corollary 1 of [5]. With these substitutions one may proceed as in Theorem 10 of [7] to show that if X is not a simple closed curve then a component of the boundary of one of its complementary domains must be an arc which leads to a contradiction.

REMARK. A Sierpinski curve (i.e., a 2-sphere with a sequence of disjoint nontangent open disks removed such that the union of the disks is dense in the sphere) is near-homogeneous, locally connected, and contains a homeomorphic image of each 1-dimensional plane continuum. Thus continuously near-homogeneity cannot be replaced by near-homogeneity in Theorems 1, 2, and 3. Since the Sierpinski curve is also invertible [3], this condition (though stronger than near-homogeneity) cannot be substituted for continuous near-homogeneity. Of course continuous invertibility implies continuous near-homogeneity and thus can be substituted to give weaker versions of Theorems 1, 2, and 3. These theorems would have as corollaries some of the theorems of [4]. By removing disks from 2-manifolds other than the sphere we obtain other 1-dimensional near-

homogeneous continuous curves; one may show that such continua are non-planar by locating primitive skew curves in them.

The locally connected case of the following theorem is considered since the proof of the misstated Corollary 8.2 of [7] establishes the result only for continua which are not locally connected.

THEOREM 4. *No locally homogeneous proper subcontinuum X of a 2-manifold M contains a simple triod.*

Proof. By local homogeneity each point of X must be of the same order. Since all but a countable number of local separating points of X must be of order 2, either X is a simple closed curve or X has no local separating points. In the latter case local homogeneity may be substituted for homogeneity in the proof of Lemma 10.2 of [7] to show that the boundary of each complementary domain of X consists of a finite number of continuous curves. Let B_i be a component of the boundary of a complementary domain D_i. If B_i contains no simple closed curve it must be a dendrite which leads to a contradiction. Suppose B_i contains a simple closed curve J and $B_i \neq J$ then B_i contains a simple triod T with branch point b. Since X is locally connected and has no local separating point we may use a result of F. B. Jones' [6] to get a connected open set D in X which contains b but not the end points of T, lies on a disk in M, and has a connected boundary. Let x, y, and z be the endpoints of T. Then there are continua K_1 and K_2 in the boundary of D such that K_1 is irreducible from the arc bx to by and K_2 is irreducible from by to bz. But since x, y, z are on the boundary of D_i the continua K_1 and K_2 prevent some points near b from being boundary points of D_i. Thus B_i is a simple closed curve.

If each point of X is on the boundary of some complementary domain, X is the union of a countable number of simple closed curves. Then by a Baire theorem some one of them contains an open set and thus a local separating point. Since we have assumed that X has no local separating points, some point q of X must fail to lie on the boundary of a complementary domain. Each point a of an arc A in X must be the branch point of some simple triod $T(a)$. Let h_a be a homeomorphism from a canonical simple triod T onto $T(a)$. The set $B = \{T(a)/a \in A\}$ becomes a separable metric space under the metric $d(T(a), T(b)) = \max_{t \in T} (h_a(T), h_b(T))$. In this metric there is some sequence of triods in B converging to a triod in B and thus a sequence of disjoint arcs converging to an arc C (details of such an argument may be found in [1]). On C one may choose a disk neighborhood D and, using the local connectedness of X and the arcs of X converging to C, find a simple closed curve J_1 in $X \cap D$. The interior of J_1 relative to D must contain some complementary domain of X with boundary J. Choosing $p \in J$ and using local homogeneity just as in the proof of Theorem 1 in [2] we find that having p on the boundary of a disk in the complementary domain and q not on the boundary of a complementary domain leads to a contradiction.

The analog of Corollary 1 for locally homogeneous continua now follows from Theorem 4. To establish the analog of Theorem 3 one need only show that a

locally homogeneous proper subcontinuum of a 2-manifold is a simple closed curve if it contains a simple closed curve.

I wish to thank Professor C. E. Burgess for calling my attention to F. B. Jones' result.

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THE INTERVAL TOPOLOGY AND ORDER CONVERGENCE AS DUAL CONVERGENCE STRUCTURES

D. C. KENT, Washington State University

In a partially ordered set $(S, \leqq)$, the *interval topology* is defined by taking for a closed subbase all sets of the form x^+ and x^*, all x in S, where x^+ and x^* denote respectively the set of all lower and upper bounds of x. (For a subset A of S, A^+ and A^* denote likewise the lower and upper bounds of A.) A filter $\mathfrak{F}$ on S *order-converges* to x if $\inf \mathfrak{F}^* = \sup \mathfrak{F}^+ = x$, where $\mathfrak{F}^+ = (\cup F^+: F \text{ in } \mathfrak{F})$ and $\mathfrak{F}^* = (\cup F^*: F \text{ in } \mathfrak{F})$. The reader is referred to [1], [4], and [5] for additional information about these concepts.

A convergence γ on a set S is called a convergence structure if: (1) For each x in S, the ultrafilter x^0 generated by x γ-converges to x, and (2) if $\mathfrak{F}$ γ-converges to x and $\mathfrak{G} \supset \mathfrak{F}$ then $\mathfrak{G}$ γ-converges to x. Order convergence is a convergence structure on any partially ordered set $(S, \leqq)$, and, needless to say, each topology defines a convergence structure. A convergence structure γ is *topological* if γ-convergence of filters coincides with convergence under some topology t. If γ and δ are convergence structures on S, then γ is *finer* than δ if a filter δ-converges to a point whenever it γ-converges to that point.

We shall now suppress all order properties of $\leqq$, regarding it as an arbitrary relation on S. If $A \subset S$ and $\mathfrak{F}$ is a filter on S, then A^*, A^+, $\mathfrak{F}^*$, $\mathfrak{F}^+$ are still meaningful in this weaker structure. (For instance, $A^* = \{x \text{ in } S: y \leqq x, \text{ all } y \text{ in } A\}$.)

DEFINITION 1. (1) *A filter $\mathfrak{F}$ on S q-converges to x if $\mathfrak{F}^{*+} \subset x^+$ and $\mathfrak{F}^{+*} \subset x^*$.*

(2) *A filter $\mathfrak{G}$ on S p-converges to x if, for each ultrafilter $\mathfrak{F} \supset \mathfrak{G}$, $\mathfrak{F}^{+*} \supset x^*$ and $\mathfrak{F}^{*+} \supset x^+$.*

THEOREM 1. (1) *The relation $\leqq$ on S is reflexive if and only if q is a convergence*

structure. (2) *The relation* $\leqq$ *on* S *is transitive if and only if* p *is a convergence structure.*

Proof. In order for $\leqq$ to be reflexive (transitive) it is necessary and sufficient that, for each x in S, $x^{+*} \subset x^*$ and $x^{*+} \subset x^+$ ($x^{+*} \supset x^*$ and $x^{*+} \supset x^+$). It is clear that $(x^0)^{+*} = x^{+*}$ and $(x^0)^{*+} = x^{*+}$. Thus the statements "x^0 q-converges to x, all x in S" and "x^0 p-converges to x, all x in S," respectively, are equivalent to the statements "$\leqq$ is reflexive" and "$\leqq$ is transitive," respectively. Next, assume that $\mathfrak{F}$ q-converges to x. If $\mathfrak{G} \supset \mathfrak{F}$, then $\mathfrak{G}^{+*} \subset \mathfrak{F}^{+*}$ and $\mathfrak{G}^{*+} \subset \mathfrak{F}^{*+}$, and thus $\mathfrak{G}$ q-converges to x. The second condition for being a convergence structure is automatically satisfied by p-convergence.

THEOREM 2. (1) *If* $\leqq$ *is transitive and reflexive, then* p *is topological, and the corresponding topology* t *has a closed subbase of sets of the form* x^+ *and* x^*, x *in* S. (2) *The convergence structure* q *is finer than* p.

Proof. (1) If $\mathfrak{F}$ is an ultrafilter on S which converges to x with respect to the topology t described above, and y is not in x^+, then the complement of y^* is in $\mathfrak{F}$. Thus y is not in $\mathfrak{F}^{*+}$, and $x^+ \supset \mathfrak{F}^{*+}$. Similarly, $x^* \supset \mathfrak{F}^{+*}$, and $\mathfrak{F}$ p-converges to x. This reasoning is reversible. (2) Let $\mathfrak{F}$ q-converge to x; $\mathfrak{G} \supset \mathfrak{F}$. Then $\mathfrak{G}^{*+} \subset x^+$. If y is in x^+, then y is a lower bound of x^*, hence of $\mathfrak{G}^{+*}$, and hence of $\mathfrak{G}^*$. Thus $x^+ = \mathfrak{G}^{*+}$. Similarly, $x^* = \mathfrak{G}^{+*}$, and $\mathfrak{G}$ p-converges to x. It follows that $\mathfrak{F}$ p-converges to x.

THEOREM 3. *If* $(S, \leqq)$ *is a partially ordered set, then* q *coincides with order convergence.*

Proof. Let $\mathfrak{F}$ order-converge to x. Since $x = \sup \mathfrak{F}^+$, each upper bound of $\mathfrak{F}^+$ is an upper bound of x, and hence $\mathfrak{F}^{+*} \subset x^*$. Also, $\mathfrak{F}^{*+} \subset x^+$, and $\mathfrak{F}$ q-converges to x. Conversely, let $\mathfrak{F}$ q-converge to x. Then $\mathfrak{F}^+ \subset \mathfrak{F}^{*+} \subset x^+$. If y is an upper bound of $\mathfrak{F}^+$, then y is in x^*, and thus $x = \sup \mathfrak{F}^+$. Similarly, $x = \inf \mathfrak{F}^*$, and $\mathfrak{F}$ order-converges to x.

We have exhibited a fundamental duality between the interval topology and the order convergence in a partially ordered set which is not at all apparent from the standard definitions of these concepts.

If $\leqq$ is an equivalence relation, then q is topological; the corresponding topology has for an open base the set of equivalence classes relative to $\leqq$. When $\leqq$ is a partial order relation, the problem of finding additional conditions on the order that make q topological is interesting and nontrivial. The only solutions (other than a chain or product of chains) known to me are found in [2] and [3].

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CLASSROOM NOTES

POWER SERIES SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

JOHN W. DETTMAN, Oakland University

It has become rather common in elementary books on ordinary differential equations to prove at least one existence and uniqueness theorem for the initial value problem for a first order equation. It is much less common to find such theorems proved for higher order equations, because the usual procedure is to put the problem in the form of a system, and the students usually do not have the linear algebra required to cope with systems. For the applications, higher order linear equations are very important. The existence and uniqueness theorem for these can be handled as easily as for the first order equation if a Green's function is introduced and the problem is cast in the form of a Volterra integral equation. For example, the initial value problem $y''+q(t)y=0$, $y(0)=c_0$, $y'(0)=c_1$, is equivalent to $y=c_0+c_1t+\int_0^t(\tau-t)q(\tau)y(\tau)d\tau$. (The equation $y''+p(t)y'+q(t)y=0$ can be put into this form by first making the change of variable $y=vw$, where $v=\exp[-\frac{1}{2}\int_0^t p(\tau)d\tau]$.) The Green's function, in this case is $G(\tau, t)=\tau-t$. Using successive approximations, it is easy to prove the following theorem:

THEOREM 1. *If $q(t)$ is continuous for $|t|\leqq r$, the initial value problem $y''+qy=0$, $y(0)=c_0$, $y'(0)=c_1$ has a unique solution valid for $|t|\leqq r$.*

For the purpose of computation it is useful to have various representations of the solution to the initial value problem. One of the most important is the power series representation. It is well known that we have the following theorem:

THEOREM 2. *If $q(t)$ has a power series representation valid for $|t|<R$, then the solution to $y''+qy=0$, $y(0)=c_0$, $y'(0)=c_1$ has a power series representation valid in the same interval.*

The purpose of this note is to give a simple proof of Theorem 2, which should be readily understood by sophomores who have had a good course in calculus including infinite series. This theorem is usually proved using complex variables, Cauchy inequalities, etc., techniques which are not usually available to sophomores. In [1] Coddington proves the theorem without complex variables but the proof involves rather elaborate juggling of inequalities. The present proof makes use of the sequence of successive approximating functions $y_n(t)$ which were shown in the proof of Theorem 1 to converge uniformly to the solution $y(t)$. It is a simple matter to show that each $y_n(t)$ has a power series expansion which agrees to $2n+1$ terms with the formal power series for $y(t)$. Using this we can show that the formal series actually converges.

Since $q(t)$ has a power series representation $q_0+q_1t+q_2t^2+\cdots$ for $|t|<R$, we may differentiate it at $t=0$ as many times as we like and hence we can obtain

the coefficients for the formal power series for $y(t)$, i.e., $y(0)=c_0$, $y'(0)=c_1$, $c_2=y''(0)/2!=-\frac{1}{2}c_0q_0$, $c_3=y'''(0)/3!=-\frac{1}{6}(c_0q_1+c_1q_0)$, etc. The only problem is to show that $c_0+c_1t+c_2t^2+\cdots$ converges for $|t|<R$.

In proving Theorem 1, we shall have introduced successive approximations, $y_0=c_0+c_1t$, and for $n=1, 2, 3, \cdots$

$$y_n(t) = c_0 + c_1t + \int_0^t (\tau - t)q(\tau)y_{n-1}(\tau)d\tau.$$

and shall have shown that $y_n(t)$ converges uniformly to $y(t)$, the unique solution, for $|t|\leqq r<R$. Now it is easy to show that $y_n(t)$ has a power series representation for $|t|<R$. This is a simple induction in which one forms the Cauchy product of y_{n-1} and q and integrates term by term.

We also can show that for $n=1, 2, 3, \cdots$ the power series for y_n agrees with the formal series for y up to $2n+1$ terms. This is because y_n satisfies $y_n''=-qy_{n-1}$, $y_n(0)=c_0$, $y_n'(0)=c_1$ and therefore its derivatives at $t=0$ up to the $2n$-th are the same as those for y.

For the next step of the proof we need the following simple lemma about power series with positive coefficients:

LEMMA. *If* $f(t)=a_0+a_1t+a_2t^2+\cdots$ *converges for* $|t|\leqq r$ *and* $a_n\geqq 0$ *for* $n=0, 1, 2, 3, \cdots$, *then* $a_n\leqq M/r^n$ *where* $M=\max f(t)$ *for* $|t|\leqq r$.

Proof. Since $a_n\geqq 0$ for all n, $M=\max f(t)=a_0+a_1r+a_2r^2+\cdots$. Hence $a_nr^n\leqq M$, and $a_n\leqq M/r^n$.

We first prove the expansion Theorem 2 for the special case of a $q(t)$ with all its coefficients negative, and it will turn out that the general case can be obtained from this special case. If all the coefficients of $q(t)$ are negative then $y_n(t)$ will have a power series representation with all its coefficients positive, provided c_0 and c_1 are positive. This is because

$$y_n(t) = c_0 + c_1t + \int_0^t (t - \tau)[-q(\tau)]y_{n-1}(\tau)d\tau$$

and the Cauchy products for $-qy_{n-1}$ will all have positive coefficients. Therefore, if

$$y_n(t) = a_{n0} + a_{n1}t + a_{n2}t^2 + \cdots$$

then $a_{nm}\leqq M_n/r^m$, where $M_n=\max y_n(t)$ for $|t|\leqq r<R$. Remembering that $y_n(t)$ converges uniformly to $y(t)$ on $|t|\leqq r$, where $y(t)$ is continuous, we see that the M_n are bounded. Hence, there exists M independent of n such that $a_{nm}\leqq M/r^m$. Thus $a_{mm}\leqq M/r^m$, and $c_m=a_{mm}$, hence $c_m\leqq M/r^m$. Now $|c_mt^m|\leqq M(|t|/r)^m$ and, therefore, the formal series converges for $|t|<r$. But r is *any* number satisfying $0<r<R$. Hence, the formal series converges for $|t|<R$.

Finally, for the general case, consider $|t|\leqq r<R$. Since $q_0+q_1t+q_2t^2+\cdots$

converges absolutely for $t=r$, there exists a K such that $|q_n| \leqq K/r^n$. Consider the problem

$$Y'' - (K + Kt/r + Kt^2/r^2 + \cdots)Y = Y'' - Kr/(r-t)Y = 0,$$
$$Y(0) = C_0 = |c_0|, \quad Y'(0) = C_1 = |c_1|.$$

By the above discussion, Y has a power series representation $C_0+C_1t+C_2t^2+\cdots$ for $|t|<r$. It is easily seen by comparison, that $c_0+c_1t+c_2t^2+\cdots$ converges for $|t|<r$, since $|c_0|=C_0$, $|c_1|=C_1$, $|c_2|=\frac{1}{2}|c_0q_0| \leqq \frac{1}{2}C_0K=C_2$, $|c_3|=\frac{1}{6}|c_0q_1+c_1q_0| \leqq \frac{1}{6}(C_0K/r+C_1K)=C_3$, etc. Therefore, since r is *any* number satisfying $0<r<R$, the power series representation of y converges for $|t|<R$.

The reader will recognize that Theorem 2 holds because $t=0$ is an *ordinary point* of the differential equation. Nevertheless, the same method of proof will work for the corresponding convergence theorem for the modified power series expansion of solutions near a regular singular point (see [2]).

The method of proof presented here can easily be extended to higher order equations. The linear equation of order n can be put in the form of an integral equation, using a Green's function which is the unique solution of an initial value problem of order $n-1$. Hence the existence and uniqueness theorem and the power series representation can be established by induction.

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MATRIX OF FINITE GROUPS

C. F. HARDING, Douglas Aircraft Company

It is well known that the elements of a finite group may be represented by a set of matrices obeying the same rules of combination under matrix multiplication. It has never been shown, however, that such a set of matrices may be obtained by inspection directly from the group multiplication table. In this paper we do just that for the finite group composed of positive integers modulo a prime, with its extension to more complex groups being obvious.

THEOREM. *If a matrix $M(\alpha)$ is constructed from the multiplication table of the positive integers modulo a prime p by placing* 1 *where α appears and zeros elsewhere, then $R(\alpha)=M(1)M(\alpha)$ is isomorphic to the group and forms a representation.*

Proof. Let us assume a finite group made up of the integers modulo a prime p so that its elements are 1, 2, $\cdots$, $p-1$. Now construct a multiplication table with the elements given in order, as shown below for the case $p=7$.

(1)

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

If we take three elements $ij=k$, define the matrices $M(i)$, $M(j)$, and $M(k)$ by putting 1 wherever i, j, or k appear in the table while leaving zeros elsewhere. It should be clear from simple group algebra that each matrix has one and only one nonzero entry for either a row or a column; for example $M(5)$

(2)
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In particular the locations (row x_i, column y_i) for $M(i)$ must be and are the only ones satisfying (3) below

(3) $$x_i y_i \equiv i \pmod{p}.$$

By a similar argument the locations of $M(j)$ are fixed by the y_i which satisfy (4) as x_j ranges over $1, 2, \cdots, p-1$.

(4) $$x_j y_j \equiv j \pmod{p};$$

we now define by $M(1)$ that special matrix such that y_1 is the inverse of x_1

(5) $$x_1 y_1 \equiv 1 \pmod{p}.$$

At this point multiply the above equations in the specified order so as to obtain the composite

(6)
$$\begin{aligned} x_i y_i x_1 y_1 x_j y_j &\equiv i \cdot 1 \cdot j \pmod{p} \\ &\equiv k \pmod{p}. \end{aligned}$$

If it is true that $R(i)R(j)=R(ij)=R(k)$ then

(7) $$M(k) = M(i)M(1)M(j).$$

As we are interested in the product $M(i)M(1)M(j)$ one can choose a specific x_i;

then, to find the column y_1 for $M(i)M(1)$, we obtain the equality

$$y_i = x_1. \tag{8}$$

By the same procedure we multiply $M(i)M(1)$, whose row and column of concern are x_iy_1, by $M(j)$ and get for the final column y_j, since it must be that

$$y_1 = x_j. \tag{9}$$

The equation (6) now reads as follows (10) telling us that the rows and columns of $M(k)$ are x_i, y_j:

$$x_iy_j \equiv k \pmod{p}. \tag{10}$$

Hence we can define a set of matrices $R(\alpha) = M(1)M(\alpha)$, for each α, which obey all the rules of the original group in a one-to-one correspondence (thus giving the desired representation) and where $R(1)$ is the identity; in fact the unit matrix.

GENERALIZED LEMMAS OF THE CALCULUS OF VARIATIONS

N. X. VINH, University of Colorado

In [1], Akhiezer gives a generalized version of the second fundamental lemma of the calculus of variations [2]. Using the same line of derivation this note generalizes to systems both of the fundamental lemmas.

Consider a real n-vector function

$$M(x) = [M_1(x), M_2(x), \cdots, M_n(x)], \quad x_1 \leqq x \leqq x_2 \tag{1}$$

and an arbitrary real n-vector function

$$\eta(x) = [\eta_1(x), \eta_2(x), \cdots, \eta_n(x)], \quad x_1 \leqq x \leqq x_2. \tag{2}$$

Let

$$(M, \eta) = \int_{x_1}^{x_2} \sum_{i=1}^{n} M_i(x)\eta_i(x)dx \tag{3}$$

then:

LEMMA I. *If $M(x)$ is continuous on the interval $[x_1, x_2]$ and if the equation*

$$(M, \eta) = 0 \tag{4}$$

holds for any arbitrary vector function $\eta(x)$ which belongs to the class C^m and which vanishes together with all its derivatives up to and including those of order k, $k \leqq m$, at the endpoints of $[x_1, x_2]$, then $M(x) \equiv 0$.

Proof. Suppose to the contrary that $M(x)$ does not vanish at a point $\bar{x}$ in $[x_1, x_2]$. Let $M_p(x)$ be a nonvanishing component of $M(x)$ at that point. By continuity there exists some interval $[a, b] \subset [x_1, x_2]$ in which $M_p(x)$ never vanishes. We consider the particular vector $\eta(x)$ such that

$$\eta_p(x) = 0 \text{ outside the interval } [a, b]$$
$$\eta_p(x) = [(x - a)(b - x)]^{m+1} \quad \text{for } a \leqq x \leqq b$$

and all the other components of $\eta(x)$ are identically zero on $[x_1, x_2]$. Then

$$(M, \eta) = \int_a^b M_p(x)[(x - a)(b - x)]^{m+1}dx \neq 0. \tag{5}$$

Inequality (5) contradicts assumption (4). Hence $M(x) \equiv 0$.

LEMMA II. *Let*

$$\phi_i(x) = [\phi_{i1}(x), \phi_{i2}(x), \cdots, \phi_{in}(x)], \quad i = 1, 2, \cdots, n \tag{6}$$

be n linearly independent vector functions, continuous on the interval $[x_1, x_2]$. *If* $M(x)$ *is a piecewise continuous vector function such that*

$$(M, \eta) = 0 \tag{7}$$

for all vector functions $\eta(x)$ *satisfying*

$$(\eta, \phi_i) = 0, \quad i = 1, 2, \cdots, n \tag{8}$$

then there exists a unique constant vector C such that at every point at which $M(x)$ *is continuous*

$$M(x) = \Phi'(x)C \tag{9}$$

where Φ' *is the transpose of the matrix formed by the vectors* ϕ_i.

Proof. Let

$$M(x) - \Phi'(x)C = \eta_0(x). \tag{10}$$

We shall prove that there exists a unique constant vector C such that $\eta_0(x)$ defined by (10) will satisfy (8). This requires that

$$C_1(\phi_i, \phi_1) + C_2(\phi_i, \phi_2) + \cdots + C_n(\phi_i, \phi_n) = (M, \phi_i), \quad i = 1, 2, \cdots, n. \tag{11}$$

The determinant of the linear system (11) in the C_i is the Gram determinant of the system (6) and is nonvanishing [1]. Hence C is uniquely determined and the vector $\eta_0(x)$ thus obtained, satisfies (8). Therefore we can use $\eta_0(x)$ in (7). Next, we have

$$\begin{aligned}(M - \Phi'C, M - \Phi'C) &= (M - \Phi'C, \eta_0) \\ &= (M, \eta_0) - \sum_{i=1}^{n} C_i(\eta_0, \phi_i) = 0.\end{aligned} \tag{12}$$

The integrand in the integral on the left is nonnegative. Hence, it follows from equation (12) that at every point at which $M(x)$ is continuous

$$M = \Phi'C.$$

It is obvious that this lemma includes the generalized lemma derived in [1] for a scalar function $M(x)$.

I wish to thank the referee and the editor for suggesting modifications.
This work is supported by ARPA-ARO Contract No. DA-31-124-ARO(D)-139.

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NONARCHIMEDEAN INTEGERS

G. H. MEISTERS, University of Colorado

Our terminology is that of McCoy [1], but we shall denote an ordered integral domain (defined on page 37 of [1]) by the ordered pair (D, P), where D denotes the integral domain under consideration and P denotes the set of "positive" elements of D which satisfies the trichotomy law and is closed under addition and multiplication.

In McCoy's treatment the integers are characterized as an ordered integral domain with the property that the set of all positive elements is well-ordered (cf. [1], pp. 39–41). Then, if (D, P) is the domain of integers, the fact that P is well-ordered is used to prove (on p. 49 of [1]) the

DIVISION ALGORITHM. *$a \in D$ and $b \in P$ implies there exist unique elements q and r in D such that $a = bq + r$ and $0 \leqq r < b$.*

In this note we answer the following rather natural

QUESTION. If (D, P) is any ordered integral domain which satisfies the Division Algorithm, does it necessarily follow that P is well-ordered? And if not, what can be said about such an ordered integral domain?

That the answer to the first part of the question is "no" can be seen by an example such as the following:

An example of a nonarchimedean ordered integral domain which satisfies the division algorithm: Let D denote the integral domain of all polynomials in x over the field of rational numbers which have integral constant term. Let P denote those members of D whose leading coefficient is positive (in the usual sense). P is not well-ordered since elements of the form $x - n$ (for $n = 0, 1, 2, \cdots$) are in P and the set S of all polynomials of this type has no least member. Proof that this ordered integral domain satisfies the above Division Algorithm is as follows. Let $a(x)$ be a member of D and $b(x)$ a member of P. Then, in particular, $a(x)$ and $b(x)$ belong to $F[x]$, where F is the field of rational numbers, and so (cf. [1] 8.16, p. 135) there exist unique polynomials $q'(x)$ and $r'(x)$ in $F[x]$ such that

$$a(x) = b(x)\cdot q'(x) + r'(x) \tag{1}$$

and $r'(x)=0$ or $\deg r'(x)<\deg b(x)$. If the constant term q'_0 of $q'(x)$ is an integer and the leading coefficient of $r'(x)$ is negative, let $s=1$. Otherwise let s denote the unique rational number satisfying $0\leqq s<1$ such that q'_0-s is an integer. Then equation (1) can be written in the form

$$a(x) = b(x)\cdot[q'(x) - s] + s\cdot b(x) + r'(x). \tag{2}$$

Thus if we denote $q'(x)-s$ by $q(x)$ and $s\cdot b(x)+r'(x)$ by $r(x)$ we obtain $a(x)=b(x)\cdot q(x)+r(x)$ and $0\leqq r(x)<b(x)$, where $q(x)$ and $r(x)$ belong to D. Furthermore, $q(x)$ and $r(x)$ satisfying these conditions are seen to be uniquely determined.

Thus in order to characterize the integers, the property that P be well-ordered cannot be replaced by the Division Algorithm. The following simple results show, however, that ordered integral domains which satisfy the Division Algorithm (but without a well-ordered P) have some properties in common with the integers (such as Corollaries 1 and 2 below) and can be characterized as "nonarchimedean."

THEOREM. *If (D, P) is an ordered integral domain satisfying the Division Algorithm and if p is a member of P which is not a positive "integer," me, then $me<p$ for all integers m.* (Here e denotes the multiplicative identity of D.)

Proof. Let S denote the set of positive integers m such that $p<me$. If S is not empty, then S contains a least member, say n. Thus $(n-1)e<p<ne$. But then $0<p-ne+e<e$. If q denotes $p-ne+e$, we obtain

(i) $0=e\cdot 0+0$ with $0\leqq 0<e$, and (ii) $0=e\cdot(-q)+q$ with $0\leqq q<e$, which contradicts the uniqueness requirement of the Division Algorithm. Hence S must be empty.

COROLLARY 1. *If (D, P) is an ordered integral domain satisfying the Division Algorithm, then D contains no member between 0 and e.*

Proof. If $0<p<e$, then p is not of the form me for any positive integer m (because $me\geqq e$). Hence, by the above theorem $p>me$ for all integers m and thus, in particular $p>e$ contrary to hypothesis.

COROLLARY 2. *If (D, P) is an ordered integral domain satisfying the Division Algorithm, then no member of D other than e and $-e$ can possess a multiplicative inverse.*

Proof. Suppose $xy=e$, $x\neq e$ and $x\neq -e$. Then $|x|\cdot|y|=e$ and $|x|\neq e$ implies $|x|>e$. But this implies $|x|\cdot|y|>e\cdot|y|>0$ or $e>|y|>0$, contrary to Corollary 1.

DEFINITION. *An ordered integral domain (D, P) is said to be archimedean if for each element a in D and each element b in P there exists an integer n such that $nb>a$.*

COROLLARY 3. *An ordered integral domain (D, P) satisfying the Division Algorithm is isomorphic to the domain of integers if and only if (D, P) is archimedean.*

Proof. Direct consequence of the above theorem.

Thus the example described above is a nonarchimedean ordered integral domain which nevertheless satisfies the Division Algorithm, and consequently could be considered a system of "nonarchimedean integers."

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REMARKS ON A SECOND ORDER DIFFERENTIAL EQUATION

J. P. Hoyt, U. S. Naval Academy

The differential equation: $xd^2y/dx^2-(x+n)dy/dx+ny=0$, n a nonnegative integer, has some interesting properties which make it very handy in teaching elementary differential equations. Some of these properties are:

1. Although, for $n>0$, it has a regular singular point at $x=0$, it is satisfied by $y=e^x$ for all values of n (no restriction here), i.e. it is satisfied by a simple series solution.

2. It is satisfied by a polynomial consisting of the first $n+1$ terms of the Maclaurin expansion of e^x.

3. Its indicial equation has roots 0, and $n+1$. Although the two roots differ by an integer, the smaller root gives a general solution.

4. In solving the equation by the method of Frobenius, both the polynomial solution and a Frobenius type series solution are obtained by using the smaller root of the indicial equation. By some small manipulation the solution $y=e^x$ is then obtained.

5. It is easy to see that $y=e^x$ is a solution and then the substitution $y=ve^x$ can be easily used to reduce the order and discover a general solution.

Finally, with particular values substituted for n, the equation provides a good source of individual problems.

ON A FIXED-POINT THEOREM FOR METRIC SPACES

T. K. Hu, Southern Illinois University

Let X, Y be metric spaces. A mapping f: $X\to Y$ for which there exists r, $0<r<1$, such that $d(f(x), f(y))\leqq rd(x, y)$ for all $x, y\in X$ will be called a *contraction* and r a *contraction constant for* f. The following fixed-point theorem is well known: *Let X be a complete metric space; if S is a nonempty closed subset of X and f: $S\to S$ a contraction, then f has a unique fixed point.* Question: What are the most general metric spaces with this fixed-point property? The answer is easy:

Proposition. *Let X be a metric space. Then the following are equivalent:*
(1) *X is complete,*

(2) *If S is any nonempty closed subset of X and f: $S \to S$ any contraction, then f has a fixed point.*

Proof. We need only show that (2) implies (1). Suppose that X contains a nonconvergent Cauchy sequence $(x_n)_{n=0}^{\infty}$. Since $(x_n)_{n=0}^{\infty}$ has the Cauchy property, it has no cluster points, i.e., it has no convergent subsequence. Since, moreover, $\{x_n: n=0, 1, \cdots\}$ must be infinite, we may assume that it consists of distinct terms; otherwise, we may select a subsequence which does. Take any $x \in X$; then $l(x) = \inf\{d(x, x_n): x_n \neq x, n=0, 1, \cdots\} > 0$ because $(x_n)_{n=0}$ has no cluster points. Choose any r such that $0 < r < 1$. We define a mapping σ of the set of nonnegative integers into itself inductively as follows: $\sigma(0) = 0$; and if $n > 1$ and $\sigma(n-1)$ is defined, let $\sigma(n)$ be an integer $> \sigma(n-1)$ such that $d(x_i, x_j) \leqq rl(x_{\sigma(n-1)})$ for all integers $i, j \geqq \sigma(n)$. Then $(x_{\sigma(n)})_{n=0}^{\infty}$ is a subsequence of distinct terms and is nonconvergent. The set $S = \{x_{\sigma(n)}: n=0, 1, \cdots\}$ is closed and the mapping f: $S \to S$ defined by $f(x_{\sigma(n)}) = x_{\sigma(n+1)}$ for $n=0, 1, \cdots$, is a contraction with no fixed point. Our proof is therefore complete.

We remark that the result is not stated in the sharpest form: (2) may be weakened by replacing "closed subset" by "countably infinite closed subset" and by restricting our attention to contractions with a fixed contraction constant r.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

COLLABORATING EDITORS: J. BARLAZ, Rutgers—The State University; L. CARLITZ, Duke University; HASKELL COHEN, Louisiana State University; H. EVES, University of Maine; M. S. KLAMKIN, Ford Scientific Laboratory; R. C. LYNDON, University of Michigan; and A. WILANSKY, Lehigh University.

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P.O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before August 31, 1967. Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 1975. *Proposed by Maurice Machover, Fairleigh Dickinson University, and H. W. Gould, West Virginia University*

Prove the following summations for all real z:

(1) $$\sum_{k=0}^{n} \binom{z}{2k}\binom{z-2k}{n-k}2^{2k} = \binom{2z}{2n},$$

(2) $$\sum_{k=0}^{n} \binom{z+1}{2k+1}\binom{z-2k}{n-k}2^{2k+1} = \binom{2z+2}{2n+1}.$$

E 1976. *Proposed by J. M. Khatri, Baroda, India*

Find all sets of four natural numbers x, y, z, w in arithmetic progression such that $\Delta_x+\Delta_y+\Delta_z=\Delta_w$, where $\Delta_a=\frac{1}{2}a(a+1)$ is a triangular number.

E 1977. *Proposed by D. R. Rao, Secunderabad, India*

Find all sets of four natural numbers x, y, z, w in arithmetic progression such that $x^3+y^3+z^3=w^3$.

E 1978. *Proposed by T. L. Saaty, U. S. Arms Control and Disarmament Agency, Washington, D.C.*

Given n collinear points in a plane, what is the minimum possible number of intersections of semicircular arcs in the plane joining all nonadjacent pairs of points? What is the minimum number of intersections if simple curves are used instead of semicircles?

E 1979. *Proposed by R. T. Hood, Franklin College, Indiana*

Professor Euclides Paracelso Bombasto Umbuggio, who will be remembered by readers of this Department, has emerged from retirement. Having heard some of the recent discussion about the independence of the continuum hypothesis, he now appears in order to demolish it forever. To reach the widest possible audience, he has chosen the most elementary means of doing so: obviously, $1<\aleph_0$. Therefore, $\aleph_0<\aleph_0^{\aleph_0}$. Moreover, $\aleph_0<2^{\aleph_0}$. Therefore, $\aleph_0^{\aleph_0}<(2^{\aleph_0})^{\aleph_0}=2^{\aleph_0\cdot\aleph_0}=2^{\aleph_0}=c$. Therefore, $\aleph_0<\aleph_0^{\aleph_0}<c$.

Deflate the professor: demolish his demolishment!

E 1980. *Proposed by R. E. Chandler, Duke University*

The street plan for a certain town is a generalization of the tic-tac-toe board with n vertical streets and m horizontal streets (with no vertical street intersecting the end of a horizontal street and vice versa). An automobile will start at the end of one (unspecified) street and travel to the end of another. What is the minimum number of traffic counters (each to be placed in the middle of a block) necessary to determine the route of the automobile? The route is to have no cycles, and the counters record the time that the automobile passes so that the direction of the route is determined when two counters have been passed.

E 1981. *Proposed by M. V. Subbarao, University of Alberta*

Let b and k be given positive numbers, $k\leqq 1$, and let B be the union of the open intervals $|x-(4n+k)b| <b$ $(n=1, 2, \cdots)$. If A is a subset of B such that $x\in A$ implies $2x\in A$, show that $A=\{4b, 8b, 12b, \cdots\}$ if $k<1$ and A is null if $k=1$.

E 1982. *Proposed by J. S. Biggerstaff, Portland, Ore.*

A rational polygon is one whose sides, diagonals and area are all rational numbers. (1) Find sets of distinct rational triangles (not right-angled or isosceles) which have the same area; and (2) Find a pair of rational quadrilaterals (not parallelograms or trapezoids) which have the same area.

E 1983. *Proposed by Robert Spira, University of Tennessee*

Show that replicas of the 3×4 L-hexomino cannot be fitted to form a rectangle. (An L-hexomino is formed when straight lines are drawn joining the following points in order: (0, 0), (3, 0), (3, 1), (1, 1), (1, 4), (0, 4), (0, 0).)

E 1984. *Proposed by Erwin Just, Bronx Community College*

If the divisors of n are $d_1, d_2, \cdots, d_k$, prove that

$$\prod_{i=1}^{k} \phi(d_i) \leqq (n/k)^k.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

Difference Equation

E 1837 [1965, 1129]. *Proposed by Yasuhiko Ikebe, University of Texas*

Solve the difference equation in two positive integers, m and n,

$$f(m, n) = n[f(m-1, n-1) + f(m-1, n)],$$

under the condition $f(m, n)=0$ if $m<n$, and $f(m, 1)=1$ for every m.

I. *Solution by M. T. L. Bizley, London, England.* Let $g(m, n)=f(m, n)/m!n!$ so that

$$mg(m, n) = g(m-1, n-1) + ng(m-1, n). \tag{1}$$

Since $g(m, n)=0$ for $m<n$, let

$$G \equiv G(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} g(m, n)x^m y^n.$$

Then, $g(m, 1)=1/m!$ for all m implies, with the difference equation, that $g(m, 0)=0$ for all $m>0$. Hence we have from (1)

$$x\frac{\partial G}{\partial x} = xyG + x\left(y\frac{\partial G}{\partial y}\right),$$

$$\frac{1}{G}\frac{\partial G}{\partial x} = y\left\{1 + \frac{1}{G}\frac{\partial G}{\partial y}\right\},$$

$$\frac{\partial P}{\partial x} = y\frac{\partial P}{\partial y}, \qquad P = y + \log G.$$

Having regard to the boundary conditions $P=ye^x$, whence $G=e^{y(e^x-1)}$, which is

$$G=\sum_{n=0}^{\infty}\frac{1}{n!}y^n(e^x-1)^n.$$

Therefore $g(m, n)$ is the coefficient of x^m in

$$(e^x-1)^n/n!=\left(e^{nx}-\binom{n}{1}e^{(n-1)x}+\binom{n}{2}e^{(n-2)x}-\cdots\right)\Big/n!$$

which is

$$\frac{1}{n!}\left(\frac{n^m}{m!}-\binom{n}{1}\frac{(n-1)^m}{m!}+\binom{n}{2}\frac{(n-2)^m}{m!}-\cdots\right),$$

and hence

$$f(m, n)=g(m, n)m!n!=n^m-\binom{n}{1}(n-1)^m+\binom{n}{2}(n-2)^m-\cdots$$
$$+(-1)^{n-1}\binom{n}{n-1}1^m.$$

II. *Solution by C. A. Church, Jr. West Virginia University.* The recurrence, as given, is well known as that for $f(m, n)=\Delta^n 0^m$, the "differences of zero." The $f(m, n)$ are also known as the exponential numbers and have the exponential generating function $(e^x-1)^n$. If one divides the recurrence by $n!$ the result is the familiar recurrence for the Stirling numbers of the second kind. That is, in Riordan's notation, $S(m, n)=f(m, n)/n!=\Delta^n 0^m/n!$.

The results stated may be found in

J. Riordan, *An Introduction to Combinatorial Analysis*, New York, (1958), pp. 13, 33, 43.

C. Jordan, *Calculus of Finite Differences*, 2nd ed., New York, (1960), (Reprint), pp. 169, 613.

G. Boole, *The Calculus of Finite Differences*, 4th ed. (Reprint), pp. 28, 29.

This MONTHLY, 56 (1949), p. 347, problem 4276. (Note misprint in the last formula: the summation should be taken over $j=0, 1, \cdots, c-1$.)

Also solved by Anders Bager (Denmark), Sylvan Greene, M. S. Klamkin, D. A. Klarner, J. A. Lambert (Australia), D. C. B. Marsh, F. D. Parker, Walter Penney, Stanley Rabinowitz, and K. L. Yocom.

Bizley and Penney note that this solves the problem: Find the number of ways of placing m different objects into n different cells with no cell empty.

Two Triangle Inequalities

E 1838 [1965, 1129]. *Proposed by A. Oppenheim, University of Malaya, Kuala Lumpur*

Suppose that ABC is an acute-angled triangle: then

$$16\prod\cos^2 A+4\sum\cos^2 B\cos^2 C\leqq 1, \tag{1}$$

(2) $$4\sum \cos^2 B \cos^2 C \leqq \sum \cos^2 A.$$

Equality occurs when ABC is equilateral or right-angled isosceles and in no other case.

Solution by C. S. Venkataraman, Trichur, India. Let A' be the midpoint of BC; D the foot of the perpendicular from A upon BC; O the orthocenter; and R the circumradius of triangle ABC. We use the following three well-known results:

(i) $\Pi \cos A \leqq \frac{1}{8}$,

(ii) $\sum \cos^2 A = 1 - 2\Pi \cos A$,

(iii) $2R \cos B \cos C = OD$.

The angles A, B, C being acute implies that O lies inside the triangle ABC. Now from the triangle ODA', right-angled at D, we have $OD \leqq OA'$. It follows that the maximum value of OD is OA'. But when $OD = OA'$, the triangle OBC is isosceles with $OB = OC$, whence $AB = AC$. Hence from (iii) $2R \cos B \cos C$ is maximum when $AB = AC$. Similarly $2R \cos C \cos A$ is maximum when $BC = AB$. Then $AC = BC$ and $2R \cos A \cos B$ is also maximum, and triangle ABC is equilateral. Thus it follows that

$$\sum (2R \cos B \cos C)^2 \leqq \sum (2R \cdot \tfrac{1}{2} \cdot \tfrac{1}{2})^2 = 3R^2/4.$$

Therefore $4\sum \cos^2 B \cos^2 C \leqq \frac{3}{4}$. Using (i) this gives

$$16 \prod \cos^2 A + 4 \sum \cos^2 B \cos^2 C \leqq 16(1/64) + 3/4 = 1,$$

which is the required result (1).

Further (i) and (ii) imply

$$\sum \cos^2 A \geqq 1 - 2(1/8) = 3/4 \geqq 4 \sum \cos^2 B \cos^2 C,$$

which is result (2). Clearly, equality arises in both cases if and only if ABC is equilateral.

Triangle with Two Perpendicular Medians

E 1839 [1965, 1129]. *Proposed by Simeon Reich, The Hebrew Reali Secondary School, Haifa, Israel*

In the triangle ABC, m_b and m_c (the medians to the sides b and c) are perpendicular to each other. Prove $\cot B + \cot C \geqq \frac{2}{3}$.

I. *Solution by J. W. Baldwin, Wayne State University.* Let AD, BE, CF be the medians with $BE \perp CF$ at G, and $OD = DC = BD$. Draw $AK \perp BC$ at K. Then

$$\cot B + \cot C = \frac{BK + KC}{AK} = \frac{BC}{AK} \geqq \frac{BC}{AD} = \frac{2 \cdot GD}{3 \cdot GD} = \frac{2}{3}.$$

II. *Solution by Walter Bluger, Dominion Bureau of Standards, Ottawa.* The medians divide each other in the ratio 2:1. Let $m_c = 3$, $m_b = 3t$. Noting the right

triangles, we easily obtain

$$\cot B + \cot C = \frac{2t^2 - 1}{3t} + \frac{2 - t^2}{3t} = \frac{1}{3}(t + t^{-1}).$$

but the minimum of $t+t^{-1}$ is known to be 2 (when $t=1$ and the triangle is isosceles). Thus the proof is complete.

Also solved by A. N. Aheart, Ted Aldrich, Dov Avishalom (Israel), P. N. Bajaj, Merrill Barnebey, M. A. Bershad, D. A. Blaeuer, R. F. Brinker, M. J. Brown, John Burslem, C. R. Conniff, N. Ganesan (India), L. M. Geer, Richard Gisselquist, P. L. Gore, M. S. Greening (Australia), Louise S. Grinstein, J. D. Haggard, J. R. Hanna, M. H. Hayamizu, Donald Kendall, M. S. Klamkin, R. N. Knoshaug, J. D. E. Konhauser, Joseph Malkevitch, D. C. B. Marsh, Lieselotte Miller, Norman Miller, R. E. Moyer, P. R. Nolan (Ireland), Stanton Philipp, Stanley Rabinowitz, G. L. N. Rao (India), L. A. Ringenberg, M. S. R. K. Sastry, Perry Scheinok, D. V. Sitaramamurthy (India), R. Sivaramakrishnan & C. S. Venkataraman (India), Sidney Spital, J. L. Standig, Jimmy Vandergriff, Simon Vatriquant (Belgium), C. S. Venkataraman, M. D. Weick, Charles Wexler, J. W. Wilson, Dale Woods, K. L. Yocom, and Leonard Yonis.

A Trigonometric Identity Corrected

E 1840 [1965, 1129]. *Proposed by Simeon Reich, The Hebrew Reali Secondary School, Haifa, Israel*

Prove, for every odd integer x:

$$\sum_{k=1}^{(x-1)/2} \cos\frac{2k\pi}{x} \Big/ \sum_{k=0}^{(x-3)/2} \sin\left[\frac{\pi}{x}(2k+1)\right] = -\cot\frac{\pi}{2x}.$$

Solution by M. S. R. K. Sastry, University of Rochester. The formula as stated is wrong. The right member should be $-\tan(\pi/2x)$ as we will prove.

Multiplying the numerator and the denominator of the left member by $2\sin(\pi/x)$ and using elementary trigonometry, we can write the left member as

$$\frac{\displaystyle\sum_{k=1}^{(x-1)/2} \left\{\sin(2k+1)\frac{\pi}{x} - \sin(2k-1)\frac{\pi}{x}\right\}}{\displaystyle\sum_{k=0}^{(x-3)/2} \left\{\cos(2k)\frac{\pi}{x} - \cos(2k+2)\frac{\pi}{x}\right\}} = \frac{-\sin(\pi/x)}{1+\cos(\pi/x)} = -\tan\frac{\pi}{2x}.$$

Also solved by A. N. Aheart, P. N. Bajaj, C. A. Church, Jr., N. Ganesan (India), Sylvan Greene, H. R. Leifer, Beatriz Margolis, D. C. B. Marsh, Liselotte Miller, Perry Scheinok, D. V. Sitaramamurthy (India), Franklin C. Smith, Sidney Spital, J. L. Standig, C. S. Venkataraman & R. Sivaramakrishnan (India), Charles Wexler, Leonard Yonis, and David Zeitlin.

An Inequality of Absolute Values

E 1841 [1965, 1129]. *Proposed by D. S. Mitrinović, Belgrade University, Yugoslavia*

If a, b, c, d ($ad \neq bc$) are complex numbers and t is a real variable, prove the following inequalities:

$$\frac{|a\bar{b}-b\bar{a}|}{|ad-bc|+|a\bar{d}-b\bar{c}|} \leqq \left|\frac{at+b}{ct+d}\right| \leqq \frac{|ad-bc|+|a\bar{d}-b\bar{c}|}{|c\bar{d}-d\bar{c}|}.$$

An algebraic proof is preferred. What will happen when $c\bar{d}-d\bar{c}=0$?

Solution by C. Donald La Budde, New York University. From the conditions of the problem either $a\neq 0$ and $c\neq 0$, or $b\neq 0$ and $d\neq 0$. We assume, without loss of generality, that $ac\neq 0$, since we can always transform the inequality and consider the variable $u=1/t$. It suffices to prove the left hand inequality. We will prove

$$(1) \qquad |a\bar{b}-\bar{a}b|\,|ct+d| \leqq |at+b|\,(|ad-bc|+|a\bar{d}-b\bar{c}|).$$

Expanding the left member of (1) we have

$$\begin{aligned}
&|a\bar{b}-\bar{a}b|\,|ct+d| = |a\bar{b}ct-\bar{a}bct+a\bar{b}d-\bar{a}bd| \\
&= |a\bar{a}dt-\bar{a}bct+a\bar{b}d-b\bar{b}c-a\bar{a}dt+a\bar{b}ct-\bar{a}bd+b\bar{b}c| \\
&\leqq |a\bar{a}dt-\bar{a}bct+a\bar{b}d-b\bar{b}c| + |-a\bar{a}dt+a\bar{b}ct-\bar{a}bd+b\bar{b}c| \\
&= |\bar{a}t+\bar{b}|\,|ad-bc| + |at+b|\,|-\bar{a}d+\bar{b}c| \\
&= |at+b|\,(|ad-bc|+|a\bar{d}-b\bar{c}|).
\end{aligned}$$

Dividing (1) by the appropriate terms yields the desired inequality.

It is easily seen from differential calculus that the minimum value of $|ct+d|$ as a function of t is $|c\bar{d}-\bar{c}d|/2|c|$. Therefore, $|ct+d|$ can vanish for some real t if and only if $|c\bar{d}-\bar{c}d|=0$.

Also solved by M. S. Klamkin.

A Finite Summation

E 1842 [1965, 1129]. *Proposed by Stanton Philipp, Long Beach State College, California*

Evaluate

$$\sum_{r=0}^{n}\binom{n}{r}\sum_{k=0}^{r}(-1)^k\frac{r!}{k!}.$$

I. *Solution by K. L. Yocom, South Dakota State University.* The given sum can be written

$$\sum_{r=0}^{n}\frac{n!}{r!}\sum_{k=0}^{r}(-1)^k\binom{r}{k}.$$

The inner sum is zero for all r except $r=0$, and hence $n!$ is the value of the given sum.

II. *Solution by F. R. Olson, State University of New York at Buffalo.* Consider n objects, $a_1, \cdots, a_n$ arranged in n cells numbered 1 through n. In any permutation of these objects a certain number, say $n-r$, $0\leqq r\leqq n$, are in their proper

cells, that is, a_i in cell i, and the remaining objects are not. Of course, if $n-1$ objects are properly placed, then all n are. Thus all $n!$ permutations can be generated by selecting proper positions for $n-r$ objects (in $\binom{n}{n-r} = \binom{n}{r}$ ways) and deranging the remaining r objects in $\sum_{k=0}^{r} (-1)^k r!/k!$ ways for each value of r. (We note the last expression has value 0 for $r=1$.) Therefore $n!$ is the value of the given sum.

Also solved by A. D. Anderson, P. N. Bajaj, R. A. Bell, M. A. Bershad, M. T. L. Bizley (England), Shirley D. Braverman & Perry Scheinok, C. A. Church, Jr., D. D. Culbertson, R. V. Esperti, R. W. Feldman, H. W. Gatzke, Jr., Sylvan Greene, M. S. Greening (Australia), Emil Grosswald, P. G. Kirmser, M. S. Klamkin, C. D. La Budde, E. S. Langford, C. P. Lawes, D. C. B. Marsh, J. W. Moon, F. D. Parker, C. B. A. Peck, Stanley Rabinowitz, Jürg Rätz (Switzerland), D. P. Roselle, F. C. Smith, Sidney Spital, Selig Starr, Carolyn C. Styles, W. A. O'N. Waugh (England), and David Zeitlin.

A Diophantine Problem

E 1843 [1965, 1130]. *Proposed by A. K. Austin, The University, Sheffield, England*

Given integers a and b, define the function $f(x) = ax^2 + bx$. Find all integers p, q, r such that $f(p) = f(q) + f(r)$.

Solution by G. J. Simmons, Sandia Corporation, Albuquerque, N. M.

We assume the nontrivial case: $a \neq 0$ and $(a, b) = 1$ (otherwise we suppress the common factor (a, b)). $f(p) = f(q) + f(r)$ can be written

$$(2ap + b)^2 + b^2 = (2aq + b)^2 + (2ar + b)^2. \tag{1}$$

This is in the form $u^2 + y^2 = v^2 + z^2$, treated in elementary texts on number theory (e.g. Dickson, *Introduction to the Theory of Numbers*, Dover, 1957, p. 43), where the general solution is shown to be of the form

$$u = \tfrac{1}{2}(m\mu + n\nu) = 2ap + b, \tag{2}$$

$$y = \tfrac{1}{2}(n\mu - m\nu) = b, \tag{3}$$

$$v = \tfrac{1}{2}\left| m\mu - n\nu \right| = 2aq + b, \tag{4}$$

$$z = \tfrac{1}{2}(n\mu + m\nu) = 2ar + b, \tag{5}$$

subject to certain restrictions on the parity of μ and ν.

Without loss of generality take $(m, n) = 1$ (otherwise, if $(m, n) = d$, the substitutions $m = dm'$, $n = dn'$, $\mu' = d\mu$, $\nu' = d\nu$ leave (2)–(5) unaltered with $(m', n') = 1$).

Suppose, for given a and b, there are integers p, q, r satisfying (1) with $m\mu > n\nu$. Elementary algebraic manipulations of (2)–(5) give

$$4ap = \mu(m - n) + \nu(m + n), \tag{6}$$

$$4aq = (\mu + \nu)(m - n), \tag{7}$$

$$4ar = \nu(2m), \qquad 4a(p - q) = \nu(2n). \tag{8}$$

Now (8) implies that $\nu = 2af$ for some integer f. Using this value of ν in (2) and

(3) we find that $m\mu$ and $n\mu$ are even, so μ is even; say $\mu=2w$. From (6) or (7), $a \mid w(m-n)$; but if c $(c>1)$ is any common divisor of a and w, (3) shows that $c \mid b$, contrary to $(a, b)=1$. Hence $a \mid (m-n)$. We put $m=ag+n$, where g is an integer. Further, $(m, n)=1$ implies that $(a, n)=1$, $(g, n)=1$. Finally, from (7), $\frac{1}{2}g(w+af)=q$, an integer.

We have shown that, if p, q, r satisfy (1), there are integers n, g, m, f, w such that, from (3), $nw-maf=b$, and

$$p = \tfrac{1}{2}g(w + af) + nf, \qquad q = \tfrac{1}{2}g(w + af), \qquad r = mf. \tag{9}$$

Conversely, for integers n, g with $(a, n)=1$, if we put $m=ag+n$, $(m, n)=1$, and let f and w be any solution pair of $nw-amf=b$ (of which there are infinitely many since $(n, ma)=1$), the numbers given by (9) will satisfy (1), as is seen by direct substitution. However, we must still restrict the choices so that $q=\frac{1}{2}g(w+af)$ is integral. For g even, all is clear.

g cannot be odd if a is even; because this requires that b and n be both odd; so $nw-afm=b$ implies w is odd, implies $w+af$ is odd, implies q is not integral.

Now if a and g are both odd and $m=ag+n$, then am, an are of opposite parity. Suppose one solution pair of $nw-amf=b$ is (w_0, f_0), then $n(w-w_0)=am(f-f_0)=amnt$ gives, for any t, another solution such that $w+af=(w_0+af_0)+a(m+n)t$. Since $a(m+n)$ is odd, $w+af$ will be even, and q integral, only for values of t having appropriate parity. So we have a process which produces all integral solutions (under the initial assumption that $m\mu>n\nu$).

If, on the other hand, $n\nu>m\mu$, entirely similar analysis gives

$$p = \tfrac{1}{2}g(w + af) - nw, \qquad q = \tfrac{1}{2}g(w + af), \qquad r = mf.$$

Also solved by Michael Goldberg, B. A. Hausmann, S. J., D. C. B. Marsh, Norman Miller, and Charles Wexler.

Editorial Note. Most of the solutions were incomplete, either because the derived formulas, although satisfying the given equation, did not provide for *all* integers which do so, or because they required knowledge of the factors of some integral expression which, if it is sufficiently large, may be impossible to find.

It is interesting to let $f(x)=ax^2+bx+c$. Clearly solutions of $f(p)=f(q)+f(r)$ are impossible if $a+b$ is even and c odd, or if c is not a multiple of (a,b). Hence, take $(a,b)=1$ without loss of generality, as above. We may then easily show that there are infinitely many integral solutions if $(a,b)=1$ and a, b, c are not all odd. In fact this is still true if we further require $p=q+1$. To find *all* integer solutions for *all* choices of a, b, c may be difficult.

Point with Irrational Distances from Three Given Points

E 1844 [1965, 1130]. *Proposed by A. K. Austin, The University, Sheffield, England*

Do there exist points A, B, C such that for all points D in the plane of A, B, C, the distances AD, BD, CD are not all rational.

I. *Solution by D. E. Crabtree and R. G. Bauer, University of Massachusetts.* The answer is affirmative. Let $A=(0, 0)$, $B=(\sqrt{2}, 0)$, $C=(\pi, 0)$. If the distances

AD, BD, CD to the point $D=(x, y)$ were all rational, then the numbers

$$\tfrac{1}{2}(AD^2 - BD^2 + 2) = \sqrt{2}\cdot x \quad \text{and} \quad AD^2 - CD^2 = 2\pi x - \pi^2$$

would both be rational. This leads to the conclusion that π is algebraic, a contradiction. (Moreover, AD, BD, CD cannot all be algebraic, as the above proof shows if we read "algebraic" for "rational" throughout.)

II. *Solution by Anders Bager, Hjorring Gymnasium, Denmark.* Yes, not only in the plane, but in a general Hilbert space. Let A and B be two fixed points, C their midpoint and P an arbitrary point. The parallelogram law tells us that

$$AB^2 = 2\cdot PA^2 + 2\cdot PB^2 - 4\cdot PC^2.$$

If we take $AB=2^{1/4}$, then PA, PB, PC cannot possibly all be rational.

Also solved by Walter Bluger, Ray Carry & J. W. Wilson, Red Cougar, Michael Goldberg, M. S. Klamkin, D. C. B. Marsh, Lieselotte Miller, P. R. Nolan (Ireland), and D. L. Silverman.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate signed sheets and should be mailed before October 31, 1967. Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5480. *Proposed by Barbara L. Osofsky, Rutgers—The State University*

Let R be a ring with unity 1. Prove that a unital right R-module M is injective if and only if every R-homomorphism from a right ideal of R into M can be lifted to an R-homomorphism from R into M. If R is commutative, show that every indecomposable R-module is irreducible if and only if every indecomposable R-module is injective.

5481. *Proposed by C. C. Lindner, Coker College, Hartsville, S. C.*

Let G be a nonabelian group of squarefree order. Suppose that for any pair of subgroups H, K of G, $\{1\}\subsetneq H\subsetneq K$ implies that $N(H)\subseteq K$, where $N(H)$ denotes the normalizer of H in G. Prove that $|G|=p\cdot q$, where p and q are distinct primes.

5482. *Proposed by R. J. Weaver, University of Massachusetts*

Let I be a closed interval of real numbers. Let f and g be continuous mappings from $I^n=I\times I\times\cdots\times I$ into itself where f is surjective. Does there always exist a point P in I^n for which $f(P)=g(P)$?

5483. *Proposed by W. O. Egerland, Edgewood Arsenal, Md.*

Let $f(x)$ be continuous, $g(x)$ differentiable on I: $[a, b]$, $g(a)=0$, and $\lambda\neq 0$ a constant. If

$$|\, g(x)f(x) + \lambda g'(x)\,| \leqq |g(x)|$$

on I, then $g(x)\equiv 0$ on I.

5484. *Proposed by W. A. McWorter, University of British Columbia.*

Let p, q be distinct primes and consider the multiplication table of a semigroup S of $n=pq$ elements. If the element a of S occurs in those and only those positions on the off-diagonal (i.e., $s_i \cdot s_j = a$, s_i, $s_j \in S$, iff $j=n-i+1$), prove that S is the cyclic group of order n.

5485. *Proposed by D. E. Daykin, University of Malaya, Kuala Lumpur*

Show that the plane may be covered by squares of sides 1, 2, 3, $\cdots$ so that no two squares have an interior point in common. The side of each square must be a positive integer i, and for each i there is to be one and only one square.

5486. *Proposed by Ludwig Bruch.*

For any positive integer n, prove

$$\sum_{k=0}^{n-1} \csc^2 \left\{\frac{\pi}{2n}(2k+1)\right\} = n^2.$$

5487. *Proposed by Roy O. Davies, The University, Leicester, England*

Suppose that $f_0(x)$ is positive and Lebesgue integrable over (0, 1), that positive functions $f_1, f_2, \cdots$ are defined successively by the relations $f_{n+1}(x) = \{\int_0^x f_n(t)dt\}^{1/2}$ $(n = 0, 1, \cdots)$, and that $f_0(x) \geqq f_1(x)$ for $0 < x < 1$. Find $\lim_{n\to\infty} f_n(x)$.

(Based on an unpublished lemma of the late E. R. Reisenberg.)

5488. *Proposed by J. Z. Hearon, National Institutes of Health, Bethesda, Md.*

For any $n \times n$ matrix A, write $A = H + D$, where $D = \text{diag}(a_{11}, a_{22}, \cdots, a_{nn})$ and define $B = H - D$. Show that, if A is a continuant and λ is any root of A then $-\lambda$ is a root of B. Thus prove that an nth order continuant with zero diagonal and distinct roots is singular if n is odd.

5489. *Proposed by A. A. Mullin, University of California, Livermore*

Problem E 1735 [1965, 912] reestablished the result that every nontrivial weakly monotone multiplicative number-theoretic function f satisfies $f(n) = n^\alpha$ for some nonnegative real α. Prove the stronger result that every weakly monotone *generalized* multiplicative (see problem 5248 [1964, 1138]) number-theoretic function $f \not\equiv 0$ satisfies $f(n) = n^\alpha$ for some nonnegative real α.

SOLUTIONS OF ADVANCED PROBLEMS

Almost Disjoint Sets

5370 [1966, 311]. *Proposed by F. S. Cater, University of Oregon*

Let X be an infinite set with $\alpha = \text{card } X$ such that either $\alpha = \aleph_0$ or α has an

immediate predecessor. Prove that there exists a family M of subsets of X such that card $M>\alpha$ and

(1) card $(A_1\cap A_2)<\alpha$ for any two distinct members A_1, A_2 of M,

(2) for any subfamily N or M with card $N=\alpha$, there exists an $A\in M-N$ such that card $[(\cup_{B\in N} B)\cap A]=\alpha$. (Can the hypothesis on α be deleted?)

Solution by E. C. Milner, the University, Reading, England. It is well known (Tarski, Fund. Math., 12 (1928)) that there is a family M of subsets of X each of cardinal α such that card $M>\alpha$ and (1) holds. If, in addition, M is a maximal family of this kind, then (2) also holds. For suppose that N (well ordered) is the set $\{A_\nu: \nu<\omega(\alpha)\}$ and N is a subset of M of cardinal α ($\omega(\alpha)$ denotes the initial ordinal of cardinal α). For $\nu<\omega(\alpha)$ we can choose

$$x_\nu \in A_\nu - \bigcup_{\mu<\nu} A_\mu$$

since card $A_\nu=\alpha$ and card $[A_\nu\cap(\cup_{\mu<\nu} A_\mu)]<\alpha$. Put $C=\{x_\nu: \nu<\omega(\alpha)\}$. Since for $B\in N$, card$(B\cap C)<\alpha$, and M is maximal, it follows that there is $A\in M-N$ such that card$(A\cap C)=\alpha$ (A may be the set C). Therefore card $[(\cup_{B\in N} B)\cap A]=\alpha$.

The *cofinality cardinal* of $\alpha\geqq\aleph_0$, denoted by α', is the smallest cardinal β such that α is the sum of β cardinals each less than α. Clearly, $\alpha'<\alpha$. The above argument works if α is any *regular* cardinal (i.e., $\alpha'=\alpha$). (It is not known if there are regular cardinals other than $\aleph_0$ and the successor cardinals.) The proof above does not involve the generalized continuum hypothesis (or any weak form of this) $2^{\aleph_0}=\aleph_{\nu+1}$. Assuming the g.c.h. we can prove the following strengthening of the stated result which is also valid if α is *singular* (i.e., if $\alpha'<\alpha$).

THEOREM. *If card $X=\alpha\geqq\beta\geqq\aleph_0$, $\alpha'=\beta'$ and the g.c.h. holds, then there is a family M of subsets of X each of cardinal β such that card $M>\alpha$ and*

(1) *card$(A_1\cap A_2)<\beta$ if A_1, $A_2\in M$ and $A_1\neq A_2$,*

(2) *for any subfamily N of M with card $N=\alpha$, there is $A\in M-N$ such that $A\subset\cup_{B\in N} B$.*

A generalization of this theorem and certain related results will be included in a paper by P. Erdös, A. Hajnal and E. C. Milner.

Also solved by Paul Erdös, G. F. Schumm, and the proposer.

For the theorem of Tarski and related results on collections of almost disjoint sets, Schumm refers us to W. Sierpinski, *Cardinal and Ordinal Numbers*, Warsaw, 1958, 448–449. Erdös refers to his papers with L. Gillman and M. Henriksen, *An isomorphism theorem for real closed fields*, Annals of Math., 61 (1955), p. 542, ff. and with A. Hajnal and R. Rado, *Partition relations for cardinal numbers*, Acta Math. Hung. Acad. Sci., 19 (1964), p. 93, ff., for the theorems when α is singular.

Dimensionality of Compact Totally Disconnected Spaces

5378 [1966, 312]. *Proposed by S. W. Williams, Lehigh University*

Is every locally compact, totally disconnected space an 0-dimensional space? (Cf. W. J. Perwin, *Foundations of General Topology*, Academic Press, p. 98, #6.)

Solution by S. P. Franklin, Carnegie Institute of Technology. No. A sequence converging to two points is easily seen to be (locally) compact and totally disconnected but, being T_1 and not Hausdorff, cannot be zero dimensional.

For locally compact *Hausdorff* spaces the equivalence of the two concepts is easily proved using the corresponding fact for compact Hausdorff space.

We note that Pervin is not the only text book author to omit the Hausdorff hypothesis; see Hocking and Young, *Topology*, p. 147 Theorem 3-54.

Also solved by Roy Daigle, J. R. Isbell, James Joseph, M. D. Mavinkurve (India), P. S. Schnare, G. F. Schumm, Al Somayajulu, and the proposer.

For the situation in T_2 spaces, Schnare gives the reference, Hewitt and Ross, *Abstract Harmonic Analysis*, p. 12.

Entire Functions

5380 [1966, 420]. *Proposed by L. P. Comden and S. S. Mitra, Clarkson College of Technology, Potsdam, N. Y.*

Let $f(z)=u(x, y)+iv(x, y)$ be an entire function which is real for real values of z and strictly complex elsewhere, i.e., $v(x, y)\neq 0$ if $y\neq 0$. Prove that $f(z)=az+b$, where $a(\neq 0)$ and b are real. Cf. E 1696 [1965, 550].

I. *Solution by Robert Breusch, Amherst College.* Let $f(z)=\sum_{k=0}^{\infty} a_k z^k$, where the a_k are real. For $z=r\cdot e^{i\theta}$,

$$v(x, y) = \sum_{k=1}^{\infty} a_k r^k \sin k\theta.$$

Since $v(x, y)\neq 0$ for $y\neq 0$, $v(x, y)$ must have the same sign for all points with $y>0$; say $v(x, y)>0$ for $y>0$. (If $v<0$ for $y>0$, we work with $-f(z)$ instead of $f(z)$.) Since $n\cdot\sin\theta\pm\sin n\theta>0$ for every $n>1$, and every θ with $0<\theta<\pi$, we will have

$$\int_0^{\pi} (n\cdot\sin\theta \pm \sin n\theta)\cdot \sum_{k=1}^{\infty} a_k r^k \sin k\theta\cdot d\theta > 0.$$

The $\{\sin k\theta\}$ are orthogonal over $(0, \pi)$. Thus

$$n\cdot a_1 r(\pi/2) \pm a_n r^n(\pi/2) > 0,$$

$$|a_n| < n\cdot a_1 r^{n-1} \quad \text{for } n > 1, \text{ and all } r,$$

thus $a_n=0$ for $n>1$, $a_1>0$, and $f(z)=a_0+a_1 z$.

II. *Comment by R. P. Boas, Northwestern University.* The theorem is true in a somewhat more general form: we need only assume that $f(z)$ is real for real z and that v does not change sign for $y\neq 0$. In this form it was proved by Chebotaryoff (=Tschebotarëw), Math. Ann., 99 (1928), 660–686. A short and elementary proof is reproduced in Math. Reviews, 20, #5283.

Also solved by P. T. Bateman & N. T. Hamilton, Peter Bundschuh (Germany), C. K. Chui, G. Di Antonio, Irving Gerst, M. O. Gonzalez, M. G. Greening (Australia), D. A. Hejhal, M. S.

Klamkin, O. A. Nickel, J. J. Roseman, Al Somayajulu, L. J. Wallen, and the proposers.

Klamkin notes that the problem duplicates 4857 [1960, 597], where the solution using Picard's theorem applies, as in solution I above, if it is merely assumed that $v(x,y)$ does not change sign in the upper (lower) half plane.

Bateman and Hamilton in their solution avoid the Picard theorem through the use of an elementary lemma akin to the Weierstrass-Casorati theorem: If $f(z)$ is holomorphic for $R<|z|$, taking no values on some fixed segment $[\zeta_1,\zeta_2]$, then for some nonnegative integer n, $f(z)/z^n$ has a removable singularity at ∞.

Nonsolvable Groups

5381 [1966, 420]. *Proposed by V. H. Keiser, University of Colorado*

Let G be an imprimitive group in which all normal subgroups are transitive. Prove that G is nonsolvable.

Solution by Richard Stanley, Jet Propulsion Laboratory. More generally we prove that if G contains a minimal normal subgroup $A\neq 1$ which is transitive and abelian, then G is primitive. (This occurs as Exercise 8.8′ in H. Wielandt, *Finite Permutation Groups.*) Let G_1 be the subgroup of G fixing a letter and assume $G_1\subset H\subseteq G$ for a subgroup H. Since A is abelian and transitive, it is regular and $G=AG_1$. Choose $h\in H$ but $h\notin G_1$ where $h=ah_1$, $a\in A$, $h_1\in G_1$. Then $1\neq a\in H$. Since A is abelian and $G=AG_1$,

$$\{g_1^{-1}ag_1: g_1\in G_1\} = \{g^{-1}ag: g\in G\}\subseteq H.$$

But $\{g^{-1}ag: g\in G\}$ generates A as A is minimal normal. Hence $A\subset H$ and $H=G$. Thus G_1 is maximal and so G is primitive.

Also solved by M. G. Greening (Australia), and the proposer.

Small Submodules

5383 [1966, 420]. *Proposed by W. W. Leonard, Susquehanna University, Selinsgrove, Pa.*

A submodule E of a left A-module F is small in F if $E+H=F$ for any submodule H of F implies $H=F$. Prove that Z, the set of integers, is small in Q, the set of rational numbers. Is $Z^{(I)}$ small in $Q^{(I)}$, where I is an infinite set and $Z^{(I)}$ (resp. $Q^{(I)}$) is the direct sum of copies of Z (resp. Q)? All modules are considered as Z-modules.

Solution by Francis Sandomierski, University of Wisconsin.

THEOREM. (Notation of Problem 5383). *$Z^{(I)}$ is small in $Q^{(I)}$ if and only if I is finite.*

Proof. If I is finite and $Z^{(I)}+H=Q^{(I)}$, then by the second isomorphism theorem for groups $Q^{(I)}/H$ is isomorphic to $D=Z^{(I)}/Z^{(I)}\cap H$. Since $Q^{(I)}/H$ is divisible and $Z^{(I)}$ is finitely generated, it follows that D is a finitely generated divisible group, so $D=0$ and therefore $H=Q^{(I)}$, so $Z^{(I)}$ is small in $Q^{(I)}$.

If I is infinite, then Q is an epimorphic image of $Z^{(I)}$ by some homomorphism f. Since Q is divisible (Z-injective), f can be extended to $f^*: Q^{(I)}\to Q$. Since f is

onto, $Q^{(I)}=Z^{(I)}+\mathrm{Ker}(f^*)$; but since $f^*\neq 0$, $\mathrm{Ker}(f^*)\neq Q^{(I)}$. So $Z^{(I)}$ is not small in $Q^{(I)}$.

Also solved by G. A. Heuer, Ka Menehune, and the proposer.

Graphs of Finite Order

5384 [1966, 420]. *Proposed by D. E. Daykin, The University, Kuala Lumpur, Malaysia*

Let k be the independence number (coefficient of internal stability) of a graph G of finite order n with no circuits of uneven length. Prove that $\frac{1}{2}\leqq k/n\leqq 1$, and that these bounds are best possible. (See C. Berge, *The Theory of Graphs*, p. 48.)

Solution by H. V. Kronk, Harpur College. That $k\leqq n$ is obvious and the empty graph on n nodes has independence number n. Since G has no circuits (cycles) of uneven length, it is a bipartite graph and hence has chromatic number $\chi(G)\leqq 2$. For any ordinary graph G, it is well known that $k\cdot\chi(G)\geqq n$. Thus for a bipartite graph $2k\geqq n$. If n is even, then the complete bipartite graph $K_{n/2,n/2}$ has independence number $n/2$.

Also solved by J. F. Dillon, Myron Goldberg, A. M. Hobbs, William Lindgren, E. C. Milner (England), E. M. Palmer, and the proposer.

Associativity in Loops of Prime Order

5386 [1966, 421]. *Proposed by S. W. Williams, Lehigh University*

Let L be a loop of prime order, S be the set $\{x \mid (ax)b=a(xb);\ a,\ b\in L\}$, and R be the set $\{x \mid (ab)x=a(bx);\ a,\ b\in L\}$. Show that $S=R$.

Solution by M. G. Greening, University of New South Wales, Australia. For $x,\ y\in S$ we have $(a(xy))b=((ax)y)b=(ax)(yb)=a(x(yb))=a((xy)b)$, and $xy\in S$. Then, for $x,\ y,\ z\in S$, $(xy)z=y(yz)$ and S is an associative loop, as $e\in S$, and so a group. Then $x\in S$ implies $x^{-1}\in S$, so $xa\in S$ implies $a\in S$. Then the sets Sa, Sb, etc., are either identical or disjoint, as $xa=yb$ implies $b=y^{-1}(xa)=(y^{-1}x)a=za$ and $Sb=Sza=Sa$. Thus $o(S)\,|\,o(L)$ and $o(S)=1$ or p.

Similarly for $x,\ y\in R$, $(ab)(xy)=((ab)x)y=(a(bx))y=a(b(xy))$, and $xy\in R$, $e\in R$; R is a group with $o(R)=1$ or p. If $o(R)=p$, then $R=L$ and $R=S$ by the associativity of R; similarly $o(S)=p$ implies $S=L$ and $S=R$. Thus there are two cases: (1) $o(S)=p=o(R)$; $S=L=R$ and (2) $o(S)=1=o(R)$; $S=e=R$.

Also solved by the proposer.

An Equivalence Relation on a Finite Abelian Group

5387 [1966, 421]. *Proposed by Michael Gemignani, University of Notre Dame and Saint Mary's College*

Let G be a commutative group with identity 1. If g_1 and g_2 are in G, then define $g_1\sim g_2$ if $g_1g_2=g^2$ for some g in G. $\sim$ is an equivalence relation on G. If G

is a finite group of n elements, determine the number of distinct equivalence classes in $G/\sim$.

I. *Solution by C. C. Lindner, Coker College, Hartsville, S. C.* Let $G_2=\{x\in G: x^2=1\}$. Then the number of distinct equivalence classes in $G/\sim$ is $|G_2|$. For, if $G^2=\{x^2: x\in G\}$, then G^2 is a subgroup of G and is the equivalence class containing 1. It is easily seen that the other equivalence classes are just the right (left) cosets of G^2 in G. Hence the number of distinct equivalence classes in $G/\sim$ is $[G: G^2]$. But $[G: G^2]=|G_2|$; see, e.g., J. Rotman, *The Theory of Groups: an Introduction*, p. 29.

II. *Solution by Charles Wells, Western Reserve University.* Let $|G|=2^m r$, r odd. Then G is the direct product of a subgroup of order r and t cyclic subgroups, each of order a power of 2. This t is an invariant of G. Then $G/\sim$ contains 2^t equivalence classes: G has a basis $\{h_1, \cdots, h_n\}$ where the order of h_i is k_i (k_i a power of 2 for $i=1, \cdots, t$; k_i an odd prime power for $i=t+1, \cdots, n$). Every element of G is expressible uniquely as $\Pi_{i=1}^n h_i^{r_i}$, where r_i is determined (mod k_i). If $\Pi h_i^{r_i}=(\Pi h_i^{s_i})^2$, then $r_i=2s_i$ (mod k_i). If $1\leqq i\leqq t$, this means r_i is even. If $t+1\leqq i\leqq n$, any r_i has the form $2s_i$ (mod k_i). Thus $(\Pi h_i^{r_i})(\Pi h_i^{u_i})$ is a square if and only if r_i+u_i is even for every i, $1\leqq i\leqq t$, which happens if and only if $r_i\equiv u_i$ (mod 2) ($i=1, \cdots, t$). It follows that there are 2^t equivalence classes in $G/\sim$.

Also solved by Anders Bager (Denmark), L. Carlitz, J. P. Celenza, W. G. Dotson, Jr., N. Ganesan, J. D. Gillam, M. G. Greening (Australia), G. A. Heuer, Agatha Himmelfarb, E. C. Hook, R. B. Hora, Erwin Just, Roman Kaluzniacki, Tetsuzo Kimura (Japan), Betty Kvarda, Arthur Lieberman, N. R. Martin, J. J. Martinez, M. D. Mavinkurve (India), Ka Menehune, Wanda J. Mourant, N. S. Natarajan (India), Vishnu Parkash (India), F. D. Parker, R. F. Prisco, Donald Quiring, S. N. Rao (India), S. D. Rhodes, Azriel Rosenfeld, Al Somayajulu, Richard Stanley, Gomer Thomas, A. M. Vaidya (India), L. R. Vermani (India), and the proposer.

A Complex Rolle's Theorem

5389 [1966, 421]. *Proposed by D. J. Newman, Yeshiva University*

Let $f(z)$ be analytic and have $n+1$ zeros in a convex region C. Prove that $\operatorname{Re} f^{(n)}(z)$ vanishes somewhere in C.

I. *Solution by Robert Breusch, Amherst College.* For $n=0$, the statement is trivially true. Assume it has been proved for $n=m$, that is, for functions with $m+1$ zeros in C. Let f have $m+2$ zeros in C: $f(z)=(z-z_1)g(z)$, where g has $m+1$ zeros in C, and z_1 is in C.

$$f^{(m+1)}(z) = (z - z_1)g^{(m+1)}(z) + (m + 1)g^{(m)}(z).$$

By assumption, $\operatorname{Re}(g^{(m)}(z_2))=0$ for some z_2 in C. If $z_2=z_1$, then $\operatorname{Re}(f^{(m+1)}(z_1))=0$. If $z_2\neq z_1$, then with $\zeta=(z-z_1)/(z_2-z_1)$, $g^{(m)}(z)\equiv H(\zeta)$, $g^{(m+1)}(z)=H'(\zeta)/(z_2-z_1)$, $(z-z_1)g^{(m+1)}(z)=\zeta\cdot H'(\zeta)$, and thus $f^{(m+1)}(z)\equiv F(\zeta)=\zeta\cdot H'(\zeta)+(m+1)H(\zeta)$, with $\operatorname{Re}(H(1))=0$.

Write $H(\zeta)=u(\xi,\eta)+iv(\xi,\eta)$, $H'=u_1+iv_1$, and it follows that $\operatorname{Re}(f^{(m+1)}(z))=\operatorname{Re}(F(\zeta))=\xi\cdot u_1-\eta\cdot v_1+(m+1)u$. Since $\xi^{m+1}\cdot u(\xi, 0)$ vanishes at $\xi=0$ and at

$\xi=1$, we have by Rolle's Theorem, $\xi_0^m[\xi_0 \cdot u_1(\xi_0, 0)+(m+1)\cdot u(\xi_0, 0)]=0$ for some ξ_0, $0<\xi_0<1$, whence Re $(F(\zeta))=0$ for some point $\zeta=(\xi_0, 0)$ on the line segment from 0 to 1. We have shown that Re $(f^{(m+1)}(z))=0$ for some z on the line segment between z_1 and z_2, and since C is convex, this completes the proof.

II. *Partial Solution by M. D. Mavinkurve, Bombay, India.* To prove the case $n=1$, let z_1 and z_2 be the two zeros. If $z_1=z_2$, then $f'(z_1)=0$. If $z_1\neq z_2$, consider the real function $\phi(z)=u \cos \beta - v \sin \beta$, β fixed, $u+iv=f(z)$. $\phi(z_1)=\phi(z_2)=0$. Applying Rolle's Theorem on the segment $\overline{z_1z_2}$ we have $\partial\phi/\partial s=0$ somewhere in the interior of the segment, i.e.,

$$\cos\beta\cdot\frac{\partial u}{\partial s}-\sin\beta\cdot\frac{\partial v}{\partial s}=\cos\beta\cdot\frac{\partial u}{\partial s}+\sin\beta\cdot\frac{\partial u}{\partial n}=0.$$

This means that the derivative of u in a direction making angle β with $\overline{z_1z_2}$ is zero. Since β is arbitrary, it follows that the directional derivative of u in any direction vanishes somewhere on $\overline{z_1z_2}$. In particular, $\partial u/\partial x=\text{Re } f'(z)=0$.

III. *Solution by the proposer.* We assume that $z_1, \cdots, z_{n+1}$ are distinct zeros of $f(z)$. (Simple modifications are possible to accommodate multiple zeros.) We prove

$$0=\int\cdots\int_{\substack{x_1,x_2,\cdots,x_n\geqq 0\\ x_1+x_2+\cdots+x_n\leqq 1}} f^{(n)}(x_1z_1+\cdots+x_nz_n+x_{n+1}z_{n+1})dx_1\cdots dx_n,$$

(where x_{n+1} is an abbreviation for $1-x_1-x_2-\cdots-x_n$), whence the required result follows immediately.

We proceed by induction, noting the validity of the result for $n=0$. Assume it holds for $n-1$; then

$$\int\cdots\int_{\substack{x_1,\cdots,x_n\geqq 0\\ x_1+\cdots+x_n\leqq 1}} f^{(n)}(x_1z_1+\cdots+x_{n+1}z_{n+1})dx_1\cdots dx_n$$

$$=\frac{1}{z_n-z_{n+1}}\int\cdots\int_{\substack{x_1,\cdots,x_{n-1}\geqq 0\\ x_1+\cdots+x_{n-1}\leqq 1}} f^{(n-1)}(x_1z_1+\cdots+x_{n-1}z_{n-1}$$

$$+(1-x_1-\cdots-x_{n-1})z_n)dx_1\cdots dx_{n-1}$$

$$-\frac{1}{z_n-z_{n+1}}\int\cdots\int_{\substack{x_1,\cdots,x_{n-1}\geqq 0\\ x_1+\cdots+x_{n-1}\leqq 1}} f^{(n-1)}(x_1z_1+\cdots+x_{n-1}z_{n-1}$$

$$+(1-x_1-\cdots-x_{n-1})z_{n+1})dx_1\cdots dx_{n-1},$$

and this last is 0 by the inductive hypothesis.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Correspondence about Reviews will be welcome.

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 15 (junior level)—18 (second graduate year).

Algebra.

Eleven Papers on Topology and Algebra. By T. M. Baranovic, M. F. Bokstein, A. I. Kostrikin, A. P. Misina, A. L. Oniscik, I. I. Pjatechii-Sapiro, A. L. Smel'kin, A. S. Svarc, E. S. Tihomirova, Ju. R. Vainberg, A. V. Zarelua. Amer. Math. Soc. Translations, Series 2, Vol. 55. American Mathematical Society, Providence, Rhode Island, 1966. iv+304 pp. $15.40. P, L.

Proceedings of Symposia in Pure Mathematics, Vol. IX. Algebraic Groups and Discontinuous Subgroups. Edited by A. Borel and G. D. Mostow. Amer. Math. Soc., Providence, Rhode Island, 1966. vii+426 pp. $10.20. "An outgrowth of the twelfth Summer Mathematical Institute of the American Mathematical Society, which was devoted to Algebraic Groups and Discontinuous Subgroups, . . . held at the University of Colorado in Boulder from July 5 to August 6, 1965, . . . lecture notes, in part slightly revised, and a few papers written somewhat later . . . linear algebraic groups and arithmetic groups, adeles and arithmetic properties of algebraic groups, automorphic functions and spectral decomposition of L^2-spaces of coset space, holomorphic automorphic functions on bounded symmetric domains and moduli problems, vector valued cohomology and deformation of discrete subgroups." P, L.

A Survey of Binary Systems. By Richard Hubert Bruck. 2nd Printing. Vol. 20 of Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, New York, 1966. vii+185 pp. $9.00. Survey of the algebraic theory of systems with a single binary operation, exclusive of group and lattice theory. Bibliography of over 450 titles. S, P, L.

Die Bewegungsgruppen der Kristallographie. 2nd rev. ed. By Johann Jakob Burckhardt. Birkhauser Verlag, Basel, 1966. 209 pp. with 67 figures. 37.50 Swiss francs. S, P.

Galois Theory and Cohomology of Commutative Rings. By S. U. Chase, D. K. Harrison, and Alex Rosenberg. Amer. Math. Soc., Providence, R. I. 1965. iii+79 pp. $1.80. P.

Universal Algebra. By P. M. Cohn. Harper and Row, New York, 1965. xi+333 pp. $9.75. "Devoted exclusively to the study of features common to different algebraic systems." Sets and mappings, algebraic structures, free algebras, varieties, relational structures and models, axiomatic model classes, applications (including a discussion of non-standard models for the integers). T (17), P.

Elements of Compact Semigroups. By Karl Heinrich Hofmann and Paul S. Mostert. Merrill books, Columbus, Ohio, 1966. xiii+384 pp. $15.00. "Our original ambition to survey the whole theory of compact semi-groups and to unify the various trends in this field was soon narrowed to establishing a few trenchant facts about the structure of compact connected topological semigroups with identity, facts which seem fundamental to any penetrating discussion of the subject." S (17–18), P.

Theory of Groups in Classical and Quantum Physics. By T. Kahan in collaboration with P. Cavailles, R. Gourne, T. D. Newton, G. Rideau, G. Lochak, R. Nataf. English translation edited by A. R. Edmonds. Vol. I. Mathematical Structures and the Foundations of Quantum Theory. Translated by H. Ingram. American Elsevier, New York, 1966, 583 pp. $37.50. Interesting historical preface. Theory of groups and axiomatized mathematics for physicists, the inhomogeneous Lorentz group, abstract groups, representations, permutation groups, axiomatics of quantum mechanics, rotations. S (15–17), P, L. Reviewed in *Science*, 10 February 1967, and in *American Scientist*, December 1966.

Lectures on Rings and Modules. By Joachim Lambek. Blaisdell, Waltham, Mass., 1966. vii+183 pp. $8.50. Fundamental concepts, commutative rings, classical theory of associative rings (including the complete ring of quotients as developed by Johnson, Utumi Goldie), injectivity and related concepts, homological algebra. "Some novel twists are offered for two old pastimes, diagram chasing and idempotent lifting." T (17), S, P.

Combinatorial Group Theory. By Wilhelm Magnus, Abraham Karrass and Donald Solitar. Interscience, New York, 1966. x+444 pp. $15.00. Parts of group theory arising from presentation in terms of generators and defining relations. "Not very much overlapping" with books by Kurosh and by Hall. T (17–18), S, P.

Theory of Categories. By Barry Mitchell. Academic Press, New York and London, 1965. xi+273 pp. $13.75. The main ideas and methods developed since 1942. Assumes groups, modules, and "a fair amount of mathematical sophistication." P.

Algebraic Extensions of Fields. By Paul J. McCarthy. Blaisdell, Waltham, Mass., 1966. ix+166 pp. $8.50. " . . . a graduate text . . . to prepare . . . for further study in the theory of fields, especially in algebraic number theory and class field theory." Algebraic extensions, Galois theory, valuation theory, extensions of valuated fields, Dedekind fields. T (17, after a first course in abstract algebra including linear algebra), S, P.

Lie Algebras and Lie Groups. By Jean-Pierre Serre. Benjamin, New York, 1965. 256 pp. $3.95. Notes from lectures given at Harvard in the fall of 1964. Part I (Lie Algebras) and Part II (Lie Groups) are almost independent. P.

Analysis.

Lectures on Functional Equations and their Applications. By J. Aczel. Translated by Scripta Technica, Inc. Supplemented by the Author. Edited by Hansjorg Oser. Academic Press, New York, 1966. xix+510 pp. $19.50. A systematic treatise based on, but much enlarged from the German edition of 1961. Bibliography of 114 pages. P, L.

Functional Analysis. By George Bachman and Lawrence Narici. Academic Press, New York, 1966. xiv+530 pp. $14.50. " . . . either a one-or-two-semester course . . . The book meets the demand for an in-depth, axiomatic, and highly detailed treatment of functional analysis in its pure aspects." Assumes only advanced calculus and linear algebra. T (16–17).

Eight Papers on Differential Equations and Functional Analysis. By F. A. Berezin, V. I. Judovič, M. A. Krasnosel'skii, O. A. Ladyženskaja, V. P. Palamodov, V. A. Solonnikov, M. G. Šur, N. N. Ural'ceva, B. R. Vainberg, P. P. Zabreiko. Amer. Math. Soc. Translations, Series 2, Vol. 56. American Mathematical Society, Providence, R. I., 1966. iv+295 pp. $15.00. P, L.

Elementary Partial Differential Equations. By Paul W. Berg and James L. McGregor. Holden Day, San Francisco, 1966. ix+421 pp. $11.75. Linear equations with constant coefficients in two independent variables, heat equations, eigenfunction expansions, inhomogeneous problems, Fourier series, existence, uniqueness, representation of solutions, wave equation, infinite and semi-infinite intervals, initial boundary value problems, Laplace equation. Based on lectures given to "majors in mathematics, the physical sciences, and engineering, at different stages in their education . . . to provide . . . techniques . . . and to prepare . . . for further study in partial differential equations and linear analysis by furnishing a basis for intuition in these subjects, namely, a thorough familiarity with their simplest typical problems." T (15–17).

Calculus of Variations and Partial Differential Equations of the First Order. By C. Caratheodory. Part I. Translated by R. B. Dean and J. J. Brandstatter. Holden-Day, San Francisco, 1965. xvi+171 pp. $8.50.

Square Summable Power Series. By Louis de Branges and James Rovnyak. Holt, Rinehart and Winston, New York 1966. vii+104 pp. $4.50. " . . . an introduction to Hilbert space and analytic function theory, . . . centered around the invariant subspace concept. Its principal feature is the extensive use of formal power series methods to obtain and sometimes reformulate, results of analytic function theory . . . developed from lecture notes of courses taught at Lafayette College (Spring 1959), Bryn Mawr College (1960–61), and Purdue University (1963–64) . . . " T (15–16), S, P.

JWKB Approximation, Contributions to the Theory. By N. Froman and P. O. Froman. North-Holland, Amsterdam, 1965. 138 pp. $4.50. This book presents a new treatment of the J.W.K.B. (after Jeffreys, Wentzel, Kramers and Brillouin) approximation. The method is applied to the detailed examination of some problems in quantum mechanics, but is also suitable for other applications. P.

Approximation of Functions. Edited by Henry L. Garabedian. American Elsevier, New York, 1965. vii+220 pp. $13.00. Proceedings of a Symposium at General Motors Research Laboratories, Warren, Michigan, 1964. This volume contains thirteen papers in the categories of fundamental problems, rational approximations, linear methods, nonlinear methods, and applications. P.

Introduction to Real Analysis. By Casper Goffman. Harper and Row, New York, 1966. xi+160 pp. $7.50. Real numbers, topology of reals, infinite series, continuous functions, special functions, sequences and series of functions, differential, integration (to Riemann-Stieltjes), power series, Fourier series. T (15–16). Reviewed by L. W. Cohen in *Science*, 17 Febr. 1967.

Variational Methods for Eigenvalue Problems. An Introduction to the Weinstein Method of Intermediate Problems. By S. H. Gould. Univ. of Toronto Press, Toronto, 1966. xvi+275 pp. $8.50. An updating of the original edition. S, P.

Lectures on Riemann Surfaces. By R. C. Gunning. Preliminary Informal Notes of University Courses and Seminars in Mathematics. Princeton Univ. Press, N.J., 1966. iv+254 pp. $3.75. " . . . notes for . . . lectures given at Princeton . . . 1965–66. The subject . . . was compact Riemann surfaces, considered as complex analytic manifolds . . . The aim of the lectures, in addition to treating a beautiful subject for its own sake, was to introduce the student to some techniques (recently developed for dealing with complex analytic functions in several dimensions) in the case of one complex variable, where things are simpler and the results more complete. S, P.

Differential Equations, Stability, Oscillations, Time Lags. By A. Halanay. Academic Press, New York, 1966. xii+528 pp. $19.50. Differential and differential-difference equations, stability of nonlinear control systems, theory of oscillations, systems with time lag. P.

Calculus of Variations and Optimal Control Theory. By Magnus R. Hestenes. Wiley, New York, 1966. xii+405 pp. $12.95. " . . . presents the basic mathematical theory for problems in the calculus of variations and optimal control theory at a fairly advanced level . . . maximum and minimum for functions of N variables, with or without inequality constraints . . . the classical fixed end point problem . . . optimal control problems, including the theory for the classical problems of Mayer and Bolza . . . interrelations between the calculus of variations, dynamic programming, and the works of Pontryagin and his school . . . an indirect sufficiency proof is given . . . " T (17–18), S, P.

Studies in Real and Complex Analysis. Edited by I. I. Hirschman, Jr. MAA Studies in Mathematics, Vol. 3. Prentice-Hall, Englewood Cliffs, N.J., 213 pp. $4.00. A worthy third volume in the well-known Studies in Mathematics Series. Contents: Several Complex Variables; Nonlinear Mappings Between Banach Spaces; What is a Semi-Group?; The Laplace Transform, the Stieltjes Transform and their Generalizations; A Brief Introduction to the Lebesgue-Stieltjes Integral; Harmonic Analysis; Toeplitz Matrices. S (16–17), P, L.

Topological Vector Spaces and Distributions. Vol. I. By John Horvath. Addison-Wesley, Reading, Mass., 1966. xii+449 pp. $12.75. " . . . an elementary introduction to topological vector spaces and their most important applications: the theory of distributions of Laurent Schwartz . . . grew out of courses taught at the Univ. of Maryland, the Université de Nancy, and the Universidad de Los Andes, Bogota . . . In preference to other convergence theories I used exclusively filters. . . . The theory of distributions is presented, as originally by Schwartz, as a theory of duality of topological vector spaces . . . Volume II, which I hope will appear someday, should start with an exposition of the theory of integration and introduce some important Banach spaces of distributions . . . " T (16–17), S (partial differential equations), P, L.

Introduction to Analytic Functions. By Wilfred Kaplan. Addison-Wesley, Reading, Mass., 1966. ix+212 pp. $7.95. " . . . an elementary introduction . . . for students of science and engineering . . . based directly on Chapter nine of the author's Advanced Calculus, and also includes a completely new chapter on analytic functions of several complex variables . . . " T (16–17).

Discourse on Fourier Series. By Cornelius Lanczos. Hafner, New York, 1966. viii+255 pp. $9.85. Developed from lectures to a mixed audience at the University of Washington in 1947. " . . . the primary aim . . . was to convey something of the excitement and enthusiasm which imbued the hundreds of mathematicians who have contributed to this remarkable chapter . . . " S (16–17), P.

Introduction to Diophantine Approximations. By Serge Lang. Addison-Wesley, Reading, Mass., 1966. vii+83 pp. $6.75. " . . . The quantitative aspects of the theory of diophantine approximations are, at the moment, still not very far from where Euler and Lagrange left them. Very recent work seems to have opened some fruitful lines of research, and in this book we shall illustrate by significant special examples three aspects from the theory of diophantine approximations. It is unusual to find a mathematical theory which is in a state as primitive and naive as the present one, and there is of course some delight in catching it in that state. In fact, this book may be used for a course in number theory, addressed to undergraduates, who will thus be put in

contact with interesting but accessible problems on the ground floor of mathematics." Bibliography. S (16–17), P.

Introduction to Transcendental Numbers. By Serge Lang. Addison-Wesley, Reading, Mass., 1966. vi+105 pp. $7.50. " . . . The theory of transcendental numbers is reaching a stage where it is ready to take its place as one of the most attractive branches of mathematics. It consists in determining the transcendence and algebraic independence of numbers obtained as values of classical functions, suitable normalized . . . The theory has applications ranging from a very elementary setting (concerning the function e^t), to rather sophisticated contexts, having to do with abelian functions and automorphic functions. The embedding of 'elementary' results in broad coherent theories has been known to cause acute cases of paranoia to persons who prefer 'simple' significant examples to the elaboration of the more extensive theories. I personally like both, but I have made an attempt to keep the more elementary portions separate from the others, and hence more accessible for the reader of limited mathematical background, . . . Historical notes at the end of each chapter serve as much to describe past work in the subject as to suggest further possibilities and conjectures. . . . it is remarkable that a mathematical theory as old as the theory of transcendental numbers (dating back to Hermite's first result of 1873, the transcendence of e) is still in what can only be called an underdeveloped state . . . " Bibliography. S (16–17), P.

Approximation of Functions. By G. G. Lorentz. Holt, Rinehart and Winston, New York, 1966. ix+188 pp. $5.00. "My purpose has been to write an easily accessible book on the approximation of functions that is simple and without unnecessary details, and is also complete enough to include the main results of the theory, including some recent ones. . . . The leitmotiv of the book is that of the degree of approximation. . . . " T (16–17), S, P.

Hill's Equation. By Wilhelm Magnus and Stanley Winkler. Interscience, New York, 1966. viii+127 pp. $8.50. An "orientation to homogeneous, linear, second order differential equations with real, periodic coefficients, which can all be reduced to the equation associated with G. W. Hill's famous memoir of 1877 on the lunar perigee." Bibliography. P, L.

Entire Functions. By A. I. Markushevich. Translated by Scripta Technica, Inc., Translation Editor Leon Ehrenpreis. American Elsevier, New York, 1966. iv+105 pp. $6.50. A brief exposition on functions equal to sums of everywhere convergent power series to assist the student who has had elementary calculus and knows the simplest facts about complex numbers "to understand better and more thoroughly the facts related to an academic course." S (15–16), P.

Calculus of Residues. By D. S. Mitrinovic. Noordhoff, Groningen, 1966. 87 pp. $1.90 (paper). Problems with solutions, classified as introductory, direct application of the residue theorem, integration along the real axis, rational functions of $\cos\theta$ and $\sin\theta$, summation of series. S, P.

Lectures on Choquet's Theorem. By Robert R. Phelps. Van Nostrand, Princeton, N.J., 1966. v+130 pp. $2.50 (paper). " . . . an important recent chapter in functional analysis . . . has applications to analysis, probability, potential theory, and functional analysis . . . presupposes a knowledge of integration theory and elementary functional analysis, including the Krein-Milman theorem and the Riesz representation theorem." In the Van Nostrand Mathematical Studies series of "lecture notes, trial manuscripts, and other informal mathematical studies . . . focusing on the living and growing aspects of mathematics." Reviewed by L. D. Young in *Science*, 2 Dec. 1966. P.

Integral Equations and their Applications Vol. I. By W. Pogorzelski. Pergamon Press, New York and Polish Scientific Publishers, Warsaw, 1966. xi+714 pp. $18.50. General properties of Fredholm and Volterra equations, systems of integral equations (linear and nonlinear), applications to differential equations, Cauchy type integrals, boundary problems, singular integral equations. In spite of the untimely death of the author, Volume II is completed and will deal with polyharmonic equations, generalized analytic functions, multidimensional singular integral equations, applications to partial differential equations and to physics and technology applications. P, L.

Linear Spaces of Analytic Functions. By Pasquale Porcelli. Rand McNally, Chicago, 1966. ix+158 pp. $8.00. " . . . about one-third of the material the author presents is a first-year graduate course in complex variables . . . has for its central theme approximation theory." Log mean, functions with positive real part, boundary values and H_P spaces, H_P algebras, ring of analytic functions, approximation, topology of pointwise convergence, the Fourier transform, infinite systems of equations in Hilbert space, moment problem. S, P.

Integral, Measure and Derivative: a Unified Approach. By G. E. Shilov and B. L.Gurevich. Translated from the Russian by Richard A. Silverman. Prentice-Hall, Englewood Cliffs, N.J., 1966. xiv+233 pp. $8.95. Uses the Daniell scheme "which starts from the concept of an elementary integral defined (axiomatically) on a family of elementary functions" rather than the more usual Lebesgue-Radon-Frechet approach. "In preparing the American edition, we gave the book a complete overhaul." T (16–17), S, P, L.

Laplace Transform Theory. By M. G. Smith. Van Nostrand, Princeton, N. J., 1966. x+123 pp. $6.00. " . . . short rigorous account of the elementary theory . . . some of its more useful applications . . . primarily for mathematics students . . . " T (16–17), S, P.

Methods of the Theory of Functions of Several Complex Variables. By V. S. Vladimirov. Translated by Leon Ehrenpreis. M.I.T. Cambridge, Mass. 1966. xi+353 pp. $12.00. " . . . a systematic exposition of . . . single sheeted (schlicht) domains of homorphy and its applications to quantum field theory, to the theory of functions, and to differential equations with constant coefficients. In recent years, the theory of functions of several complex variables, which previously had not found much application in natural science, unexpectedly found numerous and fruitful applications in quantum field theory . . . The successes achieved in quantum field theory by this route have in turn had influence on the theory of functions of several complex variables . . . the theory of generalized functions is used extensively in the monograph . . . " S, P.

Algebraic Topology. By Edwin H. Spanier. McGraw-Hill, New York, 1966. xiv+528 pp. $15.00.

The dust jacket blurb begins "This long-awaited text, suitable for graduate courses, is a comprehensive survey of the fundamental ideas of algebraic topology" and this is a very good one-sentence description of the book. The first four chapters—fundamental group, covering spaces, polyhedra, and homology—would form a good one semester basic graduate course, while the last five chapters give a very good background for much of the present research in algebraic topology and its applications to differential topology and manifold theory. This is not to say that this is an easy book to read by oneself. The material is clearly, but concisely, presented. In some sections there is more generality than some readers would prefer. However, the fine introductions to each chapter and to each

section, the well-chosen exercises which lead to extensions of the theory, the excellent choice of material, the absence of confusing notation, and the absence of very technical propositions make this by far the best text on the subject at present and probably for some time to come.

FRANKLIN P. PETERSON, M.I.T.

An Introduction to Probability Theory and Its Applications. Vol. II. By William Feller. Wiley, New York, London, 1966. xviii+626 pp. $12.00.

The first volume of the work under review was published in 1950, and a second edition appeared in 1957. In volume I, Professor Feller presented a rigorous introduction to discrete probability theory (as well as to certain types of random or stochastic processes) which was warmly received by a large general readership whose mathematical background was not too extensive, as well as by serious students of probability. Volume II is devoted, in the main, to continuous sample spaces; and the reader is required to have a knowledge and appreciation of the results and techniques of analysis (hard and soft).

This volume contains nineteen chapters, the first three of which supplement material contained in volume I. The last sixteen chapters are devoted to a very elegant presentation of material that is often presented in graduate courses and seminars under the general title "advanced probability theory." The text contains a large number of examples and problems which indicate the applications of probabilistic reasoning and methodology in various branches of analysis. The reader will also find a discussion of many topics of historical and methodological interest. The author's beautiful expository style, which made volume I such a pleasure to read and study, has remained invariant.

This book is recommended as a text or supplementary text for courses in advanced probability, and as a rich source of material for seminars devoted to topics in probability and stochastic processes. Volumes I and II of Professor Feller's book constitute a unique contribution to the literature of probability theory.

A. T. BHARUCHA-REID, Wayne State University
and Mathematics Research Center,
The University of Wisconsin

An Introduction to Computer Programming. By Henry Mullish. Gordon and Breach, New York, 1966. xi+244 pp. $5.00 (paper) $14.50 (cloth).

The preface of this book makes the statement "Whereas we do not claim that after having assimilated the contents of our text the reader will have become a professional programmer, nevertheless, we feel that a thorough understanding of our material will help create a substantial understanding of the techniques employed in solving many of today's common tasks in addition to alerting him to methods of solving other problems in areas not covered by this text."

The style of the statement is a fair indication of the style of the book. Though it could serve as a programming primer for a student with a knowledge of high school mathematics, there are serious faults even at this level. For example, there are no answers to the quizzes—an especially exasperating omission since 11 of the book's 244 pages are completely blank! Flow charts are introduced with no explanation whatsoever; the flow chart on page 133 appears with a remote connector and the standard box shape convention (almost), neither of which the author bothers to explain. The chapter on matrix algebra begins with "A matrix is no more than several columns of numbers."

The needs of readers of this MONTHLY will be better served by other texts on this subject, e.g., McCracken's "A Guide to FORTRAN Programming" or by any of the IBM student texts on FORTRAN, at a fraction of the almost unbelievable price of this book.

EDWARD SELIGMAN, United Aircraft Research Laboratories

Group Theory. By Eugene Schenkman. Van Nostrand, Princeton, N.J., 1965. 289 pp. $9.50.

Perhaps this book's most striking characteristic is its unusually lucid architecture. Of its three sections, the first is devoted to the elementary parts of the theory. The second, which is the nucleus of the book, deals with Sylow theory, free groups, free products, nilpotent groups, and solvable groups. The chapter on free groups has a particularly elegant presentation of the Subgroup Theorem. The last section of the book contains a short introduction to the theory of representations of finite groups, which goes just far enough to prove Burnside's Theorem and some interesting modern results such as Thompson's Theorem on normal p-complements. The bibliography is short and not very useful. On the whole this book is an excellent introduction to group theory, wide in scope and elegant and clear in style.

OLIVER PRETZEL, Free University, West Berlin

Differential Equations of Applied Mathematics. By G. F. D. Duff and D. Naylor. Wiley, New York, 1966. xi+423 pp. $11.95.

The book represents a course on the Laplace, heat and wave equations by the authors to senior undergraduates in applied mathematics and engineering science. Modern techniques are used, and the theories of distributions and Fourier transforms are stressed. The physical backgrounds of many of the problems are treated in detail. The material is explained well, and exercises are numerous.

MARTIN SCHECHTER, Belfer Graduate School of Science, Yeshiva University

Topology from the Differentiable Viewpoint. By John W. Milnor. Univ. Press of Virginia, Charlottesville, 1965. ix+64 p. $3.00.

In this short volume several of the most important results and techniques of algebraic and differential topology are introduced through an exposition in the spirit of "advanced mathematics from an elementary point of view." The reader is assumed to be familiar with very elementary point set topology and a few of the basic theorems of real variable theory.

The Sard-Brown Theorem is proved and the degree of a map is defined using the concept of a regular value of a smooth mapping. Hopf's Theorem is proved by showing that two maps of an n-sphere into itself are homotopic if and only if they have the same degree. The notation of cobordism is introduced and the relation between framed cobordism classes of manifolds and homotopy classes of maps of a sphere into a sphere is established.

E. H. BROWN, JR., Brandeis University

Elementary Matrices (Tutorial Text No. 3). By D. S. Mitrinović. Noordhoff, Groningen, 1965. 75 pp. $2.75.

This pleasant little book makes available to English language readers a concise resumé of elementary matrix theory together with an interesting collection of problems which had previously appeared only in Serbo-Croatian. The basic definitions and elementary theorems concerning matrices and determinants are clearly stated, but no proofs are given. Topics covered include products and powers of matrices, determinants, inverses, special matrices, rank, linear forms and equations, eigenvalues and eigenvectors, and elementary canonical forms. The 151 problems vary widely in difficulty and hints are freely given. The book should prove attractive to students wishing to instruct themselves, and it will also serve well as a supplement to regular course work.

G. N. RANEY, University of Connecticut

How to Gamble if You Must: Inequalities for Stochastic Processes. By L. E. Dubins and L. J. Savage. McGraw-Hill, New York, 1965. xiv+250 pp. $12.75.

"A gamble is, of course, a probability measure γ on subsets of fortunes. In the tradition of recent decades, such a measure would be defined only on a sigma-field of subsets of F and required to be countably additive on that sigma-field. If this tradition were followed in this book, tedious technical measurability difficulties would beset the theory from the outset." With this single remark the authors lay bare the rationale behind the book and open a whole new field of endeavor. From an interest in how they would play "red and black" has come a book on stochastic processes with probability measures which may be only finitely additive.

That such is no restriction is shown by the fact that the authors develop a considerable theory of such processes in their own right, with happy analogies to some of the more usual situations. They consider countably additive measures as particularly important special cases and, where possible, indicate similarities and differences between the two kinds of additivity. Perhaps it takes authors of this stature to break new ground in this fashion. One may foresee not only a series of papers and theses resulting from this work alone but also from similar lessening of restrictions in other fields.

Unfortunately it may take several such papers before we all understand this material. Clearly the authors have had a good time thinking and writing about gambles, but they have not done nearly enough to let us in on what they are up to. They say, "The number of technical terms and notations introduced is large in spite of economy drives, but only a few are in use in any one section and many are short-lived." (With regard to the latter: Why, then, introduce them?) A rough count produced 40 definitions and 30 theorems in the first substantive chapter, and only 4 examples. While some of the terms are surely meant to be helpful ("thrifty strategies," "miserly functions," "rich man's casino"), others sound like in-jokes: "full house" (not what you think), "income-tax casinos" and their sub-houses "primitive casinos," "houses of inequity," and the ultimate horror "casinoe" ("rhymes with 'canoe' ").

For the mathematical reader approaching this *de nouveau* (And that includes most of us; non-mathematical readers stay clear!), the best approach is through specific examples. The following list of chapter headings is arranged to indicate a possible line of immersion with minimum trauma. Chapter 5, Red-and-Black, is a binomial game. (One unhappy misprint here: In Figure 1, the abscissa measures f, not w.) Chapter 6, Primitive Casinos, is a slight generalization in which the amount to be won is some non-unit multiple of the stake. Chapter 4, Casinos with Fixed Goals, could be studied separately for its interesting inequalities. Chapter 3, Strategies, can best be digested by "flashbacks" while reading 4, 5, and 6. Chapter 8, Houses on the Real Line, deals with some specific cases and concludes with the only example likely to be of interest to non-mathematical gamblers. Chapter 9, Three Particular Kinds of Casinos, is just that. Now, since they can be postponed no longer, Chapter 2, The Abstract Problem, and Chapter 7, General Principles, might be taken together, realizing that again many flashbacks for notation have previously occurred. Chapter 10, One Lottery Strategies, on sums of independent and identically distributed random variables, will fix the general ideas nicely, as will Chapter 11, Fair Casinos. Chapter 12, The Scope of Gambler's Problems, looks at more general stochastic process problems from a gambling (finitely additive measure) point of view. Several master's theses will eventually emerge from this chapter.

The book will repay (and require) serious study at the graduate level. It is unfortunate that a major share of the impact of its ideas will have to wait for a further, more extensive, exposition.

ROBERT KOZELKA, Williams College

Topology and Order. By Leopoldo Nachbin. Van Nostrand, Princeton, N. J. 1965. vi+122 pp. $2.50.

In this paperback the author first presents the basic concepts of topological spaces and ordered sets, and then proceeds to study systematically the interdependence of these notions. Thus emerge topological ordered spaces, in particular normally ordered spaces, compact ordered spaces, and completely regular ordered spaces. Furthermore, uniform ordered spaces and locally convex ordered vector spaces are treated. Much of the material stems from the research of the author.

The book is eminently readable and should be accessible to anyone with a rudimentary knowledge of general topology. (The book is actually self-contained relative to topological notions, but some of these are presented, practically speaking, only in outline form.) Theorems are carefully stated and the proofs are easy to follow. At times the reading elicits some excitement; this occurs when the interplay of order and topology reveals some unexpected results.

A. E. DANESE, SUNY at Buffalo

Applied Complex Variables. By John Dettman. Macmillan, New York, 1965. ix+481 pp. $9.95.

This is an extremely well written book dealing with the foundations and applications of analytic function theory. It is intended primarily for undergraduate engineers and physicists who have completed a modern calculus course; but, due to its scope and contents, it could be used equally well as an introduction to complex variables for mathematics majors.

The book is divided into two parts. Part I (Chapters 1–5) is devoted to the fundamentals of analytic functions and covers the main ideas of complex integration, complex sequences and series, and residue calculus. Part II (Chapters 6–10) is a collection of applications of the ideas developed in the first part. The chapter headings of this part are Potential Theory, Ordinary Differential Equations, Fourier Transforms, Laplace Transforms and Asymptotic Expansions. Each of these subjects is treated independently, depending only on Part I. This format, as the author states, makes this book readily adaptable to either a two semester course or a one semester course consisting of Part I and any one (or more) of the five applications.

The book is well documented with illustrative examples and the discussion of each topic is complemented with a set of integrated exercises. The proofs are clear and usually well motivated. Typographical errors are scarce.

RODNEY ANGOTTI, SUNY at Buffalo

Vector Spaces of Finite Dimension. By G. C. Shephard. Oliver and Boyd, London, 1966. viii+200 pp. $3.25.

This book covers the topics which the title suggests, including multilinear algebra and unitary spaces. There are several features which distinguish this book from most others in this area. Matrices are scrupulously avoided until the final chapter, where matrix interpretations of the results obtained earlier are given. The style is concise, with a lack of heuristic discussion for motivation. The geometric interpretations are almost completely ignored. (There is only one illustration in the entire book!) In short, this is not a self-teaching book; the instructor's role in supplying motivation, specific examples, etc., would be crucial.

The book is clearly written, and would be useful as a reference or as a guide for review of the subject.

H. E. CHRESTENSEN, Reed College

Partial Differential Equations. By Lipman Bers, Fritz John, and Martin Schechter. Wiley, New York, 1964. xiii+343 pp. $10.70.

In June and July of 1957 lectures on the theory of partial differential equations were delivered at the Seminar on Applied Mathematics sponsored by the American Mathematical Society at the University of Colorado. The book is an expanded and edited version of those lectures. The preface states, "This is neither a textbook for beginners, since standard material is omitted, nor a reference book for specialists. It is aimed, as the oral presentation was, at mathematicians not necessarily familiar with the field of partial differential equations who want to acquire an understanding of some of the problems and methods in this discipline. The technical prerequisites are very modest. A student familiar with the fundamentals of real and complex function theory and with the elements of functional analysis should have no difficulty in following the presentation."

The book is at the highest level of exposition. The stated goal of the preface is authoritatively and admirably fulfilled. Despite the delay between the lectures and their publication, the book should serve as an inspiration and model for future expository writing.

A. M. WHITE, Harvey Mudd College

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor V. W. Backens, Black Hills State College, represented the Association at the inauguration of President H. R. Fraser of the South Dakota School of Mines and Technology on October 30, 1966.

Professor Foster Brooks, Kent State University, represented the Association at the inauguration of President A. L. Pugsley of The Youngstown University on November 5, 1966.

Professor R. B. Deal, Oklahoma State University, represented the Association at the inauguration of President G. C. Cothen of Oklahoma Baptist University on November 5, 1966.

Professor Winston Massey, University of Chattanooga, represented the Association at the inauguration of President W. H. Masterson of the University of Chattanooga on November 4, 1966.

Professor H. E. Taylor, Florida State University, represented the Association at the inauguration of President S. W. Martin of Valdosta State College on November 15, 1966.

Northeast Missouri State Teachers College: Assistant Professor K. J. Stilwell, King's College, has been appointed Associate Professor; Mr. D. M. Norman has been promoted to Assistant Professor.

Purdue University: Dr. David Drasin, Cornell University, has been appointed Assistant Professor; Associate Professor R. E. Zink has been promoted to Professor.

University of Rochester: Dr. E. C. Curtis, of Daniel H. Wagner Associates, has been appointed Assistant Professor; Associate Professors N. G. Gunderson and R. A. Raimi have been promoted to Professors.

Texas Western College: Assistant Professor J. W. Fitzpatrick, Trinity University, has been appointed Assistant Professor; Mr. Lawrence Huntley has been promoted to Assistant Professor.

Westmar College: Mr. D. D. DeBoer has been promoted to Assistant Professor; Dr. D. L. Deever, Ohio State University, has been appointed Assistant Professor.

Associate Professor Jeanne Agnew, Oklahoma State University, has been appointed Visiting Associate Professor at Georgia State College.

Mr. G. J. Basil, Lehigh University, has been appointed Assistant Professor at Waynesburg College.

Dr. E. J. Beltrami, Grumman Aircraft Co., has been appointed Associate Professor at State University of New York at Stony Brook.

Mr. P. D. Blanchard, Wisconsin State University, has been promoted to Assistant Professor.

Mr. S. E. Clauss, Indiana Institute of Technology, has been appointed Assistant Professor at Purdue University.

Associate Professor R. M. Conkling, Highlands University, has been appointed Head of the Department of Physics and Mathematics and Director of the Computer Center.

Mrs. Hazel Cowin, Murray State University, has been promoted to Assistant Professor.

Assistant Professor Chester Feldman, Kent State University, has been promoted to Associate Professor.

Mr. D. E. Gracey, Shippensburg State College, has been promoted to Assistant Professor.

Mr. H. F. Hassenplug, Northwest Missouri State College, has been appointed Assistant Professor at Winthrop College.

Associate Professor Melvin Hausner, New York University, has been promoted to Professor.

Mr. S. E. Hunt, Rutgers—The State University, has been appointed Assistant Professor at State University College at Geneseo.

Professor Emeritus R. E. Bruce, Boston University, died on December 27, 1965. He was a Charter Member of the Association.

Professor Emeritus A. B. Coble, University of Illinois, died on December 8, 1966. He was a Charter Member of the Association.

Mr. L. A. Graham, Graham Transmissions, Inc., died in November, 1966. He was a member of the Association for eleven years.

Dr. C. L. Perry, Director of the Computation Facility at the University of California at San Diego, died in November, 1966. He was a member of the Association for seventeen years.

Professor Emeritus L. C. Plant, Michigan State University, died on November 12, 1966. He was a Charter Member of the Association.

MEMORIAL TO CARL FRIEDRICH GAUSS

This memorial, written originally in German for the lay reader in 1856 by a Goettingen colleague and close friend, is now available in English. The 85 page monograph contains source material on Gauss' life and achievements. It has been translated and privately printed by Miss Helen W. Gauss, great-granddaughter of Carl Friedrich, and is available in limited numbers, while the supply lasts, on remittance of $1.00 per copy to Colorado College Library, Colorado Springs, Colorado 80903.

Editorial Note. Professional mathematicians as well as laymen will be interested in the article "Mathematics: International Congress," by Lee Lorch, which appears in *Science* for February 24, 1967, pp. 1038–39.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FIFTIETH ANNUAL MEETING OF THE ASSOCIATION

The Fiftieth Annual Meeting of the Mathematical Association of America was held in Houston, Texas, from Thursday to Saturday, January 26 to 28, 1967, in conjunction with the Annual Meeting of the American Mathematical Society and meetings of the National Council of Teachers of Mathematics and the Association for Symbolic Logic. The session of the Association on Saturday was a joint session with the National Council of Teachers of Mathematics. There were registered 3101 persons including 1720 members of the Association.

Sessions of the Association were held on Thursday morning and Friday morning in the Grand Ballroom of the Rice Hotel and on Saturday morning and afternoon in the Cullen Auditorium of the University of Houston. Presiding officers were Professors D. G. Bourgin on Thursday morning, Professor Jim Douglas, Jr. on Friday morning, Professor D. A. Johnson for the session on geometry and school mathematics and Professor F. B. Allen for the session on high school geometry, both on Saturday morning, Professor J. C. Egsgard for the session on school geometry of the future and Professor H. F. Fehr for the session on mathematics around the world, both on Saturday afternoon. The Program Committee for the meeting consisted of D. G. Bourgin, Chairman; Louis Brand, E. H. Connell, W. T. Guy, Jr., J. H. Hlavaty, E. C. Klipple, and W. K. McNabb.

FIRST SESSION OF THE ASSOCIATION

Session on Algebraic Topology

A Topological Study of Singularities of Varieties, by Professor A. H. Wallace, University of Pennsylvania.

Let D be a region of the plane bounded by arcs of algebraic curves. The set of real plane algebraic curves of a given order n having real points in D can be represented as a subset V of an algebraic variety W, bounded by subvarieties of W. V can then be decomposed into open sets V_i of subvarieties such that the points of any V_i represent curves homeomorphic under a homeomorphism of D onto itself, but these are not homeomorphic to curves represented by boundary points of V_i. Applied to curves with an isolated singularity in D, this gives a classification of singularities. A similar procedure can be applied to singularities of varieties of higher dimension.

Bordism and Periodic Maps, by Professor P. E. Conner, Jr., University of Virginia.

The application of bordism to the study of periodic diffeomorphisms of odd prime period on closed oriented manifolds is based upon the computation of the bordism module $\Omega(Z_p)$ together with a formula which uniquely determines the bordism class $[M^n]$ mod $p\Omega$ in terms of the normal bundle to the fixed point set of (T,M^n).

For maps of prime power period it can be shown that if (T,M^n) has no fixed points then $[M^n]$ is divisible by p, and an alternative proof of the Atiyah-Bott assertion that T cannot have exactly one fixed point can be given by these bordism techniques.

Isolated Singularities of Hypersurfaces, by Professor J. W. Milnor, Princeton University.

Consider an algebraic variety $\Gamma \subset C_n$ defined by one polynomial equation in n complex variables, $n \geq 2$. The topological behavior in the neighborhood of a singular point can be studied by intersecting Γ with a small sphere S_ϵ, centered at the singular point (Brauner, Hamburg Abh. 1928). If $n \neq 3$ then E. Brieskorn has constructed examples in which $\Gamma \cap S_\epsilon$ is a (knotted, sometimes exotic) topological sphere; and hence Γ is a topological manifold (*Proc. Nat. Ac. Sci.*, 1966, p. 1395). If $n=3$ then Γ cannot be a topological manifold unless it is completely non-singular (Mumford, I.H.E.S., Paris, Publ. Math. No. 9, 1961).

SECOND SESSION OF THE ASSOCIATION

Session on Analysis

The Nature of Quasi-Analyticity, by Professor Leon Ehrenpreis, Courant Institute.

One of the most interesting properties of an analytic function is the fact that if it vanishes on an open set, then it must vanish identically. One of the first generalizations of analyticity was in the direction of finding suitable classes of functions for which this uniqueness property, called quasi-analyticity, holds. Denjoy and Carleman studied classes C_M of indefinitely differentiable functions $f(x)$ defined by inequalities of the form

$$\left| f^{(n)}(x) \right| \leqq A^{n+1} M_n,$$

where $M = \{M_n\}$ is a sequence of positive numbers.

It is possible to generalize the Denjoy-Carleman theorem by replacing the operator d/dx, which is fundamental for their considerations, by an arbitrary constant coefficient partial differential operator.

Positive-Definite Functions on Groups, by Professor Edwin Hewitt, University of Washington.

Let G be a group. Let f be a complex-valued function on G. If the sum $\sum_{j=1}^{m} \sum_{k=1}^{m} \alpha_j \alpha_k f(x_j^{-1} x_k)$ is real and nonnegative for all subsets $\{x_1, \cdots, x_k\}$ of G and sequences $(\alpha_1, \cdots, \alpha_m)$ of complex numbers, f is called positive-definite. Such functions are intimately connected with unitary representations of groups, and play a key role in the original proof (1943) by Gel'fand and Raĭkov that locally compact groups have sufficiently many unitary continuous irreducible representations. A famous theorem of Herglotz-Bochner-Kreĭn-Weil characterizes continuous positive-definite functions on locally compact Abelian groups as Fourier-Stieltjes transforms. An analogue holds for arbitrary compact groups, as was proved by M. G. Kreĭn (1949). This address explored the structure of the algebra $\mathfrak{R}(G)$ of linear combinations of continuous positive-definite functions on a compact group G.

Annual Business Meeting of the Association; the Association's Sixth Award for Distinguished Service to Mathematics, and the Award of the 1967 Chauvenet Prize.

A Constructive Approach to Abstract Analysis, by Professor E. A. Bishop, University of California, San Diego.

Brouwer's criticisms of classical mathematics are just as valid as ever. On the positive side, we wish to give mathematics descriptive content by associating to every result a numerical interpretation. Every theorem should be an assertion that certain computations within the set of positive integers will produce certain results. This program is much easier to realize than is commonly supposed. Although most results of classical mathematics are proved with the aid of transcendent logical principles, such as the law of the excluded middle, it is usually (at least in the field of abstract analysis) possible to extract a kernel—quite a large kernel in most instances—of constructive truth.

THIRD SESSION OF THE ASSOCIATION

Joint Session with the National Council of Teachers of Mathematics

The lectures delivered in this part of the program have been published in a pamphlet

available from the National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington, D. C. 20036.

Session on Geometry and School Mathematics

Geometry in the United States, by Professor B. E. Meserve, University of Vermont.

Changes in the role and content of geometry in school mathematics curricula were identified and discussed with reference to general trends. The increased attention to geometric concepts at the elementary and junior high school levels illustrates a general recognition of the importance of geometry in mathematics curricula. The demise of the "sacred cow" attitude toward the first formal course in geometry (usually tenth grade) is part of a general breaking down of capsule approaches to several different branches of mathematics and the integration of all of these into a comprehensive mathematics curriculum in which geometry has a basic role.

The Ontario K–13 Geometry Report, by Professor H. S. M. Coxeter, University of Toronto.

The new curriculum in Ontario is expected to include geometry in each grade from Kindergarten to 13, because geometry develops and refines spatial intuition and has interactions with art as well as science. Moreover, geometry provides many opportunities for the student to exercise originality and ingenuity. Instead of an axiomatic treatment, the tendency is to use geometric transformations in an informal manner. For instance, *vector* is just another name for *translation*. Stereometry will begin in Kindergarten, because simple solids can be appreciated at an earlier age than their two-dimensional counterparts. By fitting suitable pieces together, the formula for the volume of a pyramid can be made very plausible.

Danish Geometry, by Professor P. S. Jorgensen, Carleton College.

The reform of secondary school geometry in the United States has for the most part centered on various attempts to preserve the geometry of Euclid by the introduction of the real numbers or by the adaptation of Hilbert's formulation of geometry. In Denmark, on the other hand, the reform has centered on the geometry of transformations (reflections, rotations, and translations). That such an approach is feasible in the beginning stages of geometry instruction is evidenced by some of the text books now in use in the Danish *real* school. The approach, though deductive in nature, is not a rigorous one, but instead is highly intuitive. A rigorous treatment is left to the *gymnasium* where the subject is closely tied to algebra.

This paper presented a brief summary of the Danish geometry program and gave some examples of how the geometry of transformations is adapted to the beginning geometry instruction.

British Geometry, by Professor Andrew Elliott, Royal Military College, Kingston, Ontario.

In the elementary school, ages 5 to 12, the Nuffield Mathematics Project places great emphasis on the formation of patterns with one or more basic elements, and on the symmetry properties of single geometric figures in both two and three dimensions. The School Mathematics Project and the Midland Mathematical Experiment in England and the Scottish Mathematics Group in Scotland for the 12–18 year old pupils use the fundamental transformations of translation, rotation and reflection to develop the properties of the triangle and the various parallelograms. From concrete examples in travel and navigation placed against a co-ordinate background these programs also develop vectors in two and three dimensions and use them in geometric proofs.

Session on High School Geometry

Synthetic Euclidean Geometry, by Professor Walter Prenowitz, Brooklyn College.

First, the position of synthetic Euclidean geometry in the present secondary curriculum was considered. Improved treatments of tenth grade geometry have been developed and are available for classroom use which should be stimulating and challenging to the mathematics major and the liberal arts student. Second, synthetic geometry and coordinate geometry were compared and their individual advantages described. Third, the role of synthetic geometry in the secondary curriculum was discussed, particularly its advantages as a model of the deductive organization of a body of knowledge and as a medium for presenting the relation of mathematical theory to physical reality.

Affine Geometry, by Professor G. P. Johnson, Oakland University.

In 1964 a writing team sponsored by the National Science Foundation and Wesleyan University prepared materials in metric geometry for use in high schools. Two years of class testing since then support the belief that such content can provide, at least for able students, a significantly more satisfactory course than those following traditional lines.

Vector Geometry, by Professor H. E. Vaughan, University of Illinois, Urbana.

This report concerns a two-year secondary school course in process of development and testing by UICSM since September 1963. The course presupposes a year of algebra, and is designed for able students. After spending some time developing intuitive notions concerning translations, euclidean geometry is presented as the theory of a vector space—the space of translations—*operating on* a set of points—the points of euclidean space. There are three postulates which formulate the way in which translations operate on points, followed by the usual postulates for a 3-dimensional inner product space over the real numbers. For more details, see Steven Szabo, An Approach to Euclidean Geometry Through Vectors, *The Mathematics Teacher*, vol. 59 (1966), pp. 218–235.

Geometric Transformations, by Professor Seymour Schuster, University of Minnesota.

Thoughts concerning the introduction of geometric transformations into high school mathematics are becoming more prevalent. There are several strong reasons that justify a move in this direction. (1) The ideas of congruence can be given a mathematically sound treatment by means of transformations, whereas a treatment based on superposition (still taught in many quarters) is totally unsatisfactory. (2) A use of transformations in geometry reiterates and uses what is—or at least should be—one of the basic themes of the new mathematics programs: the notion of function. (3) Transformations, *per se*, are important tools which will serve students throughout their entire mathematical lives. They are "tricks of the trade" for problem-solving at all levels; they are indispensable in both the pure and applied work in fields such as differential equations and complex variable; they are the key to understanding symmetry; finally, they are used to answer, in the sense of Felix Klein, the question "What is geometry?"

Session on School Geometry of the Future

Cambridge Committee View, by Professor E. E. Moise, Harvard University.

There were two Cambridge Conference views described in the CCSM report, one algebraic and "modern," the other more classical. In support of the classical (minority) view, the following points were made:

1. In a choice of content and style, at a given teaching level, a fundamental criterion is the availability of challenging and workable problems. At low maturity levels, calculus and classical geometry outrank linear algebra and transformation groups under this criterion.

2. A vectorial conception of geometry is on a high level of abstraction, because one cannot significantly say of a point that it is a vector; the significant statement is that a space (considered as an entity in itself) is a vector space.

Topology, by Professor G. S. Young, Tulane University.

A discussion of various natural topics in Topology that could be introduced in the school curriculum was given. Examples include such concepts as orientation, separation properties, surfaces, and what are characteristics. The very tentative notion of such ideas in the speaker's mind was stressed.

Projective Geometry, by Professor C. E. Springer.

The speaker described a series of 32 lectures on analytic projective geometry, which he gave in the summer of 1966 to a class of 40 high school students selected from 19 states. This was part of a program sponsored by the National Science Foundation at the University of Oklahoma. The material consisted of the first three chapters of Dorwart's "The Geometry of Incidence," Prentice-Hall, 1966, and of selected topics and exercises from the speaker's "Geometry and Analysis of Projective Spaces," Freeman, 1964. After a treatment of homographies on a line, the concepts of

join, intersection, and incidence in the plane were introduced with a view toward solution of problems on linkages, point loci, line envelopes, and invariance under projective mappings.

Session on Mathematics Around the World

A panel discussion with Dr. J. H. Hlavaty, New Rochelle, New York, Professor Mariano Garcia, Jr., University of Puerto Rico, and Mr. Henry Swain, New Trier Township High School, Winnetka, Illinois.

Dr. Hlavaty spoke on mathematics education in Europe and England, specifically on the two-way international contacts in mathematics education, from Royaumont (1959) to the NCTM Soviet Tour and the U.S.-U.K. Conference (1966). Many western European countries have made bold changes in their programs of secondary education. These changes are inspiring new experiments in this country in the second round of curriculum experimentation. In the Soviet Union and the eastern countries changes in programs have been later in coming than in the U.S.A., and the "new" ideas (sets, structures, mappings, vectors, etc.) are being introduced slowly and without fanfare. A significant contribution of the English experiments (and some of the European ones) come in geometry—with emphasis on transformations rather than on attempts at refining the logical presentation of Euclidean geometry.

Professor Garcia spoke on the reform in mathematical education in Central and South America. For the past four years, there have been concerted efforts in Central America to improve the mathematics programs in the countries of that area. This is in line with some of the major recommendations made in the Second Inter-American Conference on Mathematical Education held in Lima, Peru, last December. These recommendations include: to stimulate the creation of teaching and research mathematics centers; to foster interchange activities of universities; to encourage the organization of national and regional mathematics associations, conferences and symposia; and to promote the publication of materials that will upgrade the mathematical level of the various countries represented.

Mr. Swain reported on projects for improvement of mathematical education in some of the countries of Africa. For five summers a group of about 60 African, British, and American mathematicians has met in Entebbe, Uganda, and then in Mombasa, Kenya, to help improve the teaching of mathematics in ten English speaking countries across middle Africa. Administered by Educational Services Incorporated and paid for by the U. S. Agency for International Development, the workshops have produced a quantity of textbook material for primary, secondary, and teacher training schools. Teacher training institutes have also been conducted. There will probably be two more summers of writing. Several other mathematics projects are in progress in various African countries, most of them stimulated by the Entebbe Mathematics Project, some by the School Mathematics Project of England.

SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in the Crystal Ballroom of the Rice Hotel as follows: on Wednesday at 7:00 P.M., "John von Neumann" (a CEM Individual Lectures Production in b & w), at 8:10 P.M., "Can You Hear the Shape of a Drum? A Lecture by Mark Kac" (a CEM Individual Lectures Production in color); on Thursday at 7:00 P.M., "Pits, Peaks, and Passes (Part I), A Lecture on Critical Point Theory by Marston Morse" (a CEM Individual Lectures Production in color), at 8:00 P.M., CEM Animated Calculus Films (in color) including "Volumes by Shells," by George Leger, "Limits" by Robert C. Fisher, "Newton's Method" by Herbert S. Wilf, "What is Area" by Charles E. Rickart, "Area Under a Curve" by Charles E. Rickart, and "The Definite Integral" by Robert C. Fisher; on Friday at 7:00 P.M., CEM Animated Level I Films (in color) including "What is a Set?," "One-to-One Correspondence," "Counting," "Union and Intersection," "Addition and Subtraction," and "Multiplication and Division," at 8:10 P.M. Films of the College Geometry Project of the University of Minnesota (in color) including

"Orthogonal Projection" by Daniel Pedoe, "Central Similarities" by Daniel Pedoe, "Dihedral Kaleidoscopes" by H. S. M. Coxeter, "Geometric Vectors" by Wm. O. J. Moser and S. Schuster, "Inversion" by Daniel Pedoe, "Curves of Constant Width" by J. D. E. Konhauser, and "Central Perspectivities" by S. Schuster. There was also a special showing of the films of the College Geometry Project on Friday at 1:00 P.M.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Wednesday morning and afternoon in the Directors Room of the Rice Hotel, with 40 members present.

The Board approved the appointment by President Wilder of the following Nominating Committee for 1967: R. H. Bing, Chairman; P. T. Bateman, and O. G. Harrold.

The Board elected Professor R. J. Walker as Second Vice-President of the Association for the two-year term 1967–1968. Since Professor R. A. Rosenbaum, as Editor of the MONTHLY, will automatically become a member of the Board, his term as governor was, with his approval, considered terminated at the end of the Houston meeting, and the Board elected Professor R. G. Bartle of the University of Illinois, Urbana, as his replacement for the unexpired part of his term extending through 1967.

The Board elected Professor Haskell Cohen as an additional Associate Editor of the MONTHLY and Professor R. W. Bagley as an additional Associate Editor of the MATHEMATICS MAGAZINE.

The Board approved in principle proposals to the National Science Foundation by the Individual Lectures Project for continued support of the project for the two-year period starting June 1, 1967, and by the Committee on the Undergraduate Program in Mathematics for continued support of the Committee for the two-year period starting July 1, 1967.

The Board approved that, effective at once, payment of the usual $2.00 initiation fee is waived for all full-time graduate students whose application blanks are properly certified.

The Board approved the following schedule of future meetings of the Association: University of Toronto, August 28–30, 1967; San Francisco, California, January 25–27, 1968; University of Wisconsin, Madison, August 26–28, 1968; New Orleans, Louisiana, January, 1969; University of Oregon, August 25–27, 1969; Miami, Florida, January 24–26, 1970.

The Executive Director reported the membership of the Association as 17,530 individual members, an increase of 816 since the corresponding date last year, 2 corporate members, and 236 academic members.

The Board passed a number of resolutions in which it expressed its deep concern regarding the severe curtailment and threatened complete termination by the National Science Foundation of several projects vital to the maintenance of present levels of mathematical education in this country, specifically, the visiting lectureship program, the academic year institutes for secondary school teachers of science and mathematics, and other similar programs. These resolutions appear on pp. 473–475 of this issue.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual Business Meeting was held on Friday, January 27, 1967, in the Grand Ballroom of the Rice Hotel, with First Vice-President Young presiding. The Association's Sixth Award for Distinguished Service to Mathematics was made to Professor W. L. Duren, Jr. of the University of Virginia. The citation (which appeared on pages 1–2 of the January issue of this MONTHLY) was prepared and read by Professor E. J. McShane of the University of Virginia; the Award was presented by First Vice-President Young. Professor Duren, in accepting the Award, expressed his pleasure in having been voted

this Award which he honored as a symbol that the Association is committed to the principle of service complementing natural attitudes toward private motivation in our work. He thought this to be an exciting time in our history in that the Association has been brought into the plans for the expansion of science and, in particular, mathematical science because of its dedication to service. He foresaw a great enlargement of the service of the Association as a fruition of its past endeavours. He concluded: "If I have had, or I can have, any part in this, it is a most happy and joyful thing, and I certainly thank you all for the Award which I shall cherish for the rest of my life."

The 1967 Chauvenet Prize was awarded to Professor G. L. Weiss of Washington University for his paper "Harmonic Analysis," published in *MAA Studies in Mathematics*, Volume 3 (1965) 124–178. The Award was presented by First Vice-President Young. (For further details on this Award, see the January issue of this MONTHLY, page 2.)

The Secretary then announced the results of the balloting for officers in which 2112 votes were cast: Professor Victor Klee of the University of Washington and Professor N. H. McCoy of Smith College were elected governors for the three-year term 1967–69.

The Secretary announced that the book APPLICATIONS OF UNDERGRADUATE MATHEMATICS IN ENGINEERING by Ben Noble is now available. MAA Studies Volume 4, *Studies in Global Geometry and Analysis*, edited by S. S. Chern, is now scheduled for publication in July 1967. The brochure YOU'LL NEED MATH, written by Leonard Gillman and prepared with a grant from the National Science Foundation, is now available from the Buffalo office of the Association. It is directed to junior high school students and gives information on the amount of mathematics needed in a great variety of careers. Large scale free distribution of this brochure is now being planned.

The Secretary then reported on some of the actions taken by the Board of Governors on Wednesday. In reference to the meeting to be held this summer at the University of Toronto, he announced that the committee on arrangements there had reserved rooms at Loyola College in Montreal for those desiring to visit Expo 67 either before or after the meeting for the following nights: August 25–27 and September 1–3. The rates will be $7 per person for a double room and $8 per person for a single room. Early reservation of these rooms is advised. Those interested in making reservations at Loyola College should write as soon as possible to: Reservations, Department of Mathematics, University of Toronto, Toronto 5, Ontario, Canada.

The Secretary then expressed, on behalf of the Association and himself personally, deep appreciation to Professor J. W. Green, the retiring Secretary of the American Mathematical Society, for all his efforts and special consideration shown to the Association in the joint endeavors in which the Society and Association are engaged, specifically the planning of their joint national meetings. He also expressed the Association's gratitude to the University of Houston for its sponsorship of the cocktail and beer party on Wednesday evening and for providing the free bus transportation between the Rice Hotel and the University of Houston on Saturday. He conveyed special thanks to Professor D. R. Traylor, the Chairman of the Committee on Arrangements, for coordinating so successfully the many activities requiring his attention.

The Secretary then moved an amendment to the By-Laws which, effective with the 1968 dues, will increase the dues of individual members from $6 to $8. The motion was approved without dissent.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Tuesday, January 24, to Friday, January 27. The fortieth Josiah Willard Gibbs Lecture was delivered by Professor Mark Kac of the Rockefeller University on Tuesday evening at 8:00 P.M. in the Grand Ballroom of the Rice Hotel on "Some Mathematical Problems in the Theory of

Phase Transitions." Professor P. J. Cohen of Stanford University delivered an address entitled "The Modern Development of Axiomatic Set Theory" on Tuesday at 2:00 P.M.; Professor Alex Heller of the City University of New York gave an address on "Stable Homotopy Categories" on Thursday at 2:00 P.M. and Professor B. H. Neumann of the Australian National University and the University of Wisconsin delivered a lecture on "Varieties of Groups" on Friday at 2:00 P.M.

The Cole Memorial Prize of the Society was awarded to Professors J. B. Ax and S. B. Kochen of Cornell University for their sequence of three papers on Diophantine problems over local fields: "Diophantine Problems over Local Fields I," *Amer. J. Math.*, 87 (1965) 605–630; "Diophantine Problems over Local Fields II. A Complete Set of Axioms for p-adic Number Theory," *Amer. J. Math.*, 87 (1965) 631–648, and "Diophantine Problems over Local Fields III. Decidable Fields," *Annals of Math.*, 83 (1966) 437–456.

The Association for Symbolic Logic met on Monday and Tuesday, January 23 and 24. Professor Georg Kreisel of Stanford University gave an invited address on "A Survey of Proof Theory" on Monday at 11:00 A.M., Professor G. E. Sacks of Cornell University and the Massachusetts Institute of Technology one on "Recent Developments in and Applications of the Theory of Degrees" on Monday at 2:00 P.M., and Professor R. M. Solovay of the University of California, Berkeley, one on "Large Cardinals" on Tuesday at 11:00 A.M.

A brief report on the proposals and recommendations of the Cambridge Conference of Training Teachers of Mathematics was given on Thursday at 6:45 P.M. Professor J. W. Green chaired the meeting, and Professor George Springer spoke about the proposals and recommendations.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of D. R. Traylor, Chairman; Mrs. Eleanor Akers, H. L. Alder, P. T. Bateman, M. L. Curtis, W. T. Ingram, C. B. Rader, J. A. Robinson, G. L. Walker, Mrs. Margaret R. Wiscamb, Martin Wright.

Registration headquarters were located in the Grand Foyer on the second floor of the Rice Hotel on Monday through Friday. On Saturday the registration desk was moved to the Cullen Auditorium on the campus of the University of Houston. Accommodations for the meeting were handled by the greater Houston Convention and Visitors Council.

The University of Houston was host at a cocktail and beer party in the Grand Ballroom of the Rice Hotel on Wednesday from 9:00 P.M. until midnight to which all members of the participating organizations were invited.

HENRY L. ALDER, *Secretary*

RESOLUTIONS VOTED BY BOARD OF GOVERNORS

The following three resolutions were approved by the Board of Governors at its meeting on January 25, 1967, in Houston, Texas:

1. A Resolution addressed to the Congress of the United States and the officers of the National Science Foundation.

Whereas, much of the dramatic improvement in science education in the United States in the past decade can be ascribed directly to a significant increase in the concern for education on the part of the scientific community,

Whereas, a real and direct manifestation of this concern has been the fact that financial support for these national efforts toward curriculum reform has come largely from the National Science Foundation, an agency which is, in a genuine sense, part of the scientific community,

Whereas, through these years of support and leadership in this area, the National

Science Foundation has developed many effective programs for aiding and improving science education,

Whereas, comparable programs of support of science education at every level do not exist in any of the other granting agencies of the Federal Government, and

Whereas, the future of science in the United States depends as much on the education of the scientists of tomorrow as on the research of the scientists of today,

THEREFORE BE IT RESOLVED that the Board of Governors of the Mathematical Association of America

strongly urges the Congress of the United States and the officers of the National Science Foundation, even in the face of the present extraordinary pressures on the National Budget, to consider carefully the importance to our national welfare of continued and even increased support of science education,

strongly urges the Congress to continue directing this support of science education through the National Science Foundation rather than through other agencies which are not similarly parts of the scientific community, and

strongly urges the officers of the National Science Foundation, even in the face of the growing need for funds in support of research, to consider carefully the importance to science in our nation of continued and even increased support of science education by the National Science Foundation.

2. A Resolution addressed to the National Science Foundation.

The Mathematical Association of America calls attention to a National Science Foundation program whose future success is threatened by decreasing support.

The Mathematical Association of America has operated, with National Science Foundation support, its program of visiting lecturers for colleges since 1953. There is overwhelming evidence that this program has played a significant role in the improvement of instruction in mathematics which has occurred in the last fifteen years. The funds now provided for operating this program are inadequate to meet the costs of the program. The colleges visited contribute several thousand dollars each year, but NSF does not permit the use of these funds to pay the costs of the program. Although the program of visiting lecturers for colleges is as badly needed as ever—especially because of rapidly increasing enrollments, the expansion in the number of two-year colleges, and the continuing shortage of qualified instructors—the program is suffering because of inadequate support by NSF. Unless support can be returned to a more adequate level, the Mathematical Association of America may be forced to discontinue its operation of the program because it has become a drain on its own limited resources.

The Mathematical Association of America, meeting in Houston, Texas on January 25, 1967, commends the National Science Foundation for the success of its programs in the past and urges that it use every means possible to continue its support of mathematics and mathematical education. In particular, the Association hopes that NSF will continue its program of visiting lecturers for colleges and will provide adequate levels of support for this program.

3. A Resolution addressed to the Congress of the United States and the officers of the National Science Foundation.

Whereas, there is currently a great shortage of adequately trained college teachers of mathematics, due to the phenomenal increase in the number of two-year colleges with inadequately trained staffs, the rapidly increasing number of college students taking mathematics, and the drastic curriculum changes recommended by the Committee on the Undergraduate Program in Mathematics, and

Whereas, institutes have proved highly valuable in improving instruction in high schools and in colleges, and appear to be the only known means of effectively reaching a significantly large number of in-service teachers, and

Whereas, the number of institutes for college teachers of mathematics supported by NSF is totally inadequate to meet current needs, and

Whereas, there is little likelihood that funds for supporting institutes can be obtained from private sources, and

Whereas, it is in the national interest to maintain and increase the quality of mathematics instruction in our two-year and four-year colleges,

THEREFORE BE IT RESOLVED that the Board of Governors of the Mathematical Association of America strongly recommends that the Federal Government through the National Science Foundation make available the necessary funds to support a greatly increased program of unitary and sequential institutes for college teachers of mathematics.

The need for institutes for teachers in two-year colleges is particularly acute. The staffs of these institutions—and their number is increasing at an astonishing rate—are being recruited largely from the ranks of high school teachers, many of whom are not prepared to teach college level mathematics. It appears that mathematics instruction in two-year colleges may be the weakest link in the whole range of mathematics education.

Also, as the attached statistics (see table below) supplied by the National Education Association shows, there is currently and in prospect a significantly greater shortage of qualified college teachers of mathematics than in the biological and physical sciences.

These facts suggest that institutes for college teachers of mathematics, particularly teachers in two-year colleges, merit high priority in the allocation of federal funds for education.

VARIOUS INDICES OF SHORTAGES OF COLLEGE TEACHERS BY FIELDS

	Biological Sciences	*Chemistry*	*Physics*	*Physical Sciences*	*Mathematics*
Percentage of new teachers with Doctor's degrees, colleges and universities, 1964–65	50.2			59.1	28.2
Percentage of new teachers with Master's degrees or less, colleges and universities, 1964–65	29.3			27.2	51.1
Percentage of new teachers with less than a Master's degree, junior colleges 1963–64 and 1964–65	12.1	14.9	19.9		17.3
Number of colleges reporting critical shortage of qualified teachers in 1963–64 and 1964–65	167	191	406		597
Number of colleges foreseeing more acute shortage of qualified teachers	120	195	362		508
Number of budgeted positions unfilled in colleges and universities, 1963–64 or 1964–65	50	51	110		166
Number of junior colleges reporting a shortage of qualified teachers in 1963–64 and 1964–65	30	105	132		159
Number of junior colleges foreseeing a future shortage of qualified teachers	30	92	120		178

Source: *Teacher Supply and Demand in Universities, Colleges, and Junior Colleges, 1963–64 and 1964–65*, National Education Association

ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendments adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting at Houston, Texas, on January 25, 1967, elected to membership in the Association the eleventh set of applicants for academic membership (for election of the other ten sets, see the April and November issues for 1962–66). Approval for election to membership was given to the following 6 applicants for academic membership:

Bowling Green State University
University of California, Irvine
Chamberlayne Junior College, Boston, Massachusetts
University of Hawaii
Mount Holyoke College
Rhode Island Junior College, Providence, Rhode Island

HENRY L. ALDER, *Secretary*

OFFICERS AND COMMITTEES AS OF FEBRUARY 1, 1967

General Offices: SUNY at Buffalo, Buffalo, New York, 14214
Executive Director: H. M. GEHMAN

OFFICERS

President, E. E. MOISE, Harvard University (1967–68)
Past-President, R. L. WILDER, University of Michigan (1967)
First Vice-President, G. S. YOUNG, Tulane University (1966–67)
Second Vice-President, R. J. WALKER, Cornell University (1967–68)
Editor, R. A. ROSENBAUM, Wesleyan University (1967–71)
Secretary, H. L. ALDER, University of California, Davis (1965–69)
Treasurer, H. M. GEHMAN, SUNY at Buffalo (1963–67)
Associate Secretary, RAOUL HAILPERN, SUNY at Buffalo (1963–67)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

R. H. BING, University of Wisconsin, Madison (1965–70)
A. W. TUCKER, Princeton University (1963–68)
R. L. WILDER, University of Michigan (1967–72)

Elected Members of the Finance Committee

E. A. CAMERON, University of North Carolina (1966–69)
G. B. PRICE, University of Kansas (1964–67)

Governors at Large

R. G. BARTLE, University of Illinois, Urbana (1967)
P. S. JONES, University of Michigan (1965–67)
VICTOR KLEE, University of Washington (1967–69)
N. H. MCCOY, Smith College (1967–69)
C. R. PHELPS, National Science Foundation (1966–68)
M. F. SMILEY, University of California, Riverside (1966–68)

Sectional Governors (July 1, 1964–June 30, 1967)

Kansas, CALVIN FOREMAN, Baker University
Missouri, J. J. ANDREWS, St. Louis University
New Jersey, L. F. MCAULEY, Rutgers—The State University

Northeastern, D. E. CHRISTIE, Bowdoin College
Ohio, WADE ELLIS, Oberlin College
Pacific Northwest, A. T. LONSETH, Oregon State University
Southeastern, J. R. WESSON, Vanderbilt University
Southwestern, HARVEY COHN, University of Arizona
Upper New York State, D. E. KIBBEY, Syracuse University

Sectional Governors (July 1, 1965–June 30, 1968)

Illinois, J. M. H. OLMSTED, Southern Illinois University
Iowa, D. W. WALL, State University of Iowa
Louisiana-Mississippi, P. K. REES, Louisiana State University
Maryland-D.C.-Virginia, DOROTHY L. BERNSTEIN, Goucher College
Michigan, K. W. FOLLEY, Wayne State University
Minnesota, E. J. CAMP, Macalester College
Philadelphia, EMIL GROSSWALD, University of Pennsylvania
Southern California, R. C. JAMES, Harvey Mudd College
Texas, MARTIN WRIGHT, University of Houston

Sectional Governors (July 1, 1966–June 30, 1969)

Allegheny Mountain, A. F. STREHLER, Carnegie Institute of Technology
Indiana, G. N. WOLLAN, Purdue University
Kentucky, L. L. SCOTT, University of Louisville
Metropolitan New York, ABRAHAM SCHWARTZ, City University of New York
Nebraska, D. W. MILLER, University of Nebraska
Northern California, D. W. BLAKESLEE, San Francisco State College
Oklahoma, R. B. DEAL, Oklahoma State University
Rocky Mountain, F. M. STEIN, Colorado State University
Wisconsin, C. J. VANDERLIN, JR., University of Wisconsin, Madison

COMMITTEES OF THE ASSOCIATION

Terms of office of members expire, except where otherwise noted, at the Annual Meeting in January following the last year of service listed below. For temporary committees, no terms of office are listed, since they are automatically discharged at the expiration of the President's term of office, which is the Annual Meeting in January, 1969.

EXECUTIVE COMMITTEE

E. E. MOISE, *Chairman* (1966–69); H. L. ALDER (1965–69), H. M. GEHMAN (1963–67), R. A. ROSENBAUM (1967–71), R. J. WALKER (1967–68), R. L. WILDER (1964–67), G. S. YOUNG (1966–67), all *ex officio*.

FINANCE COMMITTEE

E. E. MOISE, *Chairman* (1967–68), *ex officio;* E. A. CAMERON (1966–69), G. B. PRICE (1964–67), H. L. ALDER (1965–69), *ex officio*, H. M. GEHMAN (1963–67), *ex officio*.

COMMITTEE ON ADVISEMENT AND PERSONNEL

A. B. WILLCOX, *Chairman* (1967–69); K. J. ARNOLD (1966–68), E. A. DAVIS (1967–69), J. S. FRAME (1967), Mrs. F. J. MACWILLIAMS (1967–69), C. R. PHELPS (1967–68), W. H. SCHMIDT (1966–68).

COMMITTEE ON AN INTERNSHIP PROGRAM IN MATHEMATICS EDUCATION

ROY DUBISCH, *Chairman;* MAX BEBERMAN, D. E. CHRISTIE, E. E. MOISE.

COMMITTEE ON EARLE RAYMOND HEDRICK LECTURES

E. E. FLOYD, *Chairman* (1965–67); N. J. FINE (1967–69), C. B. MORREY (1966–68).

COMMITTEE ON EDUCATIONAL MEDIA

C. O. OAKLEY, *Chairman* (1966–69); A. A. BLANK (1967–68), E. A. CODDINGTON (1966–68), L. W. COHEN (1967–69), LEON HENKIN (1965–67), J. H. HLAVATY (1965–67), P. S. JONES (1967–69), C. E. RICKART (1965–67), SEYMOUR SCHUSTER (1967–69), GEORGE SPRINGER (1966–68), F. B. WRIGHT (1965–67).

Advisory Committee for Calculus Films: H. M. MACNEILLE, *Director* (1965–67); J. H. CURTISS (1965–67), S. B. JACKSON (1965–67), C. B. MORREY (1967–69), A. W. TUCKER (1967–69).

Advisory Committee for Individual Lecture Films: L. B. WILLIAMS, *Director* (1965–67), R. C. BUCK (1965–67), L. W. COHEN (1965–67), P. D. LAX (1965–67), B. J. PETTIS (1967–69).

Advisory Committee for Level I Films: C. B. ALLENDOERFER, *Director* (1965–67); N. J. FINE (1965–67), LEON HENKIN (1965–67), J. H. HLAVATY (1965–67), P. S. JONES (1965–67), E. P. VANCE (1965–67), G. C. WEBBER (1965–67), F. B. WRIGHT (1965–67).

Advisory Committee for Programed Learning: B. H. GERE, *Director* (1965–67); D. W. BLAKESLEE (1965–67), LEONARD GILLMAN (1965–67), ROBERT KALIN (1965–67), A. B. MEWBORN (1965–67), L. W. SMITH (1965–67).

Subcommittee on Television: P. S. JONES, *Chairman;* C. B. ALLENDOERFER, R. C. FISHER.

COMMITTEE ON HIGH SCHOOL CONTESTS

Terms of office of members of this committee expire on August 31 of the last year of service listed.

C. T. SALKIND, *Chairman* (1966–69); W. H. FAGERSTROM, *Director*, L. C. DALTON (1965–68), J. M. EARL (1965–68), HARRISON GIVENS, JR. (1966–69), N. S. MENDELSOHN (1965–68), HANS SAGAN (1964–67), E. E. STROCK (1964–67).

COMMITTEE ON INSTITUTES

D. L. THOMSEN, JR., *Chairman* (1967–69); D. K. HARRISON (1966–68), MARK KAC (1967–69), JOSEPH LANDIN (1966–68), V. O. MCBRIEN (1966–69), E. R. MULLINS (1966–68).

COMMITTEE ON PUBLICATIONS

R. P. BOAS, *Chairman* (1967–69); JOSHUA BARLAZ (1967–69), H. S. M. COXETER (1965–67), MARY P. DOLCIANI (1966–68), DAVID GALE (1967–69), WALLACE GIVENS (1965–67), I. I. HIRSCHMAN, JR. (1966–68), IVAN NIVEN (1966–68), OLGA TAUSSKY (1966–68), ROY DUBISCH (1964–68), *ex officio*, H. M. GEHMAN (1963–67), *ex officio*, R. A. ROSENBAUM (1967–71), *ex officio*.

Subcommittee on Carus Monographs: IVAN NIVEN, *Chairman* (1966–68); R. P. BOAS (1967–69), OLGA TAUSSKY (1966–68).

Subcommittee on Ford Awards: R. P. BOAS, *Chairman* (1967–69), *ex officio:* C. W. CURTIS (1966–68), R. P. DILWORTH (1965–67).

Subcommittee on MAA Studies in Mathematics: WALLACE GIVENS, *Chairman* (1965–67); H. S. M. COXETER (1965–67), DAVID GALE (1967–69).

Subcommittee on Slaught Papers: I. I. HIRSCHMAN, JR., *Chairman* (1966–68); JOSHUA BARLAZ (1967–69), MARY P. DOLCIANI (1966–68), R. A. ROSENBAUM (1967–71), *ex officio*.

Subcommittee on Miscellaneous Publications: R. P. BOAS, *Chairman* (1967–69); H. L. ALDER (1966–69), H. M. GEHMAN (1966–67), all *ex officio.*

COMMITTEE ON SECONDARY SCHOOL LECTURERS

H. M. GELDER, *Chairman* (1965–67); H. M. BACON (1967–69), D. R. BEY (1965–67), B. H. BISSINGER (1966–68), J. N. EASTHAM (1967–69), H. V. HUNEKE (1965–67), J. H. WAHAB (1967–69).

COMMITTEE ON SECTIONS

L. E. MEHLENBACHER, *Chairman* (1967–70); E. M. BEESLEY (1965–68), W. T. FISHBACK (1967–70), M. E. MUNROE (1964–67), ARNOLD WENDT (1966–69), RAOUL HAILPERN (1963–67), *ex officio.*

COMMITTEE ON THE AWARD FOR DISTINGUISHED SERVICE TO MATHEMATICS

G. B. PRICE, *Chairman* (1965–67); W. L. DUREN, JR. (1967–69), H. M. GEHMAN (1966–68).

COMMITTEE ON THE CHAUVENET PRIZE

VICTOR KLEE, *Chairman* (1965–67); SAMUEL EILENBERG (1966–68), G. L. WEISS (1967–69).

COMMITTEE ON THE FUTURE ADMINISTRATIVE STRUCTURE OF THE ASSOCIATION

E. A. CAMERON, *Chairman;* R. C. BUCK, H. O. POLLAK, R. A. ROSENBAUM, A. W. TUCKER.

COMMITTEE ON THE PREPARATION OF A 50-YEAR HISTORY OF THE ASSOCIATION

K. O. MAY, *Chairman;* C. B. BOYER, D. J. STRUIK

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

R. E. GREENWOOD, *Chairman* (1965–67); J. H. MCKAY, *Director* (1965–67), N. D. KAZARINOFF (1966–68), LEO MOSER (1967–69).

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

RICHARD D. ANDERSON, *Chairman* (1965–67); DOROTHY L. BERNSTEIN (1967–69), L. W. COHEN (1966–68), M. L. CURTIS (1965–67), C. R. DEPRIMA (1965–67), M. D. DONSKER (1967–69), D. T. FINKBEINER II (1967–69), LEONARD GILLMAN (1965–67), D. B. GOODNER (1966–69), H. J. GREENBERG (1967–69), I. N. HERSTEIN (1967–69), M. GWENETH HUMPHREYS (1965–67), L. J. PAIGE (1965–67), ALEX ROSENBERG (1966–68), R. M. THRALL (1965–68), A. B. WILLCOX (1966–68), G. S. YOUNG (1965–68), LEO ZIPPIN (1967–69), E. G. BEGLE, *ex officio,* E. E. MOISE (1967–68), *ex officio.*

Advisory Group on Applications of Mathematics: M. D. DONSKER, *Chairman* (1967–68).

Advisory Group on Communications: A. B. WILLCOX, *Chairman* (1966–68); C. R. DEPRIMA (1966–67), R. M. THRALL (1966–67).

Panel on College Teacher Preparation: LEONARD GILLMAN, *Chairman* (1966–68); C. E. BURGESS (1966–68), D. W. BUSHAW (1966–68), D. E. CHRISTIE (1966–68), L. W. COHEN (1966–68), D. T. FINKBEINER II (1966–68), MEYER JERISON (1966–68), HERMAN MEYER (1966–68), ALEX ROSENBERG (1966–68), E. H. SPANIER (1966–68), J. H. WELLS (1966–68).

Panel on Computing: H. J. GREENBERG, *Chairman.*

Panel on Mathematics for the Life Sciences: R. M. THRALL, *Chairman* (1967–69).

Panel on Mathematics in Two-Year Colleges: D. B. GOODNER, *Chairman* (1966–69), JOSHUA BARLAZ (1966–69), L. J. DUNHAM (1966–68), J. N. EASTHAM (1966–67), M. GWENETH HUMPHREYS (1966–67), R. C. JAMES (1966–68), CAROL H. KIPPS (1966–68), B. E. MESERVE (1966–68), J. W. METTLER (1966–67), R. Z. NORMAN (1966–68), W. R. RICE (1966–69).

Panel on Statistics: H. O. POLLAK, *Chairman* (1967–69).

Panel on Teacher Training: G. S. YOUNG, *Chairman* (1967–68); E. G. BEGLE (1965–67), MARY FOLSOM (1966–67), C. E. HARDGROVE (1966–68), E. R. KOLCHIN (1966–68), GEORGE SPRINGER (1966–68), S. S. WILLOUGHBY (1966–68).

COMMITTEE TO STUDY THE REORGANIZATION OF ASSOCIATION JOURNALS

DOROTHY L. BERNSTEIN, *Chairman;* D. W. BLAKESLEE, D. E. CHRISTIE, C. R. PHELPS, G. S. YOUNG.

COMMITTEE ON VISITING LECTURERS

R. E. GASKELL, *Chairman* (1965–67); E. M. BEESLEY (1966–68), R. D. BOSWELL, JR. (1966–69), W. K. MOORE (1966–68), M. W. POWNALL (1965–67), J. R. WESSON (1966–67), W. L. WILLIAMS (1966–68).

JOINT COMMITTEE ON EMPLOYMENT OPPORTUNITIES

Terms of office of members of this committee expire on February 28 of the last year of service listed.

D. R. MORRISON, *Chairman* (1964–68, MAA); M. L. HENRIKSEN (1966–70, AMS), G. S. JONES (1966–67, SIAM).

JOINT COMMITTEE ON PLACES OF MEETINGS

G. L. WALKER, *Chairman;* H. L. ALDER, H. M. GEHMAN, EVERETT PITCHER, all *ex officio.*

NOMINATING COMMITTEE FOR 1967

R. H. BING, *Chairman;* P. T. BATEMAN, O. G. HARROLD.

EDITORIAL BOARDS OF THE ASSOCIATION

AMERICAN MATHEMATICAL MONTHLY (all terms expire December 31, 1971).

Editor: R. A. ROSENBAUM

Associate Editors: JOSHUA BARLAZ, J. A. BROWN, LEONARD CARLITZ, HASKELL COHEN, HOWARD EVES, HARLEY FLANDERS, RAOUL HAILPERN, M. S. KLAMKIN, R. C. LYNDON, A. P. MATTUCK, K. O. MAY, J. R. MAYOR, G. N. RANEY, GIAN-CARLO ROTA, E. P. STARKE, J. G. WENDEL, ALBERT WILANSKY.

MATHEMATICS MAGAZINE (all terms expire December 31, 1968).

Editor: ROY DUBISCH

Associate Editors: R. W. BAGLEY, D. B. DEKKER, RAOUL HAILPERN, R. E. HORTON, J. H. JORDAN, C. T. LONG, SAM PERLIS, RUTH B. RASMUSEN, H. E. REINHARDT, R. W. RITCHIE, J. M. SACHS, HANS SAGAN, C. T. SANDERS, D. E. THORO, L. M. WEINER.

REPRESENTATIVES OF THE ASSOCIATION

On the AAAS Cooperative Committee on the Teaching of Mathematics and Science:
K. O. MAY (1966–68)

On the American Council on Education:
H. L. ALDER, *ex officio*, E. E. MOISE, *ex officio*

On the Conference Board of the Mathematical Sciences:
H. L. ALDER, *ex officio*, E. E. MOISE, *ex officio.*

On the Council of the American Association for the Advancement of Science:
R. D. ANDERSON (1965–67), M. M. DAY (1966–68)
On the Governing Council of Mu Alpha Theta:
G. B. PRICE (1967–69)
On the National Council for Accreditation of Teacher Education:
G. S. YOUNG (November 1966–October 1968)
On the National Research Council:
R. H. BING (July 1, 1965–June 30, 1968)
On the U.S. Commission on Mathematical Instruction:
R. P. DILWORTH (July 1, 1966–June 30, 1970), LEONARD GILLMAN (July 1, 1965–June 30, 1969).

APRIL MEETING OF THE MINNESOTA SECTION

The Spring meeting of the Minnesota Section of the MAA was held at Macalester College, St. Paul, Minnesota on April 30, 1966. There were 108 persons in attendance including 87 members of the Association. Professor E. J. Camp of Macalester College presided over the morning sessions; Professor F. L. Wolf of Carleton College presided over the afternoon sessions. The sessions were held in the new Olin Hall of Science.

At the business meeting the following officers were elected: Chairman, Dale Varberg, Hamline University; Secretary-Treasurer, Rev. Walbert Kalinowski; Executive Committee, Frank Wolf, Carleton College; Ernest Stennes, St. Cloud State College; Sylvan Burgstahler, University of Minnesota at Duluth; E. J. Camp, Macalester College. It was decided to pay membership fees to the MAA for those students who did well in the Putnam Examination. A report on the results of the High School Examination was also given.

The program of the sessions was as follows:

1. *Some reminiscences of Raymond Clair Archibald*, by Sister M. Thomas à Kempis, College of St. Teresa.

2. *Convergence of series obtained by iteration*, by G. U. Brauer, University of Minnesota.

3. *Newton's method for transformations*, by A. W. Roberts, Macalester College.

4. *Perturbation of the integral operator with separable kernel*, by H. I. Freedman, University of Minnesota.

5. *An application of finite fields to a scheduling problem*, by Bert Fristedt, University of Minnesota.

6. *An explicit formula for* e^{At}, by R. B. Kirchner, Carleton College.

7. *Group theory and spin state quantum mechanics*, by J. W. Mentele, South Dakota State University.

8. *Composition and decomposition of polynomials*, by James Sitton and Peter Crescenzo, University of Minnesota.

9. *An uncountable collection of mutually singular Cantor functions*, by Edward Emerson, University of Minnesota.

10. *Stability in autonomous systems of differential equations*, by Lynn D. Maruska, Macalester College.

11. *Disconjugacy for a linear differential operator*, by R. M. Mathsen, Concordia College.

12. *Non-linear phenomena*, by W. S. Loud, University of Minnesota (by invitation).

13. *On a circle theorem of Descartes*, by Daniel Pedoe, University of Minnesota (by invitation).

14. *Panel discussion: The CUPM Report—a general curriculum in mathematics for colleges*, by Murray Braden, Macalester College (moderator) and H. L. Turrittin, University of Minnesota; H. M. Anderson, Gustavus Adolphus College; W. J. Thomsen, Moorhead State College; J. E. Lange, St. John's University (panelists).

W. C. KALINOWSKI, *Secretary-Treasurer*

NOVEMBER MEETING OF THE NEW JERSEY SECTION

The eleventh annual meeting of the New Jersey Section of the MAA was held at Rutgers—The State University, on November 12, 1966. Professor Joshua Barlaz, Chairman of the Section, presided at the morning session and F. A. Brooks, Jr., Senior Member of the Executive Committee, presided at the afternoon session. Seventy-seven persons attended the meeting including sixty-four members of MAA.

At the business meeting Vice-Provost and Dean A. E. Meder, Jr., of Rutgers—The State University, was elected Chairman, and Professor H. F. Trotter of Princeton University was elected Member at Large of the Executive Committee ('69). Reports were presented by F. A. Varrichio, Secretary-Treasurer, and F. A. Brooks, Jr.—High School Contest Committee.

The morning session was as follows:

1. *General Curriculum for Mathematics in Colleges*, by A. W. Tucker, Princeton University (by invitation).
Analysis and discussions of 1965 Report by CUPM.

2. *What is a Point?* by L. F. McAuley, Rutgers—The State University (by invitation).

The afternoon session was as follows:

1. *Moonlighting in Moscow*, by R. M. Cohn, C. C. Sims, and E. J. Taft, Rutgers—The State University (by invitation).
Comments and sidelights of International Congress of Mathematicians in Russia, August 1966.

2. Film: *The Kakeya Problem*, A. S. Besicovitch.

F. A. VARRICHIO, *Secretary-Treasurer*

MAA PUBLICATIONS

A new volume in the series of MAA Studies will be published during the summer of 1967: MAA Studies Volume 4, Studies in Global Geometry and Analysis, S. S. Chern, editor. It contains papers by Marston Morse, S. S. Chern, Harley Flanders, S. Kobayashi, Lamberto Cesari, L. A. Santalo.

Each individual member of MAA may purchase one copy of this book at the reduced price of $3.00. Orders accompanied by payment should be sent to the Buffalo office. Additional copies and copies for nonmembers may be purchased at $6.00 from Prentice-Hall, Inc., Englewood Cliffs, New Jersey 07631.

Effective July 1, 1967, the price to members for all Carus Monographs and MAA Studies Volumes 1, 2, and 3 will be increased to $3.00 per copy. The non-member price has been increased to $6.00. Orders from members received by the Buffalo office before July 1 will be filled at the current rate of $2.00 each.

An announcement listing all MAA publications with current and prospective prices is being mailed to all MAA members this spring. Additional copies of the announcement may be obtained from the Buffalo office.

REPORT OF THE TREASURER FOR THE YEAR 1966

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1966. The report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any MAA member who wishes a copy of the consolidated report of the Treasurer may obtain one by writing to the Buffalo office of the Association.

A member who wishes to remain anonymous has given MAA securities valued at about $5,000. This gift has been added to the so-called Anonymous Fund.

The general decrease in market values of securities during 1966 has resulted in decreases in nearly all of the special funds of the Association. Also the funds devoted to publications have been subject to fluctuations depending upon payments for printing.

During 1966 final agreement was reached with NSF about indirect costs for 1965 and preceding years. Appropriate adjustments have been made in the MAA financial records.

For several years the Current Fund has had a small deficit. These deficits and expected increases in expenditures during 1967 and 1968 have resulted in an increase in dues effective in 1968.

	January 1, 1966	*December 31, 1966*
ASSETS OF THE ASSOCIATION		
M & T Trust Company, checking account	$ 69,231	$ 18,254
M & T Trust Company, special account	113,468	25,970
M & T Trust Company, third account	2,388	1,406
Securities at market values	246,840	189,158
	$431,928	$234,789
FUNDS OF THE ASSOCIATION		
Current Fund	6,928	4,437
MATHEMATICS MAGAZINE	622	−238
Carus Fund	71,796	63,547
Chace Fund	5,886	−2,108
Houck Fund	9,954	9,086
Dunkel Fund	32,952	26,296
Anonymous Fund	4,655	9,312
Awards Fund	1,965	1,992
Greenwood Fund	8,350	8,350
General Fund	131,161	80,257
	$274,273	$200,934
High School Contests	5,636	3,161
CEM: Subcommittee on TV	2,290	1,945
1965 Cooperative Summer Seminar	4,303	668
1966 Cooperative Summer Seminar	29,568	701
NSF Grants Fund	113,468	25,970
NSF Contributions Fund	2,388	1,406
	$431,928	$234,789

CALENDAR OF FUTURE MEETINGS

Forty-eighth Summer Meeting, University of Toronto, Toronto, Ontario, Canada, August 28–30, 1967.

Fifty-first Annual Meeting, San Francisco, California, January 25–27, 1968.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1967.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1967.

INDIANA, Wabash College, Crawfordsville, May 13, 1967.

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METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, St. John's University, Collegeville, May 6, 1967.

MISSOURI

NEBRASKA, University of South Dakota, Vermillion, May 6, 1967.

NEW JERSEY

NORTHEASTERN, Mt. Allison University, Sackville, New Brunswick, June 23–24, 1967.

NORTHERN CALIFORNIA

OHIO

OKLAHOMA-ARKANSAS

PACIFIC NORTHWEST, University of Montana, Missoula, June 16–17, 1967.

PHILADELPHIA, University of Delaware, Newark, November 18, 1967.

ROCKY MOUNTAIN, Western State College of Colorado, Gunnison, May 12–13, 1967.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS

UPPER NEW YORK STATE, State University College, Plattsburgh, May 20, 1967.

WISCONSIN, St. Norbert College, DePere, May 6, 1967.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, New York, N.Y. December 26–31, 1967.

AMERICAN MATHEMATICAL SOCIETY, Toronto, Ontario, Canada, August 29–September 1, 1967.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Michigan State University, June 19–22, 1967.

ASSOCIATION FOR COMPUTING MACHINERY, Sheraton-Park, Washington, D. C., August 29–31, 1967.

ASSOCIATION FOR SYMBOLIC LOGIC, Chicago, May 4–5, 1967.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 23–25, 1967.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, McGill University, Montreal, August 21–24, 1967.

OPERATIONS RESEARCH SOCIETY OF AMERICA, New York Hilton Hotel, May 31–June 2, 1967.

PI MU EPSILON, Toronto, Ontario, Canada, August 29–30, 1967.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D. C., June 12–15, 1967 (Symposiums on applied probability and fluid dynamics).

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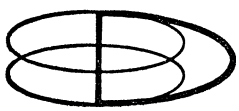
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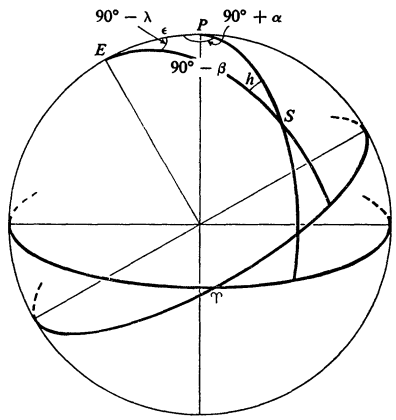
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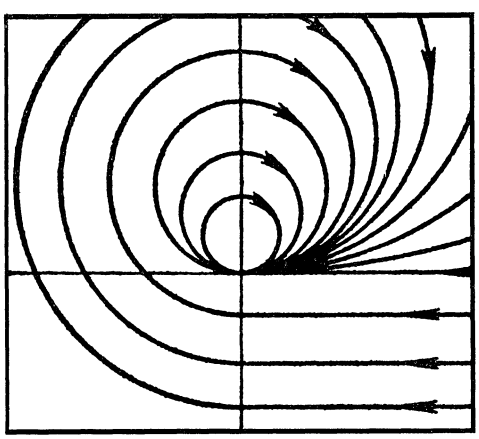
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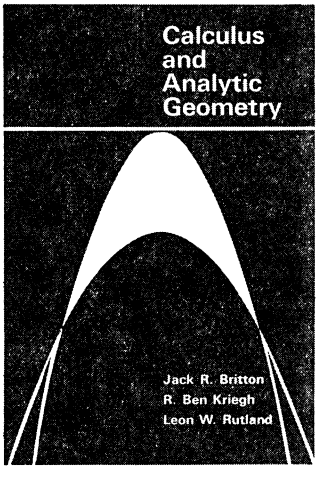
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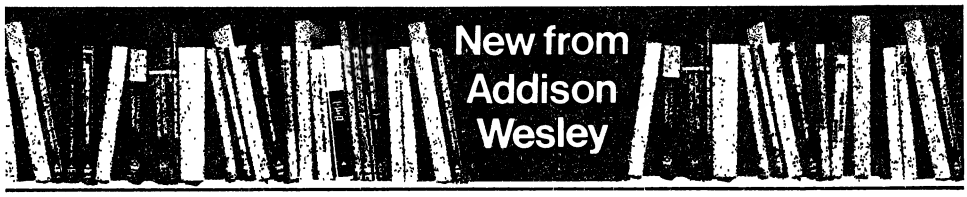
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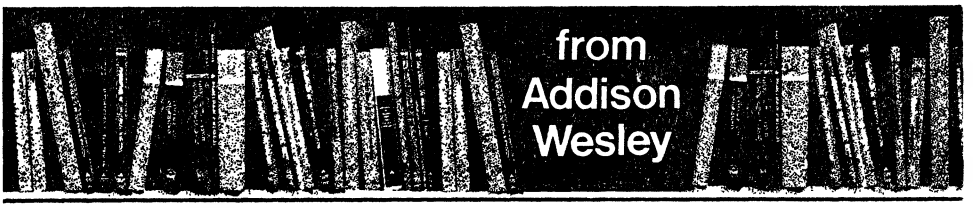
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THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 74 NUMBER 5

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MAY 1967

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$6.00 (\$8.00 in 1968). For nonmembers the subscription price is \$10.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

ON THE FUNDAMENTAL THEOREM OF ALGEBRA

HANS ZASSENHAUS, Ohio State University

Introduction. In the year 1799 C. F. Gauss [4] gave the first formal proof of the theorem that every nonconstant polynomial with real coefficients can be factored into a product of linear factors and quadratic factors. A constructive proof based on arguments of a purely algebraic nature and on assumptions about the real number field that were stated in purely algebraic terms (though the proof of these assumptions would require analytic methods) was first given by O. Perron [6]. In this paper Perron's method is further developed towards an algorithmic routine for solving algebraic equations with complex coefficients.

1. Really closed fields. The real number field has the following properties:

1.1. The negative of a nonsquare is a square.

1.2. The sum of squares is zero only if each summand is zero.

1.3. An algebraic equation of odd degree has a real solution.

The first two properties derive from the existence of an algebraic ordering of the real number field and a square root of every positive real number.

The last property follows from an application of the intermediate-value theorem of analysis to a polynomial: $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$ of odd degree n with real coefficients, in view of the inequalities:

$$1.4 \qquad f\left(1+\sum_{i=1}^{n}|a_i|\right)>0>f\left(-\left(1+\sum_{i=1}^{n}|a_i|\right)\right).$$

Fields with the properties 1.1, 1.2, 1.3 are said to be *really closed.* E.g. the real number field is really closed. For a really closed field F an algebraic ordering is given by the rule:

1.5 The element a of F is greater than the element b of F if and only if the difference element $a-b$ is a nonzero square element of F. This implies the positivity concept:

1.6 An element of F is positive if and only if it is a nonzero square element of F.

In order to deduce the ordering properties of F from 1.5 we have to verify the positivity rules:

1.7 The negative of any nonpositive nonzero element is positive.

1.8 The sum and product of two positive elements are positive.

Indeed, if a is non positive and nonzero, then a is a non square of F. It follows from 1.1 that $-a$ is a square element of F so that $-a$ is positive.

If a and b are positive elements of F then the equations

$$a=\xi^2, \qquad b=\eta^2$$

are solvable by non zero elements ξ, η of F. From 1.2 it follows that $a+b$ does not vanish. If there would hold an equation: $-(a+b)=\zeta^2$ in F then

$$\xi^2+\eta^2+\zeta^2=0$$

would hold contrary to 1.2.

Thus $a+b$ is a square element of F so that the sum of two positive elements of F always is positive. Since,

$$ab = \xi^2\eta^2 = (\xi\eta)^2 \neq 0$$

the product of two positive elements is positive.

Clearly, for any really closed field only one positivity concept exists.

2. Ordered division rings. (See [1], [2], [3], [5], [7]). For a unital ring D with positivity concept satisfying 1.7, 1.8 we define an algebraic ordering by declaring the relation

$$a > b \text{ (or: } b < a)$$

to mean that the difference of the elements a, b of D is positive.

This definition implies the rules customarily demanded of an algebraic ordering relation:

2.1 (trichotomy). For any two elements a, b of R, there holds one and only one of the three relations:

$$a > b, \qquad a = b, \qquad b > a.$$

2.2 (transitivity). If $a>b$, $b>c$ then $a>c$.

2.3 If $a>b$, $c>d$ then $a+c>b+d$ and $ac+bd>ad+bc$.

Conversely, if $a>$relation is defined in D which satisfies 2.1, 2.3 then the relation $a>0$ defines a positivity concept satisfying 1.7 and 1.8, from which the $>$relation can be derived as was done above.

The derived relation:

$$a \geqq b \text{ (or: } b \leqq a)$$

meaning that either a is greater than b or a is equal to b (or: not $a<b$) and the functions:

$$\operatorname{sign} a = \begin{cases} 1 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -1 & \text{if } a < 0 \end{cases}$$

$$|a| = a \cdot \operatorname{sign} a = (\operatorname{sign} a) \cdot a$$

satisfy the basic rules:

$$a \geqq a, \qquad a \geqq b \text{ and } b \geqq a \Rightarrow a = b, \qquad a \geqq b \text{ and } b \geqq c \Rightarrow a \geqq c,$$

$$a \geqq b \text{ and } c \geqq d \Rightarrow a + c \geqq b + d, \qquad ab + cd \geqq ad + cb$$

$$\operatorname{sign}(ab) = \operatorname{sign} a \cdot \operatorname{sign} b$$

$$|a| \cdot \operatorname{sign} a = (\operatorname{sign} a) \cdot |a| = a$$

$$|a| = |-a| \geqq 0, \qquad |a| = 0 \Leftrightarrow a = 0,$$

$$|a+b| \leqq |a| + |b|, \qquad |a-b| \geqq ||a| - |b||, \qquad |ab| = |a| \cdot |b|.$$

The positive elements of an ordered division ring D form a half-ring H (according to [8], p. 95, a halfring is a nonempty subset of a ring that is closed under addition and multiplication,) with the following properties:

2.4. The halfring H contains the square of every non zero element of the subdivision ring D_H generated by H.

2.5. The halfring H is a maximal subhalfring of D not containing the zero element of D.

Conversely, every halfring H of D satisfying 2.4 and 2.5 is associated with a positivity concept of D_H satisfying 1.7, 1.8.

Indeed, for any two non zero elements u, v of D we have the identity:

$$2.6 \qquad uvu^{-1}v^{-1} = (uv)^2(v^{-1}u^{-1}v)^2(v^{-1})^2$$

so that

$$2.7 \qquad uvu^{-1}v^{-1} \in H \quad \text{if } 0 \neq u \in D_H,\ 0 \neq v \in D_H.$$

Thus $uh = h(h^{-1}u\ hu^{-1})u \in Hu$ if $0 \neq u \in D_H$, $h \in H$; hence $hu \in uH$, $uH \subseteq Hu$. Similarly $Hu \subseteq uH$; hence $uH = Hu (u \in D_H)$.

For h of H we have

$$H = h^{-1}hH \subseteq h^{-1}H, \qquad h^{-1} = h^{-1}1^2 \in h^{-1}H,$$
$$h^{-1}H + h^{-1}H \subseteq h^{-1}H,$$
$$h^{-1}Hh^{-1}H = h^{-1}(Hh^{-1})H = h^{-1}h^{-1}HH \subseteq H \subseteq h^{-1}H,$$

hence $h^{-1}H$ is a halfring of D_H containing H, but not zero. Since $h^{-1}H$ contains every non zero square element of $D_H = D_{h^{-1}H}$ it follows from the maximal property of H that $h^{-1}H = H$, hence $h^{-1} \in H$.

If the non zero element c of D_H is not contained in H then $-c$ belongs to H. Indeed,

$$c = c \cdot 1^2 \in cH \subseteq \hat{H} = H \cup cH \cup (H + cH)$$
$$(H + cH) + (H + cH) \subseteq H + cH$$
$$HcH = cHH \subseteq cH,$$
$$cHcH = c(Hc)H = c(cH)H = c^2HH \subseteq H$$
$$(H + cH)(H + cH) \subseteq H + cH.$$

Hence, $\hat{H}$ is a halfring of D_H properly containing H, so $\hat{H}$ contains zero because of the maximal property of H.

There are elements h_1, h_2 of H for which $h_1 + ch_2 = 0$. Hence,

$$-c = h_1h_2^{-1} \in HH = H.$$

THEOREM 1. *A division ring D can be algebraically ordered if and only if the sum of finitely many finite products of square elements is zero only if all summands are zero.* (What happens if one makes the weaker assumption that the sum of finitely many squares can be zero only in the trivial way?)

Proof. That the condition is necessary was shown above. Let it be satisfied.

The set H_0 of all sums of finitely many finite products of non zero square elements of D does not contain zero. H_0 is a halfring. By Zorn's lemma [9] H_0 is contained in a maximal halfring H of D not containing zero. Hence, $2=1^2+1^2\in H_0\subseteq H$, $2\neq 0$.

Since for any element u of D we have $u=\frac{1}{2}((u+1)^2-u^2)\in D_H$ it follows that $D_H=D$ and D has indeed the algebraic ordering which is given by

2.8. $a>b$ if and only if $a-b$ is in H.

THEOREM 2. *If the field F is algebraically ordered and if F is contained in an algebraically closed field extension Ω, then there is a really closed subfield Φ of Ω such that*

2.9. *every positive element of F is a square element of Φ.*

2.10. *Ω is algebraic over Φ.*

Proof. By Zorn's lemma [9] there is a maximal halfring H of Ω containing all positive elements of F and every non zero square element of the subfield Φ generated by H, but not zero.

We have shown above that the given ordering of F can be extended to an algebraic ordering of the field Φ such that H is the halfring of the positive elements.

If there were an element ξ of Ω that would not be algebraic over Φ then the halfring formed by the sums of finitely many nonzero square elements of $\Phi(\xi)$ with coefficients in H would be larger than H and it would contain all nonzero square elements of $\Phi(\xi)$. Because of the maximal property of H it would contain zero; hence there would be an equation.

$$0=\sum_{i=1}^{n} h_i\left(\frac{P_i(\xi)}{N(\xi)}\right)^2 \quad (h_i\in H)$$

where $N(x)$, $P_1(x)$, $\cdots$, $P_n(x)$ are non zero polynomials of $\Phi[x]$. Let m be the maximum degree of the polynomials $P_1(x)$, $\cdots$, $P_n(x)$ and let a_i be the coefficient of x^m in $P_i(x)$. Then

$$0=\sum_{i=1}^{n} h_iP_i(\xi)^2=\sum_{i=1}^{n} h_iP_i(x)^2=\sum_{i=1}^{n} h_ia_i^2,$$

contrary to the construction of H. Hence Ω is algebraic over Φ.

For every element u of H the equation $u=\xi^2$ is solvable in Ω. If ξ would not belong to Φ then the elements $a+b\xi$ with a, b contained in Φ such that $a\geqq 0$, $b\geqq 0$, $a+b>0$ forms a halfring $\overline{H}$ of Ω larger than H; it would be contained in the halfring $\overline{H}\,\overline{H}^{-1}$ containing all square elements of $\Phi(\xi)$, but not zero, which is a contradiction. (It is clear that $\overline{H}$ is a halfring larger than H, not containing zero and that $\overline{H}\,\overline{H}^{-1}$ has the same properties. Moreover, if a, $b\in\Phi$, $a\geqq 0$, $b<0$, $a^2>b^2u$, then both $-bu+a\xi$ and $(a+b\xi)(-bu+a\xi)=(a^2-b^2u)\xi$ belong to $\overline{H}$; hence $a+b\xi$ belongs to $\overline{H}\,\overline{H}^{-1}$. Now, if c, $d\in\Phi$ and $c+d\xi\neq 0$, then $(c+d\xi)^2\neq 0$,

then $(c+d\xi)^2=a+b\xi$ when $a=c^2+d^2u\in H$, and $b=2cd$, $a^2-b^2u=(c^2-d^2u)^2>0$ so that indeed $(c+d\xi)^2$ belongs to $\overline{H}\,\overline{H}^{-1}$.) Hence every element of H is a square element of Φ and 1.1, 1.2 are satisfied by Φ.

If there were a polynomial $f(x)$ of odd degree with coefficients in Φ for which the equation

$$2.11 \qquad -1 \equiv \sum_{i=1}^{s} f_i(x)^2 \,(f(x))$$

could be solved by polynomials $f_1(x), \cdots, f_s(x)$ with coefficients in Φ, then we could find among these $f(x)$ one of minimal degree $2n+1$. Upon substitution of the least remainders with regard to division by $f(x)$ it would follow that a relation 2.11 would obtain in which the maximum degree k of the polynomials $f_1(x), \cdots, f_s(x)$ would be not greater than $2n$.

Moreover, since Φ has an algebraic ordering we conclude that $n>0$ and that the coefficient of x^{2k} in

$$\sum_{i=1}^{s} f_i(x)^2$$

does not vanish, therefore the polynomial $1+\sum_{i=1}^{s} f_i(x)^2=f(x)g(x)$ is of degree $2k$ and consequently the polynomial $g(x)$ is of odd degree less than $2n+1$. But this is impossible in view of the congruence:

$$-1 = \sum_{i=1}^{s} f_i(x)^2 \,(g(x))$$

and the minimal property of $f(x)$.

Hence, for a polynomial $f(x)$ of odd degree over Φ a relation of the form 2.11 never holds. On the other hand there must be a polynomial $g(x)$ of odd degree among the irreducible divisors of $f(x)$ in $\Phi[x]$. Since Ω is algebraically closed, there is a root ξ of $g(x)$ in Ω. As was shown above, no sum of square elements of the extension field $\Phi(\xi)$ can be equal to -1. This implies that the sums of finitely many nonzero square elements of $\Phi(\xi)$ form a halfring $\overline{H}$ containing H as well as all nonzero square elements of $\Phi(\xi)$. Because of the maximal property of H we conclude that $\overline{H}$ coincides with H, $\Phi(\xi)=\Phi$, $\xi\in\Phi$, Φ is really closed.

3. The ring extensions associated with an equation. Suppose that the polynomial

3.1. $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$ with coefficients $a_0=1, a_1, a_2, \cdots, a_n$ in the commutative unital ring v has the root ξ in v then

$$f(x) = f(x) - f(\xi) = \sum_{h=0}^{n} a_{n-h}(x^h - \xi^h)$$

$$= (x - \xi) \sum_{h=1}^{n} a_{n-h}\left(\sum_{j=0}^{h-1} \xi^{h-j-1}x^j \right)$$

$$= (x - \xi)\frac{f}{x - \xi}(x)$$

where

$$3.2 \qquad \frac{f}{x - \xi}(x) = \sum_{j=0}^{n-1}\left(\sum_{h=0}^{n-1-j} a_{n-j-h-1}\xi^h\right)x^j.$$

Hence every root of the polynomial 3.2 is also a root of $f(x)$. Moreover, if v is an integral domain then every root of $f(x)$ that is not equal to ξ is also a root of 3.2. And if for some polynomial $g(x)$ of degree $n-1$ over v the equation $f(x) = (x-\xi)g(x)$ holds then $g(x) = f(x)/(x-\xi)$. This is because the equation for $g(x)$ allows us to determine recursively the coefficients of $g(x)$.

The preceding remarks motivate the following formal construction.

Let us denote by $v[\xi; f]$ the v-module with basis $1, \xi, \cdots, \xi^{n-1}$ over v. The rule

$$\begin{aligned} \xi_\tau(1) &= \xi \\ \xi_\tau(\xi^i) &= \xi^{i+1} \quad (0 < i < n-1) \\ 3.3 \qquad \xi_\tau(\xi^{n-1}) &= -a_1\xi^{n-1} - a_2\xi^{n-2} - \cdots - a_n 1 \\ \xi_\tau\left(\sum_{j=0}^{n-1} b_j\xi^j\right) &= \sum_{j=0}^{n-1} b_j\xi_\tau(\xi^j) \quad (b_j \in v,\ 0 \leqq j < n) \end{aligned}$$

establishes a v-endomorphism ξ_τ of $v[\xi; f]$ satisfying the equation:

$$3.4 \qquad f(\xi_\tau) = 0$$

which is an equation for ξ_τ over v of minimal degree.

We define a multiplication on $v[\xi; f]$ by the rule:

$$\left(\sum_{h=0}^{n-1} k_h\xi^h\right)\eta = \sum_{h=0}^{n-1} k_h\xi_\tau^h(\eta)\,(\eta \in v[\xi; f])$$

which turns the v-module $v[\xi; f]$ into a commutative unital ring extension of v when the mapping $b \to b1 + 0\xi^1 + \cdots + 0\xi^{n-1}\,(b \in v)$ provides the embedding isomorphism of v into $v[\xi; f]$. The unit element of v also is the unit element of $v[\xi; f]$. Together with

$$\xi = 01 + 1\xi^1 + 0\xi^2 + \cdots + 0\xi^{n-1}$$

the ring v generates the v-ring $v[\xi; f]$ with v-basis $1, \xi, \xi^2, \cdots, \xi^{n-1}$ such that $f(\xi) = 0$. If in a commutative unital ring extension v^* of v for some element η of v the equation $f(\eta) = 0$ holds, then the mapping

$$\sum_{j=0}^{n-1} b_j\xi^j \to \sum_{j=0}^{n-1} b_j\eta^j (b_j \in v;\ 0 \leqq j < n)$$

provides a v-homomorphism of $v[\xi; f]$ into v^*.

Extending the preceding construction we define the commutative unital ring extension $v[\xi_1, \cdots, \xi_n; f]$ of v with $n!$ basis elements

$$\xi_1^{\nu_1}\xi_2^{\nu_2}\cdots\xi_n^{\nu_n} \quad (0 \leqq \nu_j < n-j; j=1, 2, \cdots, n)$$

over v as follows:

$$3.5 \qquad v[\xi_1, \cdots, \xi_n; f] = (v[\xi_1; f])[\xi_2, \cdots, \xi_n; f/(x-\xi_1)].$$

It follows that in $v[\xi_1, \cdots, \xi_n; f]$ there holds the factorization:

$$3.6 \qquad f(x) = (x-\xi_1)(x-\xi_2)\cdots(x-\xi_n)$$

of $f(x)$ into n linear factors.

THEOREM. 3 *The $n!$ permutations of the n distinct roots $\xi_1, \xi_2, \cdots, \xi_n$ of f determine automorphisms of $v[\xi_1, \cdots, \xi_n; f]$ such that the automorphism induced by the permutation π maps the polynomial expression $P(\xi_1, \xi_2, \cdots, \xi_n)$ on $P(\pi\xi_1, \cdots, \pi\xi_n)$ for any polynomial $P(x_1, \cdots, x_n)$. These $n!$ automorphisms form a group γ_n such that every element of v is fixed by every member of γ_n. The elements of v are the only elements of the ring $v[\xi_1, \cdots, \xi_n; f]$ that are fixed by γ_n.*

Proof. This is clear if $n=1$. Apply induction over n. Let $n>1$. The subset $\bar{v}$ of all elements of the ring extension that are fixed by γ_n is a subring containing v. Because of 3.5 and the induction assumption it follows that $\bar{v}$ is contained in $v[\xi_1]$. Applying a permutation automorphism interchanging ξ_1, ξ_2 to an element of $v[\xi_1]$ that is not contained in v, we obtain an element of $v[\xi_1]$ $[\xi_2]$ which is not contained in $v[\xi_1]$. Hence the theorem.

COROLLARY. (Theorem on symmetric functions.) *Let $R=v[x_1, x_2, \cdots, x_n]$ be the polynomial ring in n commuting independent variables $x_1, x_2, \cdots, x_n$ over v.* Let $s_i = \sum x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_i} \mid 1 \leqq \alpha_1 < \cdots < \alpha_i \leqq n$ *be the i-th basic symmetric function $(1 \leqq i \leqq n)$. Then every polynomial in $x_1, x_2, \cdots, x_n$ over v that is fixed by all variable permutations (symmetric polynomial in $x_1, x_2, \cdots, x_n$) is equal to a polynomial in $s_1, s_2, \cdots, s_n$ over v.*

Proof. We note that, for the polynomial $f(x) = x^n - s_1x^{n-1} + s_2x^{n-2} - \cdots + (-1)^n s_n$,

$$3.7 \qquad f(x) = (x-x_1)(x-x_2)\cdots(x-x_n),$$

$$3.8 \qquad f(x_i) = 0 \quad (1 \leqq i \leqq n).$$

If v coincides with the rational integer ring $\mathbb{Z}$, then R is an integral domain. Hence

$$3.9 \qquad \frac{f(x_2)}{x-x_1} = 0.$$

Using the homomorphism of $\mathbb{Z}[x_1, x_2, \cdots, x_n]$ into R which sends 1 into 1, x_i into $x_i (1 \leqq i \leqq n)$ we find that 3.9 always holds. By induction over n we conclude that there is a homomorphic mapping ψ of the ring extension $T = S[\xi_1 \cdots, \xi_n; f]$ over the subring S of R generated by v and $s_1, s_2, \cdots, s_n$ which maps ξ_i onto x_i. Clearly T is generated by v and $\xi_1, \xi_2, \cdots, \xi_n$, hence there is a homomorphism σ of R onto T over v mapping x_i onto $\xi_i (1 \leqq i \leqq n)$. We have $\sigma\psi = 1_T$, $\psi\sigma = 1_R$, hence σ, ψ are isomorphisms over v. Now the corollary follows from Theorem 3.

Let $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_n$ be another set of n independent commuting variables over v and let $\bar{f}(x) = x^n - \bar{s}_1 x^{n-1} + \bar{s}_2 x^{n-2} - \cdots + (-1)^n \bar{s}_n$.

There is a homomorphism ϕ of the ring extension $\overline{T} = \overline{S}[\bar{\xi}_1, \cdots, \bar{\xi}_n; \bar{f}]$ of $\overline{S} = v[\bar{s}_1, \cdots, \bar{s}_n]$ onto T over v which maps $\bar{\xi}_i$ onto $\xi_i (1 \leqq i \leqq n)$ and hence $\bar{s}_i$ onto s_i. On the other hand, there is a homomorphism κ of R onto $\overline{T}$ over v mapping x_i onto $\xi_i (1 \leqq i \leqq n)$ and we have: $\kappa\psi\phi = 1_{\overline{T}}$, $\phi\kappa\psi = 1_T$ therefore ϕ, $\kappa\psi$ are isomorphisms over v. It follows that the homomorphic mapping of $\overline{S}$ onto S over v which maps $\bar{s}_i$ onto $s_i (1 \leqq i \leqq n)$is an isomorphism. In other words the basic symmetric functions are independent over v.

From the theorem on symmetric functions it follows that the coefficients of the polynomials

$$S_j(f)(x) = \prod_{1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_j \leqq n} (x - (\xi_{\alpha_1} + \xi_{\alpha_2} + \cdots + \xi_{\alpha_j}))$$

of degree $\binom{n}{j}$ are in v. Moreover, if v is an integral domain in which the factorization $f(x) = \prod_{j=1}^{n} (x - \eta_j)$ obtains then the mapping of ξ_j onto $\eta_j (1 \leqq j \leqq n)$ can be extended in precisely one way to a homomorphism of $v[\xi_1, \cdots, \xi_n; f]$ onto v over v, and the equations $S_j(f)(\eta_{\gamma_1} + \cdots + \eta_{\gamma_j}) = 0$ hold whenever $1 \leqq \gamma_1 < \gamma_2 < \cdots < \gamma_j \leqq n$.

Assume that v is a field and that $S_2(f)$ has the root ξ in v. Let $d(x)$ be the greatest common divisor of the polynomials $f(x)$ and $f(\xi - x)$ in $v[x]$ with leading coefficient 1.

There holds an equation $d(x) = A(x)f(x) + B(x)f(\xi - x)$ with polynomials $A(x)$, $B(x)$ in $v[x]$. Hence in $v[\xi_1, \cdots, \xi_n; f]$:

$$d(x) = A(x)f(x) + B(x)f(\xi_1 + \xi_2 - x) + (\xi - \xi_1 - \xi_2)g(x, \xi_1 + \xi_2)$$

where $g(x, y)$ is a polynomial in two variables x, y over v. Upon substitution of ξ_2: $d(\xi_2) = (\xi - \xi_1 - \xi_2)g(\xi_2, \xi_1 + \xi_2)$.

If $d(x)$ would be a nonzero constant then $d(x) = 1 = d(\xi_2)$, hence $\xi - \xi_1 - \xi_2$ would be invertible in $v[\xi_1, \cdots, \xi_n; f]$. The same would apply to $\xi - \xi_i - \xi_k$ $(1 \leqq i < k \leqq n)$ and hence to the product.

But this product is equal to $S_2(f)(\xi)$ which is zero, a contradiction. Hence, $d(x)$ is not constant.

4. The fundamental theorem of algebra.

THEOREM 4. *Let F be a really closed field. The field extension E formed by the symbols $a + bi$ $(a, b \in F)$ with the operational rules*

$$
\begin{aligned}
&a + bi = c + di \Leftrightarrow a = c,\ b = d \\
&(a + bi) + (c + di) = (a + c) + (b + d)i \\
&(a + bi)(c + di) = (ac - bd) + (ad + bc)i \\
&(a, b, c, d \in F)
\end{aligned} \tag{4.1}
$$

is algebraically closed.

Proof. The field property of E is shown in the customary manner. The zero element of E is the symbol $0+0i$, the unit element of E is the symbol $1+0i$, the inverse of $a+bi$ is the symbol

$$\frac{a}{a^2 + b^2} + \left(\frac{-b}{a^2 + b^2}\right)i$$

provided not both of the elements a, b of F are zero. The mapping of a onto $a+0i$ provides an embedding isomorphism of F into E such that upon identification of a and $a+0i$ the field F becomes a subfield of E. The symbol $0+1i$ generates the quadratic extension E over F. Since $a+bi=(a+0i)+(b+0i)(0+1i)$, we are entitled to denote the symbol $0+1i$ by i, so that $a+bi$ is the actual sum of the element a of F and the product of the element b of F by i. The element i is a root of the irreducible quadratic equation:

$$i^2 + 1 = 0 \tag{4.2}$$

over F the other root being $-i$. Hence the complex conjugate mapping:

$$a + bi \to \overline{a + bi} = a - bi (a, b \in F) \tag{4.3}$$

establishes an involutoric automorphism of E over F. An element γ of E is in F if and only if its complex conjugate $\bar{\gamma}$ coincides with γ. The automorphism 4.3 is extended to an involutoric automorphism:

$$f(x) \to \bar{f}(x) \tag{4.4}$$

of $E[x]$ over $F[x]$ by setting

$$\bar{f}(x) = \bar{a}_0 x^n + \bar{a}_1 x^{n-1} + \cdots + \bar{a}_n \tag{4.5}$$

if

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n (a_0, a_1, \cdots, a_n \in E). \tag{4.6}$$

The polynomial 4.6 lies in $F[x]$ if it coincides with its complex conjugate polynomial $\bar{f}(x)$.

Let us further mention that for every element $a+bi$ of $E(a, b \in F)$ the equation

$$a + bi = (\xi + \eta i)^2 (\xi, \eta \in F) \tag{4.7}$$

is solved in E by:

$$\xi = \left| \sqrt{1/2(a + \left| \sqrt{a^2 + b^2} \right|)} \right|$$

4.8

$$\eta = (\operatorname{sign} b) \cdot \left| \sqrt{1/2(-a + \left| \sqrt{a^2 + b^2} \right|)} \right|.$$

Our task is to find a root in E for every polynomial

4.9 $$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$$

with coefficients $a_1, a_2, \cdots, a_n$ in E.

Let $n = 2^s n'$ when $0 \leqq s$, $n' \equiv 1(2)$.

If it would not always be possible to find a root in E for a given nonconstant polynomial, then there would be polynomials 4.9 with minimum value of s for which no root could be found in E and among those polynomials there would be a polynomial 4.9 with minimum value of n'.

If $s = 0$ and if all coefficients $a_1, a_2, \cdots, a_n$ are real then by assumption about F we can find a root of $f(x)$ in F, a contradiction.

If there is a root η of $S_2(f)$ in E then the greatest common divisor $d(x)$ of $f(x)$ and of $f(\eta - x)$ can be formed in $E[x]$. It is not constant as we have seen previously.

There holds an equation $f(x) = d(x)e(x)$ in $E[x]$ so that every root of $d(x)$ or $e(x)$ also is a root of $f(x)$. Since there is no root of $f(x)$ in E the same applies to the polynomials $d(x)$, $e(x)$. Because of the minimal property of $f(x)$ it follows that $d(x)$ has the same degree as $f(x)$ whereas $e(x)$ is a nonzero constant. In other words $f(x) = f(\eta - x)$ if $s > 0$. Or, setting $y = x - \eta/2$ we have

$$f(x) = g(y), \qquad g(y) = g(-y), \qquad g(y) = h(y^2), \qquad f(x) = h((x - \eta/2)^2)$$

when h is a polynomial of degree $n/2$ with coefficients in E.

It follows from this argument that $S_2(f)$ has no root in E if n is even. On the other hand, the degree of $S_2(f)$ is not divisible by 2^s, hence we can find a root of $S_2(f)(x)$ in E due to the minimal property of $f(x)$.

These arguments show that n is odd and that not all coefficients of $f(x)$ are real. But the degree of $f_1(x) = f(x)\bar{f}(x)$ is $2n$, the degree of $S_2(f_1)$ is the odd number $n(2n-1)$. Since $f_1(x)$ has real coefficients, also $S_2(f_1)$ has real coefficients. Therefore, $S_2(f_1)$ has a real root and by the argument given above $f_1(x)$ has a root ξ in E. Since $0 = f_1(\xi) = f(\xi)\bar{f}(\xi)$, but by assumption $0 \neq f(\xi)$ it follows that $0 = \bar{f}(\xi)$. Forming the complex conjugate of this equation we obtain $0 = f(\bar{\xi})$ so that $f(x)$ does have the root $\bar{\xi}$ in E which is a contradiction.

Thus we have established Theorem 4.

This existence proof can be reformulated so as to yield an algorithm:

I. If $s = 0$, $f(x) \in F[x]$ then we know by assumption how to find a root of f in F.

II. If the algorithm is defined already for polynomials of odd degree less than $2n+1$ with coefficients in E and if $f(x)$ is a polynomial of degree $2n+1$ with coefficients in E, then solve $S_2(f\bar{f})(\eta) = 0$ as in I. Form $d(x)$ as above. If $[d] \equiv 1$ (2) then $[d] < 2n+1$; apply the algorithm to find a root of d in E.

It turns out to be a root of f too. If $[d]\equiv 0(2)$, then the algorithm can be applied to $e(x)=(f/d)(x)$ and provides a root of e which turns out to be also a root of f.

Thus, the algorithm is obtained for solving in E equations of odd degree over E.

III. If $s>0$ and if the algorithm is defined for solving in E equations with coefficients in E such that either the degree is not divisible by 2^s or it is smaller than the degree of f and not divisible by 2^{s+1}, then apply the algorithm to the equation $S_2(f)(\eta)=0$ and form $d(x)$ as above. Apply the algorithm to $d(x)$ in case $2^s \nmid [d]$ or $2^{s+1} \nmid [d]$ and $[d]<[f]$. Apply the algorithm to f/d in case $[d]<[f]$, $2^{s+1} \mid [d]$. In either case, a root of f is obtained. If $[d]=[f]$, then form h as above. Apply the algorithm to h, to find a root ξ of h. As above we obtain the root $\eta/2+\sqrt{\xi}$ of f for finding a root.

However, the degrees of the auxiliary equations involved may become very large. E.g. for $n=8$ we may find the degrees 8,

$$\binom{8}{2} = 28, \qquad \binom{28}{2} = 378, \qquad \binom{378}{2} = 71253.$$

But let us note that the binomial coefficient

$$\binom{n}{2^i} = \frac{2^s n' \cdot (n-1) \cdot \cdots \cdot (n-2^i+1)}{2^i \cdot 1 \cdot \cdots \cdot (2^i-1)}$$

is divisible by 2^{s-i}, but not divisible by 2^{s-i+1} for $i=0, 1, 2, \cdots s$. Hence the same arguments also apply to the auxiliary equations:

$$f(x) = 0, \qquad S_2(f)(x) = 0, \qquad S_4(f)(x) = 0, \cdots, S_{2^{s-i}}(f)(x) = 0, \cdots$$
$$S_{2^s}(f)(x) = 0.$$

Note that polynomial $S_{2^{s-1}}(f)$ is symmetric about $-(1/2)a_1$ in case $n=2^s$. Hence in that case it is of the form $g(x+(1/2)a_1)^2$ where $g(x)$ is of odd degree.

In any event we obtain considerable economy; e.g. for $n=8$ we have to form and to solve equations of degrees 8, 28, 2, 35.

Assuming that there exists an algorithm for finding the real roots of a polynomial 4.9. with real coefficients we extend it as follows to an algorithm for finding all the roots of $f(x)$

(1) If $\gcd(f, df/dx)=d(x)$ is not constant, let $(f/d)(x)=e_0(x)$, $\gcd(e_0, (df/dx))=e_1(x)$ not constant, $(e_0/e_1)(x)=f_1(x)$, $\gcd(e_{j-1}, (d^jf/dx^j))=e_j(x)$ not constant, $(e_{j-1}/e_j)(x)=f_j(x)$

$$\gcd\left(e_j, \frac{d^{j+1}f}{dx^{j+1}}\right) = 1, \quad (j>0).$$

Hence $f(x)=f_1(x)f_2(x)^2 \cdots f_j(x)^j f_{j+1}(x)^{j+1}$ when $f_1(x), f_2(x), \cdots, f_j(x), f_{j+1}(x)$ are mutually prime, separable polynomials.

The task is now to find the roots of these polynomials.

(2) $\gcd(f, df/dx)=1$, $f(x)$ has the real roots $\alpha_1, \alpha_2, \cdots, \alpha_r$, $f(x)=(x-\alpha_1)$

$(x-\alpha_2)\cdots(x-\alpha_r)g(x)$, $g(x)$ is separable and has no real root. The task is to find the roots of $g(x)$.

(3) $f(x)$ is separable and has no real root.

Find the real roots of $S_2(f(x))$ say $\beta_1<\beta_2<\cdots<\beta_\rho$. Normally speaking each of the real roots β_j is simple and in this case we have $\gcd(f(x), f(\beta_j-x)) = (x-\beta_j/2)^2+\gamma_j^2$ where γ_j is positive.

In this case the roots of $f(x)$ are the $n=2\rho$ complex numbers

$$\beta_j/2 \pm i\gamma_j \quad (1 \leqq j \leqq \rho).$$

If the real roots β_j are not all simple, a more elaborate algorithm must be carried out.

Let B_0 be the set of all roots β_j and let $\mu_0(\beta_j)$ be the multiplicity of the root β_j of $S_2(f)(x)$. Denote by A_1 the set of all elements of B_0 that are not the arithmetic mean of two distinct elements of B_0. We find that $\gcd(f(x), f(\beta-x)) = \phi_\beta(x) = \phi_\beta(\beta-x)$ is nonconstant for any β of A_1. Hence $\phi_\beta(x) = h_\beta((x-\beta/2)^2)$ where $\phi_\beta(x)$ has degree $2\mu_0(\beta)$ and h_β is a polynomial with half the degree of $\phi_\beta(x)$ such that all roots of h_β are negative, say, they are of the form: $-\gamma_{\beta_k}$ $(1 \leqq k \leqq \frac{1}{2}\mu_0(\beta))$ when γ_{β_k} is positive. Hence the roots of $\phi_\beta(x)$ are the $\mu_0(\beta)$ complex numbers $\beta/2 \pm \gamma_{\beta_k}i$. If the total number of these roots is n then the task is completed.

If the total number is less than n then we determine for each member γ of B_0 that is not contained in A_1 the number of times, say $\nu_0(\gamma)$ that $\gamma = \frac{1}{2}(\beta+\beta')$ $(\beta, \beta' \in A_1, \beta<\beta')$ and

$$\gamma_{\beta_k} = \gamma_{\beta'_{k'}} (1 \leqq k \leqq \mu_0(\beta)/2,\ 1 \leqq k' \leqq \mu_0(\beta')/2).$$

In view of the connection between f and $S_2(f)$ we find that $\mu_1(\gamma) = \mu_0(\gamma) - 2\nu_0(\gamma) \geqq 0$ when $\mu_1(\gamma)$ is not always zero.

Let us form the set B_1 of all γ's for which $\mu_1(\gamma)$ is positive.

We proceed as above, substituting B_1 for B_0, μ_1 for μ_0, A_2 for A_1 when A_2 is the subset of all members of B_1 that are not an arithmetic mean of two distinct members of B_1. In this way some further roots of $f(x)$ will be obtained. If not yet all of them are found, proceed as before until all n roots are constructed.

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PSEUDOPOLYHEDRONS

J. R. GOTT, III, Mayme S. Waggener High School, Louisville, Kentucky

A regular generalized polyhedron may be defined as a network of congruent regular polygons connected vertex to vertex so that the number and arrangement of polygons around every vertex is the same. The five regular polyhedrons and the three regular tessellations are the familiar examples of regular generalized polyhedrons. In the regular polyhedrons the sum of the face angles around any vertex is less than 360°. In the regular tessellations the sum of the face angles around a vertex is equal to 360°. The regular polyhedrons approximate positively curved surfaces (spheres), while the regular tessellations approximate surfaces with 0 curvature (planes).

We may define a third group of regular generalized polyhedrons in such a way that they will share the properties of the regular polyhedrons and tessellations but will have the sum of the face angles around a vertex greater than 360°. I have called such a group of figures regular pseudopolyhedrons. I have chosen the name pseudopolyhedrons (or false polyhedrons) because while possessing many of the properties of polyhedrons they are in some aspects distinctly different. A similar convention was used in naming a specific negatively curved surface, the pseudosphere. The pseudosphere has similar properties to the sphere except that it is negatively, instead of positively, curved. By the same token it will be shown later that the pseudopolyhedrons possess similar properties to the polyhedrons except that they approximate negatively curved surfaces.

I have defined a regular pseudopolyhedron as follows:

A regular pseudopolyhedron is a network in space of congruent regular polygons fitted together vertex to vertex so that (1) every vertex is surrounded by the same number and arrangement of polygons; (2) the sum of the face angles around every vertex is greater than 360°; (3) just two polygons may meet at an edge; and (4) two polygons may share no more than one edge.

Except for condition (2), the definition is identical with that of the regular polyhedrons and the regular tessellations.

I have found seven regular pseudopolyhedrons which satisfy the conditions of this definition. These are diagrammed in Figures 1–7. Each is a repeating structure with an infinite number of faces. In the diagrams the structural form of each is presented. Each of the pseudopolyhedrons is a surface which divides

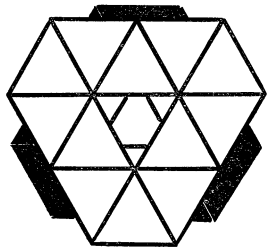


FIG. 1. Triangles, 8 around a point.

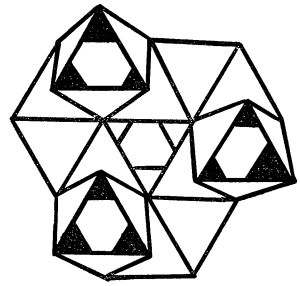


FIG. 2. Triangles, 10 around a point.

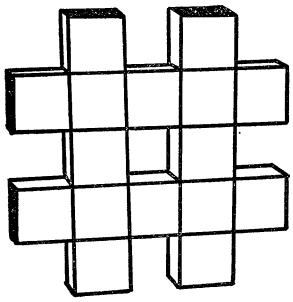


FIG. 3. Squares, 5 around a point.

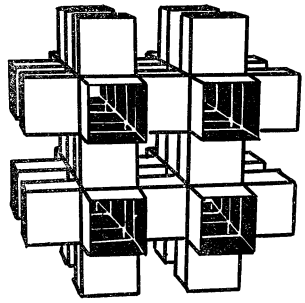


FIG. 4. Squares, 6 around a point.

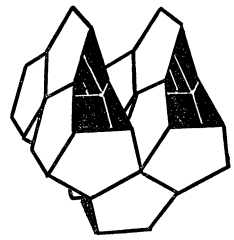


FIG. 5. Pentagons, 5 around a point.

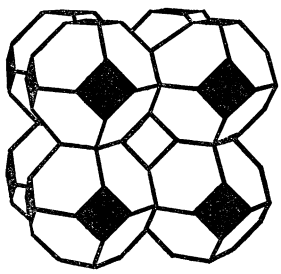


FIG. 6. Hexagons, 4 around a point.

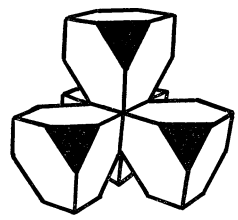


FIG. 7. Hexagons, 6 around a point.

FIGURES 1–7

REGULAR PSEUDOPOLYHEDRONS

space into two parts. All the pseudopolyhedrons possess a periodic system of holes, so that their genus is infinite. The arrangement of polygons around any vertex approximates a negatively curved or "saddle shaped" type surface.

At the time I conducted my study I was unaware that any of these figures had been constructed before. Later research revealed that in 1926 J. F. Petrie

and H. S. M. Coxeter [1] discovered and constructed the pseudopolyhedrons in Figures 4, 6, and 7. Coxeter has called these three figures "regular skew polyhedrons." Coxeter defines "regular skew polyhedrons" in terms of symmetry properties. His definition is equivalent to my definition of pseudopolyhedrons except that he requires in addition that the arrangement of polygons around every vertex be symmetrical (by rotation) around the vertex. This additional requirement insures that the dihedral angles between all adjacent faces will be equal. Coxeter has been able to show that these three regular skew polyhedrons (Figs. 4, 6, and 7) are the only constructable figures in 3-space to satisfy his definition.

Coxeter's regular skew polyhedrons form a subset of my class of regular pseudopolyhedrons. Figures 1, 2, 3, and 5 satisfy all the definition requirements of regular pseudopolyhedrons but do not satisfy the conditions of the regular skew polyhedrons. In these figures the dihedral angles between adjacent faces are not all equal (some but not all are equal to 180°).

The difference between the two approaches is essentially a difference between criteria of regularity.

For convenience the Schäfli symbol $\{p, q\}$ will be used in the future for the generalized polyhedron of regular p-gons, q surrounding each vertex [2].

The pseudopolyhedrons of type $\{4, 5\}$ and of type $\{3, 8\}$ are formed by two planes of polygons connected by a system of holes. The other five pseudopolyhedrons, however, fill the entirety of space. They all have the unusual property of dividing space into two congruent and completely interlocking parts. The "outside" and "inside" of these figures are exactly the same shape.

FIG. 8: TABLE OF PROPERTIES

Property	*Polyhedron*	*Tessellation*	*Pseudopolyhedron*
Sum of face angles around a vertex	$<360°$	$=360°$	$>360°$
Curvature of corresponding surface	Positive	0	Negative
Number of faces	Finite	Infinite	Infinite
Genus	0	0	Infinite
Euler's Formula $V-E+F=2$	Holds	Holds	Does not hold
Divides space into	2 Parts	2 Parts	2 Parts

It is possible to obtain a graphical representation of the relationships of the regular generalized polyhedrons by graphing p (the number of sides per polygon in the network) against q (the number of polygons meeting at a vertex). (See Figure 9.) Each different regular generalized polyhedron is represented by a point on the graph having integral coordinates. The three regular tessellations lie on the hyperbola:

$$(p - 2)(q - 2) = 4.$$

The regular polyhedrons lie below and to the left of this hyperbola, and the regular pseudopolyhedrons lie above and to the right of it. This is illustrated by the graph and is proven as a theorem at the conclusion of this article.

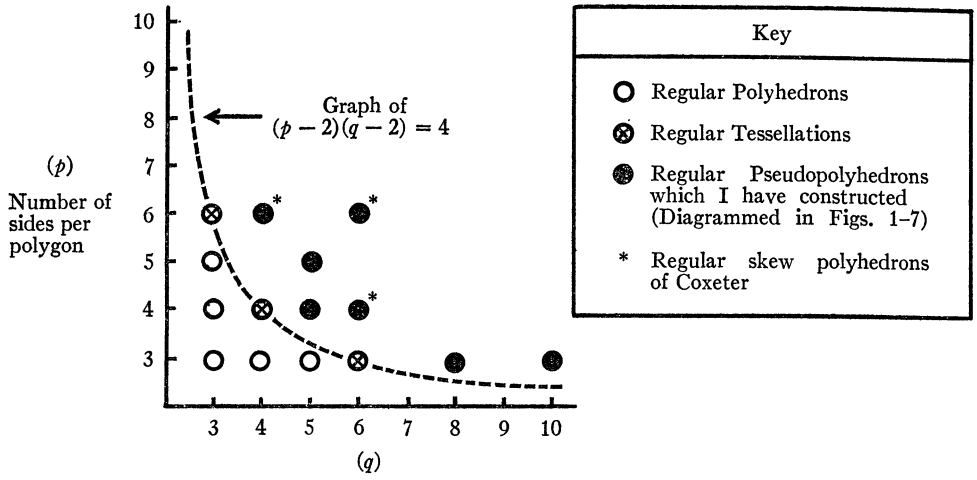


FIG. 9. Graphical representation of the Regular Generalized Polyhedrons.

While there are an infinite number of possible locations for pseudopolyhedrons on the graph, it appears that there are only a finite number of regular pseudopolyhedrons which are actually constructable in 3-space. It can be proven that no pseudopolyhedrons exist with $q=3$. (Three polygons cannot fit around a vertex to form a saddle shape surface.) An indefinitely large number of polygons of a given type may be fitted around a vertex by using a tighter and tighter zig-zag pattern. However, if a complete pseudopolyhedron is to be formed from the arrangement, the adjacent edges along the top (and bottom) of the zig-zag pattern must be separated widely enough to be connected together by edges to form the holes necessary to the network. If polygonal holes are formed, then the minimum possible separation between adjacent edges is 60° (equilateral triangular holes). The most efficient arrangement for producing these edge separations is an even zig-zag pattern (such as appears in the regular skew polyhedrons). In such a pattern q is even, but p may be either even or odd. If we construct a small unit sphere around the given vertex, we will find that the polygons cut the sphere in a series of great circle arcs which zig-zag up and down and around the sphere's equator. The critical angle between adjacent edges ($\geqq 60°$) is represented by the arc length measured on the sphere between two successive peaks in the zig-zag pattern. If we know p and q for a given network, we can compute this arc length using spherical trigonometry [5]. If the arc length is d, then:

$$\sin\frac{d}{2} = \frac{\cos\left(\frac{\pi}{2} - \frac{\pi}{p}\right)}{\cos\left(\frac{\pi}{q}\right)} \cdot \sin\left(\frac{2\pi}{q}\right).$$

Since p and q are positive, all the angles are positive. Since $d \geqq 60°$ or $\pi/3$, then $\sin(d/2) \geqq \frac{1}{2}$. The right hand side of the equation may also be simplified; thus we have

$$\frac{1}{2} \leqq \sin\frac{d}{2} = 2\sin\frac{\pi}{p}\sin\frac{\pi}{q} \quad \text{and} \quad \frac{1}{4} \leqq \sin\frac{\pi}{p}\sin\frac{\pi}{q}.$$

The only meaningful integral solutions of the equation with q even are $\{3, 8\}$, $\{3, 10\}$, $\{4, 6\}$, $\{4, 8\}$, $\{5, 4\}$, $\{5, 6\}$, $\{6, 4\}$, $\{6, 6\}$, $\{7, 4\}$, and $\{8, 4\}$. These computations were made for the most efficient type of polygon arrangement. In the actual structures any type pattern is permitted. However, if p and q are so large that even the efficient pattern does not work, then no pattern will work. We have only made computations for q even. If we remove one polygon from any zig-zag pattern and reconnect the remaining ones, there will clearly be more room between adjacent edges. For odd values of q, then the following are permissible networks: $\{3, 7\}$, $\{3, 9\}$, $\{3, 11\}$, $\{4, 5\}$, $\{4, 7\}$, $\{4, 9\}$, $\{5, 5\}$, $\{5, 7\}$, $\{6, 5\}$, $\{7, 5\}$, and $\{8, 5\}$. Thus, there can be at most 21 different pseudopolyhedrons with polygonal holes.

The difficulty with this approach is that a pseudopolyhedron may be formed which does not have polygonal holes. The example of this is in the $\{5, 5\}$ pseudopolyhedron where the holes are closed loops of edges which do not lie in one plane. Although it appears unlikely that a pseudopolyhedron could be formed with all nonpolygonal holes and all angles of separation less than 60°, the possibility does exist. No pseudopolyhedrons have as yet been discovered beyond the upper limit set by the polygonal approach.

Another puzzling phenomenon is that some possible networks which are well within the upper limits set by the angle separation problem still fail to produce pseudopolyhedrons.

What is needed is a definitive proof that will determine which combinations of sides per polygon and polygons around a vertex will actually form pseudopolyhedrons and which will not. This is an important area for future work.

The regular tessellations $\{4, 4\}$ and $\{3, 6\}$ can occur in different structural forms. In addition to covering the surface of a plane, these generalized polyhedrons may also approximate the surface of a cylinder. All the conditions of the definition are satisfied since the number and arrangement of polygons around each vertex are identical. As in several of the pseudopolyhedrons, some but not all of the dihedral angles between faces in the network are equal to 180°. Although the cylindrical structure possesses some properties different from the planar structure (the genus is 1 instead of 0), most of the properties are retained.

The curvature of the approximated surface, for example, is the same in both cases because both the cylinder and the plane have a 0 curvature.

Several pseudopolyhedrons also possess alternate structural forms. The pseudopolyhedrons of type $\{3, 8\}$ and type $\{4, 5\}$ not only can be constructed in the form of two planes connected by holes, but may also be constructed in the form of an infinite series of parallel planes of polygons connected alternately by holes and columns. The pseudopolyhedron $\{4, 6\}$ may also be constructed with rhombic holes or with alternating triangular and hexagonal holes.

Another area for future exploration is that of the semi-regular pseudopolyhedrons (defined just as regular pseudopolyhedrons only lacking the requirement that all faces in the network be congruent). One of these is a double planed, hole connected network composed of 2 squares and 2 octagons around each vertex. The semi-regular polyhedrons will possess many of the properties of the regular ones (holes, negative curvature, etc.) but should encompass a much wider range of structural forms.

In summary, pseudopolyhedrons are an interesting and unusual group of solid geometric figures. They, together with the regular polyhedrons and the regular tessellations, complete the mathematical system of constructible generalized polyhedrons. These pseudopolyhedrons create many new and challenging questions for future investigation.

THEOREMS. Let q = Number of polygons around a vertex in a given generalized polyhedron; p = Number of sides per polygon in a given generalized polyhedron; A = Vertex or face angle of polygon in generalized polyhedron; then: $A = (180° - 360°/p)$ by division of the polygon into triangles.

THEOREM I. *For every regular polyhedron:* $(p-2)(q-2)<4$. *For every regular tessellation:* $(p-2)(q-2)=4$. *For every regular pseudopolyhedron:* $(p-2)(q-2)>4$.

By definition we have: Polyhedron: $q \cdot A < 360°$; Tessellation: $q \cdot A = 360°$; Pseudopolyhedron: $q \cdot A > 360°$.

Proof for the pseudopolyhedron:

$$q \cdot A > 360° \qquad pq - 2q > 2p \text{ (since } p > 0)$$
$$q(180° - 360°/p) > 360° \qquad pq - 2(p+q) > 0$$
$$q - 2q/p > 2 \qquad pq - 2(p+q) + 4 > 4$$
$$(p-2)(q-2) > 4.$$

The other two formulas may be proven in a similar fashion. The results of this theorem are demonstrated by the graph in Figure 9.

Parts of this theorem have been proven previously [3].

THEOREM II. *The number of faces surrounding a given face is* $p(q-2)$ *in any regular generalized polyhedron.*

Proof. The given face has one face adjacent to it along each of its (p) edges. The given face also has (p) vertices; (q) polygons (faces) meet at every vertex. At each of its vertices, the given face will have ($q-3$) neighbors which touch it only at that vertex. There will thus be $p(q-3)$ neighboring faces of this type. The total number of faces surrounding a given face is thus $p+p(q-3)$ or $p(q-2)$.

THEOREM III. *Every regular pseudopolyhedron approximates a negatively curved surface.*

Proof. The approximated curved surface may be constructed so as to touch the pseudopolyhedron at every vertex. The vertices may then be connected by a series of geodesic arcs across the surface corresponding to the edges in the pseudopolyhedron. The sum of the face angles around a vertex on the pseudopolyhedron is greater than 360°, but the sum of the face angles around a vertex as measured on the surface will be equal to 360°. Thus, the angles between the geodesic arcs on the surface will be less than the corresponding angles on the pseudopolyhedron. Geodesic triangles drawn on the surface will have a sum of interior angles less than 180°. The approximated surface displays the non-euclidean geometry of a negatively curved surface and therefore has an over-all negative curvature.

In most pseudopolyhedrons the arrangement of polygons around a vertex follows the classic "saddle shape" pattern of a negatively curved surface, (as in $\{6, 4\}$). In the pseudopolyhedrons $\{4, 6\}$ and $\{6, 6\}$ the polygons around each vertex approximate a "monkey saddle" type surface. Although the curvature of a monkey saddle is 0 at the exact center, the curvature at every other point on the surface is negative. Thus, the over-all geometry of the "monkey saddle" is that of a negatively curved surface.

As an alternate idea, it has been suggested that "polygonal curvature" can be defined as $2\pi-S$, where S is the sum of face angles around a point. From this, as from the other concepts, we see that polyhedrons approximate positive curved surfaces; tessellations, 0 curved surfaces; and pseudopolyhedrons, negatively curved surfaces.

THEOREM IV. *The seven regular pseudopolyhedrons in Figures* 1–7 *are repeating structures.*

Proof. In each case it is sufficient to show that their vertex figures can be extracted from the edge system of the vertex figure of a solid tessellation or "honeycomb" of convex polyhedrons (see [6] p. 103).

For Figures 1–7.

(1) Vertex figure consists of 5 sides of a regular hexagon and three sides of a square. These edges can be extracted from the edge system of a cuboctahedron. The cuboctahedron is the vertex figure of the solid tessellation formed by regular octahedrons and tetrahedrons.

(2) Vertex figure consists of 4 sides of a regular hexagon, 3 sides of one

square and 3 sides of another square. These edges may also be extracted from the edge system of the same cuboctahedron as in 1.

(3) and (4) The vertex figures of 3 and 4 may be extracted from the edge system of a regular octahedron which is the vertex figure of the simple space filling honeycomb of cubes.

(5) The vertex figure consists of 3 sides of a trapezoid and 2 sides of an isosceles triangle. These edges form part of an irregular octahedron. Since one cross-section of this octahedron is a trapezoid with 2 parallel sides, it is indeed the vertex figure of a solid tessellation. In this case it is a space filling honeycomb of three kinds of quadrilateral prisms.

(6) The vertex figure is a skew quadrilateral which can be extracted from an irregular tetrahedron. This tetrahedron is the vertex figure for the solid tessellation of truncated octahedrons.

(7) The vertex figure is a skew hexagon which fits into a tall antiprism. This antiprism is the vertex figure of the solid tessellation of regular tetrahedrons and truncated tetrahedrons.

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ON THE DIOPHANTINE EQUATION $mX^2-nY^2=\pm 1$

D. T. WALKER, Memphis State University

This paper has two major objectives. The first is to develop a formula for all solutions of the title equation, which is done in Theorems 1 through 9. Then, using this formula, the remainder of the paper deals with the second objective, which is to establish a result concerning the prime divisors of Y and n. It turns out that for n given, with two possible exceptions, Y has a prime factor not dividing n.

First we will need some well-known theory of the closely related Pell equation $X^2-DY^2=\pm 1$, which can be found in many of the current texts in Number Theory, for example [1] and [2], where solutions of the Pell equation are defined as being the numbers $x+y\sqrt{D}$ such that $x=X$ and $y=Y$ are integers satisfying that equation, and D is a positive integer not a perfect square. At all times our "integers" are assumed to be rational integers. In particular we have the following:

DEFINITION. *The solution* $x+y\sqrt{D}$ *of* $X^2-DY^2=\pm 1$, *in which integers* X *and* Y *have their least positive values, is the fundamental solution.*

The fact that powers of the fundamental solution generate all solutions of the equation is well known and is given here without proof [1] in the following two theorems.

THEOREM 1. *If* i *is a positive integer, then the formula* $x_i+y_i\sqrt{D}=(x+y\sqrt{D})^i$ *gives all solutions with positive integers* x_i, y_i *of the equation* $X^2-DY^2=1$ *whose fundamental solution is* $x+y\sqrt{D}$.

THEOREM 2. *If* i *is a positive odd integer, then the formula* $x_i+y_i\sqrt{D}=(x'+y'\sqrt{D})^i$ *gives all solutions with positive integers* x_i, y_i *of the equation* $X^2-DY^2=-1$ *when it is solvable and has* $x'+y'\sqrt{D}$ *as its fundamental solution.*

REMARK. It follows that if $X^2-DY^2=-1$ is solvable and has fundamental solution $x'+y'\sqrt{D}$, then the even powers of this solution are the solutions of $X^2-DY^2=1$. In particular $(x'+y'\sqrt{D})^2=x'^2+Dy'^2+2x'y'\sqrt{D}$ is the fundamental solution of the equation $X^2-DY^2=1$.

In this discussion, we are concerned with positive integral solutions of $mX^2-nY^2=1$ and $mX^2-nY^2=-1$, where m and n are positive integers and neither is a perfect square. We will treat both equations together as

$$mX^2-nY^2=\pm 1. \tag{1}$$

Moreover, we will develop relations between solutions of (1), when it is solvable, and solutions of the Pell equation in the form

$$R^2-mnS^2=1, \tag{2}$$

with which we will frequently be concerned.

In a manner analogous to the way in which solutions of the Pell equation were defined, we now define solutions of (1).

DEFINITION. *The number* $x\sqrt{m}+y\sqrt{n}$ *is a solution of* (1) *if* $x=X$ *and* $y=Y$ *are integers which satisfy* (1).

DEFINITION. *A solution* $x\sqrt{m}+y\sqrt{n}$ *of* (1) *is a positive solution of* (1) *if both* x *and* y *are positive.*

REMARK. We are concerned primarily with positive solutions, and it can be shown easily that if $x\sqrt{m}+y\sqrt{n}$ and $w\sqrt{m}+z\sqrt{n}$ are positive solutions of (1) then the following statements are equivalent: (i) $x<w$, (ii) $y<z$, (iii) $x\sqrt{m}+y\sqrt{n}<w\sqrt{m}+z\sqrt{n}$. Now if (1) is solvable then it has positive solutions. Then it follows from the equivalence of (i), (ii), and (iii) that among the positive solutions of (1), when it is solvable, there is one solution in which both X and Y have their least values.

DEFINITION. *A solution* $x\sqrt{m}+y\sqrt{n}$ *of* (1) *is the smallest solution of* (1) *if it is the positive solution in which both* $X=x$ *and* $Y=y$ *have their least values.*

LEMMA 1. *If* $x\sqrt{m}+y\sqrt{n}$ *is a solution of* (1), *and* $r+s\sqrt{mn}$ *is a solution of* (2), *then the product* $(xr+nys)\sqrt{m}+(yr+mxs)\sqrt{n}$ *is a solution of* (1).

Proof. $(mx^2-ny^2)(r^2-mns^2)=(\pm 1)(1)=m(x^2r^2+n^2y^2s^2)-n(y^2r^2+m^2x^2s^2)$ $=m(xr+nys)^2-n(yr+mxs)^2=\pm 1$.

LEMMA 2. *If* $x\sqrt{m}+y\sqrt{n}$ *and* $x'\sqrt{m}+y'\sqrt{n}$ *are solutions of* (1), *then the product* $(mxx'+nyy')+(xy'+x'y)\sqrt{mn}$ *is a solution of* (2).

Proof. $(mx^2-ny^2)(mx'^2-ny'^2)=(\pm 1)(\pm 1)=(mxx'+nyy')^2-mn(xy'+x'y)^2$ $=1$. In particular if $x\sqrt{m}+y\sqrt{n}$ is a solution of (1), then $(x\sqrt{m}+y\sqrt{n})^2$ is a solution of (2).

For the remainder of this discussion we will sometimes denote solutions $x\sqrt{m}+y\sqrt{n}$ and $r+s\sqrt{mn}$ of equations (1) and (2) respectively by (x, y) and (r, s), and the product of two solutions by $(x, y)\cdot(r, s)$.

The following function $F(x, y)$ has been studied by K. Petr [3] and F. Arndt [4].

DEFINITION. $F(x, y)=(x\sqrt{m}+y\sqrt{n})^2=(mx^2+ny^2)+(2xy)\sqrt{mn}$ *for a solution* $x\sqrt{m}+y\sqrt{n}$ *of* (1).

Note that the function F, which maps solutions of (1) into solutions of (2), has the following properties:

(i) $F(-x, -y)=F(x, y)$,

(ii) $F(x, -y)=\overline{F(x, y)}$, the conjugate of $F(x, y)$,

(iii) $F((x, y)\cdot(r, s))=F(x, y)\cdot(r, s)^2$ or $=F(x, y)\cdot(r+s\sqrt{mn})^2$.

LEMMA 3. *If* $F(x, y)=1$, *then* $m=1$ *or* $n=1$.

Proof. If equation (1) has the plus sign, then

$$F(x, y) = (2mx^2 - 1) + (2xy)\sqrt{mn},$$

and $2mx^2-1=1$ implies that $m=1$. Or if equation (1) has the minus sign, then

$$F(x, y) = (2ny^2 - 1) + (2xy)\sqrt{mn}, \quad \text{and} \quad 2ny^2 - 1 = 1 \text{ implies that } n = 1.$$

THEOREM 3. *If* $F(x, y)=(u, v)\cdot(u, v)$ *for some solution* $x\sqrt{m}+y\sqrt{n}$ *of* (1) *and* $u+v\sqrt{mn}$ *of* (2), *then* $m=1$ *or* $n=1$.

Proof. Let $F(x, y)=(u, v)\cdot(u, v)$, and consider

$$\begin{aligned} F((x, y)\cdot(u, -v)) &= F(x, y)\cdot(u, -v)\cdot(u, -v) \\ &= (u, v)\cdot(u, v)\cdot(u, -v)\cdot(u, -v) = 1. \end{aligned}$$

Take $(w, z)=(x, y)\cdot(u, -v)$. Thus (1) has a solution $w\sqrt{m}+z\sqrt{n}$ such that $F(w, z)=1$. Then, by Lemma 3, $m=1$ or $n=1$.

As a consequence of Lemma 1, Lemma 2, Theorem 1, and Theorem 3 we have the following

THEOREM 4. *If* $x\sqrt{m}+y\sqrt{n}$ *is a positive solution of* (1) *for both of* $m, n>1$,

then $F(x, y) = (r, s)^{2i+1}$ *for some nonnegative integer* i, *and where* $(r, s) = r + s\sqrt{mn}$ *is the fundamental solution of* (2).

THEOREM 5. *If* (1) *has a solution for both of* $m, n > 1$, *then* (1) *has a positive solution which maps under* F *into the fundamental solution of* (2).

Proof. If equation (1) is solvable, then it has a positive solution. Let $x\sqrt{m} + y\sqrt{n}$ be a positive solution of (1) and let $r + s\sqrt{mn}$ be the fundamental solution of (2). By Theorem 1 and Theorem 4, for some non-negative integer i, $F(x, y) = (r, s)^{2i+1} = (u, v)^2 \cdot (r, s)$ for some solution $u + v\sqrt{mn}$ of (2). Now consider $(w, z) = (x, y) \cdot (u, -v)$. We have $F((x, y) \cdot (u, -v)) = (r, s) \cdot (u, v)^2 \cdot (u, -v)^2 = (r, s)$. Since $F(w, z) = (r, s)$, then w and z have the same sign. So one of (w, z) and $(-w, -z)$ is positive and the other is negative. Using property (i) if necessary, the positive one, say (w', z'), is a positive solution of (1) such that $F(w', z') = (r, s)$.

It is possible that the result of the following theorem may have been obtained by K. Petr [3] by use of continued fractions.

THEOREM 6. *If* (1) *is solvable for both of* $m, n > 1$, *and has* (x, y) *as its smallest solution, then* $F(x, y) = (r, s)$ *the fundamental solution of* (2).

Proof. By Theorem 5, there is a positive solution $w\sqrt{m} + z\sqrt{n}$ of (1) such that $F(w, z) = (r, s)$. Now if the smallest solution of (1) $x\sqrt{m} + y\sqrt{n} \neq w\sqrt{m} + z\sqrt{n}$, then by Theorem 4, for some positive integer i, $F(x, y) = (r, s)^{2i+1}$.

Then $F(x, y) = [F(w, z)]^{2i+1}$, which means that

$$(x\sqrt{m} + y\sqrt{n})^2 = [(w\sqrt{m} + z\sqrt{n})^{2i+1}]^2.$$

But $x\sqrt{m} + y\sqrt{n} = (w\sqrt{m} + z\sqrt{n})^{2i+1}$ is impossible if i is a positive integer, since $x\sqrt{m} + y\sqrt{n}$ is the smallest solution of (1). Therefore $x\sqrt{m} + y\sqrt{n} = w\sqrt{m} + z\sqrt{n}$, and $F(x, y) = (r, s)$.

The converse of Theorem 6, first noted by F. Arndt [4] is given here as

THEOREM 7. *If* (1) *is solvable for both of* $m, n > 1$, *and has a positive solution* (x, y) *such that* $F(x, y) = (r, s)$ *the fundamental solution of* (2), *then* (x, y) *is the smallest solution of* (1).

Proof. If equation (1) has the plus sign then $r = 2mx^2 - 1$, or if equation (1) has the minus sign, then $r = 2ny^2 - 1$. Let $x_0\sqrt{m} + y_0\sqrt{n}$ be the smallest solution of (1). Then

$$F(x_0, y_0) = (mx_0^2 + ny_0^2) + (2x_0y_0)\sqrt{mn} = r_0 + s_0\sqrt{mn}$$

is a solution of (2) in which both of r_0, s_0 are positive integers. Now if equation (1) has the plus sign then $x_0 < x$ implies that $r_0 = 2mx_0^2 - 1 < 2mx^2 - 1 = r$ which is impossible if (r, s) is the fundamental solution of (2). Or, if equation (1) has the minus sign, then $y_0 < y$ implies that $r_0 = 2ny_0^2 - 1 < 2ny^2 - 1 = r$; again, impossible if (r, s) is the fundamental solution of (2). Therefore (x, y) is the smallest solution of equation (1).

Although not necessary for the proofs of the main results in this discussion, the following theorem concerning the equation

$$R^2 - mnS^2 = -1 \tag{3}$$

is included as a matter of interest.

THEOREM 8. *If equation* (3) *is solvable, then equation* (1), *with both of* $m, n > 1$, *is not solvable for either sign.*

Proof. Let (3) have fundamental solution $r_1+s_1\sqrt{mn}$. Then, by the remark following Theorem 2, equation (2) has fundamental solution $(r_1^2+mns_1^2)+(2r_1s_1)\sqrt{mn}$ which can be expressed as $(2r_1^2+1)+(2r_1s_1)\sqrt{mn}$ since $r_1^2-mns_1^2=-1$. Now assume both of $m, n>1$, and suppose that equation (1) with the plus sign is solvable and has smallest solution $x\sqrt{m}+y\sqrt{n}$. Then by Theorem 6, we have

$$F(x, y) = (2ny^2 + 1) + (2xy)\sqrt{mn} = (2r_1^2 + 1) + (2r_1s_1)\sqrt{mn}.$$

But $2ny^2+1=2r_1^2+1$ implies $ny^2=r_1^2$, which is impossible if n is not a perfect square. Or, with both of $m, n>1$, suppose that equation (1) is solvable with the minus sign and has smallest solution $x\sqrt{m}+y\sqrt{n}$. Again using Theorem 6, we have

$$F(x, y) = (2mx^2 + 1) + (2xv)\sqrt{mn} = (2r_1^2 + 1) + (2r_1s_1)\sqrt{mn}.$$

Here $2mx^2+1=2r_1^2+1$ implies $mx^2=r_1^2$, which is impossible since m also is not a perfect square.

THEOREM 9. *If* (1) *is solvable for both of* $m, n>1$ *and has* $x\sqrt{m}+y\sqrt{n}$ *as its smallest solution; and if* i *is a nonnegative integer, then the formula*

$$x_i\sqrt{m} + y_i\sqrt{n} = (x\sqrt{m} + v\sqrt{n})^{2i+1} \tag{4}$$

gives all positive solutions of (1).

Proof. By Theorem 6, Theorem 1, and Lemma 1, the right member of (4), which can be expressed as $(x\sqrt{m}+y\sqrt{n})(U+V\sqrt{mn})$ for some solution $U+V\sqrt{mn}$ of (2) with positive integers U and V, always gives positive solutions of (1).

Now suppose that there is a positive solution $w\sqrt{m}+z\sqrt{n}$ of (1) which is not given by (4). Then for some nonnegative integer i, we have

$$(x\sqrt{m} + y\sqrt{n})^{2i+1} < w\sqrt{m} + z\sqrt{n} < (x\sqrt{m} + y\sqrt{n})^{2i+3}. \tag{5}$$

Assume that eq. (1) has the plus sign. Here, since $x\sqrt{m}-y\sqrt{n}=1/(x\sqrt{m}+y\sqrt{n})$ is positive, (5) becomes

$$1 < (w\sqrt{m} + z\sqrt{n})(x\sqrt{m} - y\sqrt{n})^{2i+1} < (x\sqrt{m} + y\sqrt{n})^2.$$

Now let equation (1) have the minus sign. In this case

$$x\sqrt{m} - y\sqrt{n} = \frac{-1}{x\sqrt{m} + y\sqrt{n}}$$

is negative, so from inequalities (5) we have

$$1 < [(-w)\sqrt{m} + (-z)\sqrt{n}](x\sqrt{m} - y\sqrt{n})^{2i+1} < (x\sqrt{m} + y\sqrt{n})^2.$$

By Lemma 1 and Lemma 2, for either choice of sign in equation (1) the number

$$[(\pm w)\sqrt{m} + (\pm z)\sqrt{n}](x\sqrt{m} - y\sqrt{n})^{2i+1} = u + v\sqrt{mn},$$

a solution of (2) for some pair of integers u and v. Moreover, by Theorem 6, $(x\sqrt{m}+y\sqrt{n})^2 = F(x, y) = r+s\sqrt{mn}$ is the fundamental solution of (2), so that in either case of sign for equation (1) we have

$$1 < u + v\sqrt{mn} < r + s\sqrt{mn}. \tag{6}$$

We now show that both u and v are positive integers. From (6) we have $u+v\sqrt{mn} > 1$ and $0 < u-v\sqrt{mn} = 1/(u+v\sqrt{mn}) < 1$. Now it is seen from the last three inequalities that both u and v are positive. So the existence of a positive solution of (1) not given by (4) implies the existence of a solution $u+v\sqrt{mn}$ of (2), with positive integers u, v, which is less than the fundamental solution of (2), a contradiction.

REMARK. In view of the above theorems, the equation (1) is solvable if and only if the fundamental solution of (2) is the square of a solution of (1), its smallest solution, and all other positive solutions of (1), when it is solvable, can be found from formula (4).

Example. The equation $7X^2 - 13Y^2 = \pm 1$ is not solvable for either sign, since the fundamental solution of $R^2 - 91S^2 = 1$ is $1574+165\sqrt{91}$, but $1574 \neq 2(7)x^2 \pm 1$ for any integer x.

Example. The equation $13X^2 - 29Y^2 = 1$ is solvable. The fundamental solution of $R^2 - 377S^2 = 1$ is $233+12\sqrt{377}$. Then $x=3$ and $y=2$ satisfy $2(13)x^2 - 1 = 233$, $2xy = 12$, and $13x^2 - 29y^2 = 1$. So the equation of this example has $3\sqrt{13} + 2\sqrt{29}$ as its smallest solution.

Having developed a theory of solutions of equation (1), the remainder of this paper will be concerned with positive solutions of (1) having a particular property to be defined below.

The symbol $a \mid b$ (or $a \nmid b$) means that the integer a is (or is not) a factor of the integer b. Similarly if p is a prime number, the symbol $p^c \| b$ means that, for some positive integer c, $p^c \mid b$ but $p^{c+1} \nmid b$.

DEFINITION. *A solution $x_i m + y_i \sqrt{n}$ of* (1) *has property P if, for any prime number p, $p \mid y_i$ implies that $p \mid n$.*

Störmer [5] and Skolem [6] have studied a similar property P for solutions of the Pell equation $X^2 - DY^2 = \pm 1$ with results analogous to those which we shall obtain.

In the following study we will seek solutions of equation (1) having property

P, as defined above. This study also applies to cases where the prime divisors of X are divisors of m, as can be seen by changing the sign in (1) and interchanging elements of the pairs m, n and X, Y.

Now by expanding the right member of (4) we have

$$y_i = (2i+1)m^i x^{2i} y + \binom{2i+1}{3} m^{i-1} n x^{2i-2} y^3 + \cdots + n^i y^{2i+1},$$

where we see that $y \mid y_i$.

Let $y_i = yz_i$, $ny^2 = mx^2 \mp 1 = A$, where $A > 1$, and $y\sqrt{n} = \sqrt{A}$. Then from the above and (4) we obtain $x_i\sqrt{m} + z_i\sqrt{A} = (x\sqrt{m} + \sqrt{A})^{2i+1}$, where $x_0 = x$ and $z_0 = 1$.

Now let the solution $x_i\sqrt{m} + y_i\sqrt{n}$ have property P. Then if a prime $p \mid z_i$ it follows that $p \mid A$. We first seek the values of $2i+1$ for which this is possible. If $2i+1 = 1$, and hence $i = 0$, then the solution $x_i\sqrt{m} + y_i\sqrt{n} = x\sqrt{m} + y\sqrt{n}$ the smallest solution of (1) has property P. We will assume $i > 0$ and hence $2i+1 \geqq 3$. Let p be a prime divisor of $2i+1$ so that $2i+1 = pk$ for some (odd) integer k. Then $x_i\sqrt{m} + y_i\sqrt{n} = x_i\sqrt{m} + z_i\sqrt{A} = (x\sqrt{m} + \sqrt{A})^{pk}$. Since p and k are both odd, take $p = 2j+1$ and $k = 2h+1$. Now

$$x_i\sqrt{m} + z_i\sqrt{A} = [(x\sqrt{m} + \sqrt{A})^{2j+1}]^k = (x_j\sqrt{m} + z_j\sqrt{A})^{2h+1},$$

and

$$z_i = (2h+1)m^h x_j^{2h} z_j + \binom{2h+1}{3} m^{h-1} A x_j^{2h-2} z_j^3 + \cdots + A^h z_j^{2h+1}.$$

Hence $z_j \mid z_i$, and any prime divisor of z_j also divides A. But from $x_j\sqrt{m} + z_j\sqrt{A} = (x\sqrt{m} + \sqrt{A})^{2j+1}$ we see that

$$z_j = (2j+1)m^j x^{2j} + \binom{2j+1}{3} m^{j-1} x^{2j-2} A + \cdots + A^j, \tag{7}$$

and hence each prime divisor of z_j is also a divisor of $(2j+1)m^j x^{2j}$. But, since $(x, A) = (m, A) = 1$, each prime divisor of z_j is also a divisor of $2j+1$, and since the odd prime $p = 2j+1$ we have $z_j = p^r$ for some integral $r \geqq 1$. So equation (7) becomes

$$p^r = pm^j x^{2j} + \binom{p}{3} m^{j-1} x^{2j-2} A + \cdots + A^j. \tag{8}$$

Now if $p > 3$ and $r > 1$ then the second term in the right member of (8) and all following are divisible by p^2 while the first term is only divisible by p, which is impossible. Also if $p > 3$ with $r = 1$, then equation (8) leads to

$$1 = m^j x^{2j} + \frac{1}{p}\binom{p}{3} m^{j-1} x^{2j-2} A + \cdots + \frac{1}{p} A^j,$$

where each term on the right is a positive integer since $p \mid A$. But the last equa-

tion is impossible since the right member is >1. So every prime divisor p of $2i+1$ must be $p=3$.

Thus far we have shown that if $i>0$ and the solution of (1) $x_i\sqrt{m}+y_i\sqrt{n}=(x\sqrt{m}+y\sqrt{n})^{2i+1}$ has property P, then for the exponent $2i+1$ we must have

$$2i+1=3^e \tag{9}$$

where e is a positive integer.

REMARK. It is easily seen from equation (4) that an odd positive integral power of a solution of (1) is also a solution of (1).

LEMMA 4. *Let $x_u\sqrt{m}+y_u\sqrt{n}$ be any solution of* (1) *and its third power be the solution $x_v\sqrt{m}+y_v\sqrt{n}$. Moreover, let $x_v\sqrt{m}+y_v\sqrt{n}$ have property P. Then* (i) *the solution $x_u\sqrt{m}+y_u\sqrt{n}$ also has property P.*

(ii) $3mx_u^2+ny_u^2=3^s$ *for some integer* $s>1$.

(iii) $3^1\|n$.

(iv) $3\nmid y_u$.

Proof. Since

$$x_v\sqrt{m}+y_v\sqrt{n}=(mx_u^3+3nx_uy_u^2)\sqrt{m}+(3mx_u^2y_u+ny_u^3)\sqrt{n} \tag{10}$$

then (i) follows at once from the fact that $y_v=y_u(3mx_u^2+ny_u^2)$. Now by property P for $x_v\sqrt{m}+y_v\sqrt{n}$ if a prime $p\mid 3mx_u^2+ny_u^2$ then since $p\mid n$ and $(n, mx_u^2)=1$, we must have $p=3$. This proves (ii) for some positive integer s. Then $s=1$ contradicts conditions imposed on m and n, so $s>1$. Conclusions (iii) and (iv) follow at once from (ii) and the fact that $ny_u^2=3^s-3mx_u^2$.

Accordingly, since the solution $x_i\sqrt{m}+y_i\sqrt{n}=(x\sqrt{m}+y\sqrt{n})^{3^e}$ has property P then by Lemma 4, so do all solutions $(x\sqrt{m}+y\sqrt{n})^{3^f}$ have property P for all integers f such that $0\leqq f\leqq e$. In particular, the smallest solution $x\sqrt{m}+y\sqrt{n}$ and its third power, the solution $x_1\sqrt{m}+y_1\sqrt{n}$, have property P.

We now consider the smallest solution $x\sqrt{m}+y\sqrt{n}$ of (1) and its third power given by

$$x_1\sqrt{m}+y_1\sqrt{n}=(mx^3+3nxy^2)\sqrt{m}+(3mx^2y+ny^3)\sqrt{n}. \tag{11}$$

By Lemma 4 we have $3mx^2+ny^2=3^{s_1}$ for some integer $s_1>1$. Now since $3\mid n$, then for both choices of sign in (1) we have $3mx^2+ny^2=4ny^2\pm 3=3^{s_1}$, which becomes $4n'y^2\pm 1=3^{s_1-1}$, where $n'=n/3$. For some positive integer $t=s_1-1$, therefore, we have

$$n'y^2=\frac{3^t\mp 1}{4}\cdot \tag{12}$$

Conversely, if equation (1) has smallest solution $x\sqrt{m}+y\sqrt{n}$ with property P such that equation (12) is satisfied for some positive integer t and the integer $n'=n/3$, then the solution $x_1\sqrt{m}+y_1\sqrt{n}=(x\sqrt{m}+y\sqrt{n})^3$ of (1) also has property P. This can be seen by using (12) and the fact that $mx^2=ny^2\pm 1$ in equation (11) which gives

$$y_1 = 3^{t+1}y. \tag{13}$$

Furthermore, it is impossible that any solution of (1) different from the smallest solution $x\sqrt{m}+y\sqrt{n}$ and its third power $x_1\sqrt{m}+y_1\sqrt{n}$ have property P. If such solution existed it would be, because of equation (9), $(x\sqrt{m}+y\sqrt{n})^{3^e}$ for some integer $e\geqq 2$, which can be expressed as $(x_1\sqrt{m}+y_1\sqrt{n})^{3^{e-1}}=x_v\sqrt{m}+y_v\sqrt{n}$, with $e-1\geqq 1$. Now consider the solution of (1) $x_u\sqrt{m}+y_u\sqrt{n}$, of which $x_v\sqrt{m}+y_v\sqrt{n}$ is the third power, as given in (10). Then $x_u\sqrt{m}+y_u\sqrt{n}=(x_1\sqrt{m}+y_1\sqrt{n})^{3^{e-2}}$, where $e-2\geqq 0$. Now by expanding the right member of this last equation if necessary, we see easily that $y_1 \mid y_u$. It follows from equation (13) that $3 \mid y_u$, which is impossible by conclusion (iv) of Lemma 4.

Thus we have proved the following

THEOREM 10. *Among all solutions of equation* (1), *the only solutions for which property P is possible are the smallest solution* $x\sqrt{m}+y\sqrt{n}$ *and its third power* $x_1\sqrt{m}+y_1\sqrt{n}$. *The solution* $x_1\sqrt{m}+y_1\sqrt{n}$ *will have property P if and only if* $x\sqrt{m}+y\sqrt{n}$ *has property P and the integers* $n/3$ *and* y *satisfy* $(n/3)y^2=(3^t\mp 1)/4$ *for some positive integer* t.

REMARK. It is easy to find equations (1) such as $57X^2-14Y^2=1$ for which only the smallest solution $\sqrt{57}+2\sqrt{14}$ has property P. Although less common, equations (1) do exist in which both the smallest solution $x\sqrt{m}+y\sqrt{n}$ and its third power $x_1\sqrt{m}+y_1\sqrt{n}$ have property P. For example, the equation $4921X^2-1230Y^2=1$ has smallest solution $\sqrt{4921}+2\sqrt{1230}$ and third power solution $19681\sqrt{4921}+39366\sqrt{1230}$. Both solutions have property P.

In addition to having property P as defined above with regard to integers Y and n it may also happen that every prime factor of X divides m.

DEFINITION. *A solution* $x_i\sqrt{m}+y_i\sqrt{n}$ *of* (1) *has property P for both choices of sign in* (1) *if, for any prime number* p, $p \mid y_i$ *implies that* $p \mid n$ *and, for any prime number* q, $q \mid x_i$ *implies that* $q \mid m$.

Example. The (smallest) solution $3\sqrt{33}+2\sqrt{74}$ of $33X^2-74Y^2=1$ has property P for both choices of sign.

Clearly $x_1\sqrt{m}+y_1\sqrt{n}$, the third power of the smallest solution, cannot have property P for both choices of sign in (1) since by Theorem 10 this would require that both of m and n be divisible by 3 which is impossible, since m and n are relatively prime.

THEOREM 11. *If* $x_i\sqrt{m}+y_i\sqrt{n}$ *is a solution of* (1) *with property P for both choices of sign, then* $x_i\sqrt{m}+y_i\sqrt{n}$ *is the smallest solution of* (1).

In recent years LeVeque [7] and Cassels [8] have contributed to the theory of the exponential diophantine equation

$$a^x - b^y = 1, \tag{14}$$

where for nontrivial solutions we demand integers a, b, x, y all >1 and that the

equation is not $3^2-2^3=1$. Lebesgue [9] proved that for nontrivial solutions of (14) the integer y is odd. In the special case of (14) where x also is odd the results of this paper have an application. If (14) is solvable nontrivially with both of x, y odd for some integer pair a, b then the equation $aX^2-bY^2=\pm 1$ has a solution with property P for both choices of sign. So if a and b are given, then in seeking nontrivial solutions of (14) with both of x, y odd, by Theorem 11 we need only examine the smallest solution $x_0\sqrt{a}+y_0\sqrt{b}$ of the equation $aX^2-bY^2=1$.

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ON THE COMPUTATION OF THE NUMBER OF ELEMENTS IN COMBINATIONS OF SETS

E. D. HOMER, New York University

1. Introduction. In computations involving finite sets (as in probability calculations, for example) it is frequently necessary to calculate the number of elements in one or more combinations of sets from available data on the number of elements in other combinations of these sets. This paper presents a computational scheme for performing such calculations which offers the following advantages:

a. The method is entirely numerical; that is, it is completely free of set-algebraic manipulations.

b. The method detects cases in which the available information is insufficient to provide all the values desired, but computes all the values possible from the given data.

c. The method can be programmed readily for digital computers.

The method presented here is an application of Ledley's "designation number" technique [2, 3] for Boolean computations. We employ "designation vec-

tors," formed almost identically to the "designation numbers" except that the former are based upon sets, the latter on Boolean variables. We have also used a different basis for the formation process than does Ledley, only for the purpose of forming sequences in a more conventional arrangement.

2. Definitions. D-1. We assume, given a finite number, n, of disjoint non-empty sets, called *simple sets*, and denoted by A_i $(i=0, 1, 2, \cdots, n-1)$. The (finite) number of elements of A_i will be denoted by $N(A_i)$. Our domain of interest is $S=\bigcup_{i=0}^{n-1} A_i$. The complement A_i' of a simple set A_i is defined relative to S: $A_i' = S - A_i$.

D-2. We define a *function of the sets* (or, simply, a *function*) to be one of the following:

a. a simple set, A_i, or

b. Any combination of one or more functions under the set operations of union ($\cup$), intersection ($\cap$), or complementation ($'$).

In a problem involving n simple sets the number of possible unique functions is $k=2^{(2^n)}$. We denote a function by:

$$f_j; j = 0, 1, \cdots, k-1; k = 2^m; m = 2^n,$$

and the set of all k functions by $F=\{f_j\}$.

D-3. Of the k possible functions of n simple sets, we are particularly interested in $m=2^n$ functions called the *elementary disjunctive functions*. An elementary disjunctive function, denoted by p_b, is the intersection of every set or its complement. Thus:

$$p_b = \bigcap_{i=0}^{n-1} B_i; b = 0, 1, \cdots, m-1;$$

where B_i is either A_i or A_i'. The set of all such functions, a subset of F, is denoted $F_p=\{p_b\}$.

D-4. We define a *basis arrangement of elementary disjunctive functions* as follows.

Let the index, b, of p_b, be an n-digit binary number. Let the digits of b be numbered from left to right from 0 to $n-1$. Then,

$$b = (\beta_0, \beta_1, \beta_2, \cdots, \beta_{n-1}),$$

where each β is either 0 or 1. (We will use b_{10} to refer to the decimal value of b, where necessary.) We form the elementary intersection functions by specifying the B_i of D-3 to be:

$$B_i = \begin{cases} A_i, & \text{if } \beta_i = 1, \\ A_i', & \text{if } \beta_i = 0. \end{cases}$$

Then, by listing the p_b's in ascending order of b, we obtain the usual "truth table" arrangement of the functions. This we call the basis arrangement of F_p.

D-5. Associated with each of the k functions of the n simple sets will be a unique *designation vector*, consisting of m binary components, $m=2^n$; and denoted by

$$\mathbf{d}(f_j) = (\delta_{j,0}, \delta_{j,1}, \delta_{j,2}, \cdots, \delta_{j,m-1}).$$

Designation vectors are formed according to the following rules:

R-1. For simple sets, the designation vectors consist of alternating equal length strings of 0's and 1's. The number of components in each string in $\mathbf{d}(A_i)$ is $2^{(n-1-i)}$. For example, if $n=4$, $m=16$:

$$\mathbf{d}(A_0) = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1$$
$$\mathbf{d}(A_1) = 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1$$
$$\mathbf{d}(A_2) = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1$$
$$\mathbf{d}(A_3) = 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1.$$

(In this example, as well as throughout the rest of this paper, we eliminate the customary parentheses and commas of vector notation, since no ambiguity results.)

R-1′. An alternate formation method is to view the set of vectors as a matrix, in which the columns are numbered from left to right from 0 to $m-1$. Then each column, read from top to bottom, is the binary representation of the column number.

R-2. For the complement of a single set, the designation vector is formed by performing the Boolean operation NOT ($\overline{\ \ }$) over the elements of the designation vector of the single set. Thus, continuing the previous example,

$$\mathbf{d}(A_2') = (\overline{\mathbf{d}(A_2)}) = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0.$$

R-3. The designation vector of the intersection of two or more sets is formed by performing the Boolean operation AND ($\wedge$) over the elements of the designation vectors of the sets forming the intersection:

$$\mathbf{d}(A_0 \cap A_1) = \mathbf{d}(A_0) \wedge \mathbf{d}(A_1) = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1.$$

R-4. A similar process applies for the designation vector of the union of sets, using the Boolean operation OR ($\vee$):

$$\mathbf{d}(A_0 \cup A_1) = \mathbf{d}(A_0) \vee \mathbf{d}(A_1) = 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1.$$

R-5. Of course, the above rules may be combined for more complicated functions:

$$\mathbf{d}(A_0 \cup (A_1 \cap A_2') \cup A_3) = \mathbf{d}(A_0) \vee (\mathbf{d}(A_1) \wedge \mathbf{d}(A_2')) \vee \mathbf{d}(A_3)$$
$$= 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1.$$

3. Characteristics of designation vectors. It can be shown that:

C-1. $\mathbf{d}(S) = 1\ 1\ 1\ \cdots\ 1\ 1$

C-2. $\mathbf{d}(S') = 0\ 0\ 0\ \cdots\ 0 = \mathbf{d}(\Phi)$

C-3. $\mathbf{d}(f_r) = \mathbf{d}(f_q)$ iff $f_r \equiv f_q$.

C-4. $\mathbf{d}(p_b) = \mathbf{e}_b$; *i.e.*, the unit vector such that

$$\delta_{b,i} = \begin{cases} 1, & \text{if } i = b_{10}, \\ 0, & \text{elsewhere: } i = 0, 1, \cdots, m-1. \end{cases}$$

For example:

b_{10}	p_b	$\mathbf{d}(p_b)$
0	$p_{0000} = A_0' A_1' A_2' A_3'$	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1	$p_{0001} = A_0' A_1' A_2' A_3$	0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
2	$p_{0010} = A_0' A_1' A_2 A_3'$	0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0
.		
13	$p_{1101} = A_0 A_1 A_2' A_3$	0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0
14	$p_{1110} = A_0 A_1 A_2 A_3'$	0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0
15	$p_{1111} = A_0 A_1 A_2 A_3$	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1

C-5. It is well known that the set of elementary disjunctive functions is disjoint. This fact can be verified by applying R-3 to C-4, obtaining $\mathbf{d}(\Phi)$ for the intersection of any two or more p_b's.

C-6. It is well known that any member of F may be formed by the union of elementary disjunctive functions. Let $\{p_j\}$ be a subset of F_p. Then, given a member of F, say f_j, it is always possible to find a $\{p_j\}$ such that

$$f_j \equiv \bigcup_j \{p_j\},$$

which is, of course, the disjunctive canonical form, and, by C-3,

$$\mathbf{d}(f_j) = \mathbf{d}(\bigcup_j \{p_j\}) = \bigvee_j \{\mathbf{d}(p_j)\} = \bigvee_j \{\mathbf{e}_j\}.$$

Therefore, the designation vector of any function in F is the logical sum of some of the unit vectors.

C-7. In the expression

$$\mathbf{d}(f_j) = (\delta_{j,0}, \delta_{j,1}, \delta_{j,2}, \cdots, \delta_{j,m-1}) = \mathbf{d}(\bigcup_j \{p_j\}), \qquad p_b \in \{p_j\} \text{ iff } \delta_{j,b} = 1.$$

4. Calculating the number of elements. It is well known (see [4], for example) that for arbitrary disjoint sets, X_i: $N(\bigcup_i X_i) = \sum_i N(X_i)$. Therefore, by C-6 and C-5:

$$N(f_j) = N(\bigcup_j \{p_j\}) = \sum_j \{N(p_j)\}.$$

It is convenient, however, to work with the set F_b, rather than with the subset $\{p_j\}$. We may write:

$$N(f_j) = \sum_{b=0}^{m-1} \alpha_{j,b} N(p_b),$$

where:

$$\alpha_{j,b} = \begin{cases} 1, & \text{if } p_b \in \{p_j\} \\ 0, & \text{otherwise.} \end{cases}$$

But, from C-7, $p_b \in \{p_j\}$ iff $\delta_{j,b} = 1$. Therefore $\alpha_{j,b} = \delta_{j,b}$ and

$$N(f_j) = \sum_{b=0}^{m-1} \delta_{j,b} N(p_b). \tag{1}$$

If we define the column vector

$$\mathbf{N}(P) = [N(p_0), N(p_1), \cdots, N(p_{m-1})]^t,$$

where "t" denotes "transpose," then we may write equation (1) as

$$N(f_j) = \mathbf{d}(f_j)\mathbf{N}(P). \tag{2}$$

We wish to calculate the number of elements in certain functions. Let us call the set of these functions $F_s = \{f_s\}$, a subset of F. Let us assume that there are g functions in F_s; $0 < g \leqq m$. We may solve a series of equations (2) for all $j \in s$. It is more convenient to represent the series of equations to be solved in matrix form:

$$\mathbf{N}(F_s) = \mathbf{D}(F_s)\mathbf{N}(P), \tag{3}$$

where $\mathbf{D}(F_s) = [\mathbf{d}(f_{s1}), \mathbf{d}(f_{s2}), \cdots, \mathbf{d}(f_{sg})]^t$ and $\mathbf{N}(F_s) = [N(f_{s1}), N(f_{s2}), \cdots, N(f_{sg})]^t$.

Equation (3) can be used only when the given information includes $\mathbf{N}(P)$, that is, the number of elements in all the elementary disjunctive functions. Usually, this is not the case. Therefore, let us call the set of functions for which we know the number of elements $F_a = \{f_a\}$, a subset of F such that all the f_a's are not necessarily p_b's. We may substitute in (3), obtaining $\mathbf{N}(F_a) = \mathbf{D}(F_a)\mathbf{N}(P)$. If $\mathbf{D}(F_a)$ is nonsingular, we may solve for $\mathbf{N}(P)$:

$$\mathbf{N}(P) = \mathbf{D}^{-1}(F_a)\mathbf{N}(F_a). \tag{4}$$

Substituting from equation (4) into equation (3), we get

$$\mathbf{N}(F_s) = \mathbf{D}(F_s)\mathbf{D}^{-1}(F_a)\mathbf{N}(F_a). \tag{5}$$

Equation (3) can be seen to be a special case of equation (5), for $\mathbf{D}(P) = \mathbf{D}^{-1}(P) = I$, the identity matrix.

Equation (5) may be used whenever $\mathbf{D}(F_a)$ is nonsingular. Thus a sufficient condition for the computation of $\mathbf{N}(F_s)$ is that F_a contain at least m functions whose designation vectors are linearly independent, but this condition is not necessary. If $\mathbf{D}(F_s)$ contains one or more columns which are all zero, it may be possible for F_a to contain less than m values.

Suppose that we have a set of available data containing, say, q members, with $q<m$. (Note that if we have available m values, but $\mathbf{D}(F_a)$ is singular, then we may eliminate the functions corresponding to the dependent designation vectors, and proceed with the remaining q functions.) We add $m-q$ functions to the set, to form a new set F_{a*}, subject to the constraint that the added functions, $F_d=\{f_d\}$, be such that $\mathbf{D}(F_{a*})$ be nonsingular. The number of elements in each of the f_d's is, of course, unknown. We may assume, with no loss of generality, that the added designation vectors form the *last* $m-q$ rows of $\mathbf{D}(F_{a*})$.

We are now concerned with this question: What is the set of functions, F_{s*}, for which $\mathbf{N}(F_{s*})$ can be calculated on the basis of the limited information in $\mathbf{N}(F_a)$? To determine this we compute $\mathbf{D}^{-1}(F_{a*})$. Call $\mathbf{C}$ the submatrix of $\mathbf{D}^{-1}(F_{a*})$ formed of all the rows and the last $m-q$ columns of $\mathbf{D}^{-1}(F_{a*})$. Designate by W the set of all row indices of $\mathbf{C}$ for which the row contains at least one nonzero entry. Then we may find the number of elements in any f_{s*} if $\mathbf{d}(f_{s*})$ meets this criterion:

$$\delta_{s*,j} = \begin{cases} 0, \text{ if } j \in W \\ 0, \text{ or } 1 \text{ otherwise.} \end{cases}$$

If the number of indices in W is w, then the maximum number of functions in F_{s*} is $2^{(m-w)}$. Obviously, if w equals m, F_{s*} is empty.

5. Examples.

Example 1. Let $S=A_0\cup A_1\cup A_2$. Here $n=3$, $m=8$. Let F_a consist of 6 members:

f_a	$N(f_a)$
$f_{a0} = A_0 \cup A_1' \cup A_2$	230
$f_{a1} = (A_0' \cap A_2) \cup (A_1 \cap A_2)$	100
$f_{a2} = A_0'$	140
$f_{a3} = A_0 \cup (A_1' \cap A_2')$	140
$f_{a4} = A_0 \cup A_1$	220
$f_{a5} = (A_0 \cap A_1 \cap A_2) \cup (A_0' \cap A_1' \cap A_2')$	10

Find the number of elements in:

$$f_{s0} = A_0' \cap A_1', \quad f_{s1} = A_1 \cap A_2, \quad f_{s2} = A_0' \cup (A_1 \cap A_2), \quad f_{s3} = A_2'.$$

Solution: First, form the designation numbers of the simple sets

$$\mathbf{d}(A_0) = 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1$$
$$\mathbf{d}(A_1) = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1$$
$$\mathbf{d}(A_2) = 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1.$$

Next, form the designation vectors of the functions in F_a:

$$\begin{aligned}
\mathbf{d}(f_{a0}) &= 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\\
\mathbf{d}(f_{a1}) &= 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\\
\mathbf{d}(f_{a2}) &= 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\\
\mathbf{d}(f_{a3}) &= 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\\
\mathbf{d}(f_{a4}) &= 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\\
\mathbf{d}(f_{a5}) &= 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1.
\end{aligned}$$

To this append two additional designation vectors, chosen to make $\mathbf{D}(F_{a*})$ nonsingular. We have chosen:

$$\mathbf{d}(f_{d0}) = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0, \mathbf{d}(f_{d1}) = 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1.$$

The choice is not unique. The vectors chosen are such that

$$f_{d0} = (A_0' \cap A_1) \cup (A_1 \cap A_2') \quad \text{and} \quad f_{d1} = A_1 \cup A_2,$$

although this is not important.

We now form

$$\mathbf{D}(F_{a*}) = \begin{bmatrix} \mathbf{D}(F_a) \\ \mathbf{D}(F_d) \end{bmatrix},$$

and compute its inverse:

$$\mathbf{D}(F_{a*}) = \begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1\\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1\\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0\\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{bmatrix},$$

$$\mathbf{D}^{-1}(F_{a*}) = \begin{bmatrix}
1 & -1 & 0 & -1 & 0 & 1 & 0 & 0\\
-2 & 2 & 1 & 3 & -1 & -2 & 0 & 0\\
-2 & 1 & 1 & 2 & 0 & -1 & 0 & 0\\
3 & -2 & -1 & -4 & 1 & 2 & 0 & 0\\
-2 & 2 & 1 & 3 & 0 & -2 & 0 & -1\\
3 & -3 & -1 & -4 & 1 & 2 & -1 & 1\\
-1 & 1 & 0 & 2 & -1 & -1 & 1 & 0\\
-1 & 1 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}$$

We examine the submatrix

$$\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

consisting of the last two columns of $\mathbf{D}^{-1}(F_{a*})$, and note the indices of the rows which contain non-zero entries (noting that here, as throughout this paper, indices have zero origin). We find that $W = \{4, 5, 6\}$.

We now compute the designation vectors of the functions in F_s:

$$\mathbf{d}(f_{s0}) = 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0$$
$$\mathbf{d}(f_{s1}) = 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1$$
$$\mathbf{d}(f_{s2}) = 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1$$
$$\mathbf{d}(f_{s3}) = 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0.$$

The first three functions are admissible in F_{s*}, since their designation vectors have zeros in positions j, $j \in W$, $W = \{4, 5, 6\}$. The last function, however, is not admissible, since $\delta_{s3,4}$ and $\delta_{s3,6}$ are not zero.

Solving equation (5), we get

$$\mathbf{N}(F_{s*}) = \mathbf{D}(F_{s*})\mathbf{D}^{-1}(F_{a*})\mathbf{N}(F_{a*})$$

$$\mathbf{N}(F_{s*}) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & 2 & 1 & 3 & -1 & -2 & 0 & 0 \\ -2 & 1 & 1 & 2 & 0 & -1 & 0 & 0 \\ 3 & -2 & -1 & -4 & 1 & 2 & 0 & 0 \\ -2 & 2 & 1 & 3 & 0 & -2 & 0 & -1 \\ 3 & -3 & -1 & -4 & 1 & 2 & -1 & 1 \\ -1 & 1 & 0 & 2 & -1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 230 \\ 100 \\ 140 \\ 140 \\ 220 \\ 10 \\ N(f_{a0}) \\ N(f_{a1}) \end{bmatrix}$$

$$\mathbf{N}(F_{s*}) = \begin{bmatrix} 60 \\ 40 \\ 150 \end{bmatrix} = \begin{bmatrix} N(f_{s0}) \\ N(f_{s1}) \\ N(f_{s2}) \end{bmatrix}.$$

Example 2. As a second example, we consider a problem which has appeared in [1] and in [4], and apparently stems from the 1935 Joint Associateship Examination for Actuaries.

"Certain data obtained from a study of a group of 1000 employees in a

cotton mill, as to their race, sex, and marital status, were unofficially reported as follows: 525 colored lives; 312 male lives; 470 married lives; 42 colored males; 147 married colored; 86 married males; 25 married colored males. Test this classification to determine whether the numbers reported in the various groups are consistent."

We define the sets and their designation vectors as

$$C = \text{colored lives}; \quad \mathbf{d}(C) = 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1$$
$$M = \text{male lives}; \quad \mathbf{d}(M) = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1$$
$$W = \text{married lives}; \quad \mathbf{d}(W) = 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1.$$

From the given data,

$$\mathbf{N}(F_a) = \begin{bmatrix} N(S) = 1000 \\ N(C) = 525 \\ N(M) = 312 \\ N(W) = 470 \\ N(C \cap M) = 42 \\ N(C \cap W) = 147 \\ N(M \cap W) = 86 \\ N(C \cap M \cap W) = 25 \end{bmatrix}, \qquad \mathbf{D}(F_a) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We calculate

$$\mathbf{D}^{-1}(F_a) = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Suppose that we now calculate $\mathbf{N}(P) = \mathbf{D}^{-1}(F_a)\mathbf{N}(F_a)$.

The first value calculated, $N(p_0)$, shows that the data are inconsistent, for $N(p_0) = -57$, and the number of elements in a function of sets cannot be negative.

Research sponsored by Air Force Office of Scientific Research, Office of Aerospace Research, U.S. Air Force, Grant No. AF-AFOSR-621-64.

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HAMILTON CIRCUITS OF CONVEX TRIVALENT POLYHEDRA (UP TO 18 VERTICES)

JOSHUA LEDERBERG, Stanford University School of Medicine

In a study of the graphs of chemical structures [1, 2], it became of interest to ascertain the Hamilton circuits (a closed circuit of edges through all the vertices) of trivalent graphs, and especially of the convex polyhedra. Tait [3] had conjectured that these polyhedra always had Hamilton circuits—for brevity we will now say had *circuits*. In [4], however, Tutte has demonstrated a counter-example with 46 vertices. Between about 12 or 14 vertices and 46, the territory has hardly been explored.

Grace [5] has recently presented a computer tabulation of the polyhedra through 18 vertices, affording a convenient opportunity to scan them for circuits, which were found in every instance. As Grace has noted, his criterion for isomorphism, "equisurroundedness" of the sets of faces is not strictly sufficient and his list may still be incomplete. As the isomorphism of *circuits* is fairly readily computed, this approach may be useful in further extensions of such studies.

The work needed to demonstrate a circuit is curtailed by the reducibility of any triangular face: A circuit through a triangle is equivalent to that through a node:



That is to say, to describe a circuit a triangular face can be shrunk down to a node. In effect, by induction, if all n-hedra have circuits, so will all $(n+2)$-hedra with triangular faces, and we need only examine those without. Table 1 shows that only 55 forms need to be studied.

Grace displayed the polyhedra as face-incidence lists. A computer program translated these into vertex-incidence lists. Each vertex being identified as a face-triple, those vertices are joined which share two faces. The vertex-incidence

TABLE 1. Count of Trivalent Convex Polyhedra

Vertices n	*Faces* f	*Count Total* [5]	*Count No Triangles Present*	*Count No 3-connected Regions*
4	4	1	0	0
6	5	1	0	0
8	6	2	1	1
10	7	5	1	1
12	8	14	2	2
14	9	50	5	4
16	10	233	12	10
18	11	1249	34	26
		1555	55	44

list was then processed by a binary chained search of alternative paths; hence the search is always $\ll 2^n$, in contrast to the $n!$ scope of a systematic permutation of the vertices. (A much more efficient algorithm has been discovered and is outlined as an appendix.)

Table 2 displays a circuit for each of the 55 polyhedra. The other polyhedra of order $\leqq 18$, and some of higher order can be developed from these by expanding nodes into triangles, a process that can be iterated.

The circuits lend themselves to a compact code from which the graph of a polyhedron is quickly constructed. Draw a polygon with vertices marked $1(1)n$. Each successive character of the code denotes the span of a chord drawn from the next *vacant* vertex. Thus the prism would be 2, 3, 2 or *BCB*, and Hamilton's own example, the dodecahedron, is *DJGDMJGDGD*. There will be $n/2$ characters (to be sure the last one is redundant, being fixed by its predecessors). The letters *A*, *B*, *C* · · · stand for spans of 1, 2, 3 · · · vertices. *A* and *B* do not appear in our list; *A* would connote a self-looped edge and *B* a triangular face.

Only one of the sometimes numerous circuits of each polyhedron is shown: it is merely the first one discovered by the computer search, but it has been placed in canonical form with respect to rotation and reflection of the polygon [2].

The complete list of the Hamilton circuits for each graph of Table 2 gives further insight into the composition of circuit-free graphs. Extracting one node from a polyhedron leaves a cut graph with three cut edges, e.g., a triangular region is the residue of a tetrahedron. In the first instance, in general such a residue will have the same facility for admitting a circuit as does an isolated node, depending on the set of circuits found in the polyhedron. Thus, except for *CGDIGDFD* (Grace's 16–55) all the polyhedra where $n \leqq 16$ have this property. Hence for finding circuits, any other 3-connected region can be replaced by a single node. By induction only the 44 forms counted in Table 1 need be considered for $n \leqq 18$.

CGDIGDFD, (Fig. 1a) which is the same as Tutte's graph N_2 [4], has three (symmetrically equivalent) edges that are obligatory in any Hamilton circuit. It is the simplest with such a property.

Tutte produced a 46-node circuit-free graph by replacing three nodes of a tetrahedron with 3 15-node residues so that 3 obligatory edges converged on one node in a self-contradictory way. Along the same lines a 38-node, circuit-free graph can be composed by replacing two nodes of *CFDEC*, the pentagonal prism, with two 15-node residues so as to confront two obligatory edges with two that, as pointed out by Tutte, are mutually exclusive. In an independent study, David Barnette has already discovered this graph [6].

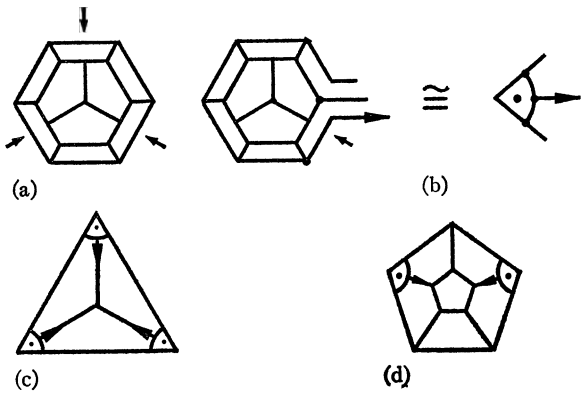


FIG. 1. Composition of non-Hamiltonian polyhedra. (a) 16*CGDIGDFD* which is Tutte's graph N_2[4]. The marked edges are obligatory in any circuit. (b) a 15-node residue with an obligatory edge as marked. (c) and (d) are two non-Hamiltonian polyhedra of 46 and 38 nodes respectively.

If we accept the enumeration of polyhedra for $n \leqq 16$, on which Grace concurs with Brückner [7], or merely that we know the 4-connected cases (as in table 2), a similar line of reasoning leads to an inferential argument for the conclusion that every 18-node polyhedron has a circuit. (We should say, more precisely, *cyclically* 4-connected as defined by Tutte [8]: "a *cyclically k-connected* graph cannot be broken up into two separate parts, each containing a polygon, by the removal of fewer than k edges.") Each of the graphs was searched by a computer program for subgraphs obtained by extracting any node-pair, and the circuit-forming properties of the subgraphs examined. None of the subgraphs was of a kind that would disqualify a pairwise combination of them from containing a circuit (cf. Tutte's arguments [8]). On the other hand, from Euler's formula, every 18-node polyhedron must (a) contain no 3-connected and at least one 4-connected subgraph which has 14 or fewer nodes, and would therefore fall within the scope of the search, or (b) is derived from a graph with at least one 3-connected region, a case already disposed of through $n=16$. The further

TABLE 2. Listing of Hamilton circuits

(Included are convex trivalent polyhedra with $n \leqq 18$ vertices. Only polyhedra with no triangular face are listed. See text for code. Each character group stands for one polyhedron.)
* marks 3-connected forms; the remainder are 4-connected.

Vertex Count	*Grace's Catalog [5] No.*	*Code*	*Vertex Count*	*Grace's Catalog [5] No.*	*Code*
8	2	CECC	18	186	CNEMFCFCD
10	5	CFDEC		195	CNDMEGECE
12	6	CGEGEC		196	COLCGEIFC
	11	CHFCFD		198	CNKHLECHE
14	6	CJHECGE		233	CNDFCGDFD
	7	CJGDHFC		326	COEMIFCFC*
	8	CIGDHFD		328	CNDMDFDFC
	15	CHFIGEC		329	CNLDHECHD
	46	CKEIECC*		347	CNLDHFDHF
16	42	CMJFDIFC		348	CNLIFDJHC
	43	CLDKDECD		350	CODGEHFDE
	44	CLIFJCGD		353	CLJGDKIGD
	52	CLIFCIGC		354	CMJGLDIGD
	54	CLDECEEC		356	CODGDHFCE
	55	CGDIGDFD		362	CLJFDKIGC
	60	CLJGECHF		376	CNJGEKIFD
	61	CJHKDHFD		383	CNIFLJGEC
	62	CKIECIGC		392	COMCIFCHC*
	70	CMKDGECF*		393	CNKHFDJHE
	88	CMDKFCEC*		401	COKHECJGC
	112	CIGKIGEC		418	CMKGEKIGC
				419	CMKHFDJHF
				426	CNLIGECIG
				427	CKIMKDHFD
				428	CMKGECJHC
				429	COCDGEHEC
				477	CNLDGECHC*
				493	COMELGECD*
				505	COCCEIECC
				508	CNGMKGECD*
				509	CODMHECFD*
				625	CODMCFCEC*
				626	CODMIECFC*
				887	CJHMKIGEC

application of this technique should make it possible to anticipate the smallest non-Hamiltonian polyhedron from the properties of the 4- and 5-connected polyhedra of the next lower order. Since these, by definition, all have circuits, it would be feasible to generate them on the computer by reasonably efficient combinatorial schemes to whatever order is required.

Appendix: Algorithm for finding Hamilton circuits of a cyclic graph.

This is illustrated for an undirected, trihedral graph but should be generalized without difficulty in an obvious way. The input is a description of the connectivity of the graph. The essence of the routine is to build a table of sets of edges so that just two edges incident on each node appear in any row of the table. The first node is chosen arbitrarily. Its three incident edges are marked *current* and *open*. The circuit-fragment table is started with three rows by listing the 3 pairwise choices among the current edges.

1. Select an open edge. The two adjacent edges become the trial edges.
2. How many trial edges match the current list: none, one, or two?
 a. If none match, close the selected edge and replace it on the current open list by the two trial edges. Scan the circuit-fragment table. Each row in which the selected edge appears is replaced by two rows, one for each trial edge. Each remaining row is replaced by one row showing both trial edges. Go to 1.
 b. If one matches, a circuit of the graph has been closed. Scan the circuit-fragment (c.f.) table contrasting the matched edge with the selected edge. Each c.f. where neither appears is deleted. If one of the two appears on a c.f., this is augmented by the trial edge. If both appear, the c.f. row stands as is unless a tracing of the c.f. shows it to be prematurely closed, whereupon it is deleted. Go to 1.
 c. If both match two adjacent faces of the graph have been closed. The preceding subroutine is revised in an obvious way to close out both matched edges: those c.f. rows are retained which are compatible with the indicated edge allocations. Go to 1.

The process is terminated when the open edge list is vacated. If this leaves some nodes unused, no Hamilton circuit is possible. Otherwise, the final closure of circuit-fragments leaves a table of circuits. This must still be scanned to separate the Hamiltonian circuits from the set of pairwise disjoint circuits.

The efficiency of the algorithm depends on keeping the current c.f. table as small as possible. This is accomplished by a lookahead routine which scans prospective choices of current edges to seek the promptest closure of a face.

For an example, Tutte's 46 node non-Hamiltonian graph has been searched exhaustively. This required a c.f. table of 12,477 rows consuming 29 seconds of a program on IBM 7090. Searches yielding all the circuits of other large Hamiltonian graphs required a comparable effort.

This procedure may have some utility for studies on classification, isomorphisms, and symmetries of abstract graphs and other network problems for which the set of Hamilton circuits is often an advantageous approach. A complete description of the computer program is available from the author.

This work has been supported by a grant from NASA (NsG 81-60) for studies on automated experimental analysis. Computations were run on an IBM 7090 at the Stanford Computation Center with support from the National Science Foundation, under grant NSF-GP-948.

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SPAN OF DERIVATIVES OF POLYNOMIALS

A. MEIR AND A. SHARMA, University of Alberta

1. We denote by π_n the class of polynomials $P(x)$ of exact degree n having n real zeros $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ and, following [1] we define the span $\sigma(P)$ of the polynomial $P(x)$ to be $x_n - x_1$. Robinson has considered the problem of determining the maximum span of the kth derivative of polynomials of π_n, when the span of the polynomial itself was given. In fact he proved, that if

$$\sigma(P) \leqq 2 \tag{1}$$

then max $\sigma(P^{(k)})$ for $1 \leqq k \leqq n-2$ is obtained when $P(x) = c(1-x)^p(1+x)^{n-p}$. For $k = n-2$, $n-3$ he determined the exact values of the corresponding spans.

The object of this note is to replace the condition (1) by restrictions on the first or second moments of the zeros $x_1, x_2, \cdots, x_n$. Related problems under similar conditions have been treated in [2; p. 140, Theorems 6.7.2, 6.7.3, and p. 379, Problem 38]. One of the problems considered therein is about the distribution of points $x_1, x_2, \cdots, x_n$ on the real line which maximize the absolute value of the Vandermondian $V(x_1, x_2, \cdots, x_n)$ under the restriction

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leqq 1.$$

2. We prove the following results:

THEOREM 1. *Let π_n^1 be the subclass of polynomials $P(x) \in \pi_n$ with zeros $\{x_i\}_{i=1}^n$ satisfying the conditions*

$$x_i \geqq 0, \quad i = 1, 2, \cdots, n; \qquad n^{-1}\sum_{i=1}^{n} x_i \leqq 1, \tag{2}$$

and let $k \leqq n-2$. Then

$$\max_{P\in\pi_n^1} \sigma(P^{(k)}) = n - k. \tag{3}$$

and the maximum is attained only for $P(x)=cx^{n-1}(x-n)$, *where* $c\neq 0$ *is an arbitrary constant.*

THEOREM 2. *Let* π_n^2 *be the subclass of polynomials* $P(x)\in\pi_n$ *with zeros* $\{x_i\}_{i=1}^n$ *satisfying the condition*

$$n^{-1}\sum_{i=1}^{n} x_i^2 \leqq 1, \tag{4}$$

and let $k\leqq n-2$. *Then*

$$\max_{P\in\pi_n^2} \sigma(P^{(k)}) = \left[\frac{2(n-k)(n-k-1)}{n-1}\right]^{1/2} \tag{5}$$

and the maximum is attained only for polynomials of the form

$$P(x) = cx^{n-2}\left(x^2 - \frac{n}{2}\right) + Q(x) \tag{6}$$

where $c\neq 0$ *is an arbitrary constant and* $Q(x)$ *is a polynomial of degree* $\leqq k-1$.

3. Proofs. Denote by $\xi_1\leqq\xi_2\leqq\cdots\leqq\xi_{n-k}$ the necessarily real zeros of $P^{(k)}(x)$, and set $P(x)=c(x^n-s_1x^{n-1}+s_2x^{n-2}-\cdots)$. Then it is well known that

$$s_1 = \sum_{i=1}^{n} x_i, \qquad s_2 = \sum_{1\leqq i<j\leqq n} x_ix_j \tag{7}$$

and easily seen that

$$\sum_{j=1}^{n-k} \xi_j = \frac{n-k}{n} s_1, \tag{8}$$

$$\sum_{1\leqq i<j\leqq n-k} \xi_i\xi_j = \frac{(n-k)(n-k-1)}{n(n-1)} s_2. \tag{9}$$

From (8) and (9) one obtains

$$\Lambda \equiv \sum_{1\leqq i<j\leqq n-k} (\xi_j - \xi_i)^2 = \frac{(n-k-1)(n-k)^2}{n(n-1)}\left\{\sum_{i=1}^{n} x_i^2 - n^{-1}s_1^2\right\}. \tag{10}$$

To prove Theorem 1 we observe that $\xi_i\geqq 0$ for all i. Then by (8) and (2)

$$\max_i \xi_i \leqq s_1\frac{n-k}{n} \leqq n-k$$

with equality only if $s_1=n$ and $\xi_1=\cdots=\xi_{n-k-1}=0$ and $\xi_{n-k}=n-k$. Hence $\sigma(P^{(k)})\leqq n-k$ with equality only in the above case. Thus the maximum span is attained for $P^{(k)}(x)=cx^{n-k-1}(x-n+k)$. Since the zeros of $P(x)$ are real and $\geqq 0$,

$P(x)$ must have $n-1$ zeros at $x=0$, so that $s_1=x_n=n$. Thus $P(x)$ is of the form $P(x)=cx^{n-1}(x-n)$. In fact

$$[x^{n-1}(x-n)]^{(k)} = \frac{n!}{(n-k)!}x^{n-k-1}(x-n+k).$$

In order to prove Theorem 2, we observe from (4) and (10) that

$$\Lambda \leqq \frac{(n-k-1)(n-k)^2}{n-1} \tag{11}$$

with equality only if $s_1=0$ and $\sum_{i=1}^{n} x_i^2=n$. Now, since $\sigma\equiv\sigma(P^{(k)})=\xi_{n-k}-\xi_1$ we find, by using the identity $A^2+B^2=\frac{1}{2}(A+B)^2+\frac{1}{2}(A-B)^2$, that

$$\Lambda \geqq \sum_{j=2}^{n-k-1}\{(\xi_{n-k}-\xi_j)^2+(\xi_j-\xi_1)^2\}+(\xi_{n-k}-\xi_1)^2 \geqq \frac{n-k}{2}\sigma^2 \tag{12}$$

with equality only if $2\xi_j=\xi_1+\xi_{n-k}$ for $2\leqq j\leqq n-k-1$. Therefore by (11) and (12)

$$\sigma^2 \leqq \frac{2(n-k)(n-k-1)}{n-1} \tag{13}$$

with equality only if

$$s_1=0, \quad \sum_{i=1}^{n} x_i^2 = n \quad \text{and} \quad 2\xi_j=\xi_1+\xi_{n-k} \quad \text{for } 2\leqq j\leqq n-k-1. \tag{14}$$

Now from (8) and (14)

$$\sum_{j=1}^{n-k}\xi_j = \frac{n-k}{2}(\xi_1+\xi_{n-k}) = 0,$$

so that $\xi_j=0$ for $2\leqq j\leqq n-k-1$, and $|\xi_1|=|\xi_{n-k}|$. Then from (13) we get

$$\xi_1^2 = \frac{(n-k-1)(n-k)}{2(n-1)},$$

so that the maximum span is attained when

$$P^{(k)}(x) = c_1x^{n-k-2}\left(x^2-\frac{(n-k-1)(n-k)}{2(n-1)}\right).$$

Integrating k times we get (6).

4. An analogous problem is to find the "minimum span" of the kth derivative of polynomials $P(x)$ having a given span. More precisely, consider the class π_n^* of polynomials of the form $(x^2-1)Q(x)$ where $Q(x)$ is of degree $n-2$ having all zeros in $[-1, 1]$. The problem is to determine

$$\min_{P\in\pi_n^*}\sigma(P^{(k)}).$$

For the special case $k=n-2$, the solution is easy to obtain from (10). Indeed in this case,

$$\Lambda = \sigma^2 = \frac{4}{n(n-1)}\left\{2 + \sum_{i=1}^{n-2} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n-2} x_i\right)^2\right\} \geqq \frac{8}{n(n-1)}$$

with equality only if $x_i=0$ for $i=1, \cdots, n-2$ which follows from Schwarz's inequality. Therefore in this case

$$\min_{P\in\pi_n^*} \sigma(P^{(n-2)}) = 2\sqrt{\frac{2}{n(n-1)}}.$$

This minimum is attained for $P(x)=cx^{n-2}(x^2-1)$.

In the general case the problem seems to be more involved, but we conjecture that, for any k, the polynomial yielding the solution of the problem will be $cx^{n-2}(x^2-1)$.

Two more related problems can be formulated as follows in the notation already used:

PROBLEM I. Let $P(x)\in\pi_n$ satisfying either of the conditions:

$$\text{(i)}\ |x_i| \leqq 1, \qquad \text{(ii)}\ \frac{1}{n}\sum_1^n x_i \leqq 1,\quad x_i \geqq 0, \qquad \text{(iii)}\ \frac{1}{n}\sum_1^n x_i^2 \leqq 1.$$

To find the polynomial in the corresponding class which maximizes

$$\min_{2\leqq j\leqq n-k} |\xi_j - \xi_{j-1}|.$$

PROBLEM II. If $P(x)\in\pi_n^*$, find the polynomial which minimizes

$$\max_{2\leqq j\leqq n-k} |\xi_j - \xi_{j-1}|.$$

When $k=n-2$, Problem I is equivalent to maximizing the span under the different conditions. In case (i) this reduces to the result of Robinson [1, p. 507] and can be obtained directly from our formula (10), since in this case we have

$$\sigma^2 = \Lambda = \frac{4}{n(n-1)}\left\{\sum_1^n x_i^2 - \frac{1}{n}\left(\sum_1^n x_i\right)^2\right\}.$$

Obviously, when $|x_i| \leqq 1$, we have

$$\sigma(P^{(n-2)}) = \frac{2}{\sqrt{(n-1)}}, \quad \text{when } n \text{ is even}$$

$$= \frac{2\sqrt{(n+1)}}{n}, \quad \text{when } n \text{ is odd.}$$

The case (ii), (iii) of Problem I and Problem II are in this special case equivalent to the case $k=n-2$ of Theorems 1 and 2.

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REPRESENTATION OF AN INTEGER AS A SUM OF FOUR INTEGER CUBES AND RELATED PROBLEMS

D. A. KLARNER, McMaster University

To my friend and teacher J. E. Householder

In 1936, L. J. Mordell [5] showed that integers belonging to all but 54 of the residue classes modulo 216 have at least one representation as a sum of four integer cubes; the proof is accomplished for a given r by writing an identity of the form

$$\sum_{i=1}^{4} (a_i k + b_i)^3 = 216k + r, \tag{1}$$

where k is a parameter and $a_1, a_2, \cdots, b_1, b_2, \cdots$ are certain integers depending on r. Sierpinski [7] seems to have been the first to conjecture that

$$k = x_1^3 + x_2^3 - x_3^3 - x_4^3 \tag{2}$$

has infinitely many solutions in nonnegative integers $x_1, x_2, \cdots$ for any given integer k and he and Schinzel [6] and Makowski [4] have proved the conjecture for all natural numbers $k \leqq 350$ except 148 and 284, as well as an infinite number of others.

It is known that every integer has infinitely many representations as a sum of five integer cubes; and considerations involving congruences reveal that numbers in the residue classes ± 4 modulo 9 cannot be the sum of three integer cubes; it has been conjectured that the numbers in the remaining classes do have such representations. The smallest number for which the conjecture is in doubt is 30; machine computations carried out by Gardiner, Lazarus and Stein [2] seem to indicate that numbers which have one representation as a sum of three integer cubes have many. It is an open question whether any number having one representation as a sum of three integer cubes has infinitely many. For example, Mordell has expressed interest in the equation

$$x_1^3 + x_2^3 + x_3^3 = 3 \tag{3}$$

for which only two essentially different solutions are known; viz., $1^3+1^3+1^3=3$ and $4^3+4^3-5^3=3$.

In this note we solve certain cubic Diophantine equations in binary quadratic forms; for example, we treat

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = kx_5^3, \tag{4}$$

with $k=a_1^3+a_2^3+a_3^3+a_4^3$, in order to prove a theorem due to Mordell [3] which plays a central role in the results of Sierpinski and Schinzel mentioned above. Solutions to (4) in binary quadratic forms when $k=0$ have been given already; for example, the identities

$$\begin{aligned}(3u^2 + 5uv - 5v^2)^3 + (4u^2 - 4uv + 6v^2)^3 + (5u^2 - 5uv - 3v^2)^3 \\ + (-6u^2 + 4uv - 4v^2)^3 = 0,\\ (6u^2 + 20uv - 56v^2)^3 + (u^2 - 7uv + 63v^2)^3 + (8u^2 - 20uv - 42v^2)^3 \\ + (-9u^2 + 7uv - 7v^2)^3 = 0,\\ (3u^2 + 16uv - 7v^2)^3 + (6u^2 - 4uv + 14v^2)^3 + (-6u^2 - 4uv - 14v^2)^3 \\ + (-3u^2 + 16uv + 7v^2)^3 = 0,\end{aligned}$$

were discovered by Ramanujan [3], Osborn, and Teilhet [1] respectively. Actually, writing $u+v$ and $u-v$ in place of u and v, respectively, in Ramanujan's identity gives Teilhet's identity, so one is just a linear transformation of the other.

To solve (4) we begin by seeking complex numbers b_1 and b_2 such that the matrix multiplication

$$\begin{bmatrix} a_1 & b_1 & a_2 \\ a_2 & -b_1 & a_1 \\ a_3 & b_2 & a_4 \\ a_4 & -b_2 & a_3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u^2 \\ uv \\ v^2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \tag{5}$$

defines forms $x_1, x_2, \cdots$ which satisfy (4). The 5×3 matrix in (5) is called a *coefficient matrix*, and multiplying this by the *parameter matrix* gives the *forms* $x_1, x_2, \cdots$. If $a_1, a_2, \cdots, b_1, b_2,$ are complex numbers such that (4) is an algebraic identity, we can make homogeneous linear substitutions $u\to\alpha_1 u+\beta_1 v$, $v\to\alpha_2 u+\beta_2 v$, where $\alpha_1, \cdots$ are complex numbers, and the relation in (4) remains an identity; only the coefficients of u^2, uv, and v^2 are changed. We say the new forms resulting from this substitution are *equivalent* to the originals, and that the corresponding new coefficient matrix is *equivalent* to the one we started with.

Equating the coefficients of u^6, u^5v, $\cdots$ in (5) gives only two equations which must be satisfied by b_1 and b_2; namely,

$$(a_1^2 - a_2^2)b_1 + (a_3^2 - a_4^2)b_2 = 0, \tag{6}$$

$$(a_1 + a_2)b_1^2 + (a_3 + a_4)b_2^2 = a_1^3 + a_2^3 + a_3^3 + a_4^3 - a_1a_2(a_1 + a_2) - a_3a_4(a_3 + a_4). \tag{7}$$

Solving this system for b_1 and b_2 gives

$$b_1 = (a_4^2 - a_3^2)h^{-1/2}, \qquad b_2 = (a_1^2 - a_2^2)h^{-1/2}, \tag{8}$$

where

$$h = (a_1 + a_2)(a_3 + a_4). \tag{9}$$

Substituting the expressions for b_1 and b_2 given in (8) into the coefficient matrix in (5) gives forms satisfying (4). Making the transformation $u \to u$, $v \to vh^{1/2}$ gives a coefficient matrix equivalent to the one just found but having the additional property that all of the elements are integers. Thus,

$$\begin{bmatrix} a_1 & a_4^2 - a_3^2 & a_2h \\ a_2 & a_3^2 - a_4^2 & a_1h \\ a_3 & a_1^2 - a_2^2 & a_4h \\ a_4 & a_2^2 - a_1^2 & a_3h \\ 1 & 0 & h \end{bmatrix} \begin{bmatrix} u^2 \\ uv \\ v^2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \tag{10}$$

is a solution of (4) in binary quadratic forms; this solution can be used to find infinitely many solutions of (4) in integers provided $a_1 \neq \pm a_2$ and $a_3 \neq \pm a_4$, otherwise $h = 0$ and the forms in (10) become trivial. Furthermore, the Pellian equation

$$x_5 = u^2 + (a_1 + a_2)(a_3 + a_4)v^2 = 1 \tag{11}$$

has infinitely many solutions u, v provided only that $(a_1+a_2)(a_3+a_4)$ is the negative of a nonsquare number, so that we have given a new proof of Mordell's

THEOREM 1. *If $k = a_1^3+a_2^3+a_3^3+a_4^3$ with $-(a_1+a_2)(a_3+a_4)$ a nonsquare and $a_1 \neq a_2$, $a_3 \neq a_4$, then k has infinitely many representations as a sum of four integer cubes.*

Equation (4) is a special case of the more general equation

$$f(x_1, x_2) + f(x_3, x_4) = kx_5^3, \tag{12}$$

where $k = f(a_1, a_2) + f(a_3, a_4)$ and $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ is a given binary cubic form. We can solve (12) in the same way we solved (4), and this result leads in turn to a more general version of Theorem 1. This time we set

$$(13)\qquad \begin{bmatrix} a_1 & b_1 & a_3 \\ a_2 & b_2 & a_4 \\ a_3 & -b_1 & a_1 \\ a_4 & -b_2 & a_2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u^2 \\ uv \\ v^2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

and again find b_1 and b_2 in terms of $a_1, a_2, \cdots$ using the system of equations found by equating coefficients of like powers of u and v in (12); this system consists of just two equations, the other five are either identities or are equivalent to these:

$$(14)\qquad \{3a(a_1^2 - a_3^2) + 2b(a_1a_2 - a_3a_4) + c(a_2^2 - a_4^2)\}b_1 + \{3d(a_2^2 - a_4^2) + ac(a_1a_2 - a_3a_4) + b(a_1^2 - a_3^2)\}b_2 = 0,$$

$$(15)\qquad \{3a(a_1 + a_3) + b(a_2 + a_4)\}b_1^2 + \{2b(a_1 + a_2) + 2c(a_2 + a_4)\}b_1b_2 + \{3d(a_2 + a_4) + c(a_1 + a_3)\}b_2^2 + \{3a(a_1^2a_3 + a_3^2a_1) + b(a_1^2a_4 + a_3^2a_2 + 2a_1a_2a_3 + 2a_1a_3a_4) + c(a_2^2a_3 + a_4^2a_1 + 2a_1a_2a_4 + 2a_2a_3a_4) + 3d(a_2^2a_4 + a_4^2a_2)\} = 0.$$

Making obvious substitutions, we can abbreviate (14) and (15) by writing $K_1, K_2, \cdots$ for constants; thus,

$$(14')\qquad K_1b_1 + K_2b_2 = 0,$$

$$(15')\qquad K_3b_1^2 + K_4b_1b_2 + K_5b_2^2 + K_6 = 0.$$

Solving for b_1 and b_2 gives

$$(16)\qquad b_1 = -K_2(p/q)^{1/2}, \qquad b_2 = K_1(p/q)^{1/2}$$

where p and q are the numerator and denominator respectively of the reduced fraction

$$(17)\qquad -K_6/(K_3K_2^2 - K_1K_2K_4 + K_1^2K_5).$$

Substituting the values of b_1 and b_2 in (16) into the coefficient matrix in (13) and making the transformation $u \to u$, $v \to v(pq)^{1/2}$ gives a coefficient matrix having integral elements:

$$(18)\qquad \begin{bmatrix} a_1 & -K_2p & a_3pq \\ a_2 & K_1p & a_4pq \\ a_3 & K_2p & a_1pq \\ a_4 & -K_1p & a_2pq \\ 1 & 0 & pq \end{bmatrix} \begin{bmatrix} u^2 \\ uv \\ v^2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

The forms $x_1, \cdots$ defined by (18) satisfy (12) and if pq is the negative of a natural number which is not a square, $x_5 = u^2 + pqv^2$ is unity for infinitely many values of u and v, and hence

$$f(x_1, x_2) + f(x_3, x_4) = k \tag{19}$$

has infinitely many solutions in integers $x_1, x_2, \cdots$.

The fact that infinitely many integers k exist such that (19) has infinitely many solutions in integers $x_1, x_2, \cdots$ leads one to guess that under the right conditions every number should have such a representation. A. Schinzel in reading the first draft of this paper made the following comment: "Clearly, if $f(x, y) = ax^2 + bx^2y + cxy^2 + dy^3$ and (21) is to have a solution for every k we must have $(a, b, c, d) = 1$, but this is not sufficient for the solvability of the congruence

$$f(x_1, x_2) + f(x_3, x_4) \equiv k \pmod{m}$$

as shown by the example $f(x, y) = x^3 + 7y^3$, $k = 3$ and $m = 7$. It should not be difficult to prove that if the congruence is solvable modulo 7 and 9 and $(a, b, c, d) = 1$ then it is solvable for every modulus. (I have proved this for prime moduli.)"

When $k = 0$, (19) can be written

$$f(x_1, x_2) = f(x_3, x_4). \tag{20}$$

Here the problem is to find numbers which have two representations in a given binary cubic form $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$. We note that if one solution, $f(a_1, a_2) = f(a_3, a_4)$ with $a_1 \neq a_3$, $a_2 \neq a_4$, of (20) is known, (18) can be used to find a solution of (20) in binary quadratic forms which in turn can be used to find infinitely many solutions in relatively prime integers. It may be of interest to see how far one can go in solving (20) in binary quadratic forms and to this end I have made an investigation of the special case

$$ax_1^3 + dx_2^3 = ax_3^3 + dx_4^3. \tag{21}$$

Starting with one solution $x_i = a_i$ of (21), a modified version of the coefficient matrix in (18) can be found which produces a solution in binary quadratic forms; thus the forms defined in

$$\begin{bmatrix} a_1 & (a_4^2 - a_2^2)dl^{-1/2} & a_3 \\ a_2 & (a_1^2 - a_3^2)al^{-1/2} & a_4 \\ a_3 & (a_4^2 - a_2^2)dl^{-1/2} & a_1 \\ a_4 & (a_1^2 - a_3^2)al^{-1/2} & a_2 \end{bmatrix} \begin{bmatrix} u^2 \\ uv \\ v^2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{22}$$

where $l = -ad(a_1 - a_3)(a_2 - a_4)$, satisfy (21).

Making the transformation $u \to u + v$, $v \to u - v$, the coefficient matrix in (22) is seen to be equivalent to

$$(23)\qquad \begin{bmatrix} a_1+a_3+(a_4^2-a_2^2)dl^{-1/2} & 2(a_1-a_3) & a_1+a_3-(a_1^2-a_2^2)dl^{-1/2} \\ a_2+a_4+(a_1^2-a_3^2)al^{-1/2} & 2(a_2-a_4) & a_2+a_4-(a_1^2-a_3^2)al^{-1/2} \\ a_1+a_3+(a_4^2-a_2^2)dl^{-1/2} & -2(a_1-a_3) & a_1+a_3-(a_4^2-a_2^2)dl^{-1/2} \\ a_2+a_4+(a_1^2-a_3^2)al^{-1/2} & -2(a_2-a_1) & a_2+a_4-(a_1^2-a_3^2)al^{-1/2} \end{bmatrix}$$

which in turn has the form

$$(24)\qquad \begin{bmatrix} R(-f_1f_2ad)^{1/2}/af_1 & 2f_1 & -\overline{R}(-f_1f_2ad)^{1/2}/af_1 \\ R & 2f_2 & \overline{R} \\ -R(-f_1f_2ad)^{1/2}/af_1 & -2f_1 & -\overline{R}(-f_1f_2ad)^{1/2}/af_1 \\ R & -2f_2 & \overline{R} \end{bmatrix}$$

where R and its conjugate $\overline{R}$ are elements of the simple extension $K[(f_1f_2ad)^{-1/2}]$ of the field of rationals. We continue as before, equating the coefficients of u^6, u^5v, $\cdots$ after substituting the forms defined by (24) into the equation under discussion. The resulting system is relatively simple and the Diophantine problems can be solved by old methods. As an example, one can solve the system associated with (24) when the coefficient matrix has the special form

$$(25)\qquad \begin{bmatrix} A & 2C & A \\ B & 2D & -B \\ A & -2C & A \\ B & -2D & -B \end{bmatrix}$$

with A and B arbitrary integers and C and D to be determined in terms of A, B, a and d. The result after a transformation is

$$(26)\qquad \begin{bmatrix} A(aA^3+dB^3) & -6dAB^3 & 3A(aA^3-dB^3) \\ B(aA^3+dB^3) & 6aA^3B & -3B(aA^3-dB^3) \\ A(aA^3+dB^3) & 6dAB^3 & 3A(aA^3-dB^3) \\ B(aA^3+dB^3) & -6aA^3B & -3B(aA^3-dB^3) \end{bmatrix}.$$

The identity corresponding to this coefficient matrix can be used to prove that every rational number a/d has infinitely many representations in the form $(x_1^3+x_2^3)/(x_3^3+x_4^3)$ where the x's are binary quadratic forms or relatively prime integers.

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A CHARACTERIZATION OF FINITE FIELDS

BROTHER JOSEPH HEISLER, C.S.C., St. Edward's University, Austin, Texas

Let R be a ring and let $f(x)\in R[x]$, say $f(x)=a_nx^n+\cdots+a_1x+a_0$. By the left functional value of $f(x)$ at $b\in R$ I shall mean $a_nb^n+\cdots+a_1b+a_0$, which will be denoted by $f_L(b)$.

If R is a ring and f is a mapping of R into R, then f is said to be left represented by a polynomial if there exists an $\bar{f}(x)\in R[x]$ such that $f(r)=\bar{f}_L(r)$ for each $r\in R$.

The purpose of this paper is to establish the following characterization of finite fields: if R is a nonzero ring then R is a finite field if and only if each function f mapping R into R is left represented by a polynomial.

If k is a finite integral domain, and so a finite field, then each function f mapping k into k is a polynomial function. This is no longer true if k is a finite ring with zero divisors. Consider for example the ring Z_4 of rational integers modulo 4 and the function $f: Z_4\to Z_4$ where $f(3)=1$ and $f(r)=0$ for $r=0, 1, 2$. Suppose that there exist elements a_i in Z_4 such that $f(x)=a_0+a_1x+\cdots+a_sx^s$ for each x in Z_4. Since $x^2(x^2-1)=0$ for each x in Z_4 we may assume that $s<4$. Specializing, we get

$$f(0) = a_0 = 0,$$

$$f(1) = a_1 + a_2 + a_3 = 0, \tag{1}$$

$$f(2) = 2a_1 = 0, \tag{2}$$

$$f(3) = 3a_1 + a_2 + 3a_3 = 1. \tag{3}$$

Equation (2) implies that $a_1=0$ or 2. This, (1), and (3) imply that $2a_3=1$, which is not solvable in Z_4.

In the above example, the ring R had zero divisors, but if R is a ring with unity such that each function mapping R into R is left represented by a polynomial, then R must be a division ring. The following theorem establishes this and more.

THEOREM 1. *Let R be a ring with unity and let R^* denote the group of units of R. Suppose that there exist a function f mapping R into R and an $\bar{f}(x)\in R[x]$ such*

that $f(0)=\bar{f}_L(0)=0$ *and* $f(r)=\bar{f}_L(r)$ *is in* R^* *for* $r\neq 0$ *and* $r\in(R-R^*)$. *Then* R *is a division ring.*

Proof. Suppose that $f(r)=a_0+a_1r+\cdots+a_mr^m$ for each $r\in(R-R^*)$. Then $f(0)=0$ implies that $a_0=0$. For $r\in(R-R^*)$ and $r\neq 0$, $f(r)=a_1r+\cdots+a_mr^m=(a_1+\cdots+a_mr^{m-1})r=u(r)$ is in R^*. Thus $(u(r))^{-1}(a_1+\cdots+a_mr^{m-1})r=1$. Hence, each nonzero element of $R-R^*$ has a left inverse. Since elements of R^* have left inverses we see that each nonzero element of R has a left inverse in R. It is now an elementary exercise to show that the nonzero elements of R form a group with respect to multiplication.

We note that each finite ring satisfying the hypotheses of Theorem 1 must be a field. A ring satisfying the hypotheses of Theorem 1 will not generally be a field. For example, the quaternions over the reals with $f(x)=x$ is such a ring. We also note that the requirement that the function f of the hypotheses be left representable on $(R-R^*)$ by a polynomial is necessary. For example, let R be the rational integers, so that $R^*=\{-1, 1\}$. Then R is not a division ring but the function $f: R\to R$ such that $f(x)=0$ for $x\in R^*$ or $x=0$, and $f(x)=1$ otherwise, satisfies the hypotheses of the theorem other than being left represented on $(R-R^*)$ by a polynomial.

For the rest of this paper we assume that R is a nonzero ring such that each function f mapping R into R is left represented by a polynomial. Then R is a finite ring. We now show that R has no zero divisors. To see this, suppose that there exist elements u and v in R, $u\neq 0$ and $v\neq 0$, such that $uv=0$. Let f be a function mapping R into R such that $f(0)=0$ and $f(u)=v$. Then there is a polynomial $\bar{f}(x)=a_mx^m+\cdots+a_1x$ in $R[x]$ such that $f(r)=\bar{f}_L(r)$ for each $r\in R$. Then $f(u)=(a_mu^{m-1}+\cdots+a_1)u=\bar{g}(u)u=v$. Hence, $0=\bar{g}(u)uv=f(u)v=v^2$ and $f(v)=\bar{g}(v)v$, and so $f(v)v=\bar{g}(v)v^2=0$. Since there are functions f such that $f(0)=0$ and $f(u)=v$, with $f(v)$ arbitrary in R, we must have that $Rv=0$. In particular, there are such f with $f(v)\neq 0$, contrary to the fact that $f(v)=\bar{g}(v)v$ which is in $Rv=0$. Thus R can have no zero divisors.

In summary, if R is a nonzero ring such that each function mapping R into R is left represented by a polynomial, then R is finite, has no zero divisors, and so is a finite field. Thus we have

THEOREM 2. *If* R *is a nonzero ring, then* R *is a finite field if and only if each function mapping* R *into* R *is left represented by a polynomial.*

SUBJECTIVE

He kept at math, ungraduated,
Unable to decide
If it were overcomplicated
Or he too simplified.

MARLOW SHOLANDER

MATHEMATICAL NOTES

SETS OF NUMBERS DEFINED BY FINITE AUTOMATA

J. HARTMANIS, Cornell University and R. E. STEARNS, G.E. Research Laboratory

1. Introduction. In this note we investigate the topological properties of sets of real numbers defined by finite automata. The concept of an automaton "defining" a number is obtained through a natural extension of the concept of an automaton accepting a finite sequence. Under this definition, we show that a finite automaton defines a closed set of numbers, we characterize those automata whose defined sets contain nontrivial intervals, and show that the end points of these intervals are rational numbers. Furthermore, we show that the measure of a set defined by an automaton is always rational.

2. Automata and defined sets. A *finite automaton* M is a quintuplet

$$M = (S, I, \delta, s_0, T),$$

where S and I are the finite nonempty sets of states and inputs,

$$\delta: \quad S \times I \rightarrow S$$

is the next state function, s_0 is the starting state and $T \subseteq S$ is the set of terminal states.

An input sequence

$$\bar{x} = x_1 x_2 \cdots x_n, \qquad x_i \text{ in } I,$$

is *accepted* by M if and only if it transfers s_0 to a terminal state, which we write symbolically as $\bar{\delta}(s_0, \bar{x}) \in T$, where $\bar{\delta}$ is the natural extension of δ to input sequences. The set of sequences accepted by M is denoted by R_M. Two states s and t of an automaton $M = (S, I, \delta, s_0, T)$ are said to be *equivalent* if and only if the automata

$$M' = (S, I, \delta, s, T) \quad \text{and} \quad M'' = (S, I, \delta, t, T)$$

accept the same set of sequences, $R_{M'} = R_{M''}$. An automaton is *reduced* if and only if no two distinct states are equivalent and all its states can be reached from the starting state. A more detailed discussion of finite automata can be found in [1].

DEFINITION 1. *Let r be an integer greater than 1 and let M be a finite automaton whose set I of inputs is the set of digits used in representing numbers in radix r notation. A real number α between 0 and 1 is said to be defined by M if and only if α has at least one representation $\alpha = \cdot\alpha_1\alpha_2 \cdots \alpha_n \cdots$ in radix r notation such that for each integer $n \geqq 1$ the prefix $\alpha_1\alpha_2 \cdots \alpha_n$ is accepted by M. The set of numbers defined by M is denoted by D_M.*

DEFINITION 2. *An automaton M is a prefix acceptor if and only if*

$$\bar{x}\bar{y} \text{ in } R_M \text{ implies that } \bar{x} \text{ is in } R_M.$$

We say that a state s of M is a trapping state if and only if for all x in I, $\delta(s, x) = s$.

We now derive some properties of prefix acceptors.

LEMMA 1. *A reduced automaton M is a prefix acceptor if and only if there is at most one nonterminal state and the nonterminal state is a trapping state.*

Proof. If M satisfies the hypotheses, then it is a prefix acceptor since once a sequence $\bar{x}$ is not accepted, all its continuations $\bar{x}\bar{y}$ are also not accepted. To prove the converse, let M be a reduced prefix acceptor. The states in $S-T$ must all be equivalent since all input sequences must transform them to nonterminals. Therefore, since M is reduced, $S-T$ can contain at most one state which can only be transformed into itself.

LEMMA 2. *A reduced prefix acceptor, M, defines a set which contains a nontrivial interval if and only if it has a trapping terminal state.*

Proof. We assume without loss of generality that $I = \{0, 1\}$. If the defined set contains an interval, then for some sequence $\bar{x} = x_1x_2 \cdots x_n$, the interval contains all α such that

$$\cdot x_1x_2 \cdots x_n0 \leqq \alpha \leqq \cdot x_1x_2 \cdots x_n1.$$

It follows that, for all finite sequences $\bar{y}$, the sequence

$$x_1x_2 \cdots x_n0\bar{y}$$

is accepted by M. Since M is reduced, there can be at most one state from which all continuations are accepted and the state

$$\bar{\delta}(s_0, x_1 \cdots x_n0) = s$$

must therefore be a trapping terminal state. Conversely, if M has a trapping terminal state, s, and

$$\bar{\delta}(s_0, \bar{x}) = s,$$

then for all $\bar{y}$, $\bar{\delta}(s_0, \bar{x}\bar{y}) = s$. But then all the α in the interval $\cdot\bar{x} \leqq \alpha \leqq \cdot\bar{x}1$ are defined by M and D_M contains a nontrivial interval.

THEOREM 1. *A prefix acceptor defines a closed set.*

Proof. We will prove that the complement of D_M is an open set. For simplicity we assume that $I = \{0, 1\}$. Let α be a point of the complement of D_M. We may speak of the left (lexicographically first) and of the right (lexicographically last) representation of α in radix r notation; these coincide in the case where α has only one representation. There exist prefixes $\bar{x}$ and $\bar{y}$ of the left and right representations, respectively, such that neither $\bar{x}$ nor $\bar{y}$ is accepted by M and we have

$$\cdot\bar{x}000 \cdots < \alpha < \cdot\bar{y}111 \cdots.$$

If

$$\cdot\bar{x}000\cdots < \beta < \cdot\bar{y}111\cdots,$$

then every representation of β has one of the sequences $\bar{x}$ and $\bar{y}$ as a prefix, and hence β is not accepted by M. This shows that the complement of D_M contains an open interval about α.

THEOREM 2. *The end points of a maximal interval contained in D_M are rational.* (The main result of [2] is a corollary of this theorem.)

Proof. We assume without loss of generality that $I = \{0, 1\}$. Let $\alpha = \cdot\alpha_1\alpha_2\alpha_3\cdots$ be the upper end point of a maximal interval in D_M. Then for some n, the interval contains the set of points

$$\{\beta \mid \cdot\alpha_1\alpha_2\cdots\alpha_n \leqq \beta \leqq \alpha\}.$$

First suppose that $\bar{\delta}(s_0, \alpha_1\alpha_2\cdots\alpha_p)$ is a trapping terminal state for some p. Then the rational number $\alpha' = \cdot\alpha_1\alpha_2\cdots\alpha_p 1\,1\,1\cdots$ must be in the interval and must be therefore the end point α since no larger value is possible.

Next assume that no prefix of α leads to a trapping terminal state. In this case, for all $p, q \geqq n$,

$$\bar{\delta}(s_0, \alpha_1\alpha_2\cdots\alpha_{p-1}) = \bar{\delta}(s_0, \alpha_1\alpha_2\cdots\alpha_{q-1})$$

must imply that $\alpha_p = \alpha_q$ because $\alpha_p = 1$ and $\alpha_q = 0$ would imply that, contrary to our hypothesis,

$$\bar{\delta}(s_0, \alpha_1\alpha_2\cdots\alpha_{p-1}0) = \bar{\delta}(s_0, \alpha_1\alpha_2\cdots\alpha_{q-1}\alpha_q)$$

is a trapping terminal because all continuations of $\alpha_1\alpha_2\cdots\alpha_{p-1}0$ must be accepted. Thus, for $m > n$, the value of α_m is determined by the state

$$s = \bar{\delta}(s_0, \alpha_1\alpha_2\cdots\alpha_{m-1})$$

and this in turn determines the next state $\delta(s, \alpha_m)$. Since there are only a finite number of states, the state sequence and the α_m must be periodic and α must be rational.

A very similar proof shows that the lower end point is rational.

We now state a result which is used to show that the measure of a set defined by a finite automaton is rational.

LEMMA 3. *Let C be a directed graph with rational transition probabilities, a starting node and a trapping node. Then the probability that the trapping node is entered is a rational number.* (This result can also be obtained from Markov Theory.)

Proof. (Outline) The proof can be done by induction on the number of nontrapping nodes.

(1) Clearly the result is true if there is only one nontrapping node.

(2) If there are $k+1$ nontrapping nodes, a new k node graph can be constructed from it which has the same probability in falling into the trapping node. This is done by removing a nontrapping, non-starting node and rerouting the arrows to perform "double node changes" with the appropriate new (rational) probabilities. Some of these "double node changes," owing to the possibility that there are immediate transitions from a node to itself, must consist of an infinite set of finite length sequences. In such a case the rational transition probability is obtained by summing a geometric series.

THEOREM 3. *The measure of a set defined by a finite automaton is rational.*

Proof. Let M be a reduced machine and again assume that $I=\{0, 1\}$. If equal probabilities are assigned to inputs, then the measure of the complement of D_M is seen to be the probability that s_0 goes into the nonterminal trapping state. This is because each sequence $\alpha_1\alpha_2 \cdots \alpha_n$ that transforms s_0 for the first time into the trapping nonterminal state must correspond to an interval of length 2^{-n} in the complement of D_M and the probability that this input sequence occurs is also 2^{-n}. By the previous lemma we know that this probability of entering the trapping nonterminal state is rational, and thus the measure of the complement of D_M is rational, and therefore the measure of D_M is rational.

For a given rational α, the set of numbers less than or equal to α can be defined by an automaton and has measure α. Thus measures are achieved for all rationals. Some interesting sets with measure zero can also be defined. For example, the Cantor middle-third set can be defined by a two-state automaton.

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ON SCHWARZ DIFFERENTIABILITY—V

S. N. MUKHOPADHYAY, Burdwan University, India

Let $f(x)$ be a real function of the real variable x defined in an open interval I of which $[a, b]$ is a closed subinterval. For $x\in[a, b]$, if the limit

$$\operatorname*{Lt}_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$$

exists, it is called the Schwarz derivative [5] or the symmetric derivative [1] of $f(x)$ at the point x and is denoted by $f^{(1)}(x)$. It is clear that if the ordinary derivative $f'(x)$ exists, then $f^{(1)}(x)$ also exists and $f^{(1)}(x)=f'(x)$; but the converse is not true.

In an interesting paper [6] Weinstock has shown that if $f(x)$ is differentiable in I, then a necessary and sufficient condition that $f'(x)$ be continuous in $[a, b]$

is that $f(x)$ should be uniformly differentiable in $[a, b]$. The purpose of the present note is to prove a result on Schwarz derivative (Theorem 1) and to study whether Weinstock's result remains true if one replaces ordinary derivative by Schwarz derivative.

We have introduced the following definition in [2] and also utilized it in [3].

DEFINITION. *Let $f(x)$ be Schwarz differentiable at each point x of I and suppose that*

$$\psi(x, h) = \frac{f(x+h) - f(x-h)}{2h} - f^{(1)}(x).$$

Then $f(x)$ is said to be uniformly Schwarz differentiable in $[a, b]$, if corresponding to the arbitrary $\epsilon > 0$, there exists a $\delta > 0$ (independent of x) such that $|\psi(x, h)| < \epsilon$, if $|h| < \delta$ for $x \in [a, b]$ and $x \pm h \in I$.

In earlier papers, the following results have been proved.

A. If $f(x)$ is continuous and uniformly Schwarz differentiable in I then $f^{(1)}(x)$ is continuous in $[a, b]$ (corollary 2, Theorem 9 [2]).

B. Let $f(x)$ be continuous and Schwarz differentiable in an open interval I of which $[a, b]$ is a closed subinterval. If $f^{(1)}(x)$ is nonnegative almost everywhere in $[a, b]$ then $f(x)$ is nondecreasing in $[a, b]$ (Theorem 3 [4]).

As a consequence of the result B, we obtain the following:

COROLLARY. *Let $f(x)$ be continuous and Schwarz differentiable in an open interval I of which $[a, b]$ is a closed subinterval. If $f^{(1)}(x) = 0$ almost everywhere in $[a, b]$, then $f(x)$ is constant in $[a, b]$.*

The result A gives the sufficient part of the analogue. Theorem 2 here contains the necessary part.

THEOREM 1. *If $f(x)$ is uniformly Schwarz differentiable in I then*
(α) $f(x)$ is continuous in $[a, b]$ if $f^{(1)}(x)$ is bounded in $[a, b]$.
(β) $f^{(1)}(x)$ is bounded in $[a, b]$ if $f(x)$ is continuous in I.

Proof of (α). Assume, if possible, that $f(x)$ is discontinuous at $\xi \in [a, b]$; then there exists $\eta > 0$ such that for each $\delta > 0$, there are points x' in $[a, b] \cap (\xi - \delta, \xi + \delta)$ such that $|f(x') - f(\xi)| > \eta$.

Since $f^{(1)}(x)$ is bounded, there exists $M > 0$ such that $|f^{(1)}(x)| \leqq M$, $x \in [a, b]$. So,

$$\left|\psi\left(\frac{x' + \xi}{2}, \frac{x' - \xi}{2}\right)\right| = \left|\frac{f(x') - f(\xi)}{x' - \xi} - f^{(1)}\left(\frac{x' + \xi}{2}\right)\right| \geqq \frac{\eta}{|x' - \xi|} - M.$$

Since x' can be chosen sufficiently near to ξ, this contradicts the hypothesis that $f(x)$ is uniformly Schwarz differentiable in I.

Proof of (β). By the result A, it follows that $f^{(1)}(x)$ is continuous in $[a, b]$. So, (β) holds.

REMARK. If $f(x)$ is uniformly Schwarz differentiable in $[a, b]$ then whether $f(x)$ may have points of discontinuity, remains an open question. The author, at the present stage, is unable to answer this. It is, however, highly possible that uniform Schwarz differentiability of $f(x)$ will imply continuity.

THEOREM 2. *If $f(x)$ and $f^{(1)}(x)$ are continuous in I then $f(x)$ is uniformly Schwarz differentiable in $[a, b]$.*

Proof. Let $\phi(x) = f(x) - \int_a^x f^{(1)}(t)dt$, $x \in I$. Then $\phi(x)$ is continuous and Schwarz differentiable in I and $\phi^{(1)}(x) = f^{(1)}(x) - f^{(1)}(x) = 0$ for $a \leqq x \leqq b$. Hence by corollary of the result B, $\phi(x) = c$, a constant, i.e.

$$f(x) = c + \int_a^x f^{(1)}(t)dt, \quad a \leqq x \leqq b.$$

Now

$$\begin{aligned} |\psi(x, h)| &= \left| \frac{1}{2h} \int_{x-h}^{x+h} f^{(1)}(t)dt - f^{(1)}(x) \right| \\ &= |f^{(1)}(x + \theta h) - f^{(1)}(x)|, \quad -1 \leqq \theta \leqq 1. \end{aligned}$$

Since $f^{(1)}(x)$ is continuous and so uniformly continuous in $[a, b]$, corresponding to arbitrary $\epsilon > 0$, there is a $\delta > 0$ (independent of x) such that

$$|f^{(1)}(x + h) - f^{(1)}(x)| < \epsilon, \quad \text{whenever } |h| < \delta.$$

So, from the above, we get $|\psi(x, h)| < \epsilon$, whenever $|h| < \delta$ and $x \in [a, b]$.

Note. The continuity of $f(x)$ is necessary. For, consider the function defined as

$$f(x) \begin{cases} = 0 & \text{if } x \neq 0 \\ = 1 & \text{if } x = 0. \end{cases}$$

Here $f^{(1)}(x)$ is continuous everywhere but $f(x)$ is not uniformly Schwarz differentiable in any interval including $x = 0$.

I am thankful to Dr. B. K. Lahiri for his kind help and suggestions in the preparation of the paper.

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SOME SPECIAL INTEGRALS

JYOTI CHAUDHURI, Jadavpur University, Calcutta, India

In 1957 Bremekamp [3] proposed the problem of showing that

$$\int_0^{\pi} (\log \sin\theta)^2 d\theta = \frac{\pi^3}{12} + \pi(\log 2)^2. \tag{1}$$

It has been solved by Beumer [1] in 1961, who has generalized it by giving the value of $\int_0^{\pi/2} (\log \sin\theta)^n d\theta$ as a Dirichlet series. In 1947 F. Bowman [2] gave the value of this integral as

$$(-1)^n \tfrac{1}{2}\pi \begin{vmatrix} \sigma_1 & -1 & 0 & 0 & \cdots & 0 \\ \sigma_2 & \sigma_1 & -2 & 0 & \cdots & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 & -3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{n-1} & \sigma_{n-2} & \sigma_{n-3} & \sigma_{n-4} & \cdots & -(n-1) \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \sigma_{n-3} & \cdots & \sigma_1 \end{vmatrix} \tag{2}$$

where

$$\sigma_r = \frac{1}{1^r} - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \cdots.$$

The object of this note is to obtain the value of a more general type of integral

$$\int_0^{\pi/2} (\log \sin^p\theta \cos^q\theta)^n d\theta, \tag{3}$$

where n is a nonnegative integer and p, q are any two positive rational numbers.

Let us write

$$D(n) = \frac{(-1)^{n-1}}{(n-1)!} \int_0^{\pi/2} (\log \sin^p\theta \cos^q\theta)^{n-1} d\theta. \tag{A}$$

Now since

$$\sum_{n=0}^{\infty} \frac{(\log \sin^p\theta \cos^q\theta)^n x^n}{n!} = \sin^{px}\theta \cos^{qx}\theta,$$

so that the series is uniformly convergent in $0<\theta<\pi/2$, term by term integration is valid. Hence

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \int_0^{\pi/2} (\log \sin^p\theta \cos^q\theta)^n d\theta = \int_0^{\pi/2} \sin^{px}\theta \cos^{qx}\theta d\theta$$

or,

$$\sum_{n=0}^{\infty}(-1)^n x^n D(n+1)=\frac{1}{2}\frac{\Gamma((px+1)/2)\Gamma((qx+1)/2)}{\Gamma((px+qx+2)/2)}. \tag{4}$$

Similarly,

$$\sum_{n=0}^{\infty}D(n+1)x^n=\int_0^{\pi/2}(\sin^p\theta\cos^q\theta)^{-x}d\theta$$

$$=\frac{1}{2}\frac{\Gamma((1-px)/2)\Gamma((1-qx)/2)}{\Gamma(1-(px+qx)/2)} \tag{5}$$

$$=\frac{\pi}{2}\frac{\sin((p+q)x\pi)/2)}{\sin((1+px)/2)\pi\sin((1+qx)/2)\pi}\frac{\Gamma((px+qx)/2)}{\Gamma((1+px)/2)\Gamma((1+qx)/2)}.$$

Due to uniform convergence, direct multiplication of (4) and (5) is justified, and hence we get

$$\left\{\sum_{n=0}^{\infty}(-1)^n D(n+1)x^n\right\}\left\{\sum_{n=0}^{\infty}D(n+1)x^n\right\}$$

$$=\frac{\pi}{2(p+q)x}\frac{\sin[(p+q)x\pi/2]}{\cos px\pi/2\cos qx\pi/2}$$

$$=\frac{\pi}{2(p+q)x}\{\tan px\pi/2+\tan qx\pi/2\}.$$

But we know ([4], p. 499) that

$$\tan\frac{\pi x}{2}=2\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}B_n\pi^{2n-1}x^n,$$

where B_n denotes the Bernoulli's numbers. Hence

$$\left\{\sum_{n=0}^{\infty}(-1)^n D(n+1)x\right\}\left\{\sum_{n=0}^{\infty}D(n+1)x^n\right\}$$

$$=\frac{\pi^2}{p+q}\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}B_n(\pi x)^{2n-2}(p^{2n-1}+q^{2n-1}). \tag{6}$$

Comparing coefficients of x^{2n-2} from both sides we have

$$D(1)D(2n-1)-D(2)D(2n-2)+\cdots+D(2n-1)D(1)$$

$$=\frac{2^{2n}-1}{(2n)!}B_n\pi^{2n}\frac{p^{2n-1}+q^{2n-1}}{p+q}. \tag{7}$$

Equating the constant term from both sides of (6)

$$\{D(1)\}^2 = \frac{\pi^2}{p+q}\,\frac{3}{2}\,B_1(p+q) = \frac{\pi^2}{4},$$

whence $D(1)=\pi/2$ as can also be seen from (A). Putting $n=2$ in (7), we obtain

$$D(1)D(3) - D(2)D(2) + D(3)D(1) = \frac{15}{4!}\,B_2\pi^4\frac{p^3+q^3}{p+q} = \frac{\pi^4}{48}(p^2 - pq + q^2).$$

From (A), $D(2) = -\int_0^{\pi/2}(\log \sin^p\theta \cos^q\theta)d\theta = (p+q)(\pi/2)\log 2$. Hence, $D(3) = (\pi/4)(p+q)^2(\log 2)^2 + (\pi^3/48)(p^2-pq+q^2)$ and so on. Writing $y = \sum_{n=0}^{\infty}(-1)^n x^n D(n+1)$, by logarithmic differentiation of (4) we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{p}{2}\,\frac{\Gamma'((1+px)/2)}{\Gamma((1+px)/2)} + \frac{q}{2}\,\frac{\Gamma'((1+qx)/2)}{\Gamma((1+qx)/2)} - \frac{p+q}{2}\,\frac{\Gamma'((px+qx+2)/2)}{\Gamma((px+qx+2)/2)}.$$

But for $\mathrm{Re}(z)>0$,

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} + \sum_{m=1}^{\infty}\left\{\log\left(1+\frac{1}{m}\right) - \frac{1}{m+z}\right\}.$$

Hence, expanding the right hand side accordingly we obtain

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} = {} & \frac{p}{2}\left[-\frac{2}{1+px} + \sum_{m=1}^{\infty}\log\left(1+\frac{1}{m}\right) - \sum_{m=1}^{\infty}\frac{1}{m+((1+px)/2)}\right] \\ & + \frac{q}{2}\left[-\frac{2}{1+qx} + \sum_{m=1}^{\infty}\log\left(1+\frac{1}{m}\right) - \sum_{m=1}^{\infty}\frac{1}{m+((1+qx)/2)}\right] \\ & - \frac{p+q}{2}\left[-\frac{2}{px+qx+2} + \sum_{m=1}^{\infty}\log\left(1+\frac{1}{m}\right)\right. \\ & \left. - \sum_{m=1}^{\infty}\frac{1}{m+1+((p+q)/2)x}\right].\end{aligned}$$

$$\frac{1}{y}\frac{dy}{dx} = \sum(-1)^{r+1}t_{r+1}x^r,$$

where

$$t_r = \sum_{m=1}^{\infty}\frac{(p^r+q^r)}{(2m+1)^r} - \sum_{m=0}^{\infty}\frac{(p+q)^r}{(2m+2)^r}.$$

Therefore,

$$\sum_{n=0}^{\infty}(-1)^n x^{n-1} n D(n+1) = \left\{\sum_{r=0}^{\infty}(-1)^{r+1}t_{r+1}x^r\right\}\cdot\left\{\sum_{n=0}^{\infty}(-1)^n D(n+1)x^n\right\}$$

whence, equating coefficients of x^{n-1}, we obtain

$$nD(n+1) = t_1D(n) + t_2D(n-1) + \cdots + t_{n-1}D(2) + t_nD(1). \tag{8}$$

It is now easy to deduce from (8) by the method of induction the general expression for $D(n+1)$ as given below:

$$(-1)^n \int (\log \sin^p \theta \cos^q \theta)^n d\theta = n!D(n+1)$$

$$= \frac{\pi}{2} \begin{vmatrix} t_1 & -1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & -2 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & -3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & -(n-1) \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 \end{vmatrix}.$$

The result (2) is easily obtained from (9) by putting $p=1$, $q=0$ because, in that case

$$t_r = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^r} - \sum_{m=0}^{\infty} \frac{1}{(2m+2)^r} = \left(1 - \frac{1}{2^{r-1}}\right)\zeta(r) = \sigma_r, \tag{10}$$

where $\zeta(r)$ denotes the zeta function of Riemann. All the results of Beumer now follow from the above by substituting $p=1$, $q=0$.

I wish to express my indebtedness to Prof. B. N. Mukherjee for his help and guidance in the preparation of this paper.

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LINEAR COMPLETELY INTEGRABLE PARTIAL DIFFERENTIAL EQUATIONS

H. H. JOHNSON, University of Washington

1. Introduction. The theory of linear ordinary differential equations [3, p. 69] illustrates how a problem in analysis can sometimes be simplified by algebraic methods. We study here an extension of this method to systems of partial differential equations which behave very much like ordinary differential equations. In particular, they have unique solutions for initial values of the variables.

Let $y^1, \cdots, y^n$ be coordinates in R^n, real Euclidean n-space, and let $x^1, \cdots,$

x^p be coordinates in R^p. We wish to study equations of the type

$$\frac{\partial y^i}{\partial x^\alpha} = f_\alpha^i(x^1, \cdots, x^p, y^1, \cdots, y^n), \qquad \begin{matrix} i = 1, \cdots, n \\ \alpha = 1, \cdots, p. \end{matrix}$$

Suppose a solution $y^i = y^i(x^1, \cdots, x^p)$, $i = 1, \cdots, n$ having continuous second partial derivatives exists passing through arbitrary points of $R^n \times R^p$. Then by the equality of the mixed partial derivatives $(\partial^2 y^i/\partial x^\alpha \partial x^\beta) = (\partial^2 y^i/\partial x^\beta \partial x^\alpha)$ and the chain rule, one must have

$$\frac{\partial f_\alpha^i}{\partial x^\beta} + \sum_{j=1}^n \frac{\partial f_\alpha^i}{\partial x^j} f_\beta^j = \frac{\partial f_\beta^i}{\partial x^\alpha} + \sum_{j=1}^n \frac{\partial f_\beta^i}{\partial y^j} f_\alpha^j \tag{1}$$

for all values of $y^1, \cdots, y^n, x^1, \cdots, x^p$. It is known that this necessary condition is also sufficient when the f_α^i have continuous second partial derivatives, and then there is a unique solution through each initial point (x_0^α, y_0^i) [1, p. 29]. See also [2, p. 104].

2. Linear completely integrable systems. Consider systems

$$\frac{\partial y^i}{\partial x^\alpha} = \sum_{j=1}^n A_{\alpha j}^i y^j, \qquad \begin{matrix} i = 1, \cdots, n, \\ \alpha = 1, \cdots, p, \end{matrix} \tag{2}$$

where $A_{\alpha j}^i \in R$ (real numbers). As in the case of ordinary differential equations we first consider how (2) behaves when we change the variables y or x by linear transformations. If $x^\alpha = \sum C_\beta^\alpha \bar{x}^\beta$, then

$$\frac{\partial y^i}{\partial x^\alpha} = \sum_{j=1}^n \sum_{\beta=1}^p A_{\beta j}^i C_\alpha^\beta y^j.$$

If $\bar{y}^i = \sum a_j^i y^j$, and $y^j = \sum b_i^j \bar{y}^i$, then

$$\frac{\partial \bar{y}^i}{\partial x^\alpha} = \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n a_j^i A_{\alpha k}^j b_l^k \bar{y}^l.$$

Thus, if we denote by A_α the $n \times n$ matrices $(A_{\alpha j}^i)$, and consider the vector space $L \subset R^{n^2}$ generated by these matrices, then a change of basis in R^p corresponds to a change of basis in L. A change of basis in R^n corresponds to a simultaneous similarity transformation $BA_\alpha B^{-1}$ of the elements of L. Thus, we may regard L as a vector space of linear transformations of R^n into R^n.

Next, the conditions (1) become

$$\sum_{j=1}^n A_{\alpha j}^i A_{\beta k}^j = \sum_{j=1}^n A_{\beta j}^i A_{\alpha k}^j,$$

for $i, j = 1, \cdots, n$; $\alpha, \beta = 1, \cdots, p$. This condition is recognized as merely $A_\alpha A_\beta = A_\beta A_\alpha$, that is, the elements of L *commute*. The algebraic problem is now

to study a vector space of commuting linear transformations on R^n. The general problem of classifying all such spaces remains open. In case one of the linear transformations in L has distinct eigenvalues, however, more can be said.

THEOREM 1. *Let L be a vector space of commuting linear transformations on R^n which contains a transformation having distinct eigenvalues. Then it is possible to choose a basis of R^n so that the matrices which represent elements of L are all diagonal matrices.*

Proof. Let $A \in L$ have distinct eigenvalues $\lambda_1, \cdots, \lambda_n$ and corresponding eigenvectors $Y_1, \cdots, Y_n$ (which are linearly independent).

If $A' \in L$, $A'(Y_j) = \sum C_j^k Y_k$, so that

$$AA'(Y_j) = \sum_{k=1}^{n} C_j^k \lambda_k Y_k,$$

$$A'A(Y_j) = \sum_{k=1}^{n} \lambda_j C_j^k Y_k.$$

Hence $(\lambda_k - \lambda_j) C_j^k = 0$ for $j, k = 1, \cdots, n$. Since the λ_j are distinct, $C_j^k = 0$ when $k \neq j$. Thus, A' is a diagonal matrix with respect to the basis $Y_1, \cdots, Y_n$.

COROLLARY. *Under the conditions of Theorem* 1 *the solutions of* (1) *may be expressed in the form*

$$y^i = C^i \exp\left(\sum_{\alpha=1}^{p} \lambda_\alpha^i x^\alpha \right), \qquad i = 1, \cdots, n,$$

where the C^i are arbitrary real numbers.

Proof. Following linear transformations on R^n and R^p, each $A_\alpha = \operatorname{diag}(\lambda_\alpha^1, \cdots, \lambda_\alpha^n)$. Then if y^i have the above form,

$$\frac{\partial y^i}{\partial x^\alpha} = \lambda_\alpha^i C^i \exp\left(\sum_{\alpha=1}^{n} \lambda_\alpha^i x^\alpha \right)$$

$$= \lambda_\alpha^i y^i = \sum_{j=1}^{n} A_{\alpha j}^i y^j.$$

Given any initial values y_0^i, x_0^α, the above solution with $C^i = y_0^i / \exp\left(\sum_{\alpha=1}^{p} \lambda_\alpha^i x_0^\alpha\right)$ passes through these initial values. Since solutions are all uniquely determined by initial values, every solution must have the stated form.

3. Exponential matrices. In ordinary linear differential equations one may express the solutions in terms of exponential matrix functions [**3**, p. 70]. Because the matrices A_α commute this is also possible here. If $C = (C_i^j)$ is a matrix of functions $C_i^j(x^1, \cdots, x^p)$, then $(\partial/\partial x^\alpha)C$ is defined to be the matrix $(\partial C_i^j / \partial x^\alpha)$. Observe that $(\partial/\partial x^\alpha)(CD) = (\partial C/\partial x^\alpha)D + C(\partial D/\partial x^\alpha)$.

LEMMA. *Let* $A_1, \cdots, A_p$ *be commuting* $n \times n$ *matrices of real numbers. If* $C = \sum_{\alpha=1}^{p} x^\alpha A_\alpha$ *then*

$$\frac{\partial}{\partial x^\alpha} C^n = nC^{n-1}A_\alpha.$$

Proof. This is true for $n=1$, and if true for $n-1$, then

$$\begin{aligned}\frac{\partial}{\partial x^\alpha} C^n &= \frac{\partial}{\partial x^\alpha} C^{n-1}C = \left(\frac{\partial}{\partial x^\alpha} C^{n-1}\right)C + C^{n-1}\frac{\partial}{\partial x^\alpha} C \\ &= (n-1)C^{n-2}A_\alpha C + C^{n-1}CA_\alpha.\end{aligned}$$

The hypothesis that $A_\alpha A_\beta = A_\beta A_\alpha$ implies that $A_\alpha C = CA_\alpha$, and the conclusion follows at once.

Let I denote the identity matrix.

PROPOSITION. *If* $C = \sum_{\alpha=1}^{p} x^\alpha A_\alpha$ *where* $A_1, \cdots, A_p$ *are* $n \times n$ *commuting matrices of real numbers, then*

$$e^C = I + C + \frac{1}{2!}C^2 + \frac{1}{3!}C^3 + \cdots$$

converges uniformly for all $(x^1, \cdots, x^p) \in R^p$, *and*

$$\frac{\partial}{\partial x^\alpha} e^C = e^C A_\alpha.$$

Proof. The uniform convergence follows as in the single-variable case, and then one may differentiate term-by-term. The previous lemma then completes the proof.

THEOREM 2. *Let the matrices* $A_\alpha = (A_{\alpha i}^j)$, $\alpha = 1, \cdots, p$ *commute, and let*

$$Y = \exp\left(\sum_{\alpha=1}^{p} x^\alpha A_\alpha\right) = (y_i^j).$$

Then if $a^1, \cdots, a^n$ *are any real numbers,* $y^i = \sum_{j=1}^{n} a^j y_j^i$, $i = 1, \cdots, n$ *is a solution of*

$$\frac{\partial y^i}{\partial x^\alpha} = \sum_{j=1}^{n} A_{\alpha j}^{\ i}\, y^j,$$

and conversely every solution can be written in this form for some $a^1, \cdots, a^n$.

Proof. By the previous proposition,

$$\frac{\partial y_i^j}{\partial x^\alpha} = \sum_{k=1}^{n} y_i^k A_{\alpha k}^{\ i}$$

and from this it follows that $y^1, \cdots, y^n$ are solutions. Conversely, every solution is uniquely determined by its initial values (y_0^i, x_0^α). Letting

$$(w_i^j) = \exp\left(-\sum_{\alpha=1}^{p} x_0^\alpha A_\alpha\right)$$

and $a^k = \sum_{k=1}^{n} w_k^i y_0^k$ $(i=1, \cdots, n)$ one obtains a solution which passes through these initial values.

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A CRITERION FOR EXTREMAL POLYGONS OF A VERTEX SET

KENNETH LEBENSOLD, The City College of New York

1. Introductory remarks and definition of rearrangeability. We consider the problem of finding the shortest and longest polygons having n distinct points as vertices (the Traveling Salesman Problem). We denote the length of a polygon P by $l(P)$. The purpose of this paper is to produce a necessary and sufficient condition on a polygon that it be extremal in the set of all polygons through its vertices; we denote the vertex set by V, and call the set of these polygons the covertex class $P(V)$.

More specifically, we wish to find a subset $P'(V)$ of $P(V)$ so that if $l(P) \leqq \min_{Q\in P'(V)} l(Q)$, then $l(P) = \min_{Q\in P(V)} l(Q)$, that is, if the length of P is not greater than the length of Q for all Q in $P'(V)$, it is not greater than the length of Q for all Q in the covertex class of V. Clearly $P'(V) = P(V)$ is such a subset. The set $P'(V)$ will depend on the particular polygon P in $P(V)$. Exactly analogous results for maxima are obtainable by reversing all the inequalities in the proof for minima.

Suppose that (P, Q) is a pair of polygons with the same vertices. Suppose that (R, S) is another pair of polygons, each having the same vertex set as P and Q. Suppose further that every edge on one of the polygons (P, Q) is on one of the polygons (R, S), and every edge on both P and Q is on both R and S, and P, Q, R, S are all different. Then (R, S) is called a rearrangement of (P, Q) and (P, Q) is said to be rearrangeable. For example, let a hexagon have its vertices numbered in order 123456, call this P and suppose Q is 136245. Then $(R, S) = (134562, 154236)$ is a rearrangement of (P, Q).

If (R, S) is a rearrangement of (P, Q), there are $2n$ edges of (R, S) and $2n$ of (P, Q). Since (R, S) contains all the edges of (P, Q) and repeats all repeated edges of (P, Q), it has all the $2n$ sides of (P, Q). Since (R, S) has only $2n$ sides,

This research was supported by NSF Undergraduate Research Participation Grant GE 6446.

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CONCERNING CONSECUTIVE DERIVED SETS

JOE TUCKER, Auburn University

Let M be a subset of a topological space S and for k a nonnegative integer let $M^{(k+1)}$ be the set of all limit points (i.e., the derived set) of $M^{(k)}$, where $M^{(0)}=M$.

DEFINITION. *For a subset M of S the sequence of derived sets of M is the sequence* $M^{(0)}, M^{(1)}, \cdots, M^{(k)}, \cdots$.

DEFINITION. *If for some positive integer k the sequence of derived sets of a subset M of S is such that $M^{(i)}\neq M^{(j)}$ for $0\leqq i<j<k$, but $M^{(0)}=M^{(k)}$, then the sequence is said to be periodic with period k.*

The primary purpose of this paper is to present one theorem concerning the period of periodic sequences of derived sets in a topological space. First, however, consider the more general Frechet space (V), [1].

For any positive integer $k>1$ a Frechet space (V) may be constructed so that for some subset the sequence of derived sets will be periodic with period k. In a Frechet space (V) neighborhoods are defined locally and thus if v is a neighborhood of p and $q\in v$, then v is not necessarily a neighborhood of q. Consider the finite point set $S=\{a_i\}_{i=1}^{k}$ and let $V=\{v(a_i)\}_{i=1}^{k}$, where $v(a_1)=\{a_1, a_k\}$ and $v(a_i)=\{a_{i-1}, a_i\}$ for $i\neq 1$, be the collection of subsets (neighborhoods) of S which are associated with the points in S. Now for the subset $M^{(0)}=\{a_i\}_{i=1}^{k-1}$ the consecutive derived sets are: $M^{(1)}=S-a_1$, $M^{(2)}=S-a_2$, $\cdots$, $M^{(j)}=S-a_j$, $\cdots$. Thus $M^{(k)}=S-a_k=M^{(0)}$ and the sequence has period k in the space (S, V).

If a space is such that every derived set is closed, (e.g., a Hausdorff space), then every periodic sequence of derived sets will have period 1. This follows from the inclusions $M^{(1)}\supset M^{(2)}\supset\cdots\supset M^{(k)}\supset\cdots$ and from the fact that if $M=M^{(k)}$, then $M^{(1)}=M^{(k+1)}$. In topological spaces in which a derived set is not necessarily closed, however, the period of a periodic sequence may be greater than 1, as is shown by the following example. Let $S=\{a, b, c\}$ and let a basis be the subsets $R_1=\{a, b\}$ and $R_2=S$. Now for $M=\{a, c\}$ the derived sets are: $M^{(1)}=\{b, c\}$, $M^{(2)}=\{a, c\}$, $\cdots$, and thus the period is 2. That 2 is the maximum period in a topological space is established as follows.

LEMMA 1. *Let (S, R) be a topological space. If $M\subset S$ and $M=M^{(k)}$ for some positive integer k, then*

$$M\cup M^{(1)}=M^{(1)}\cup M^{(2)}=\cdots=M^{(k-1)}\cup M.$$

Proof. When $k=1$, the lemma is trivially true. Now since $\mathrm{cl}(M)=\mathrm{cl}(\mathrm{cl}(M))$, $\mathrm{cl}(M)=M\cup M^{(1)}$ and $\mathrm{cl}(M\cup M^{(1)})=\mathrm{cl}(M)\cup\mathrm{cl}(M^{(1)})$, then $\mathrm{cl}(M)=\mathrm{cl}(M)\cup\mathrm{cl}(M^{(1)})$ and hence $\mathrm{cl}(M)\supset\mathrm{cl}(M^{(1)})$. Similarly $\mathrm{cl}(M^{(1)})\supset\mathrm{cl}(M^{(2)})$, $\cdots$, $\mathrm{cl}(M^{(k-1)})\supset\mathrm{cl}(M^{(k)})$ and thus

$$\mathrm{cl}(M) = \mathrm{cl}(M^{(1)}) = \mathrm{cl}(M^{(2)}) = \cdots = \mathrm{cl}(M^{(k-1)}).$$

Therefore $M\cup M^{(1)}=M^{(1)}\cup M^{(2)}=\cdots=M^{(k-1)}\cup M$.

LEMMA 2. *If k is a positive integer and M is a subset of S whose sequence of derived sets is periodic with period k, then each term in the sequence is a subset of S whose sequence of derived sets is periodic with period k.* (This follows from the definition of periodic sequences.)

THEOREM. *Let (S, R) be a topological space and let $M\subset S$. If the sequence of consecutive derived sets of M is periodic, then the period is not greater than* 2.

Proof. Suppose that $p\in M-M^{(2)}$. Then there exists an element R_1 of R such that $p\in R_1$ and $R_1\cap M^{(1)}=p$ (by Lemma 1, $p\in M^{(1)}$). But since $p\in M^{(1)}$, R_1 contains a point q of M distinct from p and hence not in $M^{(1)}$. So there exists an element R_2 of R which contains q but no point of $M-q$. In particular, R_2 does not contain p. Let R_3 be an element of R such that $q\in R_3\subset R_1\cap R_2$. But q being in $M-M^{(1)}$ is, by Lemma 1, in $M^{(2)}$. Hence, R_3 must contain a point of $M^{(1)}$, which is impossible. Therefore $M\subset M^{(2)}$.

Suppose that $p\in M^{(2)}-M$. Again by Lemma 1 $p\in M^{(1)}$. Now by Lemma 2 and the preceding paragraph,

$$M^{(1)}\subset M^{(3)}, \qquad M^{(2)}\subset M^{(4)}, \cdots, M^{(k-2)}\subset M^{(k)}.$$

So $p\in M^{(3)}$, $p\in M^{(4)}$, $\cdots$, $p\in M^{(k)}=M$. Thus $M^{(2)}\subset M$ and hence $M=M^{(2)}$; therefore, the period is not greater than 2.

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FINITE PROJECTIVE PLANES WITH NONSOLVABLE TRANSITIVE COLLINEATION GROUPS

V. H. KEISER, JR., Whitman College

Introduction. In [4] Higman and McLaughlin proved the following

THEOREM 1. *If π is a finite projective plane of odd order n with a flag-transitive collineation group G and either n is not a square or $n=m^2$ with $m\equiv 3 \pmod 4$, then π is desarguesian and G contains the little projective group of π.*

Using essentially the same methods, but applying some new results which were not available to them, one can obtain the following more general

THEOREM 2. *If π is a finite projective plane with a transitive non-solvable collineation group G and either n is not a square or $n=m^2$ with $m\equiv 2, 3 \pmod 4$, then π is desarguesian and G contains the little projective group of π.*

Before proceeding further, we first note that a flag-transitive collineation group of a finite projective plane of odd order is necessarily non-solvable from which it follows immediately that Theorem 1 is a special case of Theorem 2.

Known Results Used in the Proof of Theorem 2:

A. FEIT-THOMPSON THEOREM. *Every nonsolvable group is of even order* [2].

B. BAER'S THEOREM. *Let ϕ be an involution in a projective plane of order n. Then either* (1) *$n=m^2$ and the fixed points and lines of ϕ form a subplane of order m, or* (2) *ϕ is a perspectivity. In case* (2) *if n is odd, ϕ is a homology and if n is even, ϕ is an elation.* [1]

C. WAGNER'S THEOREM. *Let π be a projective plane and G a transitive group of collineations of π. If G has a non-trivial perspectivity, then π is desarguesian and G contains the little projective group of π.* [5]

D. *Let G be a permutation group and H a subgroup of G. Let $\mathfrak{S}$ be the set of points left fixed by H. Then the normalizer of H in G will permute the points of $\mathfrak{S}$ among themselves.*

E. *The order of a finite simple group is divisible either by* 12 *or by the cube of the smallest prime dividing its order.* (See [3] Corollary 14.3.1.)

Proof of Theorem 2. Theorem 2 follows immediately from Wagner's Theorem and the following lemma.

LEMMA. *If π is a finite projective plane with a non-solvable collineation group G and either n is not a square or $n=m^2$ with $m\equiv 2, 3 \pmod 4$, then G contains an involutory perspectivity.*

Proof. By the Feit-Thompson Theorem, G, being non-solvable, is of even order hence contains an involution. We consider two cases.

(a) There is an involution ϕ which does not fix a subplane. Then by Baer's Theorem ϕ is a perspectivity and we are done.

(b) Every involution in G fixes a subplane. (We will show that in fact this case does not occur.) By Baer's Theorem this can occur only if $n=m^2$ and the fixed subplanes are of order m. The Feit-Thompson Theorem together with result E applied to G tell us that the order of G is divisible by 4. Hence either

(b′) G contains an element of order 4 or

(b″) The Sylow 2-subgroups of G are elementary abelian.

Suppose b′ holds. Let η be of order 4. Then η^2 is an involution fixing a subplane π_0 of order m and since η is in the normalizer of η^2, it follows by result D that η permutes the points of π_0 among themselves. By Baer's Theorem η^2 fixes no points of $\pi-\pi_0$ ($\pi-\pi_0$ represents the set of points in π that are not in

π_0) and since η^2 clearly fixes all the points fixed by η, it is also true that η fixes no points of $\pi-\pi_0$. Therefore η permutes the points of $\pi-\pi_0$ in cycles of length four so that $|\pi-\pi_0| \equiv 0 \pmod 4$. Thus we must have

$$(n^2+n+1)-(m^2+m+1) \equiv 0 \pmod 4. \tag{1}$$

But if $m \equiv 2, 3 \pmod 4$, then

$$(n^2+n+1)-(m^2+m+1) \equiv (0+0+1)-(0+2+1) \quad \text{or} \quad (1+1+1)-(1+3+1) \pmod 4$$

respectively and both of these are congruent to 2 (mod 4) contradicting equation (1).

Suppose b'' holds. Let ϕ and η be elements of order two in the same Sylow 2-subgroup of G. We consider two subcases.

(b''−1) ϕ and η fix the same subplane. Since $\phi\eta$ is also of order two, it fixes the same subplane and no other points. Each nonidentity element of the group (ϕ, η) thus fixes the points of π_0 and no others. Since (ϕ, η) is of order 4, this implies that $|\pi-\pi_0| \equiv 0 \pmod 4$ contrary to the fact that, as already shown, this number is congruent to 2 (mod 4).

(b''−2) ϕ and η fix different subplanes π_ϕ and π_η. Since ϕ and η commute, each is contained in the normalizer of the other. Hence by result D ϕ permutes the points of π_η among themselves and η permutes the points of π_ϕ among themselves. Thus ϕ and η restricted to π_η and π_ϕ respectively are perspectivities. (They cannot fix subplanes of π_η and π_ϕ since m is not a square.) In particular they are elations or homologies according as $m \equiv 2$ or 3 (mod 4). Now $\phi\eta$ fixes a subplane $\pi_{\phi\eta}$ and we can assume that $\pi_{\phi\eta}$ is different from π_ϕ and π_η since otherwise this case reverts to (b''−1). Clearly $\pi_{\phi\eta} \cap \pi_\phi \cap \pi_\eta$ consists of the $m+1$ or $m+2$ points fixed by both ϕ and η. ($m+1$ if ϕ and η are elations of π_η and π_ϕ, $m+2$ if they are homologies.) As above, the group (ϕ, η) of order 4 permutes the points of $\pi-(\pi_{\phi\eta} \cup \pi_\phi \cup \pi_\eta)$ among themselves in cycles of length four so that

$$|\pi-(\pi_{\phi\eta} \cup \pi_\phi \cup \pi_\eta)| \equiv 0 \pmod 4.$$

But

$$|\pi-(\pi_{\phi\eta} \cup \pi_\phi \cup \pi_\eta)| = (n^2+n+1)-3(m^2+m+1)+2(m+1 \text{ or } m+2)$$

according as $m \equiv 2$ or 3 (mod 4). In both cases, however, this number is congruent to 2 (mod 4).

From these contradictions we conclude that case (b) does not occur. This leaves (a) and we have already shown that in this case there is an involutory perspectivity.

Added in proof: In a private communication Professor Peter Dembowski has kindly pointed out that the hypothesis of nonsolvability is stronger than necessary. In both Theorem 2 and the Lemma, "nonsolvable collineation

group G" can be replaced by the weaker "collineation group G with $|G| \equiv 0 \pmod 4$."

This paper represents part of a Ph. D. dissertation written at the University of Colorado, Boulder, under the direction of Professor Richard Roth.

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A MINIMUM PROBLEM CONNECTED WITH COMPLETE RESIDUE SYSTEMS IN THE GAUSSIAN INTEGERS

N. R. HARDMAN AND J. H. JORDAN, Washington State University

1. A Gaussian integer, γ, is a complex number that can be expressed as $\gamma = a+bi$, where a and b are real integers and i is the so-called imaginary unit. The set of Gaussian integers with the usual operations of addition and multiplication forms a unique factorization domain.

Small case Greek letters will be used to represent Gaussian integers.

Some definitions about Gaussian integers, given in [2], are:

DEFINITION. *γ divides α means there is a β such that $\beta\gamma = \alpha$. Notation:* $\gamma \mid \alpha$.

DEFINITION. $\alpha \equiv \beta \pmod \gamma$ *means* $\gamma \mid \alpha - \beta$.

This congruence relation for a fixed γ is an equivalence relation on the set of Gaussian integers. The set of equivalence classes formed with respect to this equivalence relation is called the complete residue system modulo γ. As is shown in [1] and in [2] the cardinality of a complete residue system modulo $\gamma = a+bi$ is $a^2+b^2 = N(\gamma)$ the norm of γ.

Following the notation used in [1],

(i) the symbol CRS(mod γ) for $\gamma = a+bi$ will stand for the complete residue system modulo γ, and

(ii) a representation of the CRS(mod γ) will be a set of a^2+b^2 incongruent Gaussian integers modulo γ. Several representations are exhibited in [1].

A problem suggested in [1] was that of finding for each γ a representation R_γ of CRS(mod γ) such that for any representation, T_γ, of the CRS(mod γ)

$$\sum_{\alpha \in R_\gamma} |\alpha| \leqq \sum_{\beta \in T_\gamma} |\beta| .$$

It is the purpose of this paper to settle this problem by exhibiting the R_γ desired and also to indicate which of these R_γ are unique solutions to the problem. The appropriate result is

THEOREM. *Let A be the set of Gaussian integers inside the square with vertices at $(\pm 1 \pm i)\gamma/2$ and let B be the set of Gaussian integers on the half open line segments $(\pm(-1+i)\gamma/2, (-1-i)\gamma/2)$. Then $R_\gamma = A \cup B$ is the representation with the desired "minimal" property. Further this is the unique representation with the desired "minimal" property if and only if $N(\gamma) = a^2+b^2$ is odd.*

Proof: We will prove the case when $N(\gamma) = a^2+b^2$ is odd or equivalently when $(1+i) \nmid \gamma$.

LEMMA 1. *If $m \not\equiv n \pmod 2$ and a line, L, with slope m/n passes through a point $(t+\frac{1}{2}, s+\frac{1}{2})$ with t and s integers then L has no points which have both coordinates integers.*

Proof: The equation for L is

$$ny - ns - n/2 = mx - mt - m/2$$

which does not permit x and y to be integers simultaneously.

COROLLARY. *If $(1+i) \mid \gamma$ then $B = \varnothing$.*

Proof: $\gamma = a+bi$ where $a \not\equiv b \pmod 2$.

LEMMA 2. *If $(1+i) \nmid \gamma = a+bi$ then A has a^2+b^2 elements in it.*

Proof: Separate the square $(\pm 1 \pm i)\gamma/2$ into the five areas; the four triangles A_j with vertices $i^j(a-b+(a+b)i)/2$, $i^j(-a-b+(a-b)i)/2$, $i^j(a-b+(a-b)i)/2$, $j = 0, 1, 2, 3$; and the square with vertices $i^j(a-b+(a-b)i)/2$. Now the sides of each area are void of Gaussian integers by Lemma 1. By the similarity of the four triangles it suffices to count the Gaussian integers in one and multiply by 4. To count the Gaussian integers in A_0 consider the rectangle with the vertices of A_0 and the other vertex at $(-a-b+(a+b)i)/2$. This rectangle contains twice as many Gaussian integers as does A_0 since the diagonal is void of Gaussian integers. There are, however, ab Gaussian integers in this rectangle; hence A_0 contains $ab/2$ Gaussian integers. The number of Gaussian integers in $\bigcup_{j=0}^{3} A_j$ is therefore $2ab$. The number of Gaussian integers in the square is $|a-b|\,|a-b|$ or $a^2-2ab+b^2$. Hence the number of elements in A is a^2+b^2.

This proof reminds one of a proof of the Pythagorean Theorem.

LEMMA 3. *Let $\alpha \in A$ then $|\alpha| < |\beta|$ for all $\beta \equiv \alpha \pmod \gamma$, $\beta \neq \alpha$.*

Proof. Let $\alpha/\gamma = r+si$. Now $-\frac{1}{2} < r < \frac{1}{2}$ and $-\frac{1}{2} < s < \frac{1}{2}$. Since $\beta = \alpha + \delta\gamma$, it follows that $\beta/\gamma = \alpha/\gamma + \delta = r+c+(s+d)i$. But since c and d are integers $|r+c| > |r|$ unless $c = 0$ and $|s+d| > |s|$ unless $d = 0$. It now follows that

$$|\beta/\gamma| = \sqrt{(r+c)^2 + (s+d)^2} > \sqrt{r^2+s^2} = |\alpha/\gamma|$$

or equivalently $|\beta| > |\alpha|$, unless c and d are 0 in which case $\beta = \alpha$.

Now the proof of the theorem follows immediately from Lemmas 2 and 3 in case $(1+i) \nmid \gamma$, since $B = \varnothing$ and A is a CRS (mod γ) with the minimal property.

The proof for the case when $(1+i) \mid \gamma$ is slightly more involved and is omitted. When $B \neq \varnothing$ then the uniqueness fails since for each $\beta \in B$ there is an α such that $\alpha \equiv \beta \pmod{\gamma}$ and $|\alpha| = |\beta|$, in particular either $\beta+\gamma$ or $\beta+i\gamma$ has that property.

This research was supported in part by N.S.F. Grant No. G22765/0009 (Undergraduate research) and Washington State University Grant in Aid for Research—Project No. 728.

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ON A PROPERTY OF THE b-ADIC EXPRESSION OF INTEGERS

GUY HIRSCH, Université de Bruxelles

Let $N = \beta_n b^n + \beta_{n-1} b^{n-1} + \cdots + \beta_1 b + \beta_0$ $(0 \leqq \beta_i \leqq b-1;\ b > 1)$ be the b-adic expression of the (positive) integer N. The sum

$$w_b(N) = \beta_0 + \beta_1 + \cdots + \beta_n$$

of the "b-adic digits" will be called the *b-adic weight* of N.

Denote by B_1 the set of the numbers $m - w_b(m) + (b-1)$, where m is any number divisible by b; denote by B_q $(q \geqq 1)$ the set of the numbers

$$m - w_b(m) + (b^q - 1),$$

where m is any number divisible by b^q.

REMARK. $m = 0$ being divisible by any number, with $w_b(0) = 0$, the number $b^q - 1$ belongs to B_q.

THEOREM. *Every multiple of $(b-1)$ appears exactly once in (exactly) one of the sets B_q.*

Denoting by $c_i = (b^i - 1)/(b-1) = 1 + b + \cdots + b^{i-1}$ the number whose b-adic expansion consists of i one's, another way of stating the theorem is as follows:

Let m_q be any (nonnegative) number divisible by b^q $(q \geqq 1)$; subtract from m_q the sum of its b-adic digits, and divide this difference by $b-1$. Add to the quotient the number c_q whose b-adic expansion consists of q one's. Every nonnegative integer is obtained just once in this way.

REMARKS. For $b = 2$, the theorem states that for every positive integer t there exists a unique (positive) integer q such that t is expressible uniquely in

the form $m-w_2(m)+(2^q-1)$, where m is divisible by 2^q and $w_2(m)$ denotes the dyadic weight of m, i.e. the number of one's in the dyadic expression of m.

Under this form, the theorem plays a role in algebraic topology (where it is used to prove that elements of a certain form in the Steenrod algebra yield exactly one class for every dimension).

Proof of the theorem. (The author expresses his thanks to the referee, who suggested a shorter and simpler proof, which is essentially given below.)

From $m_q-w_b(m_q)=\beta_n(b^n-1)+\beta_{n-1}(b^{n-1}-1)+\cdots+\beta_q(b^q-1)$, it follows that

$$(m_q - w_b(m_q))/(b-1) = \beta_n c_n + \cdots + \beta_q c_q.$$

The theorem states that each nonnegative integer a is expressible uniquely in the form

$$(*)\quad (m_q - w_b(m_q))/(b-1) + c_q = \beta_n c_n + \cdots + \beta_{q+1}c_{q+1} + (\beta_q + 1)c_q,$$

with $0\leqq\beta_i\leqq b-1$.

We first prove the existence of m_q for a given a. Because $a\geqq 1$ and $c_1=1$, there exists $n\geqq 1$ such that

$$c_1 < \cdots < c_n \leqq a < c_{n+1}.$$

Write $r_{n+1}=a$, and denote by r_i the remainder of the division of r_{i+1} by c_i, α_i denoting the quotient; let r_{q+1} be the last nonzero remainder.

One has

$$\begin{aligned} a &= \alpha_n c_n + r_n \\ r_n &= \alpha_{n-1}c_{n-1} + r_{n-1} \\ &\cdots\cdots\cdots \\ r_{i+1} &= \alpha_i c_i + r_i \\ &\cdots\cdots\cdots \\ r_{q+1} &= \alpha_q c_q \end{aligned}$$

with $1\leqq r_i<c_i$, $\alpha_i\geqq 0$ for $i=n, \cdots, q+1$; $\alpha_q\geqq 1$. From

$$bc_i + 1 = c_{i+1} > r_{i+1} = \alpha_i c_i + r_i \geqq \alpha_i c_i + 1$$

we get $\alpha_i<b$ for $i=n, \cdots, q+1$; from $bc_q+1=c_{q+1}>r_{q+1}=\alpha_q c_q$ we get $\alpha_q\leqq b$. Let $\beta_i=\alpha_i$ for $i>q$, and $\beta_q=\alpha_q-1$; then $0\leqq\beta_i\leqq b-1$ for all i and

$$\beta_n b^n + \beta_{n-1}b^{n-1} + \cdots + \beta_q b^q$$

is the b-adic expansion of m_q, divisible by b^q.

We now prove the uniqueness of the expression (*). Suppose that we could have

$$\begin{aligned} a &= \beta_n c_n + \beta_{n-1}c_{n-1} + \cdots + \beta_q c_q + c_q \\ &= \beta'_n c_n + \beta'_{n-1}c_{n-1} + \cdots + \beta'_{q'}c_{q'} + c_{q'} \end{aligned}$$

with $0 \leqq \beta_i \leqq b-1$, $0 \leqq \beta_i' \leqq b-1$, where, for at least one i, $\beta_i' \neq \beta_i$.

Let j be the largest index for which $\beta_j' \neq \beta_j$; suppose that $\beta_j' > \beta_j$; then $j > q$. Equating the two expressions of a, we would have

$$c_j \leqq (\beta_j' - \beta_j)c_j = (\beta_{j-1} - \beta_{j-1}')c_{j-1} + \cdots + (\beta_i - \beta_i')c_i + \cdots + c_q - c_{q'}$$

(where $\beta_i = 0$ for $i < q$, $\beta_i' = 0$ for $i < q'$). Dropping all negative terms on the right hand side, we would have $c_j \leqq \beta_{j-1}c_{j-1} + \cdots + \beta_q c_q + c_q$ or

$$\beta_{j-1}c_{j-1} + \cdots + \beta_q c_q \geqq c_j - c_q = (b^{j-1} + \cdots + b + 1) - (b^{q-1} + \cdots + b + 1)$$
$$= b^{j-1} + b^{j-2} + \cdots + b^q.$$

But $\beta_i \leqq b-1$, hence the lefthand side is not larger than

$$(b-1)c_{j-1} + (b-1)c_{j-2} + \cdots + (b-1)c_q = (b^{j-1} - 1) + (b^{j-2} - 1) + \cdots$$
$$+ (b^q - 1) = b^{j-1} + b^{j-2} + \cdots + b^q - (j-q) < b^{j-1} + b^{j-2} + \cdots + b^q,$$

which proves that the above inequality is impossible.

ONE MORE REASON WHY SEQUENCES ARE NOT ENOUGH

BERNARD KRIPKE, University of California, Berkeley

It is common knowledge that many interesting properties of nonmetric topologies cannot be characterized by sequential convergence. In particular, sequences are inadequate to characterize closure, compactness, and the like in the weak * topologies on infinite-dimensional Banach duals. One could easily conclude that one must always operate with nets in weak * topologies—that sequences will never do. Friends of Sequential Convergence will find solace from such extremism in [1]. The present note, however, strikes one more blow at Sequentialism by exhibiting a Banach space B and a linear functional Φ on its dual B^* which is weak * sequentially continuous, but not weak * continuous. That is, although Φ carries weak * convergent *sequences* in B^* into convergent sequences of real numbers, the inverse image under Φ of a point is not closed in the weak * topology of B^*.

Let S be an uncountable set, and let B be the family of all bounded real-valued functions on S which vanish at all but countably many points of S. If $x \in B$, its *support* is $\sigma(x) = \{s \in S: x(s) \neq 0\}$. Norm B by setting $\|x\| = \sup\{|x(s)| : s \in S\}$. This makes B into a Banach space, for if $x_n \to x$ uniformly on S, x vanishes outside $U_{n=1}^{\infty} \sigma(x_n)$.

Let $f \in B^*$ be a continuous linear functional on B. A subset T of S is a *supporting set* for f if $f(x) = 0$ whenever $\sigma(x)$ is disjoint from T. If $T \subseteq U$, then U is also a supporting set for f. Let us show that each f in B^* has a *countable* supporting set. Since f is bounded, there is a sequence $\{x_n\}$ in B such that $\|x_n\| = 1$ for each n and $\lim f(x_n) = \|f\|$. Put $T = U_{n=1}^{\infty} \sigma(x_n)$. If $\sigma(y)$ does not interest T and $\|y\| \leqq 1$, then $\|y + x_n\| = 1$ for each n. Suppose that $f(y) > 0$. Choose an n

so large that $f(x_n) > \|f\| - f(y)$. Then

$$f(y) + f(x_n) \leqq \|f\| \, \|y + x_n\| \leqq \|f\|,$$

a contradiction. Therefore, T is a supporting set for f.

If T is a countable subset of S, let $\chi_T \in S$ be the function which is 1 on T and 0 outside T. Suppose $T \subseteq U$ and both are countable supporting sets for f. Then the support of $\chi_U - \chi_T = \chi_{U-T}$ is disjoint from T, so that $f(\chi_U - \chi_T) = 0$. It follows that if T and U are any two countable supporting sets for f,

$$f(\chi_U) = f(\chi_{U \cup T}) = f(\chi_T).$$

We can thus define a unique function Φ from B^* to the reals as follows: if $f \in B^*$ and T supports f, put $\Phi(f) = f(\chi_T)$. It is clear that Φ is linear.

Let us first show that Φ is weak * sequentially continuous. If $\{f_n\}$ is a sequence in B^* which is weak * convergent to f, then

$$T = \sigma(f) \cup U_{n=1}^{\infty} \sigma(f_n)$$

is a countable supporting set for f and for each f_n. Then

$$\Phi(f) = f(\chi_T) = \lim f_n(\chi_T) = \lim \Phi(f_n),$$

where the central equation follows from the definition of weak * convergence for functionals in B^* as pointwise convergence on B.

Next, let us show that Φ is not weak * continuous, by showing that $C = \Phi^{-1}(1) = \{f \in B^* : \Phi(f) = 1\}$ is not weak * closed. Indeed, it will be shown that the zero functional Z ($Z(x) = 0$ for all $x \in B$) is in the weak * closure of C, although it clearly is not in C. Any neighborhood N of Z in B^* contains a basic neighborhood of the form

$$N(x_1, \cdots, x_n) = \{f \in B^* : |f(x_j)| < 1 \quad \text{for} \quad j = 1, \cdots, n\},$$

where $x_1, \cdots, x_n$ are some elements of B. Take any point $s \in S$ outside of $\sigma(x_1) \cup \cdots \cup \sigma(x_n)$. If we put $f(x) = x(s)$, then f is an element of B^*, and moreover, f is in $N(x_1, \cdots, x_n) \subset N$ because $x_1, \cdots, x_n$ all vanish at s. But $\Phi(f) = f(\chi_{\{s\}}) = 1$, so that f is also in C. That is, every weak * neighborhood of Z intersects C, so that Z is in the weak * closure of C.

Finally, it is amusing to remark that although we have shown that every element of B^* has a countable supporting set (by implicitly using the Axiom of Choice), there is no way to pick a *smallest* countable support for each element of B^*. Consider the subspace L of B consisting of functions on S which vanish at ∞. That is, x is in L if and only if for each $\epsilon > 0$, there are only finitely many points in S at which $|x(s)| \geqq \epsilon$. It is easy to see that L is closed, and is not all of B. According to the Hahn-Banach Theorem [2, page 64], there is a nonzero element f of B^* which vanishes on L. If T is any supporting set for f and s is in T, then $T - \{s\}$ also supports f. Indeed, $\{s\}$ is countable, so that $\chi_{\{s\}} \in L$, and $f(\chi_{\{s\}}) = 0$. If x vanishes outside $T - \{s\}$, then $x - x(s) \cdot \chi_{\{s\}}$ vanishes outside T, and

$$f(x) = f(x) - x(s)f(\chi_{\{s\}}) = f(x - x(s)\chi_{\{s\}}) = 0.$$

No matter what supporting set for f we choose, we can always find a smaller one.

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A THEOREM ON SPACES OF CONSTANT CURVATURE

C. M. FULTON, University of California, Davis

To obviate comments on notation we will conform throughout to the one used in [1]. We wish to prove the following

THEOREM. *If in an n-dimensional Riemannian space there is a coordinate system x^i such that the Christoffel symbols* [1, p. 55] *satisfy the relation*

$$[k, ij] = p_k g_{ij} \tag{1}$$

for some functions p_k, then the space has constant curvature.

This theorem may be proved by using the Weyl tensor and/or Schur's theorem. Since this would require a special method for the case $n=2$, we prefer a treatment valid for all n. First of all we notice that the p's are components of a gradient because we may infer [1, p. 56] from (1) that $p_i=\partial_i \log \sqrt{g}$. It is now a matter of straightforward manipulation [1, pp. 55, 110] to derive the curvature tensor from (1). We find

$$R^a_{ijk} = g_{ik}\partial_j p^a - g_{ij}\partial_k p^a. \tag{2}$$

Changing to the covariant curvature tensor and making use of one of its symmetry properties [1, p. 112], we have

$$R_{hijk} = g_{ik}g_{ha}\partial_j p^a - g_{ij}g_{ha}\partial_k p^a = R_{ihkj} = g_{hj}g_{ia}\partial_k p^a - g_{hk}g_{ia}\partial_j p^a. \tag{3}$$

On multiplication by g^{hk} we obtain two expressions for the Ricci tensor, namely $g_{ia}\partial_j p^a - g_{ij}\partial_a p^a = g_{ia}\partial_j p^a - n g_{ia}\partial_j p^a$. Writing A for $(1/n)\partial_a p^a$ this gives rise to

$$g_{ia}\partial_j p^a = g_{ij}A \quad \text{and} \quad \partial_j p^a = \delta^a_j A. \tag{4}$$

With the aid of the covariant derivative $p^a_{,j}=\partial_j p^a + p^a p_j$ and (4) we get $p_{i,j} = g_{ij}A + p_i p_j$. Further covariant differentiation yields

$$p_{i,jk} - p_{i,kj} = g_{ij}A_{,k} - g_{ik}A_{,j} + g_{ik}A p_j - g_{ij}A p_k.$$

This is rewritten [1, p. 110] by means of a Ricci identity and (2) and (4) as follows:

$$p_a R^a_{ijk} = p_j g_{ik}A - p_k g_{ij}A.$$

Hence, by comparison, $g_{ij}A_{,k}=g_{ik}A_{,j}$, which implies that A is a constant. We now conclude on account of (4) that $p^i=Ax^i$ neglecting the constants of integration without loss of generality. Also, (3) shows that

$$R_{hijk} = A(g_{hj}g_{ik} - g_{hk}g_{ij})$$

and so [1, p. 117] $A=K$ is the constant Riemannian curvature of the space.

To conclude we determine the contravariant metric tensor from the relation for its partial derivatives [1, p. 56] which is equivalent to $g^{ik}_{,j}=0$. Thus, from (1), $\partial_j g^{ik} = -Kx^i\delta^k_j - Kx^k\delta^i_j$, whence it follows that $g^{ik} = -Kx^ix^k + c^{ik}$, the c's being arbitrary symmetric constants. If their reduced cofactors are denoted by c_{ik}, it is easily verified that

$$g_{ij} = c_{ij} + Kc_{ir}x^rc_{js}x^s/(1 - Kc_{ab}x^ax^b).$$

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GENERALIZATION OF A RELATION INVOLVING RIGHT TRIANGLES

W. J. BLUNDON, Memorial University of Newfoundland

1. Introduction. Let R, r, s represent respectively the circumradius, inradius and semiperimeter of a triangle with sides a, b, c. It is well known that one of the angles of a triangle is a right angle if and only if $s=2R+r$. The purpose of this paper is to derive the corresponding equation for an angle of any given measure. It turns out that the relation is again linear in R, r, s. For an interesting account of related problems, see Coxeter [2]. We prove the following theorem.

THEOREM. *At least one of the angles of a triangle has measure θ if and only if* $s=2R\sin\theta+r\cot\theta/2$.

To prove this we need expressions for the elementary symmetric functions of the cosines of the angles in terms of R, r, s, as set forth in the following lemma.

LEMMA. *In every triangle*

$$\cos A\cos B\cos C = (s^2 - 4R^2 - 4Rr - r^2)/4R^2,$$

$$\sum \cos A\cos B = (s^2 - 4R^2 + r^2)/4R^2, \tag{1}$$

$$\sum \cos A = (R + r)/R.$$

2. *Proof of the Lemma.* From the basic identities, namely,

$$a + b + c = 2s$$

$$ab + bc + ca = s^2 + 4Rr + r^2$$

$$abc = 4Rrs,$$

we obtain after some computation the following identities. (See [1].)

$$\sum a^2b - \sum a^3 = 8rs(R + r), \tag{2}$$

$$2\sum a^2b^2 - \sum a^4 = 16r^2s^2. \tag{3}$$

With the help of the identities

$$\sum a^3b^3 = (\sum ab)(\sum a^2b^2) - abc\sum a^2b,$$

$$\sum a^5b = (\sum ab)(\sum a^4) - abc\sum a^3,$$

the identity $\sum ab(b^2+c^2-a^2)(c^2+a^2-b^2) = 2\sum a^3b^3 - \sum a^5b + abc\sum a^3$ becomes

$$\sum ab(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) = 16r^2s^2(s^2 - 4R^2 + r^2). \tag{4}$$

Also, from (2) it follows easily that

$$\sum a(b^2 + c^2 - a^2) = 8rs(R + r). \tag{5}$$

Applying the law of cosines to (4) and (5), we have the second and third identities of the lemma.

Finally, the identity $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$ gives

$$\begin{aligned}\cos A \cos B \cos C &= \tfrac{1}{2}(1 - \cos^2 A - \cos^2 B - \cos^2 C)\\ &= \tfrac{1}{2}(\sin^2 A + \sin^2 B + \sin^2 C - 2)\\ &= (a^2 + b^2 + c^2)/8R^2 - 1\end{aligned}$$

which reduces to the first identity of the Lemma.

3. *Proof of the Theorem.* At least one of the angles of the triangle ABC has measure θ if and only if

$$(\cos\theta - \cos A)(\cos\theta - \cos B)(\cos\theta - \cos C) = 0. \tag{6}$$

Using (1), we may put (6) in the form

$$4R^2\cos^3\theta - 4(R^2+Rr)\cos^2\theta + (s^2-4R^2+r^2)\cos\theta - (s^2-4R^2-4Rr-r^2) = 0.$$

Solving for s^2 and taking the square root, we have

$$s = 2R(1 - \cos^2\theta)^{1/2} + r(1 + \cos\theta)^{1/2}/(1 - \cos\theta)^{1/2},$$

which reduces to $s = 2R\sin\theta + r\cot\theta/2$. This completes the proof of the theorem.

COROLLARY 1. *At least one of the angles of a triangle is* 60° *if and only if* $s = (R+r)\sqrt{3}$.

COROLLARY 2.

$$\begin{vmatrix} 1 & \sin A & \cot A/2 \\ 1 & \sin B & \cot B/2 \\ 1 & \sin C & \cot C/2 \end{vmatrix} = 0.$$

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AN INFINITE GROUPOID WHICH ADMITS ONLY TRIVIAL TOPOLOGIES

J. R. HANSON, Madison College and Virginia Polytechnic Institute

Let $G=(a, b, c, \cdots)$ be a groupoid, i.e. a set closed under a binary operation. If U_x represents an open set containing the element x, the topology T is said to be admissible for the groupoid G provided for every product $a\cdot b=c$ in G and for every open set U_c in T there exist open sets U_a and U_b in T such that $U_a\cdot U_b\subseteq U_c$. A groupoid G will be called topologically trivial if the only admissible topologies for G are the trivial discrete and indiscrete topologies.

Let $I^+=(1, 2, 3, \cdots)$ be the positive integers with multiplication defined by $x\cdot y=1$ if $x\neq y$, and $x\cdot x=x+1$. The purpose of this note is to show that this infinite groupoid introduced by Doyle and Warne [1] is topologically trivial.

In what follows all small letters will represent elements of I^+.

LEMMA 1. *If (p) is an open set in an admissible topology for I^+, then (q) is open for all $q<p$.*

Proof. If $p>1$, $(p-1)(p-1)=(p)$. Thus there exists an open set U_{p-1} such that $U_{p-1}\cdot U_{p-1}=(p)$. This implies $U_{p-1}=(p-1)$, or $(p-1)$ is open. Proceed by induction.

LEMMA 2. *If I^+ admits a topology T which is not the discrete topology, the following statements are valid:*

(1) *There exists j in I^+ such that if U_k is in T and $k>j$, then 1 is in U_k.*

(2) *If U_1 is in T, then 2 is in U_1.*

(3) *If U is in T, U cannot consist of a single point.*

Proof. Clearly, there exists j in I such that each open set with respect to T containing j contains two or more elements. $j\cdot j=j+1$. Thus for U_{j+1} there exists U_j such that $U_j\cdot U_j\subseteq U_{j+1}$. But U_j contains elements r and j with $r\neq j$. Therefore, $r\cdot j=1$ belongs to U_{j+1}. Proceed by induction to complete the proof of (1).

(2) There are elements m and n in I each greater than the j of (1), with $m\neq n$ so that $m\cdot n=1$. Therefore, for any U_1 there exist U_m and U_n such that $U_m\cdot U_n\subseteq U_1$. By (1) 1 is in U_m and U_n so that $1\cdot 1=2$ is in U_1.

(3) If (p), $p>1$, were an open set, (1) would be an open set by Lemma 1. But (1) is not an open set by (2).

THEOREM. *I^+ with the given multiplication is topologically trivial.*

Proof. Suppose I^+ is not topologically trivial and T is a nontrivial admissible topology. Then by (3) no single point is an open set in T.

First, 1 is in U_j for all j in I^+. Clearly, 1 is in U_1. For $j>1$, $(j-1)\cdot(j-1)=j$ implies that there exists U_{j-1} such that $U_{j-1}\cdot U_{j-1}\subseteq U_j$. Since $(j-1)$ is not open, there exist r and s, $r\neq s$, in U_{j-1} so that $r\cdot s=1$ is in U_j.

Now suppose that k belongs to U_j for every j in I^+. Then $k+1$ belongs to U_j for every j. For U_j $(j\geqq 2)$ there exists U_{j-1} such that $U_{j-1}\cdot U_{j-1}\subseteq U_j$. Since k is in U_{j-1}, $k\cdot k=k+1$ is in U_j. For U_1 there exist U_r and U_s, $r\neq s$, such that $U_r\cdot U_s\subseteq U_1$. But k is in U_r and U_s, so $k\cdot k=k+1$ is in U_1. Hence the induction is complete, and $U_j=I^+$ for every j. But T was assumed to be nontrivial, a contradiction. Therefore, I^+ is topologically trivial.

The interested reader is invited to investigate, as the author has, which groups, semigroups, and groupoids can be topologically trivial.

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A COINCIDENCE THEOREM

RAUL MACHUCA, Fairleigh Dickenson University, Teaneck

Let X be a T_1 topological space satisfying the first axiom of countability, Y a complete metric space and $f\colon X\to Y$ and $g\colon X\to Y$ continuous. A *compact* mapping is a mapping k such that $k^{-1}(K)$ is compact for each compact set K contained in the image of k.

THEOREM. *If $f(X)\subset g(X)$, f is a compact mapping with $f(X)$ closed, or g is a compact mapping with $g(X)$ closed, and for some $K<1$, $\delta(f(x), f(x'))\leqq K\delta(g(x), g(x'))$ for all x, $x'\in X$, then $f(t)=g(t)$ for some $t\in X$.*

Proof. Define a sequence $\{x_n\}\subset X$ such that $f(x_{n-1})=g(x_n)$ for $n=2, 3, \cdots$. Since $\delta(g(x_n), g(x_{n+1}))\leqq K\delta(g(x_{n-1}), g(x_n))$ with $K<1$, $\{g(x_n)\}$ is Cauchy in Y. Since Y is complete there exists $y\in Y$ such that $g(x_n)\to y\leftarrow f(x_n)$. By the hypothesis on compactness of f or g we have $f^{-1}(A)$ or $g^{-1}(A)$ compact in X, where A is the compact subset of Y consisting of the points of the sequence $\{f(x_n)\}$ plus its limit y. Thus $\{x_n\}$ has a convergent subsequence. It follows that $f(t)=g(t)$ where t is the limit of any such subsequence.

COROLLARY. *If $f\colon X\to X$ is a continuous mapping of a complete metric space X onto itself such that $\delta(f(x), f(x'))\geqq K\delta(x, x')$ for all x, $x'\in X$ and some $K>1$, then f has a fixed point.*

The corollary follows from an investigation of f^{-1}.

ONCE MORE THE BRAUER-RADEMACHER IDENTITY

P. Szüsz, Pennsylvania State University

The Brauer-Rademacher identity is the following:

$$\phi(r) \sum_{\substack{d \mid r \\ (d,n)=1}} d\mu\left(\frac{r}{d}\right)\frac{1}{\phi(d)} = \mu(r) \sum_{d \mid (r,n)} d\mu\left(\frac{r}{d}\right). \tag{1}$$

There are several proofs and generalizations of (1); see Subbarao [1] where references are given. In the present note I give a very short and simple proof of (1). It would be easy to give some generalizations in the same way (for instance that of Subbarao), but the essential idea is contained in the proof of (1).

First I show that if r is not square-free, then

$$\phi(r) \sum_{\substack{d \mid r \\ (n,d)=1}} d\mu\left(\frac{r}{d}\right)\frac{1}{\phi(d)} = 0. \tag{2}$$

Namely let p be a prime factor of r contained at least to the second power. The divisors of r, d and d' are called pairs, when r/d doesn't contain p, r/d' contains p exactly to the first power and $d/d' = p$. Since p is contained in r at least to the second power, we have $d/\phi(d) = d'/\phi(d')$ and

$$\mu\left(\frac{r}{d}\right) = -\mu\left(\frac{r}{d'}\right),$$

that is

$$d\mu\left(\frac{r}{d}\right)\frac{1}{\phi(d)} + d'\mu\left(\frac{r}{d'}\right)\frac{1}{\phi(d')} = 0. \tag{3}$$

Now we can separate all the divisors of r into three classes, the first two being pairs and the third, the terms of which are denoted by d'' consisting of the divisors of r for which $p^2 \mid (r/d'')$.

From (3) and $\mu(r/d'') = 0$ we obtain (2).

Now let r be square-free. Then we have

$$\phi(r) \sum_{d \mid r} d\mu\left(\frac{r}{d}\right)\frac{1}{\phi(d)} = 1; \tag{4}$$

namely, since r is square-free, $\phi(r) = \phi(d)\phi(r/d)$ for each divisor of r, that is

$$\phi(r) \sum_{d \mid r} d\mu\left(\frac{r}{d}\right)\frac{1}{\phi(d)} = \sum_{d \mid r} d\mu\left(\frac{r}{d}\right)\phi\left(\frac{r}{d}\right) = r \sum_{d \mid r} \mu(d)\frac{\phi(d)}{d} = 1.$$

Let (r, n) be

$$(r, n) = q_1 q_2 \cdots q_k \qquad (q_1, \cdots, q_k \text{ distinct primes}). \tag{5}$$

Then

$$\phi(r) \sum_{\substack{d \mid r \\ (d,n)=1}} d\mu\left(\frac{r}{d}\right)\frac{1}{\phi(d)} = \phi(r)\left[\sum_{d \mid r} - \sum_{q \mid (r,n)} \sum_{\substack{d \mid r \\ q \mid d}} + \cdots\right]. \tag{6}$$

By (4)

$$\phi(r) \sum_{\substack{d \mid r \\ c \mid d}} d\mu\left(\frac{r}{d}\right)\frac{1}{\phi(d)} = \phi(r) \sum_{d \mid (r/c)} cd\mu\left(\frac{r}{cd}\right)\frac{1}{\phi(c)\phi(d)} = c;$$

hence, by (6)

$$\phi(r) \sum_{\substack{d \mid r \\ (d,n)=1}} d\mu\left(\frac{r}{d}\right)\frac{1}{\phi(d)} = 1 - q_1 - q_2 - \cdots - q_k + q_1q_2 + q_1q_3 + \cdots$$

$$= \prod_{j=1}^{k} (1 - q_j),$$

which is equal to the right side of (1) and our proof is complete.

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CLASSROOM NOTES

Materials for this department should be sent to George Raney, Department of Mathematics. University of Connecticut, Storrs, CT 06268.

THE VANDERMONDE MATRIX

ALLEN KLINGER, The RAND Corporation, Santa Monica, California

1. The explicit representation of the inverse to Vandermonde's matrix seems unavailable in the mathematical literature and generally inaccessible, though simple in form. Inverting this matrix is necessary in many areas of numerical analysis (in exponential approximation and quadrature sums, in addition to the applications we will present). The material which follows outlines a derivation for the inverse and relates matrices of Vandermonde-form to polynomial interpolation and least-square fitting of data, in a way suitable for classroom exercises.

The $n \times n$ Vandermonde matrix V is given by

$$V = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix} = [t_i^{j-1}]. \tag{1}$$

There are two well-known facts concerning V which appear in texts on matrix theory ([7] pp. 78 and 80; [1] p. 186):

(2) V is nonsingular if $\{t_i\}_{i=1}^{n}$ are distinct;

(3) $$\det V = \prod_{n \geq j > i \geq 1} (t_j - t_i).$$

As a consequence of (2), an equation with m distinct specified roots, $\{r_i\}_{i=1}^{m}$, can be written

(4) $$\det \begin{bmatrix} 1 & r_1 & r_1^2 & \cdots & r_1^m \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & r_m & r_m^2 & \cdots & r_m^m \\ 1 & x & x^2 & \cdots & x^m \end{bmatrix} = 0.$$

2. Inverse of the Vandermonde matrix. The determinant formed from an $n \times n$ Vandermonde determinant, det V, by deleting the $(k+1)$-th column (kth powers) and adjoining as a new nth column the nth powers of the t_i, det V_k, satisfies

(5) $$\det V_k = (a_{n-k})(\det V),$$

where a_{n-k}, is the $(n-k)$-th elementary symmetric function of the n values of t_i ([2] pp. 144–145; [8] p. 99). a_r is the coefficient of y^{n-r} in the expansion of $\prod_{s=1}^{n}(y+t_s)$; thus

$$a_r = \sum_{1 \leq s_1 < s_2 < \cdots < s_r \leq n} \prod_{q=1}^{r} t_{s_q}.$$

Let a_{n-k}^{i} be the $(n-k)$-th elementary symmetric function of the $(n-1)$ arguments t_l, $l=1, 2, \cdots, n$, $l \neq i$.

Making use of (3), (5), and noting that the cofactors appearing in the expression for the inverse matrix are $(n-1) \times (n-1)$ determinants of the same form as det V_k, leads to

(6) $$V^{-1} = \left[\frac{(-1)^{i+j} a_{n-j}^{i}}{\prod_{k=i+1}^{n} (t_k - t_i) \prod_{l=1}^{i-1} (t_i - t_l)} \right]^T,$$

where T denotes transpose. With $\sum_{i=1}^{n_j} a_i$ for $\sum_{i=1, i \neq j}^{n} a_i$ and $\prod_{i=1}^{n_j} a_i$ for $\prod_{i=1, i \neq j}^{n} a_i$, equation (6) can easily be rewritten as

(7) $$V^{-1} = \left[\frac{(-1)^{i+1} \sum_{1 \leq s_1 < s_2 < \cdots < s_{n-i} \leq n}^{j} \prod_{q=1}^{n-i} t_{s_q}}{\prod_{k=1}^{n_j} (t_k - t_j)} \right].$$

(This result appears, without proof, in [4] p. 312.) The inverse for the special case $t_i = x + ih$, $i=0, 1, \cdots, n-1$, and an outline of an alternate derivation of

the inverse for the general Vandermonde matrix (but not the explicit representation of V^{-1}) may be found in [5].

3. Rectangular Vandermonde matrices. The concept of Vandermonde matrices generalizes to the rectangular case. Thus, we define

$$V_{nm} = \begin{bmatrix} 1 & t_1 & \cdots & t_1^{m-1} \\ 1 & t_2 & \cdots & t_2^{m-1} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & t_n & \cdots & t_n^{m-1} \end{bmatrix} = [\mathbf{T}_1\mathbf{T}_2 \cdots \mathbf{T}_n]^T, \tag{8}$$

where $n \neq m$ and $\mathbf{T}_i$ is the vector

$$\mathbf{T}_i = [1, t_i, t_i^2, \cdots, t_i^{m-1}]^T, \qquad i = 1, \cdots, n. \tag{9}$$

Now put $\mathbf{U} = [u_i] = [u(t_i)]$ for brevity. If we seek an $(m-1)$-th degree polynomial which yields a least squares fit to $\{u_i\}_{i=1}^n$, we want an m-dimensional constant vector $\mathbf{C}$, such that $\mathbf{C}$ attains

$$\operatorname*{Min}_{\mathbf{C}} \sum_{i=1}^{n} (u_i - \mathbf{C}^T\mathbf{T}_i)^2. \tag{10}$$

Then, the usual procedure of differentiation with respect to the components of $\mathbf{C}$, and equating the resulting expressions to zero to obtain the normal equations ([6] pp. 343–355), yields a result which may easily be expressed by means of a rectangular Vandermonde matrix

$$V_{nm}^T V_{nm}\mathbf{C} = V_{nm}^T\mathbf{U}. \tag{11}$$

4. Interpolating polynomials. The interpolating polynomial through $(t_i, u(t_i))$, $i = 1, 2, \cdots, n$ is ([3] p. 193):

$$P(t) = \sum_{i=1}^{n} \frac{\omega(t)}{(t - t_i)\omega'(t_i)} u(t_i) = \sum_{i=1}^{n} u(t_i) \prod_{j=1}^{ni} \frac{t - t_j}{t_i - t_j} = \mathbf{L}^T\mathbf{U}, \tag{12}$$

where prime signifies derivative with respect to t,

$$\mathbf{L} = \left[\prod_{j=1}^{ni} \frac{t - t_j}{t_i - t_j} \right]^T, \text{ and } \omega(t) = \prod_{j=1}^{n} (t - t_j).$$

By setting $m = n$ in (11) we can obtain a matrix expression for this polynomial:

$$P(t) = [V_{nn}^{-1}\mathbf{U}]^T\mathbf{T} = \mathbf{T}^T V_{nn}^{-1}\mathbf{U}. \tag{13}$$

That $\mathbf{L}^T \equiv \mathbf{T}^T V_{nn}^{-1}$ is easily established by use of (6) or (7).

Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of the RAND Corporation or the official opinion or policy of any of its governmental or private research sponsors.

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A GENERALIZATION OF HJEMSLEV'S THEOREM

P. C. JOSEPH, St. Thomas College, Palai, India

Hjemslev's Theorem [1, p. 47; 2, p. 268] states that, in the Euclidean plane, when all the points P on one line are related by an isometry to all the points P' on another, the midpoints of the segments PP' are distinct and collinear or they all coincide.

A natural generalization is the following

THEOREM. *If S, S' are two figures in a plane, which are congruent (related by an isometry) and X is the midpoint of the segment joining any point P in S to the corresponding point P' in S', then the set of points X either all coincide or are collinear or they form a figure similar to S and S'.*

The proof is an application of Coxeter [1, p. 46, 3.51]: Any direct isometry is either a translation or a rotation. Any opposite isometry is either a reflection or a glide reflection.

Suppose, first, that the figures S, S' are two triangles ABC, $A'B'C'$.

(1) If the isometry is opposite, the midpoints X all lie on the line which is the mirror of the reflection or glide reflection.

(2) If the isometry is direct and is a translation, it is immediately evident that the figure formed by the points X is congruent (therefore certainly similar) to S, being related to S by a translation through half the displacement of the original translation.

(3) The only remaining possibility is a rotation about some point O through an angle α. If $\alpha=\pi$, the points X all coincide at O. Otherwise, we can choose the sense of rotation suitably so that $\alpha<\pi$.

If X, Y are the midpoints of AA', BB', it is easy to see that $OX=OA\cos(\alpha/2)$, $OY=OB\cos(\alpha/2)$, $\angle XOY=\angle AOB$. It follows that $XY=AB\cos(\alpha/2)$.

This relation holds for any two of the points A, B, C and leads to the conclusion that the triangles XYZ and ABC are similar.

The extension to any two congruent figures S, S' is immediate and this completes the proof.

The following concrete interpretation of the theorem is interesting.

If two identical maps of a country, drawn on transparent sheets, be thrown at random on a plane floor, the midpoints of the segments joining corresponding places on the two maps will either (A) be all collinear or (B) form a map of the same country, not bigger than the original, the possibility that the new map collapses into a single point being included.

(A) occurs if the two maps fall different sides up and (B) if both have the same side up.

Editorial Note. Before Julius Petersen changed his name to Hjelmslev, he discovered the "Petersen-Schoute Theorem" [3, p. 74], which shows that the same conclusion can be obtained if S and S', instead of being congruent, are directly similar.

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A NOTE ON HERMITIAN FORMS

RABE VON RANDOW, Otago University, Dunedin, New Zealand

Introduction. It is well known that the standard form of a quadratic form f on a finite-dimensional vector space over (a) the field R of real numbers and (b) the field C of complex numbers is (a) $f(x) = \sum_1^p \xi_i^2 - \sum_{p+1}^r \xi_i^2$, where r is the rank and $2p-r$ is the signature of f, and (b) $f(x) = \sum_1^r \xi_i^2$, where r is the rank of f, and that the standard form of a hermitian form f on a finite-dimensional vector space over C is

$$f(x) = \sum_1^p |\xi_i|^2 - \sum_{p+1}^r |\xi_i|^2,$$

where again, r is the rank and $2p-r$ the signature of f. In fact, the theory of hermitian forms over C is completely analogous to that of quadratic forms over R, which, upon extension of the scalar field, gives rise to the theory of quadratic forms over C. Thus the following question presents itself: is it possible to develop a theory of hermitian forms over some division algebra containing C, which is completely analogous to that of quadratic forms over C? Now it is known that the only division algebras over R are C (cf. [1]), and the skew-field H of quaternions (cf. [2]). In this note we give a definition of hermitian forms over H answering the above question. The usual theory of hermitian forms over H utilizing the ordinary conjugation of quaternions is again analogous to that of quadratic forms over R, but a repetition of the above question at this level

obviously leads no further, as no other division algebras are available. We may summarize the situation schematically as follows:

$$\begin{array}{ccc} \text{(Quadratic Forms over } R) & \longrightarrow & \text{(Quadratic Forms over } C) \\ \text{analogous to} & & \text{analogous to} \\ \text{(Hermitian Forms over } C) & \longrightarrow & \text{(Hermitian Forms over } H) \\ \text{analogous to} & & \text{(to be defined)} \\ \text{(Usual Hermitian Forms over } H) & & \end{array}$$

The theory of hermitian forms differs essentially from that of quadratic forms in that it involves an involutory antiautomorphism of the scalar field. Furthermore, extra care is needed, as we are dealing with a skew-field. Let V be an n-dimensional (right) vector space over K, where K denotes C or H, let ϕ be a map $V\times V\to K$, and let $\alpha\to\bar{\alpha}$, $\alpha\in K$, be an involutory antiautomorphism of K, i.e., an automorphism of the vector space K over R satisfying $\overline{\alpha\beta}=\bar{\beta}\cdot\bar{\alpha}$ and $\bar{\bar{\alpha}}=\alpha$, $\alpha, \beta\in K$.

DEFINITION. *ϕ is a (left) sesquilinear form over K if*

$$\begin{aligned} \phi(x_1+x_2, y) &= \phi(x_1, y)+\phi(x_2, y), \quad (x_1, x_2, y\in V), \\ \phi(x, y_1+y_2) &= \phi(x, y_1)+\phi(x, y_2), \quad (x, y_1, y_2\in V), \\ \phi(x\alpha, y) &= \bar{\alpha}\cdot\phi(x, y), \quad (x, y\in V, \alpha\in K), \\ \phi(x, y\beta) &= \phi(x, y)\cdot\beta, \quad (x, y\in V, \beta\in K). \end{aligned}$$

ϕ is hermitian if $\phi(y, x)=\overline{\phi(x, y)}$, $x, y\in V$.

A few moments' trial and error using $\phi(x\alpha\beta, y)$, and $\phi(x\alpha, y\beta)$ in the last relation, show that $\alpha\to\bar{\alpha}$ must be an involutory antiautomorphism and that, apart from interchanging left and right throughout, the above definition is the only sensible one.

With a fixed basis (x_i) in V, we have a 1-1 correspondence between the set of hermitian sesquilinear forms over K and the set of hermitian $(n\times n)$-matrices over K (a square matrix (a_{ij}) over K is *hermitian* if $a_{ij}=\overline{a_{ji}}$ for all i, j): if $x=\sum x_i\xi_i$, $y=\sum x_j\eta_j$, then $\phi(x, y)=\sum\sum\xi_i\cdot\phi(x_i, x_j)\cdot\eta_j=\underline{\xi}^*\cdot M_\phi\cdot\underline{\eta}$ in the usual vector notation, where $M_\phi=(\phi(x_i, x_j))$ is the matrix of ϕ.

DEFINITION. *Take a fixed basis (x_i) in V. Then a hermitian form f over K is a map $V\to K$ given by a hermitian $(n\times n)$-matrix A over K: $f(x)=\underline{\xi}^*\cdot A\cdot\underline{\xi}$, where $x=\sum x_i\xi_i$.*

For the fields dealt with here, we have a 1-1 correspondence between the set of hermitian forms over K and the set of hermitian $(n\times n)$-matrices over K.

We now have to fix the involutory antiautomorphisms to be considered. For $K=C$, it follows readily from the relations $\bar{1}=1$ and $(\bar{i})^2=-1$, that the usual conjugation $z=x+yi\to\bar{z}=x-yi$ is the only involutory (anti)automorphism apart from the identity map. For $K=H$, it has been shown in [3] that, up to

change of basis in H, there are only two involutory antiautomorphisms, namely

$$q = a + bi + cj + dk \to a - bi - cj - dk,$$
$$q = a + bi + cj + dk \to a - bi + cj + dk,\ q \in H.$$

The first of these is the usual conjugation of quaternions, and its use would give the usual theory of real-valued hermitian forms over H. For our purposes we define $\bar{q}=a-bi+cj+dk$. Furthermore, we embed C in H by means of the monomorphism ν: $x+yi\to x+yi+0j+0k$, and it follows that $\overline{\nu(z)}=\nu(\bar{z})$. Note that if $K=C$, $f(x)$ is real for all $x\in V$, i.e., $f(x)\in\{z\in C\,|\,z=\bar{z}\}$, and if $K=H$, $f(x)\in N=\{q\in H\,|\,q=\bar{q}\}$ for all $x\in V$. This is similar to the situation in quadratic form theory: a quadratic form over R is real-valued, whereas a quadratic form over C is complex-valued.

THEOREM. *Given a hermitian form f over K, there exists a basis for V such that $f(x)=\sum_1^r \bar{\eta}_i c_i \eta_i$, where the $c_i\in R-\{0\}$ if $K=C$ and the $c_i\in N-\{0\}$ if $K=H$, and r is the rank of f.*

The proof for $K=C$ is standard (cf. [4]), and is readily adapted to the case $K=H$.

COROLLARY 1. *If $K=C$, f can be reduced further to the form $f(x)=\sum_1^p |\eta_i|^2 - \sum_{p+1}^r |\eta_i|^2$.*

The proof is again standard (cf. [4]), and furthermore, the Sylvester law of inertia holds for hermitian forms over C, i.e., two hermitian forms over C are equivalent if and only if they have the same rank and signature.

COROLLARY 2. *If $K=H$, f can be reduced further to the form $f(x)=\sum_1^r \bar{\eta}_i\cdot\eta_i$. Hence two hermitian forms over H are equivalent if and only if they have the same rank.*

Proof. We need only prove that for each $c_i\in N-\{0\}$, $\exists q_i\in H-\{0\}$ such that $\bar{q}_i c_i q_i=1$. We will prove the following stronger result: given $q'\in N-\{0\}$, $\exists q\in N-\{0\}$ such that $q'=q\bar{q}$.

Let $q'=\alpha+\zeta j$, $\alpha\in R$, $\zeta\in C$, and set $q=a+zj$, $a\in R$, $z\in C$. Then $q\bar{q}=q^2=(a^2-|z|^2)+2azj=q'$ gives $\alpha=a^2-|z|^2$ and $\zeta=2az$.

Clearly $a=0$ if and only if $\zeta=0$ and $\alpha<0$, i.e., $q'\in R$ and $q'<0$. In this case $|z|=\sqrt{-\alpha}$, hence we choose $z=\sqrt{-\alpha}$, i.e., $q=\sqrt{-\alpha}\cdot j$.

For all other q', $a\neq 0$. Then $z=\zeta/2a$, hence $\alpha=a^2-|\zeta|^2/4a^2$ or $4a^4-4\alpha a^2-|\zeta|^2=0$. Solving for a (which is real), we get $a=\pm\sqrt{\tfrac{1}{2}\{\alpha+\sqrt{(\alpha^2+|\zeta|^2)}\}}$ and $z=\zeta/2a$ in this case.

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FUNCTIONS SATISFYING A CERTAIN TYPE OF RECIPROCITY OF DERIVATIVES

C. J. BROOKES, University of Aston, Birmingham, England

Although in general

$$\frac{d^n y}{dx^n} \neq \frac{1}{d^n x/dy^n}, \quad n \geqq 2,$$

it can be shown that a particular solution of the family of nonlinear differential equations

$$\frac{d^n y}{dx^n} = \frac{1}{d^n x/dy^n}$$

is the family of rectangular hyperbolae

$$(x + a)(y + b) = (n!)^{2/(n-1)}$$

where a and b are arbitrary constants.

Moreover, the procedure involved also leads to a particular solution of the more general equation

$$\frac{d^n y}{dx^n} \cdot \frac{d^m x}{dy^m} = 1, \quad n \geqq 2, m \geqq 2,$$

in the form

$$(x + a)^{n-1}(y + b)^{m-1} = (-1)^{n+m} \frac{\Gamma((nm - 1)/(n - 1))\Gamma((nm - 1)/(m - 1))}{\Gamma((n - 1)/(m - 1))\Gamma((m - 1)/(n - 1))}.$$

Γ signifying the usual gamma function.

Proof. Consider first the case $n=2$, i.e.

$$\frac{d^2 y}{dx^2} \cdot \frac{d^2 x}{dy^2} = 1.$$

With $p=dy/dx$, $d^2y/dx^2=p(dp/dy)$ and $d^2x/dy^2=-(1/p^2)dp/dy$, the equation can be solved to give $(x+a)(y+b)=4$, where a and b are arbitrary constants.

General solutions for $n \geqq 3$, however, are extremely difficult, if not impossible, to determine. For example consider $n=3$, i.e.,

$$\frac{d^3 y}{dx^3} \cdot \frac{d^3 x}{dy^3} = 1. \tag{1}$$

With $p=(dy/dx)$, as before, we find that equation (1) reduces to:

$$\left\{\left(\frac{dp}{dy}\right)^2 + p\frac{d^2p}{dy^2}\right\}\left\{2\left(\frac{dp}{dy}\right)^2 - p\frac{d^2p}{dy^2}\right\} = p^2.$$

With $u=dp/dy$ and $d^2p/dy^2=u(du/dp)$, this equation reduces to:

$$u^2p^2\left(\frac{du}{dp}\right)^2 - u^3p\left(\frac{du}{dp}\right) + (p^2 - 2u^4) = 0.$$

With $p=z^2$ and $u=rz$, we obtain $3r^2 \pm \sqrt{(9r^4-4)} = a_1z^6$ or a_2/z^6 where a_1, a_2 are constants.

Reverting to the original variables p and y, the equations become:

$$\frac{dp}{dy} = \pm \frac{\sqrt{(a_1^2p^6 + 4)}}{p\sqrt{(6a_1)}}, \qquad \frac{dp}{dy} = \pm \frac{\sqrt{(a_2^2 + 4p^6)}}{p\sqrt{(6a_2)}}$$

solutions of which lead to transcendental equations in (dy/dx) in terms of elliptic integrals. Further progress seems impossible.

From symmetry, however, it is suggested that a particular solution to the general equation

$$\frac{d^ny}{dx^n}\cdot\frac{d^nx}{dy^n} = 1 \tag{2}$$

might take the form

$$(x + a)(y + b) = k, \tag{3}$$

where k is to be determined. From (3) it can be seen that

$$\frac{d^ny}{dx^n} = \frac{(-1)^nk(n!)}{(x + a)^{n+1}}, \qquad \frac{d^nx}{dy^n} = \frac{(-1)^nk(n!)}{(y + b)^{n+1}}.$$

Equations (2) and (3) now give

$$\frac{d^ny}{dx^n}\cdot\frac{d^nx}{dy^n} = \frac{k^2(n!)^2}{(x + a)^{n+1}(y + b)^{n+1}} = \frac{(n!)^2}{k^{n-1}}.$$

Choosing $k=(n!)^{2/n-1}$, $n\geqq 2$, gives the required result.

Now consider the equation

$$\frac{d^ny}{dx^n}\cdot\frac{d^mx}{dy^m} = 1 \tag{4}$$

and suppose that

$$(x + a)^\alpha(y + b)^\beta = k \tag{5}$$

where α, β and k are to be determined. From equation (5) we find that

$$\frac{d^ny}{dx^n} = (-1)^n\frac{\Gamma\left(\frac{\alpha}{\beta} + n\right)}{\Gamma\left(\frac{\alpha}{\beta}\right)}\cdot\frac{(y + b)}{(x + a)^n}$$

and that

$$\frac{d^m x}{dy^m} = (-1)^m \frac{\Gamma\left(\frac{\beta}{\alpha} + m\right)}{\Gamma\left(\frac{\beta}{\alpha}\right)} \cdot \frac{(x+a)}{(y+b)^m}.$$

Hence

$$\frac{d^n y}{dx^n} \cdot \frac{d^m x}{dy^m} = (-1)^{n+m} \frac{\Gamma\left(\frac{\alpha}{\beta} + n\right)\Gamma\left(\frac{\beta}{\alpha} + m\right)}{\Gamma\left(\frac{\alpha}{\beta}\right)\Gamma\left(\frac{\beta}{\alpha}\right)} \cdot \frac{1}{(x+a)^{n-1}(y+b)^{m-1}}.$$

Choosing $\alpha = n-1$, $\beta = m-1$ now gives the result—readily seen to hold for $m = n$.

A NOTE ON PARTITION-INDUCING RELATIONS

CHARLES GREEN, University of Maine

Let R be a relation in the set S. Let us say that *R induces a partition of S* if there exists a partition of S such that for all x, $y \in S$ with $x \neq y$, x and y belong to the same set in the partition iff $(x, y) \in R$. Harary [1, corollary, p. 215] has proved that R induces a partition of S iff R is symmetric and distinctly transitive. The proof depends on concepts from graph theory. In this note we point out another proof which has the advantage of characterizing the sets in the induced partition.

LEMMA. *Let R be a relation in S and let I be the identity relation on S. Then R is symmetric and distinctly transitive iff $R \cup I$ is an equivalence.*

Proof. It is clear that neither adding nor deleting pairs of the form (x, x) affects symmetry and distinct transitivity. Thus $R \cup I$ is reflexive, symmetric, and distinctly transitive iff R is symmetric and distinctly transitive. But a reflexive, symmetric, and distinctly transitive relation is an equivalence [1, p. 215].

THEOREM. *R induces a partition of S iff R is symmetric and distinctly transitive. If R does induce a partition, its elements are the equivalence classes of S with respect to $R \cup I$.*

Proof. The first statement follows from the lemma and the one-to-one correspondence between equivalences and partitions; the second from the fact that for $x \neq y$, $(x, y) \in R \cup I$ iff $(x, y) \in R$.

Reference

1. F. Harary, A parity relation partitions its field distinctly, this MONTHLY, 68 (1961) 215–217.

A NOTE ON HOMOTOPY THEORY

J. M. ADAMS, New Mexico State University

If A and B are topological spaces, let $C(A, B)$ denote the set of all continuous functions from A to B with the compact-open topology. If X and Y are topological spaces and I is the closed unit interval $[0, 1]$, then $C(X, C(I, Y))$ is homeomorphic to $C(I\times X, Y)$ (cf. [2]). Using this fact, it is easy to prove that the usual definition of homotopy between two continuous functions is equivalent to the following definition.

DEFINITION. *Let f and g belong to $C(X, Y)$; f is homotopic to g iff there exists an element h in $C(X, C(I, Y))$ such that $h(x)(0)=f(x)$ and $h(x)(1)=g(x)$, for each x in X.*

A development of homotopy theory including this equivalent definition along with the usual definition, has the following important advantages:

1. Geometrical appeal, since $h(x)(t)$, for $0\leqq t\leqq 1$ and fixed x in X, is actually the deformation of the point $f(x)$ into the point $g(x)$,
2. Obvious relationship with the relatively recent development of homotopy theory in R-modules [1] or abelian categories [3], where the usual definition is analogous to i-homotopy (based on injectives) and the equivalent definition is analogous to p-homotopy (based on projectives), and
3. Simplification of proofs of some theorems (see the theorem below).

THEOREM. *If Y is a contractible topological space, then every continuous function f of X into Y is homotopic to a constant function.*

Proof. By hypothesis, the identity function on Y is homotopic to a constant function by means of $h\in C(Y, C(I, Y))$. Now f is homotopic to a constant function by the composition $h\circ f: X\to C(I, Y)$.

References

1. P. J. Hilton, Homotopy Theory of Modules and Duality, Symposium Internacional de Topologia Algebraica, Universidad Nacional Autonama de Mexico y UNESCO, (1958) 273–281.
2. Sze-Tsen Hu, Elements of General Topology, Holden-Day, San Francisco, 1964.
3. Heinrich Kleisili, Homotopy theory in abelian categories, Canad. J. Math., 14 (1962) 139–169.

ON BEING MEAN TO THE MEAN VALUE THEOREM

LEON W. COHEN, University of Maryland

With characteristic vigor, L. Bers announced in a recent conversation: "Who needs the mean value theorem! All we want as a start in elementary calculus is the proposition that if $f'(x)=0$ for all x in $[a, b]$, then f is constant." A subsequent conversation indicated that the proof given here is different from that of Professor Bers.

REMARK. If $P=(x, y)$, $Q=(u, v)$, $R=(s, t)$, $x\leqq s\leqq u$, and m, m', m'' are the

slopes of PQ, PR, RQ respectively, and $x<u$, then $m \leqq \max\{m', m''\}$. (Proof later.)

Now suppose $f'(x)=0$ on $[a, b]$, $a \leqq a_1 < b_1 \leqq b$, and $f(a_1) < f(b_1)$. Let $c_1 = (a_1+b_1)/2$, $P_1=(a_1, f(a_1))$, $Q_1=(b_1, f(b_1))$, and $R_1=(c_1, f(c_1))$. By the Remark, one of $[a_1, c_1]$, $[c_1, b_1]$, which we call $[a_2, b_2]$, is such that

$$m_2 = \frac{f(b_2) - f(a_2)}{b_2 - a_2} \geqq m_1 = \frac{f(b_1) - f(a_1)}{b_1 - a_1} > 0.$$

By induction, we find for each k an interval $[a_k, b_k]$ such that

(1) $$[a_{k+1}, b_{k+1}] \subset [a_k, b_k], \qquad b_{k+1} - a_{k+1} = (b_k - a_k)/2 = (b_1 - a_1)/2^k$$

and $m_k = (f(b_k) - f(a_k))/(b_k - a_k)$ such that

(2) $$m_{k+1} \geqq m_k \geqq m_1 > 0.$$

Since by (1), $\lim_k (b_k - a_k) = 0$, $\bigcap_k [a_k, b_k]$ contains a single point x_0. Since $a_k \leqq x_0 \leqq b_k$, the Remark assures us that we can define x_k as a_k or b_k so that

(3) $$\frac{f(x_k) - f(x_0)}{x_k - x_0} \geqq m_k.$$

Now $\lim_k x_k = x_0$ since $|x_{k+1} - x_0| \leqq b_{k+1} - a_{k+1} = (b_1 - a_1)/2^k$. By the hypothesis, (2) and (3),

$$0 = f'(x_0) = \lim_k \frac{f(x_k) - f(x_0)}{x_k - x_0} \geqq m_1 > 0.$$

This is impossible. In case $f(a_1) > f(b_1)$, we put $g(x) = -f(x)$.

Proof of Remark. If $s=x$, $m=m''$ and if $s=u$, $m=m'$. If $x<s<u$, $s = x + \alpha(u-x)$, where $0<\alpha<1$. If $t \geqq y + \alpha(v-y)$, i.e. R is on or above PQ, then

$$m' - m = \frac{t - y}{\alpha(u - x)} - \frac{v - y}{u - x} = \frac{t - y - \alpha(v - y)}{\alpha(u - x)} \geqq 0.$$

If $t \leqq y + \alpha(v-y)$, i.e. R is on or below PQ, then

$$m'' - m = \frac{v - t}{(1 - \alpha)(u - x)} - \frac{v - y}{u - x} = \frac{y + \alpha(v - y) - t}{(1 - \alpha)(u - x)} \geqq 0.$$

In either case $m \leqq \max\{m', m''\}$.

In connection with the construction used above, the reader may be interested in recent work on ordinary differential equations in a vector space by A. K. Aziz and J. B. Diaz. Their paper [1] contains an extensive bibliography.

Reference

1. A. K. Aziz and J. B. Diaz, On a mean-value theorem of the differential calculus of vector-valued functions, and uniqueness theorems for ordinary differential equations in a linear-normed space, Contributions to Differential Equations, I (1963) 251–273.

ON AVOIDING THE MEAN VALUE THEOREM

LIPMAN BERS, Columbia University

The arrangement (hardly a new one*), which I had in mind when talking to Leon Cohen [1], is as follows.

THEOREM. *If* $f'(x)>0$ *for* $a<x<b$, *then* f *is increasing.*

This is intuitively obvious and easy to prove. Indeed, assume there is a p, $a<p<b$, such that the set S of all x, $a<x<p$, with $f(x)\geqq f(p)$ is not empty. Set $q=\sup S$; since $f'(p)>0$, we have $a<q<p$. If $f(q)\geqq f(p)$, then since $f'(q)>0$, there are points of S to the right of q. If $f(q)<f(p)$, then q is not in S and, by continuity, there are no points of S near and to the left of q. Contradiction.

COROLLARY 1. *If* $f'(x)\geqq 0$ *for* $a<x<b$, *then* f *is nondecreasing.* (Apply the theorem to $f(x)+\epsilon x$, $\epsilon>0$.)

This implies that a function with an identically vanishing derivative is constant.

COROLLARY 2. *If* $f'(x)\leqq K$ *for* $a\leqq x\leqq b$, *then* $f(b)\leqq f(a)+K(b-a)$. (Apply Corollary 1 to $K(x-a)-f(x)$.)

Using the intermediate value theorem and either Corollary 2 or the fundamental theorem of calculus, one obtains at once the mean value theorem for continuously differentiable functions. That's all one needs in calculus.

The "full" mean value theorem, for differentiable but not continuously differentiable functions, is a curiosity. It may be discussed together with another curiosity, Darboux' theorem that every derivative obeys the intermediate value theorem.

* I would be grateful to any reader who may know of a reference.

Reference

1. Leon Cohen, On being mean to the mean value theorem, the preceding article in this MONTHLY.

NONNORMALITY OF LINEAR COMBINATIONS OF NORMALLY DISTRIBUTED RANDOM VARIABLES

G. E. ALBERT AND R. L. TITTLE, University of Tennessee

Rosenberg [1] has given an example of a bivariate distribution function which is not a normal distribution but for which the marginal distributions of the individual variables x_i, $i=1, 2$, are normal. Also the distribution of the sum x_1+x_2 is not normal. In this note we present such an example for which the demonstration of the stated properties is much simpler than in Rosenberg's example.

Let a bivariate probability density function $f(x_1, x_2)$ be defined over the plane by

$$(1)\qquad f(x_1, x_2) = \begin{cases} \frac{1}{\pi}\exp[-\frac{1}{2}(x_1^2 + x_2^2)] & \text{if } x_1 \geqq 0,\ x_2 \geqq 0, \text{ or if} \\ & x_1 \leqq 0,\ x_2 \leqq 0, \\ 0 \quad \text{if } x_1 > 0,\ x_2 \leqq 0 \quad \text{or if} \quad x_1 \leqq 0,\ x_2 > 0. \end{cases}$$

Note that $f(x_1, x_2)$ is zero in the second and fourth quadrants of the plane of (x_1, x_2) and so does not define a bivariate normal distribution. It is easy to show that the marginal density functions $f_i(x_i)$, $i=1, 2$, of the two variables are standardized normal density functions. For example,

$$f_1(x_1) = \begin{cases} \frac{1}{\pi}\int_0^\infty \exp[-\frac{1}{2}(x_1^2 + x_2^2)]dx_2 = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x_1^2) & \text{if } x_1 \geqq 0 \\ \frac{1}{\pi}\int_{-\infty}^0 \exp[-\frac{1}{2}(x_1^2 + x_2^2)]dx_2 = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x_1^2) & \text{if } x_1 \leqq 0. \end{cases}$$

The two random variables u_1 and u_2 defined by

$$(2)\qquad \begin{cases} u_1 = (x_1 + x_2)/\sqrt{2} \\ u_2 = (x_1 - x_2)/\sqrt{2} \end{cases}$$

are not jointly normally distributed. To see this we need only notice that (2) defines a 45° rotation of axes in the plane such that the joint density function $g(u_1, u_2)$ of u_1 and u_2 will have the form $\pi^{-1}\exp[-\frac{1}{2}(u_i^2+u_2^2)]$ in two quarter planes which are bisected by the axis of u_1 and in the other two quarter planes $g(u_1, u_2)=0$.

The marginal distribution of the random variable u_1 is not normal, since its density function is given by

$$g_1(u_1) = \begin{cases} \frac{1}{\pi}\int_{-u_1}^{u_1} \exp[-\frac{1}{2}(u_1^2 + u_2^2)]du_2 & \text{if } u_1 \geqq 0 \\ \frac{1}{\pi}\int_{u_1}^{-u_1} \exp[-\frac{1}{2}(u_1^2 + u_2^2)]du_2 & \text{if } u_1 < 0. \end{cases}$$

From the above one obtains

$$(3)\qquad g_1(u_1) = \sqrt{\frac{2}{\pi}}\exp(-\tfrac{1}{2}u_1^2)[\Phi(|u_1|) - \Phi(-|u_1|)],$$

where $\Phi(x)$ denotes the cumulative of the standardized normal distribution. Since $E(u_1)=0$, if (3) were a normal density function, the bracketed expression

would reduce to the form $c\cdot\exp(-ku_1^2)$ for some constants c and k with $c>0$ and $k\geqq 0$. But

$$\Phi(|u_1|) - \Phi(-|u_1|) = (2\pi)^{-1/2}\int_{-|u_1|}^{|u_1|}\exp(-\tfrac{1}{2}u_2^2)du_2$$
$$= \sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n|u_1|^{2n+1}}{n!(2n+1)2^n}$$

and this power series does not have the appropriate form to be the expansion of $c\cdot\exp(-ku_1^2)$.

Reference

1. Lloyd Rosenberg, Nonnormality of linear combinations of normally distributed random variables, this MONTHLY, 72 (1965) 888–890.

INTEGRATION, ANTI-DIFFERENTIATION AND A CONVERSE TO THE MEAN VALUE THEOREM

HOWARD LEVI, Hunter College, City University of New York

While preparing a revision of his experimental course in analysis [1] the writer came upon a theorem which could also be of use in other revisions of the calculus course. The theorem is the following converse to the Mean Value Theorem.

THEOREM. *Let F and f be real valued functions defined on the open interval I and let f be continuous there. Suppose that for every x and y of I there is a z between x and y such that $F(x)-F(y)=(x-y)f(z)$. Then F is differentiable on I and f is its derivative.*

While this theorem does not seem to be in the literature no detailed proof is needed here, since the result is a direct consequence of the definition of derivative and continuity. The writer does wish to comment on some of its implications for the teaching of calculus.

The principal implication is that it makes it possible to bypass the definite integral in the first study of calculus. At present the usual sequence of events is: (a) The definite integral is defined and studied; (b) the fundamental theorem about its evaluation by the indefinite integral is taken up, and (c) applications are given. These take the form of recognizing that certain quantities such as area, arc length, mass, work, etc., can be defined or otherwise expressed as definite integrals and hence can often be evaluated by anti-differentiation. The theorem in question makes it possible to omit steps (a) and (b) and begin at once with the applications. This is so because the same information which permits the formulation of the quantity in question as the definite integral $F(x)-F(y)=\int_y^x f(t)dt$ implies that $F(x)-F(y)$ is between $(x-y)$ max f and $(x-y)$ min f.

Hence, if f is continuous, the theorem identifies f as the derivative of F directly.

Of course even if it is conceded that bypassing the definite integral is possible it might be maintained that such a step is undesirable. The writer's considered opinion on this question is: (a) the material in this topic is so difficult that the student is almost always presented with a frankly incomplete treatment, an incorrect treatment or an incomprehensible treatment; (b) at this level all the functions are piece-wise analytic and require neither the generality nor the subtlety of the usual approach.

The theorem can also be applied meaningfully if F is given and f is unknown. For instance, to identify speed as ds/dt we usually tell our students that this is how the physicists do or should define it. While it is true that many physicists are adept in the use of limits, it is not reasonable to presuppose this skill in practitioners of other disciplines in which calculus can play a role. An alternative approach, using speed as an example, would be to note that we could plausibly expect $\Delta t \cdot \min$ speed to underestimate distance and $\Delta t \cdot \max$ speed to overestimate it. The theorem then asserts that, however speed is defined, ds/dt is a correct formula for it in the usual situations.

Reference

1. H. Levi, An experimental course in analysis for college freshmen, this MONTHLY, 70 (1963) 877–879.

DEFINITION OF RADIUS OF CONVERGENCE

D. E. MYERS, University of Arizona

In most calculus texts, the standard definition of the radius of convergence of a power series

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \tag{1}$$

is

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

With this definition the problems are carefully selected so that the limit always exists. In addition to obtaining a certain skill in computing this limit, the student is asked to grasp two more subtle ideas. On the one hand, r is represented intuitively as maximal such that the series converges for all x with $|x| < r$ and then, somewhat in contrast, the student is cautioned to check independently for convergence at the end points of the interval. Often, to enforce the student's retention of this latter concept, the examples and problems are so chosen that divergence occurs in at least one of the end points. As a consequence, many students expect that this will always occur.

At the next level, perhaps in an advanced calculus or introductory analysis

course, the radius of convergence is generally defined by the root-test whereby the problems of existence of the limit are avoided. The computation of r still seems to rely, however, on the ratio test with "Lim" replaced by "Lim Sup"; consequently, the student still thinks of radius of convergence in terms of maximal region of convergence or the value of

$$\operatorname*{Lim\,Sup}_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

The student next encounters the same problem in the context of complex variable theory complicated by the study of singularities. At this point, the circle of convergence becomes identified with the location of the singularities of the function represented by the series.

We come now to a very interesting example, intended to justify the preceding discussion. Let $f(z)=\sum_{n=1}^{\infty} z^n/n^2$. It is easily seen that the series converges not only for $|z|<1$ but for $|z|=1$. The student wonders where the singularities of f are. If r represents the maximum such that convergence occurs for $|z|<r$, why is the radius for this example 1 and not $1+\epsilon$ ($\epsilon>0$)? The answer is that in this case the singularity is manifested not at a point of divergence, but rather in the nondifferentiability of f. The singularity is easily found by testing the convergence of the differentiated series. It occurs at $z=1$. Of course, if one differentiates the series twice then the resulting series diverges at every point on the circle $|z|=1$.

A FOURIER SERIES DERIVATION OF THE EULER-LAGRANGE EQUATION

RICHARD MENZEL, Haile Sellassie I University

THEOREM. *Suppose that* $y=\bar{y}(x)$ *extremizes* $\int_a^b f(x, y(x), y'(x))dx$ *with respect to all functions which are continuously differentiable on* $[a, b]$ *and which satisfy* $y(a)=y_1$, $y(b)=y_2$. *Then there is a constant* c *such that*

$$f_{y'}(x, \bar{y}(x), \bar{y}'(x)) - \int_a^x f_y(x, \bar{y}(x), \bar{y}'(x))dx = c \qquad \text{for all } x \text{ in } [a, b].$$

Proof. (We assume that $f(x, y, y')$, $f_y(x, y, y')$ and $f_{y'}(x, y, y')$ are continuous for all x in $[a, b]$ and for all y and y'.) We may work over the interval $[-\pi, \pi]$ rather than over $[a, b]$ without loss of generality. Let $I_n(\alpha)=\int_{-\pi}^{\pi} f(x, \bar{y}+\alpha \sin nx, \bar{y}'+\alpha n \cos nx)dx$ for $n=1, 2, 3, \cdots$. Since $\bar{y}(x)$ extremizes $\int_{-\pi}^{\pi} f(x, y(x), y'(x))dx$ with respect to all continuously differentiable functions having the appropriate end-point conditions, it extremizes $I_n(\alpha)$ with respect to functions of the form $\bar{y}+\alpha \sin nx$ for $n=1, 2, 3, \cdots$. Hence

$$I_n'(0) = 0 = \int_{-\pi}^{\pi} [f_y(x, \bar{y}, \bar{y}') \sin nx + f_{y'}(x, \bar{y}, \bar{y}')n \cos nx]dx$$

$$= \int_{-\pi}^{\pi} \left[- \int_{-\pi}^{x} f_y(x, \bar{y}(x), \bar{y}'(x))dx + f_{y'}(x, \bar{y}(x), \bar{y}'(x)) \right] n \cos nx dx.$$

The function

$$- \int_{-\pi}^{x} f_y(x, \bar{y}(x), \bar{y}'(x))dx + f_{y'}(x, \bar{y}(x), \bar{y}'(x))$$

has a corresponding Fourier series

$$a_0/2 + \sum_{1}^{\infty} (a_i \cos ix + b_i \sin ix)$$

and

$$I_n'(0) = \int_{-\pi}^{\pi} \left[a_0/2 + \sum_{i}^{\infty} (a_i \cos ix + b_i \sin ix) \right] n \cos nx dx = n a_n \pi = 0$$

for $n = 1, 2, 3, \cdots$.

Hence $a_n = 0$ for $n = 1, 2, 3, \cdots$. Similarly we define

$$J_n(\alpha) = \int_{-\pi}^{\pi} f(x, \bar{y} + \alpha(\cos nx - (-1)^n), \bar{y}' - \alpha n \sin nx)dx \quad \text{for } n = 1, 2, 3, \cdots$$

and note that

$$\begin{aligned} J_n'(0) &= \int_{-\pi}^{\pi} [f_y(x, \bar{y}, \bar{y}')(\cos nx - (-1)^n) + f_{y'}(x, \bar{y}, \bar{y}')(-n \sin nx)]dx \\ &= \int_{-\pi}^{\pi} \left[- \int_{-\pi}^{x} f_y(x, \bar{y}, \bar{y}')dx + f_{y'}(x, \bar{y}, \bar{y}') \right] (-n \sin nx)dx \\ &= \int_{-\pi}^{\pi} \left[a_0/2 + \sum_{1}^{\infty} b_i \sin ix \right] (-n \sin nx)dx = - n b_n \pi = 0 \end{aligned}$$

and hence that $b_n = 0$ for $n = 1, 2, 3, \cdots$. Since

$$- \int_{-\pi}^{x} f_y(x, \bar{y}(x), \bar{y}'(x))dx + f_{y'}(x, \bar{y}(x), \bar{y}'(x))$$

is a continuous function on the closed interval $[-\pi, \pi]$, it equals its constant Fourier series there.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

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Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P. O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted legibly on separate, signed sheets and should be mailed before September 30, 1967. Any contributor wishing acknowledgement of his solution should include a self-addressed, stamped postcard. (Solvers outside the United States may omit the stamp.)

E 1985. *Proposed by E. O. Buchman, University of California, Los Angeles*

The functions $t(m)$ are defined recursively by $t(0)=1$, $t(m+1)=\sum_{i=0}^{m}\binom{m}{i}t(i)$, $i=1, 2, \cdots$. Show that $t(m)$ is the number of elements in the set $T_m=\{(a_1, \cdots, a_m)\colon a_i \text{ is a positive integer, } a_1=1,\ a_i\leqq 1+\max_{k<i}a_k\}$. See E 1785 [1966, 672].

E 1986. *Proposed by H. S. Shapiro, New York University*

Let B denote the set $\sum_{i=1}^{n}x_i^2\leqq 1$ in Euclidean n-space. Suppose f is a differentiable function on an open set containing B, and that $|f(x)|\leqq 1$ for $x=(x_1, \cdots, x_n)\in B$. Then show that there is a point interior to B at which grad f has a magnitude less than 4.

E 1987. *Proposed by Joseph Arkin, Suffern, N. Y.*

Show that there are infinitely many solutions of

$$(a/b)^2+(b/a)^2-(x/y)^2-(y/x)^2=L^2,$$

where $L(\neq 0)$ is a rational number and each of a, b, x, y is the sum of squares of two integers.

E 1988. *Proposed by R. S. Luthar, Colby College, Waterville, Me.*

Given any integers a, k, $a\geqq k\geqq 0$, show that, for each positive integer n there exists an integer b such that

$$\lfloor\sqrt{a}-\sqrt{(a-k)}\rfloor^n=\sqrt{b}-\sqrt{(b-k^n)}.$$

E 1989. *Proposed by W. A. McWorter, University of British Columbia*

Define a generalized map on the plane to be a partition of the plane into a finite family of pairwise disjoint, nonempty, connected sets. These sets are called countries. We say two countries are adjacent if their union is connected.

Prove or disprove that such a map on the plane can be colored in five colors so that no two adjacent countries have the same color. What happens if we replace "connected" with "arcwise connected"?

E 1990. *Proposed by V. F. Ivanoff, San Carlos, California*

Given a pentagon with the sides 1, 2, $\cdots$, 5 in that order. Denoting by $p(q:r)$ the segment on the side p intercepted by the sides q and r, prove that

$$\text{(a)}\ \prod[1(3:5)] = \prod[5(1:3)], \qquad \text{(b)}\ \prod[1(4:5)] = \prod[5(1:2)],$$

where each factor is obtained by increasing the numbers of its predecessor by 1, the resulting numbers to be taken modulo 5. (Note: Both statements have been proved for cyclic pentagons. See W. B. Carver, *Cyclic Polygons*, this MONTHLY, 68 (1961) 537, with two illustrations.)

E 1991. *Proposed by J. M. Khatri, Baroda, India*

Find integral solutions of $(a^2-x)(b^2-x)=c^2-x$.

E 1992. *Proposed by Eric Mendelsohn, University of Manitoba*

If we define a generalized nth derivative as $f^{(n)}(x)=\lim_{h\to 0}\Delta^n f/h^n$, where $\Delta f=f(x+h)-f(x)$, we have

$$f''(x) = \lim\,[f(x+2h) - 2f(x+h) + f(x)]/h^2,$$

etc. Is there a function which is infinitely differentiable at a limit point of its discontinuities?

E 1993. *Proposed by R. O. Davies and A. J. Wiseman, the University, Leicester, England*

Prove that for $n=1, 2, \cdots$, the following $(n+1)\times(n+1)$ determinant vanishes:

$$\begin{vmatrix}
1 & 2\binom{2n-1}{1} & 2^2\binom{2n-1}{2} & \cdots & 2^n\binom{2n-1}{n} \\
1 & \binom{n+1}{1} & \binom{n+1}{2} & \cdots & \binom{n+1}{n} \\
0 & 1 & \binom{n+1}{1} & \cdots & \binom{n+1}{n-1} \\
0 & 0 & 1 & \cdots & \binom{n+1}{n-2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \binom{n+1}{1}
\end{vmatrix}.$$

E 1994. *Proposed by T. M. K. Davison, University of Waterloo, Ontario*

Let $\{\alpha_n\}$ be a sequence of positive numbers. Put $a_1=1$, $a_{n+1}=a_n+\alpha_n a_n^{-1}$ for $n\geqq 1$. Prove that as $n\to\infty$, $\lim a_n$ exists if and only if $\sum_{n=1}^{\infty}\alpha_n$ is finite.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Variant of Langford's Problem

E 1845 [1966, 81]. *Proposed by R. S. Nickerson, Hanscom Field, Bedford, Mass.*

Consider a sequence of integers in which: (a) every integer, i, $0<i\leqq n$, occurs exactly twice, and (b) the second occurrence of the integer, i, follows its first occurrence by exactly i integers. For example, for $n=5$, the sequence 3, 5, 2, 3, 2, 4, 5, 1, 1, 4 satisfies these conditions.

For arbitrary n, (1) state a rule determining whether or not at least one such sequence exists, (2) describe a procedure for generating such a sequence if it exists, and (3) state a rule for determining how many such sequences exist.

Solution by D. C. B. Marsh, Colorado School of Mines. (1) At least one such sequence exists if and only if $n\equiv 0$ or 1 (mod 4). For, if the positions are labelled from 1 to $2n$, then if "1" occupies positions p_1 and p_1+1, and if "2" occupies positions p_2 and p_2+2, $\cdots$, and if "n" occupies positions p_n and p_n+n, summing we find that $2\sum p_i+\frac{1}{2}n(n+1)=n(2n+1)$, whence $\frac{1}{4}(3n^2+n)=\sum p_i$ must be integral, whence $n\equiv 0$ or 1 (mod 4) is necessary. Sufficiency is shown in (2).

(2) For $n=4t$, possible positions for the integers 1, 2, $\cdots$, n are as shown:

Integer	*Position*	*Integer*	*Position*	*Integer*	*Position*
$4t$	$1,\ 4t+1$	$2t-1$	$4t+2,\ 6t+1$	$2t-3$	$5t+2,\ 7t-1$
$4t-2$	$2,\ 4t$			$2t-5$	$5t+3,\ 7t-2$
$4t-4$	$3,\ 4t-1$	$4t-3$	$4t+3,\ 8t$		
.		$4t-5$	$4t+4,\ 8t-1$	3	$6t-1,\ 6t+2$
2	$2t,\ 2t+2$				
		$2t+1$	$5t+1,\ 7t+2$		
$4t-1$	$2t+1,\ 6t$				
1	$7t,\ 7t+1$				

Note: if $t=2$, omit the third column; if $t=1$, omit both the second and third columns.

E.g., 42324311, 8642724683573115, (12)(10)8642(11)2468(10)(12) 5973(11) 531179, (16)(14)(12)(10)8642(15)2468(10)(12)(14)(16)7(13)(11)953(15)735119 (11)(13).

For $n=4t+1$. $n=5$ has been illustrated. For $t>1$,

Integer	*Position*	*Integer*	*Position*	*Integer*	*Position*
$4t$	$1,\ 4t+1$	$4t-1$	$4t+2,\ 8t+1$	$2t-3$	$5t+4,\ 7t+1$
$4t-2$	$2,\ 4t$	$4t-3$	$4t+3,\ 8t$	$2t-5$	$5t+5,\ 7t$
.					
2	$2t,\ 2t+2$	$2t+1$	$5t+1,\ 7t+2$	3	$6t+1,\ 6t+4$
		$2t-1$	$6t+3,\ 8t+2$		
$4t+1$	$2t+1,\ 6t+2$				
		1	$5t+2,\ 5t+3$		

where, for $t=2$, the third column is to be omitted.

E.g., 864292468751193573, (12)(10)8642(13)2468(10)(12)(11)97113(13)5379(11)5.

(3) A finite algorithm for determining the number of such sequences is the following. Let S be the set $\{1, 2, \cdots, 2n\}$. Choose $p_1 \in S$ (letting p_1 run from 1 through $2n$). Compute p_1+1. If p_1+1 is not in S, no sequence exists. If $p_1+1 \in S$, choose any p_2 in $S_1 = S - \{p_1, p_1+1\}$. Compute p_2+2. If $p_2+2 \notin S_1$, no sequence exists; if $p_2+2 \in S_1$, choose any p_3 in $S_2 = S_1 - \{p_2, p_2+2\}$. Continue until S is exhausted.

NOTE: Langford's problem (which differs from the above in requiring i integers in the series *between* the two occurrences of i) was proposed in Math. Gaz., 42 (1958), p. 228. Solutions were given by Priday and Davies in the next volume, pp. 250–255. A generalization, for which no solutions have been discovered, was proposed by Gillespie and Utz in the Fibonacci Quarterly, **4** (1966), p. 184–186. (References supplied by Douglas Lind.)

Also solved by Michael Goldberg.

Fibonacci Polynomials

E 1846 [1966, 81]. *Proposed by M. N. S. Swamy, University of Saskatchewan*

Show that the Fibonacci polynomials defined by

$$f_n(x) = x \cdot f_{n-1}(x) + f_{n-2}(x), \qquad f_1 = 1, \qquad f_2 = x,$$

satisfy the inequality $f_n^2(x) \leqq (x^2+1)^2 \cdot (x^2+2)^{n-3}$, $n > 2$.

Solution by R. E. Giudici, Student, University of Pittsburgh. For every real x, the relation

$$x(x^2+2)^{-1/2} + (x^2+2)^{-1} < 1, \tag{1}$$

follows immediately from $x^4+2x^2 < (x^2+1)^2$. Since $f_3 = x^2+1$, the proposed inequality is plainly satisfied for $n=3$. We proceed by induction assuming

$$f_k^2(x) \leqq (x^2+1)^2 \cdot (x^2+2)^{k-3}, \qquad k > 2.$$

We then have

$$f_{k+1}^2(x) = [xf_k(x) + f_{k-1}(x)]^2$$
$$\leqq [x(x^2+1)(x^2+2)^{(k-3)/2} + (x^2+1)(x^2+2)^{(k-4)/2}]^2$$
$$\leqq (x^2+1)^2(x^2+2)^{k-2}[x(x^2+2)^{-1/2} + (x^2+2)^{-1}]^2.$$

Taking into account (1), this proves $f_{k+1}^2(x) \leqq (x^2+1)^2(x^2+2)^{k-2}$ and the proof is complete.

Also solved by C. A. Bridger, J. L. Brown, Jr., John Burslem, M. S. Demos, Mrs. A. C. Garstang, E. D. Gingerich, Sylvan Green, Emil Grosswald, Richard Guilfoyle, M. H. Hayamizu, A. E. A. Hunt, Erwin Just, D. A. Klarner, Kenneth Kramer, Jeffrey Kuester, R. N. Leggett, Jr., G. A. Luester, Beatriz Margolis, D. C. B. Marsh, M. F. Neuts, Stanton Philipp, Robert Prielipp, George Purdy (England), Stanley Rabinowitz, L. A. Ringenberg, V. K. Rohatgi, P. A. Scheinok, M. J. Schweitzer, A. Sharma & A. Meir, R. J. Snelling, W. J. Sonsin, Stephen Spindler, Sidney Spital, J. E. Wilkins, Jr., Dale Woods, David Zeitlin, and the proposer.

Guilfoyle shows that $f_n(x)$ is the $(n-1)\times(n-1)$ determinant

$$\begin{vmatrix} x & -1 & 0 & 0 & \cdots & & & & \\ 1 & x & -1 & 0 & \cdots & & & & \\ 0 & 1 & x & -1 & \cdots & & & & \\ \cdots & & & & & & & & \\ & & & & & 1 & x & -1 & \\ & & & & & 0 & 1 & x & \end{vmatrix}$$

from which the result follows by Hadamard's inequality. See: P. Davis, *Interpolation and Approximation*, New York, 1963.

Bridger gives a reference to P. F. Byrd, *Expansion of analytic functions in polynomials associated with Fibonacci numbers*, in The Fibonacci Quarterly, I(1963), pp. 16–28, for those wishing to pursue the subject further.

Lagrange Polynomials

E 1848 [1966, 82]. *Proposed by J. F. Traub, Bell Telephone Laboratories, Murray Hill, N. J.*

Let $P(t)$ be a polynomial of degree n with distinct zeros $\rho_1, \rho_2, \cdots, \rho_n$. We know from interpolation that

$$P'(t) = \sum_{i=1}^{n} P'(\rho_i)L_i(t),$$

where $L_i(t)$ is the Lagrange polynomial

$$L_i(t) = \frac{P(t)}{P'(\rho_i)(t-\rho_i)}.$$

Prove also that $P'(t) = \sum_{i=1}^{n} P'(\rho_i)L_i^2(t)$.

Solution by David Zeitlin, Minneapolis, Minn. Let $P(t) = \prod_{i=1}^{n}(t-\rho_i)$. Then, using partial fractions,

$$\frac{1}{P(t)} = \sum_{i=1}^{n} \frac{1}{P'(\rho_i)} \cdot \frac{1}{(t-\rho_i)}. \tag{1}$$

If we differentiate (1) with respect to t, we obtain

$$P'(t) = \sum_{i=1}^{n} \frac{P^2(t)}{P'(\rho_i)(t-\rho_i)^2} \equiv \sum_{i=1}^{n} P'(\rho_i) L_i^2(t),$$

since $P^2(t) \equiv L_i^2(t)(P'(\rho_i))^2(t-\rho_i)^2$.

Also solved by L. Carlitz, M. M. Chawla (India), H. E. Fettis, P. G. Kirmser, Kenneth Kramer & Joel Spruck, R. N. Leggett, Jr., D. C. B. Marsh, Stanton Philipp, Stanley Rabinowitz, A. Sharma, Sidney Spital, I. Vrečko (Yugoslavia), J. E. Wilkins, Jr., K. L. Yocom, P. H. Young, and the proposer.

Identity in sinh x and cosh x

E 1849 [1966, 82]. *Proposed by George Purdy, The University, Reading, England*

Let $f_0(x) = \cosh x$, $f_1(x) = \sinh x$. Then for $r = 0$, 1 and $m \geqq 1$,

$$f_r(x_1 + x_2 + \cdots + x_m) = \sum f_{s_1}(x_1) \cdots f_{s_m}(x_m)$$

identically, where the summation is taken over all choices of $s_1, \cdots, s_m$, each being 0 or 1, with $s_1 + \cdots + s_m \equiv r \pmod 2$. A proof is desired which does not employ mathematical induction.

Solution by M. F. Neuts, Purdue University. We have

$$f_0(x) + f_1(x) = e^x, \qquad f_0(x) - f_1(x) = e^{-x}.$$

So

$$e^{\pm(x_1+\cdots+x_m)} = \prod_{i=1}^{m} [f_0(x_i) \pm f_1(x_i)].$$

By adding and subtracting these two equations, the required identities are obtained, since all terms for which the subscripts do not satisfy the stated congruence cancel out.

Also solved by John Burslem, L. Carlitz, D. C. B. Marsh, R. C. Fisher, Stanley Rabinowitz, I. Vrečko (Yugoslavia), and the proposer.

An Integral Equal to $\sigma(n)$

E 1850 [1966, 82]. *Proposed by George Purdy, The University, Reading, England*

Prove the following formula in which $\sigma(n)$ is the sum of the divisors of the positive integer n, and $[x+1]$ denotes the integral part of $x+1$:

$$\sigma(n) = \sum_{m=1}^{n} \int_0^m \cos\left(\frac{2\pi n[x+1]}{m}\right) dx.$$

Solution by J. L. Brown, Jr., Pennsylvania State University. We have

$$\int_0^m \cos\left(\frac{2\pi n[x+1]}{m}\right)dx = \sum_{k=1}^{m}\int_{k-1}^{k} \cos\left(\frac{2\pi n}{m}\right)k\,dx$$
$$= \sum_{k=1}^{m} \cos k\left(\frac{2\pi n}{m}\right).$$

Using the well-known result

$$\sum_{k=1}^{m} \cos kx = \cos\left(\frac{m+1}{2}x\right)\sin\frac{mx}{2} \Big/ \sin\frac{x}{2},$$

we find

$$\int_0^m \cos\left(\frac{2\pi n[x+1]}{m}\right)dx = \begin{cases} m & \text{if } m \mid n \\ 0 & \text{if } m \nmid n, \end{cases}$$

from which the result follows.

Also solved by I. K. Abroub, M. S. Demos, R. B. Eggleton (Australia), M. G. Greening (Australia), Stephen Hoffman, Peter Landesman & Sylvie Vandermousen, R. N. Leggett, Jr., C. C. Lindner, D. C. B. Marsh, Haynes Miller, Lieselotte Miller, Jorge More, M. F. Neuts, Stanton Philipp, Stanley Rabinowitz, S. M. Sarma & A. M. Vaidya (India), Perry Scheinok, R. J. Snelling, Al Somayajulu, Stephen Spindler, Sidney Spital, P. C. Yang, and the proposer.

Diagonal in a Rectangle of Unit Squares

E 1851 [1966, 82]. *Proposed by W. E. Patten, South Boston, Va.*

In problem E 1704 [1965, 669] it was found that if an $m\times n$ rectangle is partitioned into mn unit squares in the natural way, then the number of these which meet a diagonal of the rectangle is $m+n-(m, n)$, where (m, n) is the g.c.d. of the positive integers m and n. Given the positive integer k, for how many rectangles does a diagonal meet precisely k unit squares?

Solution by Miltiades S. Demos, Drexel Institute of Technology. Set $(m, n)=d'$, $m=xd'$, $n=yd'$; $k=dd'$ since $d' \mid k$. Then the problem is equivalent to finding the number of solutions of $x+y-1=d$, $(x, y)=1$, for all divisors d of k. Then $(x, d+1)=1$, $(y, d+1)=1$ and the number of solutions is $\sum_{d\mid k}\phi(d+1)$ if we distinguish between rectangles $m\times n$ and $n\times m$. Otherwise, since there is exactly one square $(k\times k)$, the number is $\frac{1}{2}\{1+\sum_{d\mid k}\phi(d+1)\}$.

Also solved by Walter Bluger, John Burslem, L. Carlitz, R. B. Eggleton (Australia), Michael Goldberg M. G. Greening (Australia), D. C. B. Marsh, Norman Miller, and C. B. A. Peck.

Summations of Binomial Coefficients

E 1854 [1966, 83]. *Proposed by Rolf Schneider, The University, Frankfurt am Main, Germany*

Let p be an odd prime and k any positive integer. Show that

$$\sum_{i=0}^{(p^k-1)/2} (-1)^i \binom{2i}{i} \equiv 5^{k(p-1)/2} \pmod{p},$$

$$\sum_{i=0}^{(p^k-1)/2} \binom{2i}{i} \equiv (-3)^{k(p-1)/2} \pmod{p}.$$

Solution by Leonard Carlitz, Duke University. We have

$$5^{k(p-1)/2} \equiv 5^{(p^k-1)/2} \equiv (1+4)^{(p^k-1)/2} \equiv \sum_{j=0}^{\frac{1}{2}(p^k-1)} \binom{\frac{1}{2}(p^k-1)}{j} 4^j \pmod{p},$$

$$(-3)^{k(p-1)/2} \equiv (-3)^{(p^k-1)/2} \equiv \sum_{j=0}^{\frac{1}{2}(p^k-1)} (-1)^j \binom{\frac{1}{2}(p^k-1)}{j} 4^j \pmod{p}.$$

Hence it suffices to show that

$$4^j \binom{\frac{1}{2}(p^k-1)}{j} \equiv (-1)^j \binom{2j}{j} \pmod{p}.$$

We recall that if

$$a = a_0 + a_1 p + \cdots + a_r p^r \qquad (0 \leqq a_t < p)$$

$$b = b_0 + b_1 p + \cdots + b_r p^r \qquad (0 \leqq b_t < p),$$

then

$$(*) \qquad \binom{a}{b} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1} \cdots \binom{a_r}{b_r} \pmod{p}.$$

Thus $j = j_0 + j_1 p + \cdots + j_{k-1} p^{k-1}$ $(0 \leqq j_t < p)$ implies

$$\binom{\frac{1}{2}(p^k-1)}{j} \equiv \binom{\frac{1}{2}(p-1)}{j_0}\binom{\frac{1}{2}(p-1)}{j_1} \cdots \binom{\frac{1}{2}(p-1)}{j_{k-1}}.$$

But

$$\binom{\frac{1}{2}(p-1)}{j} \equiv \binom{-\frac{1}{2}}{j} \equiv (-1)^j 2^{-2j} \binom{2j}{j} \pmod{p},$$

so that

$$\binom{\frac{1}{2}(p^k-1)}{j} \equiv (-1/4)^{j_0+\cdots+j_{k-1}} \binom{2j_0}{j_0}\binom{2j_1}{j_1} \cdots \binom{2j_{k-1}}{j_{k-1}}.$$

Note that unless each $j_t \leqq \frac{1}{2}(p-1)$, the product $\equiv 0$. Therefore, again using (*), we get, as required,

$$\binom{\frac{1}{2}(p^k-1)}{j} \equiv (-1/4)^j \binom{2j}{j} \pmod{p}.$$

Also solved by M. G. Greening (Australia).

Summation of Greatest Integer Functions

E 1856 [1966, 198]. *Proposed by S. W. Williams, Lehigh University*

Given n any positive integer, prove the identity

$$\sum_{k=1}^{n} [kx] = n[x] + \sum_{i=1}^{n-1} \sum_{k=1}^{i} \left[x + \frac{k}{i+1} \right],$$

where, as usual, $[x]$ is the greatest integer in x.

Solution by Henry Ricardo, Yeshiva University. It is known that

$$\sum_{i=0}^{k-1} \left[x + \frac{i}{k} \right] = [kx],$$

(easily proved; see, e.g., Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, II, sec. viii, problem 9). Therefore

$$\sum_{k=1}^{n} [kx] = \sum_{k=1}^{n} \sum_{i=0}^{k-1} \left[x + \frac{i}{k} \right]$$

$$= n[x] + \sum_{k=2}^{n} \sum_{i=1}^{k-1} \left[x + \frac{i}{k} \right] = n[x] + \sum_{k=1}^{n-1} \sum_{i=1}^{k} \left[x + \frac{i}{k+1} \right].$$

Also solved by J. C. Abad, A. N. Aheart, R. B. Eggleton (Australia), A. L. Epstein, R. W. Feldmann, M. G. Greening (Australia), Stephen Hoffman, J. E. Homer, Jr., Edward Hook, F. T. Howard, M. S. Klamkin, J. D. E. Konhauser, E. S. Langford, Arthur Lieberman, D. C. B. Marsh, Lieselotte Miller, Norman Miller, Stanton Philipp, Stanley Rabinowitz, Jürg Rätz (Switzerland), Simeon Reich (Israel), Judy Richman, M. S. R. K. Sastry, Perry Scheinok, Robert Schreiner, Al Somayajulu, Sidney Spital, David Zeitlin, and the proposer.

Commutativity of Addition

E 1858 [1966, 198]. *Proposed by S. E. Dickson, University of Nebraska*

Assume the usual properties of the real numbers except for commutativity of addition. Given that the equation $x+y=y+x$ holds for real numbers x and y less than 10^{-26} apart, show that this latter restriction can be removed.

Solution by Margaret W. Taft, student, Clark University. Suppose $|a-b| < k$, a and b real; and let $\epsilon > 0$ be arbitrarily small. We have $\epsilon|a-b|/(k+1) < \epsilon$. By the distributive law, $|\epsilon a/(k+1) - \epsilon b/(k+1)| < \epsilon$, whence, by hypothesis,

$$\frac{\epsilon a}{k+1} + \frac{\epsilon b}{k+1} = \frac{\epsilon b}{k+1} + \frac{\epsilon a}{k+1}, \qquad \frac{\epsilon}{k+1}(a+b) = \frac{\epsilon}{k+1}(b+a).$$

Thus $a+b=b+a$ for all real a and b.

Also solved by Joe Beckely, J. L. Brown, Jr., J. P. Celenza, C. W. Dodge, G. F. Feissner, Arne Garness & G. A. Heuer, E. D. Gingerich, Anton Glaser, H. D. Gunghenn, Erwin Just, R. S. Kaluzniacki D. C. B. Marsh, Norman Miller, D. E. Nixon & J. P. Thomas, Robert Prener, Azriel Rosenfeld, G. F. Schumm, E. J. Wegman, John Wessner, P. C. Yang, and the proposer.

Rosenfeld notes that the conclusion follows even if the second sentence of the statement is omitted; see Problem E 1812 [1967, 199]. Dodge found an equivalent result in H. Eves, *An Introduction to the Foundations and Fundamental Concepts of Mathematics*, Rev. ed., p. 313.

Characteristic Roots of a Matrix

E 1859 [1966, 198]. *Proposed by Judith H. Hallman, University of North Carolina and Western Electric Company*

Let $k_1, k_2, \cdots, k_n$ be any real numbers different from zero and let $A=(a_{ij})$ be the matrix $a_{ij}=k_i/k_j$. Find the n characteristic roots of A.

I. *Solution by D. E. Crabtree, Amherst College.* Let D be the diagonal matrix $D=\text{diag}\ (k_1, k_2, \cdots, k_n)$. Then A is similar to the matrix $D^{-1}AD$, each entry of which is 1. Thus the characteristic roots of A are 0 (with multiplicity $n-1$) and n. Of course, the restriction that the k_i be real is unnecessary.

II. *Solution by P. G. Kirmser, Kansas State University.* The matrix A is the matrix M of Problem E 1685 [1965, 321], where the characteristic roots are found to be 0, of multiplicity $n-1$, and n, of multiplicity 1.

Also solved by H. D. Abramson, W. R. Allaway, J. F. Burke, J. P. Celenza, P. L. Claypool, R. W. Feldmann, R. O. Fisher, A. V. Geramita, M. G. Greening (Australia), R. S. Kaluzniacki, E. S. Langford, Arthur Lieberman, D. C. B. Marsh, Dave Nixon, Stanton Philipp, Stanley Rabinowitz, July Richman, Steven Russ, R. Sivaramakrishnan (India), J. B. Skinner, Sidney Spital, Rory Thompson, John Wessner, H. H. Wong, and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of advanced problems in this issue should be submitted on separate, signed sheets and should be mailed before November 30, 1967. Any contributor wishing acknowledgement of his solution should include a self-addressed stamped postcard. (Solvers outside the United States may omit the stamp.)

5490. *Proposed by J. T. Darwin, Jr., Auburn University, Alabama*

Let L_1 and L_2 denote those sets of measurable real-valued functions which are Lebesgue integrable and square integrable, respectively, on the interval $[0, 1]$. Prove that $f\in L_2$ if and only if (1) $f\in L_1$ and (2) there exists a nondecreasing function, h, such that for every

$$[r, s] \subset [0, 1], \qquad \left| \int_r^s f \right|^2 \leqq [h(s) - h(r)][s - r].$$

5491. *Proposed by Roy O. Davies and Roy Buckley, The University, Leicester, England*

Determine the coefficients $C_{n\nu}$ in the expansion

$$\int_\alpha^{\pi/2} \frac{\sin\,(2n+1)\psi}{\sqrt{(\cos^2\alpha - \cos^2\psi)}}\,d\psi = \sum_{\nu=0}^{n} C_{n\nu}\cos^{2\nu}\alpha \qquad (0 < \alpha < \tfrac{1}{2}\pi).$$

5492. *Proposed by S. B. Akers and R. D. Berlin, General Electric Company, Syracuse, N. Y.*

A $4k-1$ step symmetric random walk is performed by A along the x-axis starting at the origin. He is accompanied by B through the first $2k-1$ steps; thereafter each of B's steps is taken opposite in direction to the corresponding step taken by A. What is the probability that at the termination of the walk A and B will be on opposite sides of the origin?

5493. *Proposed by H. S. Shapiro, New York University*

Let $f(x, y)$ denote a cubic polynomial in x, y (i.e. a combination of monomial terms x^iy^j with $i+j\leqq 3$). Suppose the surface $z=f(x, y)$ has three saddle points and a local maximum at points S_1, S_2, S_3, and M respectively. Then M lies inside the triangle $S_1S_2S_3$.

5494. *Proposed by József Dénes and Raisz Klára, Central Research Institute for Physics, Budapest, Hungary*

If two permutations of degree $n(n\geqq 3)$ having at least two fixed elements are similar then they belong to the same conjugate class of the alternating group of degree n.

5495. *Proposed by Erwin Just, Bronx Community College, New York*

Let f and g be real-valued continuous functions defined on $[0, 1]$. If no linear combination of f and g has more than one zero in $(0, 1)$, and $f(0)g(1)\neq g(0)f(1)$, prove that some linear combination of f and g does not have any zeros in (0. 1).

5496. *Proposed by R. H. Shudde and E. S. Langford, U. S. Naval Postgraduate School*

Let B be any $m\times m$ matrix over the integers, with B^* the adjoint matrix of B. Prove that every $s\times s$ subdeterminant of B^* is divisible by $(\det B)^{s-1}$.

5497. *Proposed by Peter Yff, American University, Beirut, Lebanon*

Prove or disprove the following conjecture: If $A+B+C=\pi$, and $\cot\omega=\cot A+\cot B+\cot C$, then $8\omega^3\leqq ABC$, with equality only when $A=B=C$. ω is the Brocard angle.

5498. *Proposed by J. A. Hildebrandt, Louisiana State University*

Let S be a semigroup with at least three elements and $x\in S$ such that $S\backslash x$ is a group. Prove that S is regular if and only if x is a zero or an identity.

5499. *Proposed by D. E. Daykin, University of Malaya, Kuala Lumpur, and J. S. Biggerstaff, Portland, Oregon*

(1) Are there rational numbers a, b such that no rational numbers c, k exist for which there is a triangle having sides a, b, c and area k?

(2) Prove or disprove the conjecture: for every positive rational number k

there exist positive rational a, b, c such that a triangle with sides a, b, c has area k (e.g. triangle with sides 3/2, 5/3, 17/6 has area 1).

SOLUTIONS OF ADVANCED PROBLEMS

Converse of Weierstrass' M-test?

5390 [1966, 547]. *Proposed by D. W. Hight, Kansas State College, Pittsburg*

Is there a series $\sum_{i=0}^{\infty} u_i(x)$ of functions continuous on a closed interval $[a, b]$ that converges absolutely and uniformly for which the Weierstrass M-test fails?

I. *Solution by B. L. D. Thorp, York University, Toronto, Canada.* Let C denote the Banach space of continuous real or complex-valued functions on $[a, b]$ with the norm $\|u\| = \sup \{ |u(x)| : a \leqq x \leqq b \}$ $(u \in C)$. Since C is an infinite dimensional space there exists, by a theorem of A. Dvoretzky and C. A. Rogers (*Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. (U.S.A.), 36(1950), 192–197) a series $\sum_{n=1}^{\infty} U_n(x)$ in C satisfying

(a) the series is convergent with respect to the norm in C (i.e. the series is uniformly convergent),

(b) any rearrangement of the series is uniformly convergent (i.e. the series is uniformly and absolutely convergent),

(c) $\sum_{n=1}^{\infty} \|U_n\|$ diverges.

For a simple example on $[0, 1]$ let the function $u_k(x)$ be defined by

$u_k(x) = 0$ if $0 \leqq x \leqq 1/(2k+1)$ or $1/(2k-1) \leqq x \leqq 1$;

$u_k(x) = 1/k$ if $x = 1/2k$;

$u_k(x)$ is defined linearly in the intervals $[1/(2k+1), 1/2k]$, $[1/2k, 1/(2k-1)]$.

The series $\sum_{k=1}^{\infty} u_k(x)$ satisfies the required conditions.

II. *Solution by P. R. Meyer, Hunter College.* Yes. A series $\sum u_k(x)$ such that $u_k'(x)$ approaches zero uniformly may be achieved as follows. We define first $v_1(x)$ by:

$$v_1(0) = v_1(1) = 0, \qquad v_1\left(\frac{1}{2}\right) = \frac{1}{2 \log 2},$$

$v_1(x)$ is linear in $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. $v_2(x)$ is a roof function on $[0, \frac{1}{2}]$ with maximum height of $1/4 \log 4$ at $x = \frac{1}{4}$, and zero for the rest of $[0, 1]$. $v_3(x)$ is a roof function on $[\frac{1}{2}, 1]$ with maximum height of $1/4 \log 4$ at $x = \frac{3}{4}$. Similarly v_4, v_5, v_6, v_7 are defined as roof functions on $[0, \frac{1}{4}], \cdots, [\frac{3}{4}, 1]$, respectively, with maximum heights of $1/8 \log 8$ at the midpoints of these intervals. We see that

$$\sum \|v_k\| = 1/2 \log 2 + (\tfrac{1}{4} + \tfrac{1}{4})/\log 4 + \cdots = \infty,$$

$$\sum v_k(x) \leqq 1/2 \log 2 + 1/4 \log 4 + 1/8 \log 8 + \cdots < \infty.$$

The $u_k(x)$ are now obtained by applying a simple smoothing operation to v_k.

Also solved by Andreas Blass, R. P. Boas, Béla Bollobás (Hungary), Robert Breusch, T. J. Burke, Yu Chang, P. R. Chernoff, Louis Comtet (France), Stephen Curtis, R. J. Driscoll, John Fridy, Robert Goldstein (England:, G. A. Heuer, D. A. Hejhal, D. W. Hight, M. S. Klamkin, E. S. Langford, D. C. Lewis, Jr., Arthur Lieberman, W. S. Loud, M. D. Mavinkurve (India), Gideon Peyser, Stanton Philipp, Jürg Rätz (Switzerland), Ronald Rietz, J. T. Rosenbaum, W. F. Trench, Bertram Walsh, and the proposer.

NOTE. The second solution above was motivated by the question: Do uniform convergence and some additional condition on u_k imply the applicability of the Weierstrass M-test? We see, of course, from the theorem quoted in the first solution that an example with $\|u_k'(x)\| \to 0$ exists; any conditions on u_k would have to imply finite dimensionality of the resulting space, as noted by Walsh in his solution.

Klamkin refers to the literature for other examples—e.g. E. W. Hobson, *Theory of Functions of a Real Variable* v. II, 1957, p. 119. He also quotes a theorem of Baire to the effect that the Weierstrass M-test will apply to any uniformly convergent series after an appropriate insertion of brackets; this may be easily verified in the examples above.

Convex Hull of a Twisted Cubic

5391 [1966, 547]. *Proposed by J. H. Halton and M. Skibinsky, Brookhaven National Laboratory*

Give an explicit characterization of the convex hull of the arc from $t=0$ to $t=1$ of the twisted cubic $x=t$, $y=t^2$, $z=t^3$.

Solution by the proposers. Let $a_i=(i, i, i)$, $i=0, 1$, and define the truncated cones

$$C_i = \bigcup_{q\in S} \{((1-\theta)a_i + \theta q) : 0 \leqq \theta \leqq 1\}.$$

The convex hull of S is the bounded set which has the boundary $C_0 \cup C_1$. Let $*$ denote the convex hull; then $C_i \subset S^*$ and $S \subset C_i \subset C_i^*$, $i=0, 1$, so that $C_i^* = S^*$, $i=0, 1$.

On the other hand, if we denote by $\hat{C}_i$ the untruncated cone corresponding to C_i, it is easily shown that

$$(\hat{C}_0)^* \cap (\hat{C}_1)^* = \left\{(u, v, w) : 0 \leqq u \leqq 1,\ u^2 \leqq v \leqq u,\ \frac{v^2}{u} \leqq w \leqq v - \frac{(v-u)^2}{1-u}\right\},$$

where left and right hand sides, in the inequality on w inside the brackets, are to be interpreted for $u=0$ and 1, respectively, as the corresponding limits, namely, 0 and 1. But this set is also C_0^* (alternatively, C_1^*) and its boundary is $C_0 \cup C_1$.

NOTE. For generalizations and further interpretations for problems of this nature, John R. Isbell refers us to Memoir no. 12 of the American Mathematical Society, S. Karlin and L. S. Shapley, *Geometry of Moment Spaces.*

A Function of the Base in the Representation of Real Numbers

5392 [1966, 547]. *Proposed by Hwa Hahn, Pennsylvania State University*

Let $f(x)$ be the number obtained from a real x by reading the representation of x in the base p as the representation in the base $q (1 < p \leqq q)$. Evaluate $\int_0^1 f(x)dx$.

Solution by C. R. Mac Cluer, University of Michigan. For each x in $[0, 1]$ let $a_m(x)$ be defined by $x = \sum_{m=1}^{\infty} a_m(x)/p^m$. Then except for some rational values of x, $a_m(x)$ is a well defined periodic, integral valued, measurable, staircase type function, and

$$\int_0^1 a_m(x)dx = \frac{0 + 1 + 2 + \cdots + (p-1)}{p} = \frac{p-1}{2}.$$

But $f(x) = \sum_{m=1}^{\infty} a_m(x)/q^m$. Thus

$$\int_0^1 f(x)dx = \sum_{m=1}^{\infty} \frac{1}{q^m} \int_0^1 a_m(x)dx = \frac{p-1}{2q} \sum_{m=1}^{\infty} \frac{1}{q^m} = \frac{p-1}{2(q-1)}.$$

Also solved by D. W. Bailey, P. T. Bateman, Andreas Blass, J. R. Blum, Robert Breusch P. R. Chernoff, Addison Fischer, G. J. Foschini, R. G. Hart, D. A. Hejhal, G. A. Heuer, E. C. Hook, James Kiefer, M. S. Klamkin, E. S. Langford, D. A. Prener, Ronald Rietz, Alan Saleski, Sidney Spital, W. F. Trench, and the proposer.

The Matrix $a_{ij} = (i, j)$

5393 [1966, 547]. *Proposed by A. A. Mullin, University of California, Livermore*

Does there exist an integer for which the $n \times n$ matrix whose elements are $a_{ij} = (i, j)$ is singular? (i, j) is the greatest common divisor of the natural numbers i and j.

Solution by W. J. Blundon, Memorial University of Newfoundland. Let X_n be the given matrix. Define two $n \times n$ matrices B and C as follows.

$$B = [b_{ij}] \quad \text{where} \quad b_{ij} = \begin{cases} \phi(i) & j = i, \\ 0 & \text{otherwise;} \end{cases}$$

$$C = [c_{ij}] \quad \text{where} \quad c_{ij} = \begin{cases} 1 & j \mid i, \\ 0 & \text{otherwise;} \end{cases}$$

and let C' denote the transpose of C. It follows that $X_n = CBC'$ since the ijth term of this matrix is equal to

$$\sum_{k=1}^{n} c_{ik} b_{kk} c'_{kj} = \sum_{k \mid (i,j)} \phi(k) = (i, j)$$

from the well-known summation of the ϕ-function. Now let $\det C = \det C' = 1$. Thus $D_n = \det X_n = \det B = \phi(1)\phi(2) \cdots \phi(n)$. Since $\det X_n$ is always positive, X_n is never singular.

[Note the recurrence relation: $D_n/D_{n-1} = \phi(n)$.]

Also solved by Eugene Allgower, Anders Bager (Denmark), P. T. Bateman, Andreas Blass, Robert Breusch, E. P. Emerson, N. J. Fine, J. R. Isbell, James Kiefer, E. S. Langford, Andrzej Makowski (Poland), Ka Menehune, C. B. A. Peck, Stanton Philipp, Bob Prielipp, Jürg Rätz (Switzerland), Henry Riccardo, and D. P. Roselle.

NOTES. Bateman, Rätz and Riccardo found the problem in Polya und Szegö, *Aufgaben und Lehrsätze*, V.2, part 8, number 57.

Peck refers to Muir, *Theory of Determinants in the Historical Order of Development*, V.3, pp. 116–121, for generalizations. The first known appearance of this result seems due to H. J. Smith—as noted by Bager and Roselle; see, e.g., Dickson, *History of the Theory of Numbers*, V.1, pp. 122–123.

Makowski refers us to B. Gyires *On a generalization of Smith's determinant theorem*, Publ. Math. Debrecen, 5 (1957–8) 162–171.

Finally, the merits of this problem were recognized previously in this Department, when a generalization appeared as no. 4101 [1945, 165] and was subsequently inserted in the *Otto Dunkel Memorial Problem Book*, this MONTHLY, V. 64 #7, Part 2, p. 35.

An Inequality in the Complex Plane

5394 [1966, 547]. *Proposed by P. J. O'Hara, Jr., University of Miami, Coral Gables, Florida*

Show that the functions, $(1-z)/(1-z^n)$, $n=1, 2, \cdots$, are uniformly bounded in modulus on the disk $\{z: |z-\frac{1}{2}| \leqq \frac{1}{2}\}$ in the complex plane; and that in fact for all n the bound is attained at $z=0$.

I. *Solution by Mary R. Embry, University of North Carolina at Charlotte.* Let $D=\{z: |z-\frac{1}{2}|<\frac{1}{2}\}$ and $C=\{z: |z-\frac{1}{2}|=\frac{1}{2}\}$. Define

$$f_n(z) = \begin{cases} \dfrac{1-z}{1-z^n} & \text{for } z \neq 1, \\ \dfrac{1}{n} & \text{for } z = 1, \end{cases} \qquad n = 1, 2, \cdots.$$

The f_n are analytic on D and continuous on $C \cup D$. For $n=1$, $|f_n(z)| \equiv 1$; for $n>1$, f_n is nonconstant and, with $z \in D$, $|f_n(z)| < \sup(|f_n(z)| : z \in C)$.

If we assume that $\sup(|f_n(z)| : z \in C) > 1$ for some $n>1$, then there exists a z on C, $z \neq 0$, $z \neq 1$ and $|f_n(z)| = 1$. Write $z = re^{i\theta}$ where $r = \cos\theta$. It follows in turn that

$$r^2 - 2r\cos\theta = r^{2n} - 2r^n \cos n\theta,$$
$$1 = 2r^{n-2}\cos n\theta - r^{2n-2}.$$

Since $0<r<1$ and $\cos n\theta \leqq 1$, we have $1 \leqq 2r^{n-2} - r^{2n-2} < 2r^{n-1} - r^{2n-2}$ or $(1-r^{n-1})^2 < 0$. This contradiction implies that $\sup |f_n(z)| \leqq 1$ in $C \cup D$.

II. *Solution by Stanton Philipp, University of California at Riverside.* In Problem 4975 [1962, 927] it is shown that $|z^{n+1}-1| > |z^n||z-1|$ if $\mathrm{Re}(z) \geqq 1$. Replacing z by $1/z$, it follows that $|(1-z)/(1-z^n)| \leqq 1$ if $|z-\frac{1}{2}| \leqq \frac{1}{2}$ and $\mathrm{Re}(1/z) \geqq 1$.

Also solved by P. G. Bugl, Robert Breusch, D. Ž. Djoković (Yugoslavia), James Kiefer, M. S. Klamkin, Al Somayajulu, and the proposer.

Games with a Winning Strategy

5395 [1966, 547], *Proposed by Roy Westwick and W. A. McWorter, University of British Columbia*

Suppose G is a two-person game satisfying:

(1) The players play alternately.

(2) There is no draw.

(3) There are at least one and at most $N(<\infty)$ plays a player can make. Prove that either there is a strategy insuring a win for the first player, or there is a strategy insuring a win for the second player.

I. *Solution by John Lawrence, Eastern Ontario Institute of Technology.* This theorem obviously does not apply to games of chance. If $N=1$, either the first player, A, has a play such that the second player, B, has no counterplay, or he does not. In the former case, A will win by the use of proper strategy, while in the second case, B will win. Thus the theorem is true for $N=1$.

Suppose the theorem is true for $N \leqq M-1$ and suppose that H is a game with $N=M$. After $M-1$ plays there will be some arrangement such that in the next play, A has a move such that B has no counter-play or there will be no such arrangement. Define game H' as follows:

(1) The plays are the same as for H.

(2) A is the winner in H' if a winning arrangement has been achieved in the first $M-1$ plays of H.

(3) B wins if A does not win.

(4) There are at most $M-1$ plays.

From the definition, if A can win H' then he can win H. The same applies to B. Thus there is a strategy insuring a win of H for a given player if there is a strategy insuring a win of H' for that player. The theorem is thus proved by induction.

II. *Solution by Donald Quiring, Las Vegas, Nevada.* Suppose game G satisfies the hypothesis but there does not exist a winning strategy for either player. We first establish that at some stage in the game a situation is reached in which a winning strategy does exist for one of the players. Call the players A and B without regard to who moves first. Suppose the game must end in victory for A after a move by A. Immediately before A made that move he had a simple winning strategy—namely, to make the critical move. If the game must end in victory for A after a move by B, then A has a winning strategy of doing nothing (if this is not properly regarded as a strategy, the solution merely requires rewording).

Now consider the first point in the game at which A was in possession of a winning strategy. If A has just moved, then he clearly had a winning strategy immediately before he made that move. We conclude that B has just moved.

Immediately before B's last move, there was no winning strategy for A. Therefore B must have been able to make a move which would not give A a winning strategy. In other words, a win by A requires a mistake by B, and the

game will never end if neither player makes a mistake. This contradicts the hypothesis and we conclude that the required winning strategy must exist at the outset.

Also solved by R. G. Hart, James Kiefer, D. L. Silverman, and the proposer.

Editorial Note. As was pointed out by Silverman, the key word "play" in hypothesis (3) must be interpreted as "round" and there are to be a finite number of these rounds. The actual number of possibilities in each round may indeed be infinite. Moreover, the solution depends on players seeing opponent's choices. If these conditions are not fulfilled, the result of the problem is not true as seen in counterexamples provided by D. E. Daykin (Malaysia), Scott Guthery, and John R. Isbell.

An Identity for Bessel Functions

5397 [1966, 548]. *Proposed by R. E. Shafer, University of California at Livermore*

Show, for $|\operatorname{Re}(\nu)| < \frac{1}{2}$ and $|\arg z| < \pi$, that

$$K_\nu(z) = e^{-z} z^\nu \cos \pi\nu \int_0^\infty \frac{e^{-t}}{t+z}\, t^{-\nu} I_{-\nu}(t)\, dt,$$

where I_ν, K_ν are modified Bessel functions.

Solution by E. L. Koh, State University of New York at Stony Brook.

From the *Tables of Integral Transforms* by Erdelyi *et al.*, Vol. 2 (McGraw-Hill, 1954), p. 229, eq. 36, we have the Stieltjes transform:

$$\int_0^\infty \frac{t^\nu e^{-t} I_\nu(t)}{t+z}\, dt = \frac{z^\nu e^z K_\nu(z)}{\cos \pi\nu}; \qquad |\operatorname{Re}(\nu)| < \tfrac{1}{2},\ |\arg z| < \pi.$$

This yields

$$K_\nu(z) = e^{-z} z^{-\nu} \cos \pi\nu \int_0^\infty \frac{e^{-t}}{t+z}\, t^\nu I_\nu(t)\, dt.$$

Replacing the order ν by $-\nu$ and noting that $K_\nu(z) = K_{-\nu}(z)$ and $\cos \pi\nu = \cos(-\pi\nu)$ we have the desired result. (Note the correction $I_{-\nu}(t)$ instead of $I_\nu(t)$ as originally printed.)

Also solved by S. P. Singh, and the proposer.

Limit Points of Partial Sums of Rearranged Series

5398 [1966, 548]. *Proposed by G. A. Heuer, Concordia College, Moorhead, Minnesota*

Let $\sum a_n$ be a conditionally convergent series, S the set of its partial sums, and S' the derived set (set of accumulation points) of S. As is well known, rearranging the series $\sum a_n$ may alter S'. Consider all possible arrangements of the fixed series $\sum a_n$. What are the sets S' which occur?

Solution by Hugh Noland, University of California at Davis. Let $\sum a_{k(n)}$ be a rearrangement of the conditionally convergent series $\sum a_n$, and let α and β

be real numbers (extended) which are accumulation points of the set of partial sums of $\sum a_{k(n)}$. Since $\lim_{n\to\infty} a_{k(n)} = 0$, it is clear that every real number between α and β is also an accumulation point of the above-mentioned set.

Thus, using Riemann's theorem, the sets S' (in the notation of the problem) consist of all closed intervals (including singleton sets) of the extended real line.

Also solved by Béla Bollobás (Hungary), D. E. Daykin (Malaysia), D. A. Hejhal, L. R. King, M. S. Klamkin, M. D. Mavinkurve (India), Sidney Spital, Yu Chang, and the proposer.

Klamkin offers the following addendum for complex series. The "range of summation" of a series Σa_n of complex terms, i.e., the set of values which may be obtained as sums of convergent rearrangements of Σa_n, is either a definite point, or a definite straight line, or the entire plane. Other cases cannot occur. A proof is given by P. Levy (Nouv. Annales (4), Vol. 5 (1905), p. 506), but an unexceptional statement of the proof is not found earlier than in E. Steinitz (*Bedingt konvergente Reihen und konvexe Systeme*, J. Reine Angew. Math., Vol. 143, 1913; Vol. 144, 1914; Vol. 146, 1915. The case that is still open is: what is the set of accumulation points when the deranged sums in the complex case lie on a definite straight line?

Spectral Radius of (0, 1) Matrices

5399 [1966, 548]. *Proposed by L. F. Shampine, Sandia Corporation, Albuquerque, N. M.*

Let $p(n)$ be the largest spectral radius attained by any (0, 1) matrix with n nonzero entries. Prove $p \sim \sqrt{n}$.

Solution by R. A. Brualdi, University of Wisconsin. If $A = [a_{ij}]$ is any (0, 1) matrix with n 1's and $\lambda_1, \cdots, \lambda_m$ are the eigenvalues of A, then by Schur's inequality,

$$\sum_{i=1}^{m} |\lambda_i|^2 \leqq \sum_{i,j=1}^{m} a_{ij}^2,$$

we see that $\max_{1\leqq i\leqq m} |\lambda_i| \leqq \sqrt{n}$. Thus $p(n) \leqq \sqrt{n}$. Let k be the unique positive integer such that $k^2 \leqq n < (k+1)^2$. Then the matrix

$$\begin{bmatrix} J_k & 0 \\ 0 & I_{n-k^2} \end{bmatrix},$$

where J_k is the $k \times k$ matrix of all 1's and I_{n-k^2} is the identity matrix of order $n-k^2$, is a (0, 1) matrix having n 1's and with maximal characteristic root equal to k. Hence $p(n) \geqq k$. But then

$$\frac{k}{k+1} \leqq \frac{p(n)}{\sqrt{n}} \leqq 1.$$

Since as $n \to \infty$, $k \to \infty$ and thus $k/(k+1) \to 1$, we have the desired result

$$\lim_{n\to\infty} \frac{p(n)}{\sqrt{n}} = 1.$$

Also solved by R. Dormaty (Austria), P. J. Nikolai, J. T. Rosenbaum, and the proposer.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Correspondence about Reviews will be welcome.

Notes on Differential Geometry. By Noel J. Hicks. Van Nostrand, Princeton, N. J., 1965. vi+183 pp. $2.95.

This fine book is the third in a series of mathematical studies published in paperback form by Van Nostrand. The topics discussed include the main results from classical differential geometry, given a fresh look, and some results, such as the Weil homomorphism, which have not appeared before in an introductory text. Both the language and the techniques used are modern.

The contents by chapter are: 1. Manifolds. 2. Hypersurfaces of R^n. 3. Surfaces in R^3. 4. Tensors and Forms. 5. Connexions. 6. Riemannian Manifolds and Submanifolds. 7. Operators on Forms and Integration. 8. Gauss-Bonnet Theory and Rigidity. 9. Existence Theory. 10. Topics in Riemannian Geometry.

The treatment is quite brisk. Chapters 4 and 7, in particular, are almost too brief. The one hundred problems in the text are crucial to the development. The book has an extensive bibliography and is well documented, but is marred by many typographical errors and misspelled names.

This book is highly recommended to anyone who wants an introduction to modern differential geometry or a modern treatment of results from classical geometry. Prerequisites for the book are linear algebra, advanced calculus or elements of real analysis including some calculus of vector functions and the basic existence theorems in ordinary differential equations. It can be used for courses at either the undergraduate or graduate level and should be useful to those graduate students who are preparing for qualifying exams in the area. At $2.95, it is a very good buy.

G. F. FEEMAN, Williams College

Mathematical Discourses: The Heart of Mathematical Science. By Carroll V. Newsom. Prentice-Hall, Englewood Cliffs, N. J., 1963. 121 pp. $5.50.

Euclidean geometry consists of a set of primary propositions, which are unproved postulates, a set of secondary propositions, which are logical consequences of the primary propositions, and a set of terms, some of which are necessarily undefined. These propositions and terms constitute what is known as a mathematical discourse, and this is what this book is about. There are many kinds of mathematical discourses mentioned and some are discussed in detail. There are discourses for various types of non-Euclidean geometry, for projective geometry, for various algebras, for Newtonian mechanics, and so on.

In general the historical approach is used, and after an interesting chapter on how man began to be mathematical the author introduces Euclid's list of ten primary propositions. These are criticized and then replaced by Robinson's (1940) list. Since the secondary propositions are logically included in the primary propositions, although this is often not immediately obvious, it would appear that in the last analysis mathematics is an argument in a vacuum. The author notes that Aristotle recognized this, and also quotes a number of modern authors, some of whom assert that this is true not only of mathematical science but of other sciences as well. "Formal sciences are characterized by the fact that the theorems are not 'about' some phase of the existing world but are 'about' what is *postulated* by thought" (Nagel 1939). Mathematics is "a game played on paper with meaningless symbols" (D. Hilbert 1862–1943). "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true" (B. Russell 1901), but mathematics is not the only subject of this sort.

Mathematical science becomes as we say useful as one intuitively believes that certain observed phenomena satisfy very closely the conditions imposed in the discourse. This is called an interpretation of the discourse, and the discourse may have been expounded initially without any interpretation having been thought of. For example there is a geometry due to Riemann which was without interpretation until it was used in general relativity theory, which in turn is another mathematical discourse, and which again becomes useful if one believes certain natural phenomena satisfy its conditions. On the other hand quite frequently there may be an interpretation desired and then the discourse invented. The theory of quaternions is mentioned as an example of this.

With the author's rich use of historical material he has made his account absorbingly interesting and his carefully phrased sentences have made it easily understandable by the nonprofessional mathematicians for whom the book was written, not of course by all nonmathematicians, but surely by those with a high school education, a philosophic temper and a willingness to make a minor mental effort. In the preface it is stated that "an understanding of the subject of this book is essential to mathematical literacy." This may be putting it a bit strong. There have been many literate mathematicians who have been interested not in the foundations of mathematics but in the superstructure, but the book is certainly necessary to an understanding of the philosophy of mathematics and of the significance of much modern work. It would seem that many college students will find it both interesting and stimulating.

B. H. CAMP, Wesleyan University

Non-Standard Analysis. By A. Robinson. Edited by L. E. J. Brouwer, A. Heyting, A. Robinson and P. Suppes. North Holland, Amsterdam, 1966. xi+293 pp. $10.00. (Teleg. Rev. March 1967.)

Aristotle held that in mathematics, as in nature, nothing actually is infinitely large or infinitely small. A potentiality for indefinite increase or decrease in magnitude was all that was permitted in peripatetic philosophy or in classical mathematics based on the axioms of Eudoxus and Archimedes. The differentials of Leibnizian calculus seemed to point to the possibility of infinitely small magnitudes, but this notion lost support with the rise of the limit concept of D'Alembert, Cauchy, and Weierstrass. The infinitely large, by contrast, achieved canonical status in mathematics through Cantor's *Mengenlehre.* The history of mathematics shows, however, that no advance continues indefinitely in a given direction, nor is any achievement established beyond question. Consequently it should occasion no surprise that the book under review presents a closely-reasoned elaboration of the non-conforming thesis that "the infinitely small and infinitely large numbers of a non-standard model of Analysis are neither more nor less real than, for example, the standard irrational numbers."

The volume is the outcome of a view developed by the author in seminars and papers since 1960, inspired in part by Skolem's non-standard models of arithmetic. It purports to demonstrate "that the concepts and methods of contemporary Mathematical Logic are capable of providing a suitable framework for the development of the Differential and Integral Calculus by means of infinitely small and infinitely large numbers." Following substantial sections on such themes as general topology, functions of real and complex variables, linear spaces, and groups, the book closes with a chapter "Concerning the History of the Calculus" which effectively summarizes developments from Leibniz to 1961. More than a third of this well-written chapter is devoted to an analysis of the concepts and language of Cauchy, the intention being to show that these are not so far removed from the non-standard approach as usually is supposed. Admitting that Cauchy's differentials were, at least in the earlier stage, variables rather than numbers, Robinson argues that subsequent definitions by Cauchy can readily be interpreted in the sense of non-standard analysis. Such interpretation appears to be somewhat tendentious, but one

can readily subscribe to the author's conclusion that "Cauchy stands in the history of the calculus not as a man who broke with tradition and swept away old and rotten foundations to make room for new and sound ones but rather as a link between the past and the future." Whether the future will belong to the customary Weierstrassian view or to the unwonted approach of non-standard analysis—or to both—only time can tell. In any case, the present book will warrant attention from mathematicians for its provocative thesis and from historians for the *prêcis* in the final chapter.

C. B. BOYER, Brooklyn College

Probability Theory. By K. Krickeberg. Addison-Wesley, Reading, Mass., 1965. x+230 pp. $8.95.

The book under review is an English translation of the original German edition, titled *Wahrscheinlichkeitstheorie*, published by Teubner, Stuttgart, 1963. The author states that this book is suitable for an advanced undergraduate or first-year graduate course in probability. The only prerequisites specified are calculus and the elementary theory of real finite-dimensional linear spaces, as some measure-theoretic concepts are introduced and used throughout the book. Chapter titles: I. Random variables, II. Expectations and distributions, III. Sequences of independent random variables, IV. Conditional expectations, V. The Brownian motion and the Poisson process.

The chapter titles listed above give some indication of the general scope of this book; and, in general, we feel that this book is a very well written, mathematically rigorous introduction to probability theory. Our only somewhat negative comment concerns the suitability of this volume as an introductory text for senior and first-year graduate students at universities in the United States. In many universities, graduate students specializing in probability are encouraged to have a more comprehensive knowledge of both measure theory and probability than that which could be obtained from this book; hence the book would not, in general, be suitable for this group of students. It does, however, appear to be an ideal text for (1) mature seniors, (2) graduate students with a background in measure theory, or (3) the more mathematically oriented graduate students in the sciences and engineering, who want a modern and mathematically sound introduction to probability theory.

A. T. BHARUCHA-REID, Wayne State University

Elements of the theory of functions. Vol. 90. By R. S. Guter, L. D. Kudryavtsef and B. N. Levitan. Translated by H. F. Cleaves. Pergamon, Oxford, 1966. xii+215 pp. $8.50. (Teleg. Rev. January 1967.)

The intent of the Russian edition of this book was to present to the nonexpert some of the basic definitions and results in the theory of functions of a real variable, approximation and interpolation of functions, and the theory of almost periodic functions.

In carrying out such a purpose it is necessary that special care be taken to be precise and correct since there are relatively few internal checks on the material.

I have not seen the Russian original but this translation is bad. It appears to have been done by a nonmathematician and appears never to have been proof-read. A catalog of the errors in translation, garbled text, odd terminology, bad punctuation, typographical errors, and just plain bad English would take many pages.

A few random samples may suffice to indicate what is meant in the preceding paragraph. On p. 66 the word *complement* is used when *completion* is meant, on p. 83 the *inequality of the triangle* is referred to, on p. 81 a function which is zero almost everywhere is said to be "equivalent to identical zero," on p. 173 a function which is different from zero *at most* on a denumerable set is said to "differ from zero *very considerably* on a denumerable set."

There are symbols missing, inequalities reversed, continual confusion between the use of the definite and indefinite articles, words like *lower* translated into their antonyms etc., etc.

Finally, as if the above were not enough, the book is badly cross indexed and the bibliography fails to mention the availability in English of several titles given in Russian.

I cannot recommend the book to anyone.

J. B. ROBERTS, University College Dar Es Salaam and Reed College

Theory of Probability. By Bernard Harris. Addison Wesley, Reading, Mass., 1966. ix +294 pp. $9.75. (Teleg. Rev. March 1967.)

This is a closely-written, businesslike account of the mathematical aspects of probability theory, starting from scratch with the Kolmogorov axioms, covering combinatorial probabilities, random variables on discrete and continuous sample spaces, characteristic functions, a certain amount of distribution theory and a selection from the more important limit theorems of probability theory. The one chapter devoted to random experiments is so compressed and abbreviated that it could have been left out without appreciable loss.

Through the continued use of distribution functions and the Riemann-Stieltjes integral, the author is able to cover in a connected and complete fashion a substantial core of probability theory in a brief 250 pages. I think a treatment of this sort is bound to seem a bit dull to a beginner, but the large collection of quite superior problems gives life to the book as a whole.

Presumably the book was written as a textbook to accompany a lecture course to students with some mathematical attainments, at least to the level of advanced calculus. It should serve this purpose well. It is a fairly complete skeleton, well wired together, leaving to the lecturer the task of putting flesh and perhaps some other adornments on the bones. This, to my way of thinking, is what a textbook should do.

D. B. DE LURY, University of Toronto

A Treasury of Mathematics. A collection of source materials, edited with biographical and historical sketches by Henrietta O. Midonick, Philosophical Library, New York, 1965. xxiv+820 pp. $15.00.

One of the constant needs of anyone interested in the history of mathematics is having good source materials. This collection includes 54 selections, 11 Chinese, 10 Greek, 9 English, 5 German, 4 Indian, 2 Hebrew, 2 Arabian, 2 French, 2 Egyptian, 2 American, and one each of Babylonian, Mayan, Incan, Danish and Flemish.

It is unfortunate that the book is arranged alphabetically by author, rather than by natural groupings, such as subject or period. The few modern selections could just as well be transferred to a separate volume devoted to the mathematics of the post-Renaissance period, and replaced by selections from authors who have been omitted, such as al-Kashi and Leonardo of Pisa. The bias toward Chinese mathematical texts is overextended but understandable, considering the relative neglect of Chinese contributions to mathematics in ancient and medieval times.

This is a book which belongs in all mathematic libraries, even at the inflated price.

CHARLES ARONSON, Berkeley, California

Mathematics in the Social Sciences and Other Essays. By Richard Stone. M.I.T. Press, Cambridge, Mass. 1966. xiii+291 pp. $12.00.

Professor Stone of Cambridge University is perhaps today the most outstanding econometrician in England. This collection of essays published in various journals 1960–64 should also be of interest to mathematicians, who concern themselves with the more advanced applications of mathematics in the social sciences.

We now list the essays and indicate the mathematical methods used: 1. Mathematics

in the social sciences. 2. The *a priori* and the empirical in economics (matrices). 3. The housekeeper and the steersman (cybernetics). 4. Misery and bliss (calculus). 5. Three models of economic growth (calculus of variations). 6. Models of the national economy for planning purposes. 7. A demonstration model for economic growth (matrices). 8. Transitional planning (difference equations). 9. A model of the educational system (difference equations). 10. Social accounts at the regional level: a survey (matrices). 11. A comparison of the economic structure of regions based on the concept of "distance" (multivariate methods of statistics). 12. Population mathematics, demand analysis and investment planning (Markov chains). 13. A dynamic model of demand (difference equations). 14. The changing pattern of consumption (matrices, least squares). 15. Private savings in Britain, past, present and future (difference equations, statistical estimation). 16. Multiple classifications in social accounting (matrices). 17. British economic balances in 1970: a trial run on rocket (matrices, difference equations).

GERHARD TINTNER, University of Southern California

Topological Structures. By Wolfgang J. Thron. Holt, Rinehart and Winston, New York, 1966. ix+231 pp. $9.50.

This book is designed as a text for a full-year course on general topology at the senior or first-year graduate level. The standard material for such a course is well covered, except for the topic on topology of function spaces. Thus neither the Stone-Weierstrass Theorem nor Ascoli's Theorem are presented. A special feature of the book is a chapter devoted to a thorough account of the proximity spaces. There is also a chapter on topological groups. Both filters and nets are used. The use of filters in this book is more frequent than in other texts on general topology in the English language.

The author gives a good deal of attention to the historical development of the basic ideas. The exposition is well motivated by numerous historical remarks and intuitive observations. The book is divided into 23 short chapters, with exercises provided at the end of each chapter.

KY FAN, University of California, Santa Barbara

Brief Numerical Methods. By Wendell E. Grove. Prentice-Hall, Englewood Cliffs, N.J., 1966. xi+117 pp. $6.75.

This book is a very lucid introduction to numerical methods. It presents computer-oriented techniques for solving a relatively broad scope of problems.

Topics included in the text are: a rather exhaustive treatment of the solution of algebraic and transcendental equations by iteration (some simplified flow diagrams are shown); methods for calculating complex roots of equations; the solution of simultaneous equations; the least-squares fitting of curves (including the use of orthogonal polynomials); numerical integration; and the solution of ordinary differential equations. A technique is given for solving an ordinary differential equation with boundary values given at both ends of an interval, and a chapter, dedicated to interpolation techniques, introduces the E and Δ operators. The book presents many less well-known methods which are suitable for machine coding (Wegstein's methods for simultaneous equations, Lin's method for complex roots, Adams-Bashforth and Nystrom methods for solution of differential equations). It gives helpful tips as to the limitations of the techniques. There is only a modicum of misprints in the text (e.g., pp. 44 and 76).

J. C. ROGERS, Omnimetrics

Combinatorial Group Theory. By Wilhelm Magnus, Abraham Karrass and Donald Solitar. Interscience, New York, 1966. x+444 pp. $15.00.

One of the subtleties which the beginning student of group theory has to master is this: When a group is given by generators and defining relations, there is no obvious way

of telling when different expressions represent the same element (even the elements represented by different generators need not be distinct). This is essentially the word problem and from the work of Novikov, Boone and Britton we know that groups with unsolvable word problem exist. But from the point of view of the working group theorist, and even more, of the general mathematician using groups, it is more important to have a good knowledge of methods of solving the word problem. The book under review is devoted to a study of such methods.

The main topics (after a brief review of the basic concepts) are: The presentation of subgroups (Reidemeister-Schreier method) and the Schreier Subgroup Theorem. This leads on to Nielsen transformations, which are applied to automorphisms of free groups, and to knot and braid groups. Free products (also with an amalgamated subgroup) are defined in terms of presentations of the factors and are studied by standard cancellation methods. The theory is developed far enough to yield the Kurosh Subgroup Theorem (but not Grushko's Theorem,which is stated without proof); other applications include the independence theorem and the word problem for groups with one defining relation. A chapter on commutator calculus establishes the connection between groups and Lie rings, proves the existence of basic commutators in the free case and applies the result to study the lower central series, especially for free groups. The final chapter reports (without proofs) on recent results.

Throughout the book the emphasis is on techniques, rather than general results and this outlook is supported by the very substantial collection of exercises, many of interest in their own right. The tone is elementary throughout; the authors are at pains to make themselves understood and at no point do they presuppose any specific knowledge. Of course to profit from the book the reader should have some acquaintance with group theory as well as a liking for combinatorial methods. Given these prerequisites (as well as pencil, paper and patience), the reader will acquire a valuable tool for handling groups, and a most useful collection of specimens.

No doubt the series book has come to stay, but the publishers should desist from putting a book in *two* series: this book is described as vol. 13 of a series of Texts and Monographs, and also as vol. 20 of the Interscience Tracts (of which vol. 13 is Nagata's Local Rings), a most confusing situation which should not be perpetuated.

P. M. COHN, University of London

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (Teacher Training), L (Library purchase), 15 (junior level)—18 (second graduate year).

Sêminaire Bourbaki. December 1948–May 1965. Benjamin, New York, 1966. Twelve volumes of from 200–350 pp. $15.00 per vol., $150 for the set. Facsimile (photo-offset) reproduction copied directly from the second mimeographed edition in seventeen volumes of the original notes of 294 lectures, almost entirely in French. Each volume contains a table of contents and an index of authors cited in all twelve volumes. The last has also an alphabetical index of lecturers. An additional volume is planned annually. Libraries should purchase at this high price per page only if they seek completeness. Virtually all the content has appeared or will appear elsewhere. (For a contrary view see *Science* 16 Dec. 1966, p. 1440.)

Guidebook to Departments in the Mathematical Sciences in the United States and Canada. Edited by Raoul Hailpern. The Committee on Advisement and Personnel, Mathematical Association of America, Buffalo, 2nd ed., 1966. 80 pp. 50¢. An invaluable

reference for those who are selecting a place to do undergraduate or graduate work in mathematics. P, L.

German-English Mathematical Vocabulary, 2nd ed. By Sheila Macintyre and Edith Witte. Wiley, New York, 1966. ix+95 pp. $2.95.

NOTE: We are informed by Chelsea Publishing Co., that a new edition of *Commutative Normed Rings* by Gelfand, Raikov, and Shilov will appear this summer "incorporating new material especially written for this edition by Professor Shilov, and eliminating the offending theorem and erroneous proof" mentioned in the review by Gelbaum in this Monthly of February 1967.

Computers and Computing

Introduction to Cybernetics. By Viktor M. Glushkov. Translated by Scripta Technica. Academic Press, New York, 1966. x+322 pp. $11.75. Abstract theory of algorithms, Boolean functions, automata, self-organizing systems, electronic digital computers and programming, the predicate calculus and the automation of the processes of scientific discovery. P, L.

Analysis of Numerical Methods. By Eugene Isaacson and Herbert Bishop Keller. Wiley, New York, 1966. xv+541 pp. $11.95. ". . . we stress techniques for the development of new methods. This requires knowing why a particular method is effective on some problems but not others. Hence we are led to the analysis of numerical methods rather than merely their description and listing." Linear systems, matrix inversion, nonlinear equations, eigenvalues, polynomial approximation, differences, integration, ordinary and partial differential equations. (Reviewed in *Science* 24, February 1967.) T (15–16) P, L.

Logic and Algorithms. With Applications to the Computer and Information Sciences. By Robert R. Korfhage. Wiley, New York, 1966. xii+194 pp. $7.95. ". . . evolved from the basic logic course for students in mathematics and the computer sciences . . . and a similar but more elementary course presented to an industrial group of computer users . . . logic books might be grouped into three main categories: those written by philosophers . . . the mathematical logic books . . . and the books written for engineers. . . . This book is written on the premise that computer users and designers need to have some knowledge of all three aspects of logic as well as the fundamentals of algorithm theory." Set algebras, relations, mappings, Boolean algebras, propositional calculus, quantification theory, internal symbol representations of a binary vector, formal languages, history. T (14–15), S, P.

Programming Real-Time Computer System. By James T. Martin. Prentice-Hall, Englewood Cliffs, N. J., 1965. xii+386 pp. $12.00. This book discusses the design, programming, testing and implementation of "real time" computer systems, paying special attention to management problems. P.

Interval Analysis. By Ramon E. Moore. Prentice-Hall, Englewood Cliffs, N. J., 1966. xi+145 pp. $9.00. ". . . addressed to the mathematician, the numerical analyst, and the programmer. The title refers to a set of concepts and techniques based on treating an *interval* of real numbers as a new kind of number, represented by a pair of real numbers, namely, its left and right endpoints. The techniques can be programmed for computers in order to obtain simultaneously upper and lower bounds to exact solutions of equations of various types . . . a number of interesting and challenging research problems concerned with ways in which the methods and programs can be

extended, refined, and improved. The material in this book was presented as a graduate course in the Computer Sciences Department of the University of Wisconsin . . ." T (17), S, P.

Theory of Self-Reproducing Automata. By John von Neumann, edited and completed by Arthur W. Burks. University of Illinois Press, Urbana, 1966. xix+388 pp. $10.00. Part I: "Theory and Organization of Complicated Automata," five lectures at the University of Illinois in 1949 (pp. 31–87). Part II: The Theory of Automata (pp. 91–380). Both are carefully edited from manuscripts left at von Neumann's untimely death, and they are introduced by a 28 page analysis of all von Neumann's work on computers and automata. As the editor points out, the unfinished nature of the manuscripts allows us "to see von Neumann's powerful mind at work." Clearly this is an important mathematical and historical document, a must for anyone interested in computers, automata or cybernetics. P, L.

PL/IProgramming Primer. By Gerald M. Weinberg. McGraw-Hill, New York, 1966. ix+278 pp. $5.95. ". . . a text for an introductory course in computer programming . . . also as an introduction to PL/I . . . one of the first higher level languages which shows promise of being reasonably universal in its application, at least for the applications that exist today . . . " T, S, P.

A Short Introduction to Numerical Analysis. By M. V. Wilkes. Cambridge University Press, 1966. 76 pp. $4.25. Elementary treatment of iteration, interpolation, numerical integration and differential equations, simultaneous equations, linear systems. T (14–15), S.

Tables and Formulae for the Spherical Functions $P^m_{-\frac{1}{2}+ir}(z)$. By M. I. Zhurina and L. N. Karmazina. Translated by E. L. Albasiny. Pergamon, New York, 1966. vii+107 pp. $3.50. Forty pages on properties of the functions, seven describing the tables. L.

Mathematical Education

The Role of Axiomatics and Problem Solving in Mathematics. Edited by E. G. Begle for the Conference Board of the Mathematical Sciences. Ginn and Co., Boston, 1966. v+137 pp. $1.60 (paper). A collection of essays solicited by the CBMS at the request of the United States Commission of the International Commission on Mathematical Instruction.

Preface. E. G. Begle. The Use of the Axiomatic Method in Teaching High School Mathematics. F. B. Allen. Lyons Township High School. Western Springs, Illinois. The Use and Abuse of the Axiomatic Method in High School Teaching. A. A. Blank. New York University. The Role of a Naive Axiomatics. R. C. Buck. University of Wisconsin. Mathematics: Its Structure, Logic, and Method. Irving Allen Dodes. Kingsborough Community College of the City University of New York. Axioms, Postulates, and the Teaching of Elementary Mathematics. A. M. Gleason. Harvard University. The Axiomatic Method in Mathematics Courses at the Secondary Level. Leon Henkin. University of California, Berkeley. Mathematics and Axiomatics, Morris Kline, New York University. The Axiomatic Method and School Mathematics. M. E. Shanks. Purdue University. The Axiomatic Method in High School Mathematics. Patrick Suppes. Stanford University. A Use of the Axiomatic Method in Teaching Algebra. H. E. Vaughan. University of Illinois. The Role of Postulates in School Mathematics. G. S. Young. Tulane University.

The Role of Problems in Mathematical Education. R. P. Dilworth, California Institute of Technology. Some Thoughts on Problem Solving. N. J. Fine. Pennsylvania State University. The Role of Problems in the Development of Mathematical

Activity. F. D. Jacobson, Albertus Magnus College. The Role of Problems in Secondary School Mathematics. P. S. Jones. University of Michigan. The Role of Problems in the High School Mathematics Curriculum. P. D. Lax. New York University. On Individual Exploration in Mathematics Education. H. O. Pollak. Bell Telephone Laboratories. On Teaching Problem Solving. George Polya, Stanford University. Problem Making and Problem Solving. P. C. Rosenbloom. Columbia University. Problems in the Teaching of Mathematics. František Wolf. University of California, Berkeley.

Obviously this is a must for everyone interested in mathematical education. Instead of soliciting a review, we invite readers to comment. S, P. L.

Mathematics and the Elementary Teacher. By Richard W. Copeland. Saunders, Philadelphia, 1966. xvii+355 pp. $6.50. Part I: Arithmetic. Part II: Geometry. A reviewer to whom this book was sent writes that its objectives are commendable, but that its use should be discouraged because of much unsound material and a confusing effort to mix modern ideas with "the imprecise, fragmented and mechanistic approaches" of the past.

Algèbre linéaire et géométrie élémentaire. By Jean Dieudonné. Hermann, Paris, 1964. 223 pp. 36 F. An interesting, contentious book on secondary education by a close collaborator of N. Bourbaki. Euclidean geometry appears as an interpretation of linear algebra. The introduction ends as follows: ". . . I have not the slightest personal interest in these matters of secondary education, and I care little if, when or how there is a reform at this level. I have simply wished to place on file for the future historian an example of what one might do in this matter if one tried to act in a *rational* way." (Editor's free translation.) S, P.

Introduction to Contemporary Mathematics. By Sze Tsen Hu. Holden-Day, San Francisco, 1966. viii+202 pp. $7.95 in U.S.A. ". . . for the first fundamental course for mathematical majors after two years of calculus . . . a guided tour . . . emphasizing the austere aspect of abstraction and precision without . . . deeper results . . . calculus isn't necessary." T (13–15) S.

The Number Systems and Operations of Arithmetic. By Orval M. Klose. Pergamon Press, Oxford, London, 1966. xi+265 pp. $3.95. ". . . for the single purpose of explaining to elementary school teachers (both in-service and in-training) the nature of those basic principles of mathematics which form the foundations and structural framework of arithmetic, and how the familiar formal algorithms . . . stem from these principles . . . frankly intuitive approach . . . " TT.

The Study of Arithmetic. By L. Clark Lay. Macmillan, New York, 1966. xv+590 pp. $8.00. TT (year course for future elementary teachers).

Report of the International Clearinghouse on Science and Mathematics Curricular Developments. Compiled under the Direction of J. David Lockard. A joint project of the American Association for the Advancement of Science and the Science Teaching Center, 1966. 291 pp. Free from the Science Teaching Center, University of Maryland, College Park, Md. 20740. The first of an annual series from the International Clearinghouse on Science and Mathematics Curricular Developments, this volume describes current projects in the U.S.A. and 28 other countries. P, L.

Prelude to Analysis. By Paul C. Rosenbloom and Seymour Schuster. Prentice-Hall, Englewood Cliffs, N.J., 1966. 473 pp. $8.25. Not just another precalculus book. It

deals with the prerequisites of a strong calculus course (real numbers, algebraic structures, coordinates, functions, vector algebra, approximation, sequences, induction) in a modern spirit (abstraction and rigor, motivated and related to applications) without the trappings of "modern" mathematics (excessive preoccupation with terminology, logic, set theory, and unmotivated formalism). T (13), S, P, L.

A Path to Modern Mathematics. By W. W. Sawyer. Penguin books, Baltimore, 1966. 223 pp. $1.25 (paper). Written for secondary teachers. Largely on concepts from linear algebra leading to a final chapter on "metric and Banach Spaces." S, TT.

Trends in Elementary School Mathematics. By Lloyd Scott. Rand McNally, Chicago, 1966. ix+215 pp. $3.50 (paper). "This book was written out of empathy for teachers of elementary school mathematics. . . . The expressed need was for an overview of the modern elementary school mathematics program with particular attention to the practical interest of teachers." S, TT, P, L.

History

The Role of Mathematics in the Rise of Science. By Salomon Bochner. Princeton Univ. Press, Princeton, N.J., 1966. x+386 pp. $9.00. Rather philosophical essays written 1961–1965 and accompanied by comments and 150 brief biographies of mathematicians and others. P, L.

Mind and Cosmos. Essays in Contemporary Science and Philosophy. Edited by Robert G. Colodny. Univ. of Pittsburgh Press, Pittsburgh, 1966. xviii+362 pp. $8.00. The third in a series published under the auspices of the Center for the Philosophy of Science at the University of Pittsburgh, this volume contains essays by Herbert Simon (thinking by computers, discovery and problem solving), D. Shapere (Meaning and scientific change), S. Bromberger ("Why-Questions"), C. G. Hempel (induction), W. C. Salmon (inference), J. T. Clark (physics, components of intellectual culture), T. Gold (cosmic processes, time), H. Margenau (quantum theory). S, P, L.

Research Papers in Statistics. Edited by F. N. David and Evelyn Fix. Wiley, New York, 1966. viii+468 pp. $19.30. A volume of research and review papers in honor of Neyman's 70th birthday, including "The Neyman-Pearson Story: 1926–34." By E. S. Pearson. (See review by Wolfowitz in *Science* 20, Jan. 1967.) P, L.

Collected papers of G. H. Hardy. By G. H. Hardy including joint papers with J. E. Littlewood and others. Edited by a committee appointed by the London Mathematical Society. Vol. I. Oxford Univ. Press, London, 1966. 700 pp. $16.80. This first of seven planned volumes contains papers on Diophantine approximation and additive number theory (combinatory analysis and sums of squares, Waring's problem, Goldbach's problem, Inaugural Lecture) with introductory comments to each group of papers, the obituary by Titchmarsh, a table showing the arrangement of papers in all seven volumes, and a complete bibliography of Hardy's papers beginning with a question to the Educational Times in 1899 and ending with a note on Fourier series in 1949. There is a superb informal photograph of Hardy peering over his glasses at the reader. P, L, B.

The Birth of Mathematics in the age of Plato. By Francois Lasserre. World, New York, 1964. 191 pp. $1.65 (paper). Based on analysis of surviving classic texts without, reference to historical studies of van der Waerden, Neugebauer, etc. P.

Measure and the Integral. By Henri Lebesgue. Edited with a biographical essay by

Kenneth O. May. Holden Day, San Francisco, 1966. xii+194 pp. $6.95. Part I: Measure of Magnitudes (*La Mesure des Magnitudes*, 1931–1935), Part II: The development of the Integral Concept (*Sur le développement de la notion d'intégrale*, 1926). Selected bibliography on Lebesgue. S, P, L.

The Algebra of Abu Kamil. By Martin Levey with foreword by Marshall Clagett. Hebrew text, with English translation and commentary with special reference to the Arabic text. Univ. of Wisconsin Press, Madison, 1966. xi+226 pp. $10.00. This earliest (ca. 850–930 A.D.) Arabic algebra after Al-Khwarizmi known to be extant has survived best in this fifteenth century Hebrew version. S, P, L.

The Collected Mathematical Works of J. H. and A. G. M. Michell. Edited by F. W. Niedenfuhr and J. R. M. Radok. Foreword by T. M. Cherry, Noordhoff, Groningen-Holland, 1965. 338 pp. $7.10. Papers in applied mathematics and mechanical engineering by the Australian brothers, J. H. Michell (1863–1940) and A. G. M. Michell (1870–1959). P, L.

De Proportionibus proportionum and Ad pauca respicientes. By Nicole Oresme. Translated and edited by Edward Grant. Univ. of Wisconsin Press, Madison, 1966. xxii+466 pp. $10.75. The first scholarly edition and translation of treatises on proportions and exponents and on the kinematics of circular motion. The original Latin and English translations are on facing pages. "All the tortuous turns and complexities of the text are unraveled and explained . . . and an introduction examines Oresme's career and discusses the meaning and significance of the treatises in the context of medieval mathematics." Well reviewed by Curtis Wilson in *Science* 23, December 1966. P, L.

Logic and Reality in Leibniz's Metaphysics. By G. H. R. Parkinson. Oxford Univ. Press, Oxford, 1965. 159 pp. $5.60. P.

The Mathematical Practitioners of Hanoverian England. 1714–1840. By E. G. R. Taylor. Cambridge Univ. Press, New York, 1966. xv+502 pp. $16.50. This is a sequel to the author's *Mathematical Practitioners of Tudor and Stuart England* (Cambridge University Press, 1954: soon to be reprinted). In spite of some weaknesses (see the review by D. T. Whiteside in *Science*, 30, September 1966), this is an important and useful historical reference. P, L.

Papers presented to J. E. Littlewood on his eightieth birthday. Edited by J. D. Weston. Oxford Univ. Press, New York, 1966. 314 pp. $12.80. A collection of 26 research papers contributed by friends and colleagues. P.

Number Theory

Number Theory. By Z. I. Borevich and I. R. Shafarevich. Translated by Newcomb Greenleaf for Scripta Technica. Academic Press, New York, 1966. x+435 pp. $12.95. ". . . we start from concrete problems . . . General theories arise as tools for solving these . . . these theories are developed . . . so that the reader can see . . . their strength and beauty, and so that he learns to apply them." Congruences, representation by decomposable forms, divisibility, local methods, analytic methods, algebraic supplement. T (15–17), S, P, L.

Theory of the Linear Diophantine Equation. By P. Laborde. Published by the author, Univ. of Puerto Rico, 1965. vii+59 pp. (paper). S, P.

Elementary Number Theory, 2nd ed. By Edmund Landau. Translated by Jacob Goodman, with exercises by Paul Bateman and Eugene Kohlbecker. Chelsea, New York, 1966. 256 pp. $4.95.

Elementary Methods in the Analytic Theory of Numbers. By A. O. Gel'fond and Yu. V. Linnik. Translated by D. E. Brown. Translation edited by I. N. Sneddon. Pergamon Press, New York, 1966. xi+232 pp. $9.80. "Many important problems in the modern analytic theory of numbers can be stated in terms of elementary mathematics and simple concepts such as that of a limit or just of an indefinitely increasing parameter. At the present time very many problems of the analytic theory of numbers are capable of solution by elementary methods and solutions have been published in journals and monographs. In this volume many of these solutions are collected, systematized to a certain extent and simplified where possible. . . . " Schnirelmann's method, Mann's theory, Erdös' theorem, Waring's problem, Hilbert-Kamke problem, distribution of primes, law of distribution of Gaussian primes, almost-prime Gaussian numbers, sieve of Eratosthenes, Atle Selberg's method, distribution of fractional parts of numerical sequences, integral points within contours, power residues, Hasse's theorem, Siegel's theorem, transcendence of certain number classes. S (16–18), P, L.

Topics from the Theory of Numbers. By Emil Grosswald. Macmillan, New York, 1966. xiv+299 pp. $8.50. ". . . rather little overlap with any other text I know of, except, of course, for the most fundamental topics . . . emphasis given to the historical point of view and the inclusion of references to a large amount of source material. . . . " Historical background, elementary number theory (divisibility, congruences, quadratic residue, arithmetical functions,) and topics from analytic and algebraic number theory (distribution of primes and Riemann zeta functions, proof of the prime number theorem, diophantine equations and Fermat's conjecture, ideal theory, proof of Fermat's conjecture for regular primes, partitions). Appendices on topics from analysis and algebra. T (15–17, one or two semesters), S, P.

Probability and Statistics

Elements of Probability Theory. By J. Bass. Translated by W. McKay. Translation editor A. Jeffrey. Foreword by E. W. Barankin. Academic Press, New York, 1966. xiv +249 pp. $9.75. Six chapters on the usual introductory topics lead to the law of large numbers and central limit theorem—described by the author as "the focal point of the course." There follow two chapters on random functions, two on statistics (estimation, confidence intervals, maximum-liklihood, tests, etc.), and 53 graded problems. Barankin, in his interesting foreword, praises the book for bringing to our students "some European sense of caution and discrimination" and for building up slowly with extensive explanation and illustration. T (15–16, especially for students in science and engineering).

Measurement and Analysis of Random Data. By Julius S. Bendat and Allan G. Piersol. Wiley, New York, London, 1966. ". . . a reference for practicing engineers and scientists." P.

Principles of Statistics. By M. G. Bulmer. M.I.T. Press, Cambridge, Mass., 1965. 214 pp. $7.50. "The aim of this book is to describe as simply as possible the fundamental principles and concepts of statistics. I hope that it may be useful both to the mathematician who requires an elementary introduction to the subject and to the scientist

who wishes to understand the theory underlying the statistical methods which he used in his own work." The approach is eclectic. Both frequency and degree of belief are considered and the style is discursive. A rather balanced presentation of the controversies over inference is given, but in the bibliography R. A. Fisher is cited seven times, while neither J. Neyman or E. Pearson appear at all! There are no exercises. S (14–15), P.

The Monte Carlo Method. The Method of Statistical Trials. By N. P. Buslenko, D. I. Golenko, Yu. A. Shreider, I. M. Sobol and V. G. Sragovich. Edited by Yu. A. Shreider. Translated from the Russian by G. J. Tee. Translation edited by D. M. Parkyn. Pergamon Press, New York, 1966. xii+381 pp. $12.50. "The book is intended for mathematicians, physicists and engineers who need to solve problems in applied mathematics." P, L.

Bibliography of Statistical Literature 1940–49. By Maurice G. Kendall and Alison G. Doig. Hafner, New York, 1965. 190 pp. $10.50. This is the second of three projected volumes covering the literature of statistics and probability from the sixteenth century. L.

Elements of Probability and Statistics. By Elmer B. Mode. Prentice-Hall, Englewood Cliffs, N.J. 1966. xii+356 pp. $8.75. Includes multinomial, bivariate binomial, and chi-square distributions, estimation, testing hypotheses (very little), regression, correlation, and Markov chains. T (precalculus).

Sequential Methods in Statistics. By G. Barrie Wetherill. Wiley, New York, 1966. x+218 pp. $5.50. ". . . a survey . . . with emphasis on practical importance. Detailed mathematical proofs are not given, but the logical basis of the methods is presented. . . . The chapter on sequential estimation brings out the difference between the Neyman-Pearson and Bayesian approaches . . . a set of problems at the end of each chapter, some of which lead to further research." S, P.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor C. H. Gladman, Texas Western College, represented the Association at the dedication of the Mathematical Sciences Building at New Mexico State University on December 27, 1966.

University of Alberta, Edmonton: Assistant Professor Nadhla Al-Salam, University of Alberta, Calgary, and Dr. D. W. Boyd, University of Toronto, have been appointed Assistant Professors; Associate Professor W. A. Al-Salam, University of Alberta, Calgary, has been appointed Associate Professor; Assistant Professor J. W. Moon has been promoted to Associate Professor; Professor A. L. Dulmage has been appointed Associate

Dean, Faculty of Arts and Science, at the University of Manitoba; Professor A. E. Livingston has been appointed Professor at Oregon State University; Associate Professor D. W. Willett has been appointed Associate Professor at the University of Utah.

Arlington State College: Dr. B. B. Williams, University of Texas, has been appointed Assistant Professor; Professor B. M. Wall has accepted the position of Head of the Mathematics Department.

Bowling Green State University: Assistant Professor J. F. Leetch has been promoted to Associate Professor; Dr. C. E. Harrell, University of Texas, and Mr. J. F. Ramaley, Carnegie Institute of Technology, have been appointed Assistant Professors.

Brigham Young University: Dr. V. H. Miller, Montana State University, has been appointed Assistant Professor; Assistant Professor K. M. Larsen has been promoted to Associate Professor.

Brooklyn College: Dr. Howard Allen, Polytechnic Institute of Brooklyn, and Miss Marion S. Weiner, New York University, have been appointed Assistant Professors; Assistant Professor Margaret Y. Woodbridge has been promoted to Associate Professor.

California State Polytechnic College: Assistant Professors C. F. Gerald and C. T. Haskell have been promoted to Associate Professors.

University of Illinois: Dr. J. W. Brown, U.C.L.A., Dr. L. E. Claborn, Cornell College, Dr. R. L. Finney, Princeton University, and Dr. Phyllis G. Parr, Indiana State University, have been appointed Assistant Professors; Assistant Professors F. B. Knight, Hiram Paley, and N. J. Rothman have been promoted to Associate Professors; Dr. S. B. Bank has been promoted to Assistant Professor; Associate Professors Mary E. Hamstrom and J. H. Walter have been promoted to Professors.

Kenyon College: Associate Professor R. M. McLeod, University of Tennessee, has been appointed Associate Professor; Professor W. R. Transue has been appointed Professor at SUNY at Binghamton; Assistant Professor R. M. Fesq has been promoted to Associate Professor; Associate Professor W. D. Lindstrom has been promoted to Professor.

Michigan State University: Dr. G. D. Ludden, University of Notre Dame, Dr. C. R. MacCluer, University of Michigan, Assistant Professor R. C. O'Neill, University of Michigan, and Dr. C. E. Weil, University of Chicago, have been appointed Assistant Professors; Associate Professor John Wagner has been promoted to Professor.

University of Nebraska: Dr. M. D. Larsen, University of Kansas, and Dr. D. L. Skoug, University of Minnesota, have been appointed Assistant Professors; Assistant Professors S. E. Dickson and A. M. Fink have been promoted to Associate Professors.

New Mexico State University: Associate Professor J. E. Kist, Pennsylvania State University, has been appointed Visiting Associate Professor; Professor R. B. Crouch has been appointed Professor and Chairman of the Mathematics Department at Drexel Institute.

Northwestern State College: Mr. E. F. Ratliff, Jr., Oklahoma Baptist University, has been appointed Assistant Professor; Mr. D. R. Pitt, Jr., has been promoted to Assistant Professor.

Professor J. J. Gergen, Duke University, died on January 16, 1967. He was a member of the Association for thirty-three years.

Professor J. D. Hill, Michigan State University, died on January 4, 1967. He was a member of the Association for thirty-seven years.

Professor H. T. Muhly, University of Iowa, died on December 22, 1966. He was a member of the Association for sixteen years.

Dr. J. H. Waddell, Kitt Peak National Observatory, died in November, 1966. He was a member of the Association for seven years.

Professor Emeritus A. K. Waltz, Steubenville College, died on December 5, 1966. He was a member of the Association for forty-seven years.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the MAA was held at the University of Manitoba in Winnipeg, Manitoba on October 29, 1966. There were 75 persons in attendance, including 53 members of the association. Professor Bernard Noonan of the University of Manitoba presided over the morning sessions; Professor Dale Varberg of Hamline University presided over the afternoon sessions. The sessions were held in University College.

At the business meeting a nominating committee for next year's officers was appointed. The following proposal was made: that each year the Minnesota Section send a letter of commendation to all Putnam contestants in the section who score within the top 25 percent in the national competition, inviting them to the spring meeting with a free luncheon at the courtesy of the section. In addition, the top 10 of these contestants will be awarded a year's subscription to the Mathematical Monthly.

The program of the sessions was as follows:

1. *A combinatorial problem*, by J. A. Ewell, University of Manitoba, introduced by H. Finlayson.
2. *Groups and graphs*, by C. T. Benson, University of Manitoba, introduced by Nora Losey.
3. *On subalgebra lattices of universal algebras*, by George Grätzer, University of Manitoba, introduced by Nathan Mendelsohn.
4. *Maximal binary determinants by electronic computer*, by Bernard Noonan, University of Manitoba, introduced by Roy Dowling.
5. *Critical cases of implicit function theory*, by H. I. Freedman, University of Minnesota.
6. *The international congress of mathematicians at Moscow*, by K. W. Wegner, Carleton College.
7. *Combinatorial problems in group theory*, by N. S. Mendelsohn, University of Manitoba (by invitation).
8. *Des films mathématiques (Avec des exemples)*, by Seymour Schuster, Minnemath Center at the University of Minnesota (by invitation).

WALBERT KALINOWSKI, *Secretary-Treasurer*

DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual fall meeting of the Maryland-District of Columbia-Virginia Section of the MAA was convened at Georgetown University, Washington, D. C. on December 10, 1966. Chairman T. L. Reynolds of the College of William and Mary, Williamsburg, Virginia, presided. Approximately 120 persons attended. The morning was devoted to short papers presented in two concurrent sessions.

During morning session I the following papers were presented:

1. *Minimum number of subsets to distinguish individual elements*, by P. R. Meyers, National Bureau of Standards, Washington, D. C.

Given a set S of cardinality m, the author determines the minimum cardinality $f(m)$ for a family F of subsets of S such that each $s \in S$ can be expressed as the intersection of some subfamily of F. The problem is solved in the following inverse form. For a given number n of subsets of S, find $g(n)$: the maximum number of elements of S which can be written as the intersection of some of these subsets. It is shown that $g(n)$ is the largest binomial coefficient for combinations of n things.

2. *A note on the Pell equation*, by Paul Devine, Jr., George Washington University.

The author proves that a necessary condition for the solvability of the Pell Equation, $x^2 - Ny^2$

$=-1$, N a positive, nonsquare integer, is that (1) $N=\alpha^2+\beta^2$, gcd $(\alpha, \beta)=1$, (2) α is odd, (3) α is a quadratic residue of N. There are 294 values of N, less than 10,000, for which (1) alone is necessary but not sufficient; the least value is $N=34$. The author observes that there are only 70 values of N, out of these 294, for which (1)–(3) are necessary but not sufficient; the least value is $N=505$.

3. *An extension of Rainville's classification of simple polynomial sets*, by Sister Mary Seraphine Bennett, Mt. Saint Agnes College.

4. *A note on abelian p-groups and their endomorphism rings*, by Eleanor Green Dawley, Hampton Institute.

This note gives an alternate proof of the fact that the direct sum of a bounded abelian p-group and a divisible abelian p-group is determined by its endomorphism ring. The proof is more simple than Kaplansky's classical proof of this result and does not involve the extensive ring theory utilized by Kaplansky.

5. *Ideals and filters*, by C. E. Aull, Virginia Polytechnic Institute.

DEFINITION 1. *An ideal A is an f-ideal in a commutative semigroup S if for $a, b \in A$, there exists $c \in A$ such that $cx=a$ and $cy=b$ have solutions in S.*

All semigroups and rings are commutative. The author proves that the filters are identical with the f-ideals of the semigroup of all nonnull subsets of a set with the operation of set union; an f-ideal in a semigroup which is an ideal of another semigroup is also an f-ideal of the latter semigroup; a cancellation semigroup has an identity iff all its principal ideals are f-ideals; an f-ideal in the semigroup under multiplication of a ring is always an ideal in the ring.

During morning session II, the following papers were presented:

1. *The solution of the Dirichlet problem for Poisson's equation by finite difference methods*, by B. E. Hubbard, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.

With the advent of modern high speed computers it has become feasible to calculate approximate solutions to boundary value problems with great accuracy.

Difference methods are quite often used because of their simplicity and generality. It thus becomes a matter of great interest to study the rates of convergence of such solutions to the unknown function as the mesh parameter approaches zero. It is of particular interest to study the effect on such convergence rates under various assumptions on the regularity of the data (and hence of the solution itself). This paper concerns itself principally with this latter question.

2. *Asymptotic analysis of a first-moment integral and its derivatives*, by A. J. Goldman, National Bureau of Standards, Washington, D. C.

Let $f(p)$ be the first moment, with respect to point P, of a planar mass distribution about which one knows only that it is concentrated in a disk D of radius R around its (known) center of gravity C. For P far from D, $f(P)$ and its derivatives (with respect to the coordinates of (P) look like the corresponding expressions for an equivalent point mass at C. Error bounds are derived for this asymptotic relation. (Joint work with C. Witzgall, arising in a study of optical location methods for the Post Office Department.)

3. *Stability of nonlinear nonautonomous systems by the stroboscopic method*, by Abolghassem Ghaffari, Goddard Space Flight Center, Greenbelt, Maryland.

The stroboscopic method (N. Minorsky, Nonlinear oscillations, Van Nostrand, 1962) is particularly advantageous in connection with nonlinear nonautonomous systems inasmuch as it permits replacing the problem of stability of a periodic solution by the problem of stability of equilibrium.

Applying the stroboscopic criterion to the nonlinear non-autonomous differential equation $\ddot{x}+(a+bx^2)\dot{x}+x=c\cos t$ and limiting the analysis to the first-order approximation only, the author finds that the conditions of stability are: $a<0$, $b>0$ (case of van der Pol equation with an external periodic excitation $c\cos t$), and the instability condition is $b<0$.

4. *Let's modernize the teaching of modern mathematics*, by G. L. Engel, U. S. N. W. L., Dahlgren, Virginia.

A review and summary of available and proposed systems in Computer Assisted Instruction is presented. The author indicates the interest of CAI as a tool in training and makes an appeal for the study of CAI applications in the teaching of mathematics in the colleges of the district.

5. *Systems of non-linear equations*, by M. A. Hyman, IBM Corporation, Gaithersburg, Maryland.

The author is concerned with the problem of computing best (in the sense of least squares) parameters to fit non-linear functions to observational data. In the classical approach one solves a sequence of linear problems hoping that the sequence of solutions converges to the solution vector of the non-linear problem; in practice this sequence often does not converge. Further, the intermediate linear problems are ill-conditioned. K. Levenburg's work on "damped least squares" (Qu. Appl. Math., 1944) assures convergence and avoids ill-conditioning; this is related to recent work by E. Weiss and W. Goodyear. The paper concludes with brief mention of (a) the "quasi-inverse" of F. Ditto, (b) recent functional-analytic results of Robert-Shipman, and a zero-finding algorithm (Hyman, 1961).

Following a short business meeting during which Prof. M. G. Humphreys, Prof. S. S. Saslow and Dr. Daniel Shanks were appointed to the Nominating Committee, the afternoon session focused on *A General Curriculum in Mathematics for Colleges*. Professor M. Gweneth Humphreys of Randolph-Macon Woman's College, Lynchburg, Virginia, reviewed the GCMC, after which brief comments on its reception by the universities and colleges were presented by Professors W. L. Duren, University of Virginia, J. A. Hummel, University of Maryland and G. S. Quinn, The American University.

G. N. TRYTTEN, *Secretary*

FEBRUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The annual meeting of the Northern California Section of the MAA was held at the University of California, Davis on February 4, 1967. Professor L. H. Lange, Chairman of the Section, presided; Professor E. M. Beesley was Program Chairman. There were 120 persons in attendance, 85 of whom were members of the Association.

At the business meeting the following officers were elected: Chairman, Professor C. A. Hayes of the University of California, Davis to succeed Professor L. H. Lange of San Jose State College; Vice-Chairman, Dr. Henry Osner of Modesto Junior College to succeed Professor Hayes; Secretary-Treasurer, Professor G. L. Alexanderson of the University of Santa Clara to succeed Dr. Joel Brenner of Stanford Research Institute.

Also at the business meeting Professor H. L. Alder reported on the national meeting in Houston and Professor D. W. Blakeslee reported on the meetings of the Board of Governors. Professor Hayes presented the findings of the Reaction Committee to the CUPM Report on *A General Curriculum in Mathematics for Colleges*. The Committee's report will be forwarded to CUPM. A report on the activities of the High School Contest Committee was submitted by Mr. William Landis, Contest Committee Chairman.

The following papers were presented:

1. *Some results on Sturm-Liouville differential systems defined on continua and discrete sets*, by J. F. Schipper, Sacramento State College.

2. *Some Properties of Hahn Polynomials*, by R. J. Levit, San Francisco State College.

From the orthogonality property it follows that the Hahn polynomial of degree m, $P_m^{(\alpha,\beta,\gamma)}(x)$, has m real distinct zeros in the interval $(0, \gamma-1)$, provided $\alpha, \beta > -1$ and γ is an integer $> m$. In the present paper this is extended to all real $\gamma > m$. A new symmetry property then leads to a similar result for $\gamma < -(m+\alpha+\beta)$ with the zeros in $(\beta+\gamma, -\alpha-1)$. Examples show that, if the restrictions on the parameters are further relaxed these conclusions may fail. Other results concern separation of zeros and interlacing of zeros of Hahn polynomials with appropriately related degrees or parameter values.

3. *Nonroutine problems for naive programmers*, by Dmitri Thoro, San Jose State College.

Computer-oriented problems of a number-theoretic or combinatorial nature were considered; some led to a machine-language program, others a pencil and paper solution.

4. *Sifting for answers*, by D. H. Lehmer, University of California at Berkeley, (invited address).

5. *Predicativity: mathematics of the definable*, by Solomon Feferman, Stanford University, (invited address).

The meeting concluded with a panel discussion on "The Role of Axiomatics and Problem Solving." Panel members were E. G. Begle, Stanford University; H. M. Bacon, Stanford University; C. M. Larson, San Jose State College; L. D. Hawkinson, Gunn High School, Palo Alto; Leon Henkin, University of California at Berkeley.

G. L. ALEXANDERSON, *Secretary-Treasurer*

PROPOSED AMENDMENT TO THE BY-LAWS OF THE MAA

At the meeting of the Board of Governors held on January 25, 1967, in Houston, the Secretary was instructed to submit to a vote of the membership an amendment to Article V of the By-Laws, which will spell out more specifically the procedures for the creation and modification of Sections of the Association. In accordance with these instructions, a motion will be made at the business meeting of the Association to be held at the University of Toronto on Tuesday, August 29, 1967, to amend Article V of the By-Laws to read as follows:

ARTICLE V—SECTIONS

1. In the interest of more effective promotion of the objectives of the Association on a local level, the United States, Canada and their possessions shall be subdivided by the Board of Governors into non-overlapping geographical areas, and a Section of the Association shall be established in each of these areas. The subdivision into non-overlapping areas may be changed by the Board, upon recommendation by the Committee on Sections (see paragraph 7).

2. Each member of the Association residing in the United States, Canada or their possessions shall belong to one and only one Section. He will belong to the Section in whose geographic area he resides, except that a member who resides in one area and is employed in a different area may elect the Section to which he prefers to belong. Any member may petition the Committee on Sections for reassignment of his membership to another Section.

3. Each Section shall adopt a set of By-Laws which, along with any subsequent changes, must be approved by the Board. The geographic area covered by a Section shall be described in the By-Laws for the Section.

4. If there are members of the Association residing in a geographic area in which no Section has been organized, any ten or more members of this Association residing or employed in this area may petition the Board for authority to organize a Section covering that area.

5. A group of not less than twenty-five members of an existing Section may petition the Board to partition the area and the Section into two or more Sections. The Board shall have authority to approve or deny this petition. The Board may specify conditions under which such action may be accomplished. It may conduct a poll of some or all members of the Association in the Section to determine the advisability of allowing the proposed partition of the Section. If separate Sections are approved then each new Section must prepare its own set of By-Laws to be approved by the Board.

6. A group of not less than twenty-five members residing or employed in that part of the area of an existing Section which they desire to become part of another existing Section may petition the Board to redefine the geographic boundaries of the Sections affected. The Board shall have authority to approve or deny this petition. It may conduct a poll of some or all members of the Sections involved to determine the advisability of permitting such action.

7. There shall be a standing Committee on Sections through which the Board shall maintain general supervision over the activities of all Sections. This Committee, in particular, shall study all matters involving creation of Sections or modification of boundaries of Sections and make appropriate recommendations to the Board.

8. The Association shall not be obligated to pay from its treasury any of the expenses of a Section except as the Board may provide.

HENRY L. ALDER, *Secretary*

THE EMPLOYMENT REGISTER

During the summer meeting at the University of Toronto, the Mathematical Sciences Employment Register will be open for the scheduling of interviews between applicants and employers.

The Employment Register, which is sponsored by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be located on the first floor of Sidney Smith Hall, and will be open from Tuesday, August 29, through Thursday, August 31, from 9:00 A.M. to 5:00 P.M. on each of the three days.

Registration for the Employment Register is separate and apart from meeting registration, and it is, therefore, most important that both applicants and employers sign in at the Employment Register Desk as early as they can on Tuesday morning. A separate visual index will be maintained for Employment Register use only. Appointments may be requested only with applicants and employers who have actually signed in at the Register.

There is no charge for registration except when the late registration fee of $5.00 for employers is applicable. Provision will be made for anonymity of applicants upon payment of $5.00 to defray the cost involved in handling such a listing.

Applicants and employers who wish to be listed should write to the Mathematical Sciences Employment Register, P.O. Box 6248, Providence, Rhode Island 02904, for applicant qualification forms or position description forms. These forms must be completed and returned to the Employment Register not later than July 15, 1967, in order to be included in the August lists. Those forms which arrive too late to be included in the printed lists are taken to the meeting where they may be seen by applicants and/or employers who are interested in them. The printed lists will be mailed to subscribers during the first week in August. Lists can be ordered from the Employment Register office in Providence. They will also be available at the meeting.

A subscription to the lists, which includes three issues (January, May, and August) of both the applicants list and the positions list is available for $15.00 a year; individual issues of both lists may be purchased for $7.50; single copies of the positions list only may be purchased for $3.00. Checks should be made payable to the American Mathematical Society and sent to the address given above.

CALENDAR OF FUTURE MEETINGS

Forty-eighth Summer Meeting, University of Toronto, Toronto, Ontario, Canada, August 28–30, 1967.

Fifty-first Annual Meeting, San Francisco, California, January 25–27, 1968.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Mississippi State College for Women, Columbus, February 16–17, 1968.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY
NORTHEASTERN, Mt. Allison University, Sackville, New Brunswick, June 23–24, 1967.
NORTHERN CALIFORNIA
OHIO
OKLAHOMA-ARKANSAS
PACIFIC NORTHWEST, University of Montana, Missoula, June 16–17, 1967.
PHILADELPHIA, University of Delaware, Newark, November 18, 1967.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Loyola University of Los Angeles, March 9, 1968.
SOUTHWESTERN, New Mexico State University, University Park, Spring 1968.
TEXAS
UPPER NEW YORK STATE
WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, New York, N. Y., December 26–31, 1967.
AMERICAN MATHEMATICAL SOCIETY, Toronto, Ontario, Canada, August 29–Sept. 1, 1967.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Michigan State University, June 19–22, 1967.
ASSOCIATION FOR COMPUTING MACHINERY, Sheraton-Park, Washington, D. C., August 29–31, 1967.
ASSOCIATION FOR SYMBOLIC LOGIC
CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 23–25, 1967.
INSTITUTE OF MATHEMATICAL STATISTICS
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, McGill University, Montreal, August 21–24, 1967.
OPERATIONS RESEARCH SOCIETY OF AMERICA
PI MU EPSILON, Toronto, Ontario, Canada, August 29–30, 1967.
SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D. C., June 12–15, 1967. (Symposiums on applied probability and fluid dynamics.)



A Modern Approach to Mathematics . . .

CALCULUS WITH ANALYTIC GEOMETRY AND LINEAR ALGEBRA

by LEOPOLDO V. TORABALLA, *New York University*

This new text, designed for use in an introductory undergraduate course, deals with the standard material of elementary calculus. In accordance with recent proposals of the Committee on the Undergraduate Program in Mathematics, which state, "The basic concepts should be introduced in the same spirit in which they are used by working mathematicians, and proofs ought to have the same clarity and elegance which distinguishes all first rate mathematics," the author uses a thorough and direct approach on the level of rigor demanded by modern mathematics.

The book begins with a thorough postulational presentation of the real number system and follows this with a well-motivated, simple, yet logically adequate development of the topology of the space of real numbers.

SPECIAL FEATURES:

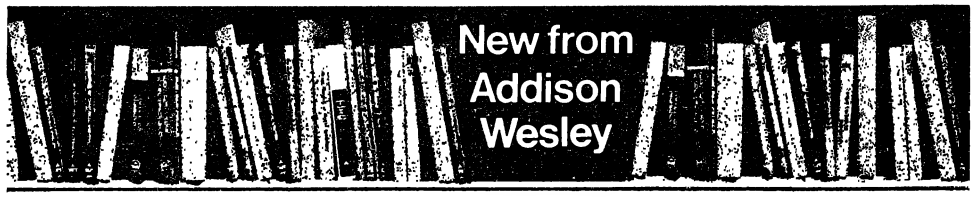
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THE OFFICIAL JOURNAL OF
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JUNE-JULY 1967

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$6.00 (\$8.00 in 1968). For nonmembers the subscription price is \$10.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

ON A THEOREM IN GEOMETRY

DANIEL PEDOE, University of Minnesota

If four circles in a plane touch each other externally, and if ϵ_1, ϵ_2, ϵ_3, and ϵ_4 denote their curvatures (that is, the reciprocals of their radii), then the following relation holds between them:

$$2(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2) = (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)^2.$$

A statement of this theorem, without proof, but with a descriptive poem attached, appeared in *Nature*, an English scientific journal, in 1936 [12]. The author was Frederick Soddy, distinguished by his pioneering work on isotopes. A year later a generalization of the theorem to $n+2$ spheres in mutual contact, lying in Euclidean space of n dimensions, was published by Thorold Gossett in the same journal [7]. Again, no proof was given, but an extra verse for the Soddy poem was suggested.

The title of the Soddy poem, "The kiss precise," is probably enough to indicate its flavour. Coxeter [3 p. 15] quotes a complete verse.

(It is perhaps worth noting, as a warning to aspiring poetasters, that *Nature* is averse to *belles lettres* these days. A few years ago Sir Alexander Oppenheim submitted the statement of an attractive theorem in geometry, accompanied by a graceful descriptive poem which my wife had written at his suggestion, but *Nature* rejected the joint offering.)

Two proofs of what, for the moment, we shall call the Soddy theorem have appeared in textbooks. One is elementary, uses trigonometry, and does not lend itself to a possible generalization [Coxeter, **3**, p. 15]. The second is apparently simple, but it is not elementary, and uses the methods of the Grassmann calculus as applied to circles by Homersham Cox in 1883 [2]. But this proof in Forder's book [6] can be generalized. We shall return to it later. There is no reference to Soddy in the Forder proof, and in fact the theorem just occurs as one of a number of worked examples, on p. 339.

The Soddy theorem is an interesting one because it can be viewed from a number of different mathematical angles. We approach it heuristically in the first instance, using arguments which may need amplification, but which do suggest possible lines of attack on the proof.

We do not know *a priori* that a relationship of the Soddy type holds between four circles which touch each other. We do know, from the Apollonius theorem, that eight circles touch three given circles in general position [Pedoe, **11**, p. 23]. When these circles are specialized, some of the eight touching circles may coincide, and, as in all enumerative problems, the coincident solutions of the problem must be counted with a certain multiplicity [Hodge and Pedoe, **9**, vol. II, p. 127].

Suppose now that the three given circles C_1, C_2, C_3 all touch each other. Then since C_1 touches itself, it is a solution of the Apollonius problem. So is C_2, and

so is C_3. The multiplicity with which each of these circles must be counted in the solution of the Apollonius problem is (a) greater than 1, and (b) the same for each circle. Let it be n. Then since $3n$ must not exceed 8, we must have $n=2$, and therefore we expect only two other circles to touch three circles which touch each other.

If there is an algebraic relation connecting the curvatures of four circles which touch each other, the above reasoning shows that it must be in the form of a quadratic polynomial equated to zero. In fact, it must be a quadratic *form*, since if we multiply the curvatures of three of the circles by a factor k, by a similarity transformation, the curvature of the fourth circle increases likewise by a factor k. Finally, the form must remain unchanged if we interchange any two of $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$, the curvatures of the four circles. It must therefore equal

$$p\,(\sum \epsilon_i^2) + q \sum \epsilon_i\epsilon_j \quad (1 \leqq i < j \leqq 4).$$

Now, if two circles touch each other externally, and t is the length of a common tangent, we have at once

$$t^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2 = 4r_1r_2,$$

where r_1 and r_2 are the radii. Let three circles of radii r_1, r_2 and r_3 respectively touch each other, and a given line at P, Q and R respectively. Then we have

$$PQ^2 = 4r_1r_2, \quad QR^2 = 4r_2r_3, \quad \text{and} \quad PR^2 = 4r_1r_3.$$

Since $PQ \pm QR = PR$, this leads to

$$1/\sqrt{r_3} \pm 1/\sqrt{r_1} = 1/\sqrt{r_2},$$

which, in terms of the curvatures is $\sqrt{\epsilon_1} \pm \sqrt{\epsilon_2} \pm \sqrt{\epsilon_3} = 0$. This, in rational form, is

$$\sum \epsilon_i^2 - 2 \sum \epsilon_i\epsilon_j = 0 \quad (1 \leqq i < j \leqq 3).$$

Now, the line has curvature $\epsilon_4 = 0$, and this relationship in the case $\epsilon_4 = 0$ indicates (a) that there is an algebraic relation between the four curvatures ϵ_i $(i=1, \cdots, 4)$ of four circles which touch each other and (b) it must be

$$\sum \epsilon_i^2 - 2 \sum \epsilon_i\epsilon_j = 0 \quad (1 \leqq i < j \leqq 4),$$

and this is the Soddy theorem.

This method can be generalized, if we use the theorem that there is a polynomial relation between the squares of the mutual distances of $n+2$ points in Euclidean space of n dimensions. In two dimensions the relation for four points P_i $(i=a, b, c, d, 4)$ can be written:

$$\sum (P_aP_b)^2(P_bP_c)^2(P_cP_d)^2 - \sum (P_aP_b)^2(P_bP_c)^2(P_cP_a)^2 - \sum (P_aP_b)^2(P_cP_d)^4 = 0,$$

where the summations are over all permutations of unordered pairs, and the first summation contains 12 terms, the second 4, and the third 6. This relation

expresses the fact that the volume of the tetrahedron formed by the points is zero, and can be traced back to L. N. M. Carnot [Guggenheimer, **8**].

We can use this relation to indicate the form of the Soddy theorem for five spheres in mutual contact in Euclidean space of three dimensions. We assume that one sphere is a plane, so that the other four spheres touch this plane, and, as above, using the formula $4(P_aP_b)^{-2}=\epsilon_a\epsilon_b$, which is also true for spheres of curvatures ϵ_a, ϵ_b touching a line at P_a, P_b, we obtain the relation

$$\sum \epsilon_i^2 - \sum \epsilon_i\epsilon_j = 0 \quad (1 \leqq i < j \leqq 4),$$

which indicates that the Soddy relation for 5 spheres in mutual contact in Euclidean space of 3 dimensions is

$$\sum \epsilon_i^2 - \sum \epsilon_i\epsilon_j = 0 \quad (1 \leqq i < j \leqq 5),$$

or

$$3\left(\sum \epsilon_i^2\right) = \left(\sum \epsilon_i\right)^2 \quad (i = 1, \cdots, 5).$$

It is worth noting that if we attempt to generalize the method used for circles to show that two spheres touch each of four spheres (in space of 3 dimensions) which are in mutual contact, we run into a serious snag. We may assume, from algebraic considerations that the number of spheres which touch four spheres in general position in space of 3 dimensions is $2^4=16$. If the four given spheres touch each other, and each counts with multiplicity n in the total count, we have $4n+x=16$, where we know that n exceeds 1, and we should like x to equal 2. This will not work. The reason why this method breaks down is that the Soddy theorem is a theorem of *real* Euclidean space, and we cannot expect an algebraic treatment, which gives complex solutions, always to fit the real case.

If we take four sufficiently small spheres round the vertices of a tetrahedron, they will be touched by a sphere that approximates to the circumsphere of the tetrahedron. Noting that each small sphere can lie inside or outside the approximating circumsphere, we see that there are 16 such Apollonian spheres. If the algebraic details are carried out (Prof. Coxeter has done this), it is found that when the radii of the small spheres increase steadily towards the situation of mutual contact, 6 of the Apollonian spheres cease to be real, and 8 of the remaining 10 coincide in pairs with the 4 given spheres, leaving 2 real spheres which touch the 4 given spheres. We wish to find these without having recourse to the algebraic manipulation.

We recollect that our derivation of the Soddy theorem in the plane used properties of the real line. We accordingly begin our investigations again, using a method that proves the Soddy theorem in the plane, generalizes to n dimensions, and makes essential use of real Euclidean space.

To show that two circles touch three given circles in a plane which already touch each other, we invert, with center of inversion at a point of contact. The

inverse figure consists of two parallel lines, touched by a circle. Any circle which touches the two lines must have the same radius as the circle already touching the lines, and there are two such circles which touch the circle already touching, *one on each side* of this circle.

We observe that the quadratic equation

$$p(\sum \epsilon_i^2) + q \sum \epsilon_i \epsilon_j = 0 \quad (1 \leqq i < j \leqq 4),$$

must be such that if $\epsilon_3 = \epsilon_4 = 0$ (the two lines each having curvature zero) the equation is identically satisfied by $\epsilon_1 = \epsilon_2$, since the circles have the same radius. Hence

$$p(2\epsilon_1^2) + q(\epsilon_1^2) = 0;$$

we may take $p = 1$ and $q = -2$, and obtain the Soddy theorem.

For spheres in three dimensions, we suppose that we are given four spheres which touch each other, and we wish to find how many spheres touch these four spheres. Inverting with respect to a point of contact of two of the spheres, we obtain two parallel planes, each of which is touched by spheres C_1 and C_2 which touch each other. Any sphere which touches the two planes must have radius equal to that of the equal spheres C_1 and C_2. We can construct two equilateral triangles, of which one side is the join of the centers of C_1 and C_2, the triangles lying in the plane through the centers parallel to the two parallel planes in the figure, and the new vertices, *on either side* of the join of the centers, are centers of spheres which touch the two planes and C_1, C_2.

Again the properties of real Euclidean space have been used, and we deduce that there is a form

$$p(\sum \epsilon_i^2) + q \sum \epsilon_i \epsilon_j = 0 \quad (1 \leqq i < j \leqq 5)$$

connecting the curvatures of five spheres which touch each other in real Euclidean space of three dimensions. Again, this form is to be identically satisfied if we put $\epsilon_4 = \epsilon_5 = 0$, and $\epsilon_1 = \epsilon_2 = \epsilon_3$. This leads to

$$p(3\epsilon_1^2) + q(3\epsilon_1^2) = 0;$$

we may take $p = 1$, $q = -1$, and deduce once more that the Soddy theorem in three dimensions is

$$3 \sum \epsilon_i^2 = (\sum \epsilon_i)^2 \quad (i = 1, \cdots, 5).$$

The extension to real Euclidean space of n dimensions is immediate. We assume that we have $n+1$ spheres which touch each other, and we invert with respect to a point of contact of two spheres. The inverse figure consists of two parallel hyperplanes, each of which is touched by $n-1$ spheres which also touch each other. These spheres must all have the same radius, and any sphere which touches the two hyperplanes also has this radius. If such a sphere also touches

the $n-1$ equal spheres, its center must be at one of two points, and these points are mirror images of each other in the flat space of $n-2$ dimensions spanned by the centers of the $n-1$ spheres. Each of the two centers completes, with the $n-1$ centers, a regular simplex in the unique $(n-1)$-dimensional space which contains the $n-1$ centers and is parallel to the two hyperplanes.

If the curvatures of the $n+2$ spheres in real Euclidean n-space are $\epsilon_1, \cdots, \epsilon_{n+2}$, then if all these spheres touch each other, we have a relation

$$p(\sum \epsilon_i^2) + q(\sum \epsilon_i\epsilon_j) = 0, \quad (1 \leqq i < j \leqq n+2).$$

If we put $\epsilon_{n+1} = \epsilon_{n+2} = 0$, this relation is identically satisfied by $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n$, so that

$$pn\epsilon_1^2 + q\binom{n}{2}\epsilon_1^2 = 0,$$

and this is satisfied by taking $p = n-1$, and $q = -2$, so that the statement of the Soddy theorem for n dimensions is

$$(n-1)(\sum \epsilon_1^2) - 2(\sum \epsilon_i\epsilon_j) = 0, \quad (1 \leqq i < j \leqq n+2)$$

or

$$n(\sum \epsilon_i^2) = (\sum \epsilon_i)^2 \quad (i = 1, \cdots, n+2).$$

This is the generalization given by Gossett without proof [7].

It was at this point in my investigations that I received a postcard from Prof. Makowski of Warsaw, Poland. He had seen a short reference to my work in *Mathematical Notices*, and his card drew my attention to a paper in Danish by David Fog entitled "The position of a point with reference to certain regular point sets" [5]. This paper was inspired by a paper by M. E. Wise [14] entitled "On the radii of five packed spheres in mutual contact." While working for a Pneumoconiosis Research Unit in Penarth, Wales, Wise had investigated the radii of five spheres in mutual contact, and had discovered the Soddy theorem in three dimensions without knowing about the theorem in two dimensions!

Wise uses inversion, and so does Fog, and as far as this goes both procedures are exactly the same as my own. But Fog then obtains the Soddy theorem in n dimensions by obtaining the polynomial relation which connects the squares of the distances of $n+2$ points in real Euclidean space of n dimensions in the special case when $n+1$ of the points form a regular simplex. Our previous work shows how a regular simplex enters on the scene. Both the Wise and Fog papers are excellent. Let us see how inversion produces the original Soddy theorem in two dimensions without any assumptions about polynomial relations. The proof will illustrate the mechanism underlying the Wise proof for three dimensions.

We have four circles in mutual contact, touching externally, and we are go-

ing to invert the figure, choosing the center of inversion at the point of contact of two of the circles. With this point as origin of coordinates, let the equations of the two circles be

$$(x - 1/\epsilon_1)^2 + y^2 - 1/\epsilon_1^2 = 0,$$

and

$$(x + 1/\epsilon_2)^2 + y^2 - 1/\epsilon_2^2 = 0.$$

Take the circle of inversion as

$$x^2 + y^2 - 1 = 0.$$

The inverse of each circle is merely the chord of intersection with the circle of inversion, so that the inverses are

$$x = \epsilon_1/2 \quad \text{and} \quad x = -\epsilon_2/2,$$

two parallel lines. The remaining two circles invert into two circles which touch these lines and each other. Their centers must lie on the line $x=(\epsilon_1-\epsilon_2)/4=p$, say, and the radius of each of these circles is

$$r = (\epsilon_1 + \epsilon_2)/4.$$

If one of these circles is $(x-p)^2+(y-q)^2-r^2=0$, it is the inverse of the circle

$$x^2 + y^2 - \frac{2(px + qy)}{p^2 + q^2 - r^2} + \frac{1}{(p^2 + q^2 - r^2)^2} = 0,$$

so that

$$\frac{1}{\epsilon_3^2} = \frac{p^2 + q^2}{(p^2 + q^2 - r^2)^2} - \frac{1}{(p^2 + q^2 - r^2)} = \frac{r^2}{(p^2 + q^2 - r^2)^2},$$

and therefore

$$\pm r\epsilon_3 = p^2 + q^2 - r^2.$$

Since $p^2+q^2-r^2=-\epsilon_1\epsilon_2/4+q^2$, we assume that q is sufficiently large for r to be positive. We now have

$$r\epsilon_3 = p^2 + q^2 - r^2, \tag{1}$$

and therefore also

$$r\epsilon_4 = p^2 + (q + 2r)^2 - r^2, \tag{2}$$

since the center of the fourth *inverted* circle is at a distance $2r$ from that of the third inverted circle, touching it externally.

All we have to do now is to eliminate q from (1) and (2) to obtain the Soddy theorem. We note, however, that (1) is merely a statement of what should be a well-known formula in inversion:

$$(R/R')^2 = (D^2 - R^2)^2, \tag{3}$$

which gives the radius R' of the inverse of a circle of radius R in a unit circle, D being the distance of the center of inversion from the center of the circle of radius R. Using this formula, (1) and (2) are obtained immediately.

To eliminate q, subtract (2) from (1), obtaining

$$r(\epsilon_3 - \epsilon_4) = -4qr - 4r^2;$$

substituting for q^2 in (1) gives

$$16(r\epsilon_3 + r^2 - p^2) = 16q^2 = (4r + \epsilon_3 - \epsilon_4)^2;$$

the substitution of $p = (\epsilon_1 - \epsilon_2)/4$ and $r = (\epsilon_1 + \epsilon_2)/4$ in this equation leads directly to the Soddy theorem. This is probably the simplest elementary proof for the case $n = 2$.

For the case $n = 3$, we should find after inversion that we need a relation between the distances of a given point from the vertices of an equilateral triangle of given side, the point being at a given distance from the plane of the triangle, and so on. This is the method of Wise, extended to n dimensions by Fog.

The assertion we have not yet justified is that there exists an algebraic relation between the curvatures of four circles in a plane which touch each other. This is derived immediately from the identical relation which we have already used connecting the lengths d_{ij} of the six joins P_iP_j of the four centers P_i $(i = 1, \cdots, 4)$ if we put

$$d_{ij} = r_i + r_j,$$

assuming the circles all touch each other externally. The identical relation occurs most naturally in determinantal form as

$$\begin{vmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 & 1 \\ d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 & 1 \\ d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 & 1 \\ d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 & 1 \end{vmatrix} = 0,$$

(see, for instance, Kowalesky [10], p. 242). By simple manipulation this determinant can be reduced to the form

$$\begin{vmatrix} -1 & 1 & 1 & 1 & \epsilon_1 \\ 1 & -1 & 1 & 1 & \epsilon_2 \\ 1 & 1 & -1 & 1 & \epsilon_3 \\ 1 & 1 & 1 & -1 & \epsilon_4 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & 0 \end{vmatrix} = 0,$$

where $\epsilon_i = r_i^{-1}$, and the existence of an algebraic relation between the ϵ_i is estab-

lished. Moreover it is clearly a symmetric quadratic form in the ϵ_i. This method extends immediately to the case of $n+2$ spheres in mutual contact in Euclidean space of n dimensions, leading to the relation

$$\begin{vmatrix} -1 & 1 & 1 & \cdots & 1 & \epsilon_1 \\ 1 & -1 & 1 & \cdots & 1 & \epsilon_2 \\ 1 & 1 & -1 & \cdots & 1 & \epsilon_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & -1 & \epsilon_{n+2} \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \cdots & \epsilon_{n+2} & 0 \end{vmatrix} = 0,$$

and our previous methods show that the expanded form of this determinant is

$$n \sum \epsilon_i^2 - \left(\sum \epsilon_i\right)^2 = 0 \quad (i = 1, \cdots, n+2).$$

This method of showing there is an algebraic relation between the ϵ_i is due to the late Prof. A. Aeppli [1]. His method for obtaining the coefficients in the expansion of the determinant differs from that given above. I am indebted to his son, my colleague Prof. Alfred Aeppli, for drawing my attention to his father's paper, and also to the fact that his father had traced the theorem we have been discussing (for the case $n=2$) back to Descartes! It appears that Descartes had mentioned the theorem in a letter to the Princess Elizabeth (p. 49 of [4]). The theorem was also known to Steiner (p. 63 of Vol. 1 of [13]). Evidently Soddy revived the theorem, and certainly obtained a lot of publicity for it, but this is hardly enough for a lasting claim to be laid to the theorem, and we shall now refer to the two-dimensional case as "The Descartes circle theorem," and to the n-dimensional case as "The extension of the Descartes circle theorem." If the theorem is to be named after everyone who has rediscovered it, we should add the names of Steiner, Wise and others. Perhaps one day a Mathematical Council will meet and decide on such matters!

We now approach the configuration of four circles in mutual contact in a plane from another point of view. We represent the circles by points in Euclidean space of three dimensions, E_3. This representation is fully worked out in my book [11]. We give a few details here since the same representation will be of use in our next look at the Descartes circle theorem, using the Grassmann calculus.

A normalized circle, with coefficient of x^2+y^2 unity,

$$C \equiv x^2 + y^2 - 2\xi x - 2\eta y + \zeta = 0$$

is represented by the point (ξ, η, ζ) in E_3. This gives a mapping, if we use ordinary rectangular cartesian coordinates, with the point (ξ, η, ζ) immediately above the center of the circle, or, less informally, the center of the circle is the orthogonal projection of the representative point onto the (x, y)-plane. Circles

of radius zero (called point-circles) are mapped on the points of the paraboloid of revolution

$$\Omega \equiv z - x^2 - y^2 = 0.$$

Circles of imaginary radius are mapped onto points which lie inside Ω, and are treated equally with ordinary circles, which are mapped onto points outside or on Ω. Circles of the coaxal system $\lambda_1 C_1 + \lambda_2 C_2 = 0$ are mapped onto the points of the line in E_3 which joins the points representing C_1 and C_2. The real limiting points of the coaxal system, if they exist, are given by the orthogonal projection of the points of intersection of this line with Ω. If C_1 and C_2 touch, this line *touches* Ω.

If we erect parallels to the z axis at the points of the circle C, these will intersect Ω in a plane section, of equation

$$z - 2\xi x - 2\eta y + \zeta = 0.$$

This is the *polar plane* of the representative point (ξ, η, ζ) with respect to the quadric Ω. Two points (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) are conjugate with regard to Ω, so that

$$2\xi_1\xi_2 + 2\eta_1\eta_2 - \zeta_1 - \zeta_2 = 0$$

if and only if the circles C_1 and C_2 which they represent are *orthogonal.* The point circles on the circumference of C are orthogonal to C. Hence we have a representation of circles by plane sections of Ω, or by points, the *poles* of the plane sections with respect to Ω.

Finally, inversion of circles in a fixed circle C in the plane is represented in E_3 by a *harmonic homology* of the points of E_3, the center of the homology being the point which represents C, and the plane of fixed points being the polar of this point with respect to Ω. The points of Ω which lie in pairs on lines through the center of the harmonic homology are interchanged by the mapping, and project into inverse points in C.

Now, four circles in the plane which touch each other are mapped onto the four vertices of a tetrad in E_3, the sides of which all touch Ω. Harmonic homology applied to such a tetrad produces a tetrad with similar properties in relation to Ω, and there must be some invariant of such a tetrad under a harmonic homology which is the expression of the Descartes theorem for circles in a plane. The discovery of such an invariant would perhaps produce the most satisfactory proof of the Descartes theorem, but, failing this, let us use this theorem to produce a theorem for a tetrad whose six edges all touch a paraboloid of revolution!

We note that if the points A, B of E_3 represent the circles C_1, C_2 of radii ϵ_1^{-1}, ϵ_2^{-1}, which we suppose touch externally at P, then P is the orthogonal projection of the point X on AB where the line AB touches Ω, and we have the simple relation $AX:XB = \epsilon_1^{-1}:\epsilon_2^{-1} = AP:PB$. If we write the Descartes relation as

$$2\left\{1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 + \left(\frac{\epsilon_3}{\epsilon_1}\right)^2 + \left(\frac{\epsilon_4}{\epsilon_1}\right)^2\right\} = \left(1 + \frac{\epsilon_2}{\epsilon_1} + \frac{\epsilon_3}{\epsilon_1} + \frac{\epsilon_4}{\epsilon_1}\right)^2,$$

and A, B, C and D represent the four circles, where AB touches Ω at X, AC touches Ω at Y and AD touches Ω at Z, using the relations $AX:XB=\epsilon_1^{-1}:\epsilon_2^{-1}$, $AY:YC=\epsilon_1^{-1}:\epsilon_3^{-1}$, and $AZ:ZD=\epsilon_1^{-1}:\epsilon_4^{-1}$, we obtain the relation:

$$2\left\{1+\left(\frac{AX}{XB}\right)^2+\left(\frac{AY}{YC}\right)^2+\left(\frac{AZ}{ZD}\right)^2\right\}=\left(1+\frac{AX}{XB}+\frac{AY}{YC}+\frac{AZ}{ZD}\right)^2,$$

connecting the vertices A, B, C, and D and the points of contact X, Y and Z of AB, AC and AD respectively of a tetrad $ABCD$ *all* of whose six edges touch a paraboloid of revolution.

If we had some simple way of proving *this* theorem, the Descartes circle theorem would follow immediately!

Our last look at the Descartes circle theorem is via the Grassmann calculus. The methods of Grassmann were extended and applied to circles by Homersham Cox in 1883 [2], and are given by Forder in his interesting book [6] on the Grassmann calculus. These methods have always smacked of the theologio-mathematical, since higher entities are invoked from time to time, and these can be persuaded to work apparent miracles! But as far as the applications lead to a simple proof of the Descartes theorem, we believe we can explain the calculus in terms of our representation of circles given in the previous section. The same methods will then lead to the generalization of the theorem to n dimensions.

We have seen how circles can be combined linearly, using the mapping:

$$C\equiv x^2+y^2-2\xi x-2\eta y+\zeta=0\rightarrow(\xi,\eta,\zeta).$$

We can also map a circle C on its plane section representation:

$$C\equiv x^2+y^2-2\xi x-2\eta y+\zeta=0\rightarrow z-2\xi x-2\eta y+\zeta=0,$$

and in the present context this is the more convenient mapping. These plane sections can also be combined linearly, but we note that since our circles are normalized, we are essentially using barycentric coordinates in E_3, and we can therefore find *four* linearly independent circles, but any set of five circles is linearly dependent. For instance, although we have not yet mentioned improper circles (of infinite radius), it is clear that the four circles $x=0$, $y=0$, $x^2+y^2-1=0$ and $x^2+y^2+1=0$ are linearly independent, and that any given circle can be expressed linearly in terms of this set of four circles.

There is no difficulty with improper circles. If we consider

$$C_1-C_2\equiv-2(\xi_1-\xi_2)x-2(\eta_1-\eta_2)y+\zeta_1-\zeta_2=0,$$

this line, the radical axis of the two circles, is represented without difficulty by the plane section of Ω which has the same equation, the plane being parallel to the z-axis. The point representation in this case breaks down. But if we consider the radical axis of two concentric circles, both representations break down. If the common center is (ξ_1,η_1),

$$C_1 - C_2 \equiv \zeta_1 - \zeta_2 = 0,$$

and there is no finite plane section representation. But we are accustomed to the paradoxical equation: *constant* $=0$ representing the plane at infinity, and we introduce this special plane section of Ω, and call it θ. We shall normalize it in a moment.

We observe that θ can always be used as one of the plane sections in E_3 which form a linearly dependent set with any given four planes. In other words, given any four circles, there always exists a linear relation between them of the form

$$x_1C_1 + x_2C_2 + x_3C_3 + x_4C_4 = \theta,$$

where θ is merely some definite non-zero constant.

We now introduce an *inner product* for two circles C_1, C_2. We write

$$[C_1 \cdot C_2] = \tfrac{1}{2}(2\xi_1\xi_2 + 2\eta_1\eta_2 - \zeta_1 - \zeta_2).$$

This looks familiar, and of course it arises very naturally from our point of view. Cox introduces it as

$$[C_1 \cdot C_2] = \tfrac{1}{2}\{r_1^2 + r_2^2 - (O_1 - O_2)^2\},$$

where r_1, r_2 are the respective radii, O_1 and O_2 the respective centers of the two circles. This is an equivalent expression (see [11], p. 32).

The inner product is bilinear and commutative. (This is true for normalized circles; some detail has been omitted here.) Hence its usefulness. It is zero for two orthogonal circles, and $[C_1 \cdot C_1]$, which we write C_1^2, is r_1^2 (hence the introduction of the factor 1/2). If C_1, C_2 touch externally

$$[C_1 \cdot C_2] = \tfrac{1}{2}(r_1^2 + r_2^2 - (r_1 + r_2)^2) = -r_1r_2,$$

and if C_1, C_2 are concentric, $[C_1 \cdot C_2] = (r_1^2 + r_2^2)/2$.

We have one more piece of machinery to set up, and then we can prove our theorem and its generalization. We investigate the inner product of θ with proper circles and with itself. If C_1 and C_2 are concentric,

$$C_1 - C_2 = \zeta_1 - \zeta_2 = k\theta,$$

where k has to be determined. This can be written

$$C_1 - C_2 = (\xi_1^2 + \eta_1^2 - \zeta_2) - (\xi_1^2 + \eta_1^2 - \zeta_1)$$

and we choose $k = (r_1^2 - r_2^2)/2$, thus normalizing θ as the constant -2. We now have $C_1 - C_2 = \tfrac{1}{2}(r_1^2 - r_2^2)\theta$. Hence

$$\begin{aligned}[C_1 \cdot (C_1 - C_2)] &= [C_1 \cdot (\tfrac{1}{2}(r_1^2 - r_2^2)\theta)] = C_1^2 - [C_1 \cdot C_2] \\ &= r_1^2 - (r_1^2 + r_2^2)/2 = (r_1^2 - r_2^2)/2.\end{aligned}$$

Since $[C_1\cdot(\frac{1}{2}(r_1^2-r_2^2)\theta)]=\frac{1}{2}(r_1^2-r_2^2)[C_1\cdot\theta]$, we deduce that

$$[C_1\cdot\theta] = 1$$

for any proper circle C_1. Finally, since

$$\begin{aligned}\tfrac{1}{4}(r_1^2 - r_2^2)^2\theta^2 &= [(C_1 - C_2)\cdot(C_1 - C_2)]\\ &= C_1^2 - 2[C_1\cdot C_2] + C_2^2\\ &= r_1^2 - (r_1^2 + r_2^2) + r_2^2 = 0,\end{aligned}$$

we find that $\theta^2=0$.

We reassure ourselves about this result by remembering that θ represents a special plane section of Ω, by the plane at infinity, and this touches Ω at infinity, Ω being a paraboloid, and the point of contact, being on Ω, represents a zero circle, so that $C_1^2=r_1^2$ would lead us to expect that $\theta^2=0$. But we could hardly have proved the property this way!

If we weight our circles in advance by multiplying each of them by its curvature, the reciprocal of its radius, then two circles C_1, C_2 touching externally, have the inner product $[C_1\cdot C_2]=-1$ (instead of $-r_1r_2$), and $[C_1\cdot\theta]=1/r_1$ (instead of 1).

We can now prove the Descartes circle theorem. Take four weighted circles of curvatures ϵ_i $(i=1, \cdots, 4)$ which touch each other externally. We can write, the weighted circles being C_i $(i=1, \cdots, 4)$,

$$x_1C_1 + x_2C_2 + x_3C_3 + x_4C_4 = \theta.$$

Forming the inner product of both sides with C_1, we have

$$x_1C_1^2 + x_2[C_1\cdot C_2] + x_3[C_1\cdot C_3] + x_4[C_1\cdot C_4] = [C_1\cdot\theta],$$

that is $x_1-x_2-x_3-x_4=\epsilon_1$. Similarly we obtain three more equations:

$$\begin{aligned}-x_1 + x_2 - x_3 - x_4 &= \epsilon_2,\\ -x_1 - x_2 + x_3 - x_4 &= \epsilon_3,\\ -x_1 - x_2 - x_3 + x_4 &= \epsilon_4.\end{aligned}$$

We now form the inner product with θ, and we obtain

$$\epsilon_1x_1 + \epsilon_2x_2 + \epsilon_3x_3 + \epsilon_4x_4 = 0.$$

From the first four equations we find that

$$\sum \epsilon_i^2 = 4\sum x_i^2, \quad \text{and} \quad \sum \epsilon_i\epsilon_j = 4\sum x_ix_j.$$

But, substituting for the ϵ_i from the first four equations in the last we find that

$$\sum x_i^2 - 2\sum x_ix_j = 0,$$

so that we have

$$\sum \epsilon_i^2 - 2 \sum \epsilon_i \epsilon_j = 0 \quad (1 \leqq i < j \leqq 4),$$

and this is the Descartes circle theorem.

The method of representing circles by points in E_3 can evidently be extended to the representation of spheres in E_n by points in E_{n+1}. The details correspond exactly, and we can at once say that if we have $n+2$ weighted spheres touching each other externally in E_n, we can write

$$x_1 C_1 + x_2 C_2 + \cdots + x_{n+2} C_{n+2} = \theta,$$

and forming the inner products we have a set of equations

$$\begin{aligned} x_1 - x_2 - \cdots - x_{n+2} &= \epsilon_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ -x_1 - x_2 - \cdots + x_{n+2} &= \epsilon_{n+2}, \end{aligned}$$

and the equation

$$\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_{n+2} x_{n+2} = 0.$$

The subsequent procedures are the same; merely the coefficients are different. We obtain

$$\sum \epsilon_i^2 = (n+2) \sum x_i^2 + (2n-4) \sum x_i x_j,$$

$$\sum \epsilon_i \epsilon_j = (n+1)(\tfrac{1}{2}n - 1) \sum x_i^2 + (2n+2) \sum x_i x_j,$$

from the first set of equations, and substituting in the final equation for the ϵ_i gives again

$$\sum x_i^2 - 2 \sum x_i x_j = 0,$$

so that now we have

$$(n-1) \sum \epsilon_i^2 - 2 \sum \epsilon_i \epsilon_j = 2n \sum x_i^2 - 4n \sum x_i x_j = 2n\left(\sum x_i^2 - 2 \sum x_i x_j \right) = 0,$$

so that the Descartes theorem in n dimensions is

$$n \sum \epsilon_i^2 = \left(\sum \epsilon_i \right)^2, \qquad (i = 1, \cdots, n+2).$$

In conclusion, it may be said of geometry, as has been said of other more popular and less abstract entities, that it is a many-splendoured thing. In fact, since geometry is not at all easy to define, we might hazard the definition that any part of mathematics which can be regarded interestingly from more than one point of view qualifies to be regarded as geometry!

Part of this paper was written under NSF Grant GE-3848 to the University of Minnesota.

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MATRICES WITH SPECIFIED EIGENVALUES AND ASSOCIATED EIGENVECTORS, PROPER OR GENERALIZED

LOUIS BRAND, University of Houston

1. Introduction. We shall consider $n \times n$ matrices with elements in the field of reals and limit our discussion to the case when all eigenvalues are real. If λ_i is an eigenvalue and e_i an associated *proper eigenvector*, $Ae_i = \lambda_i e_i$ or

$$(A - \lambda_i I)e_i = 0. \tag{1}$$

We shall regard all members of the equivalence class formed by the scalar multiples of e_i as the same eigenvector.

For a multiple eigenvalue λ_1 various cases arise which depend on the rank of the matrix $A - \lambda_1 I$. If λ_1 is k-tuple and $\operatorname{rank}(A - \lambda_1 I) = n - 1$, there will be a single proper eigenvector e_1 that satisfies (1) and $k-1$ *generalized eigenvectors* $e_2, e_3, \cdots, e_k$ which satisfy the $k-1$ equations

$$(A - \lambda_1 I)e_i = e_{i-1}, \qquad i = 2, 3, \cdots, k. \tag{2}$$

We shall show that these equations are consistent, obtain explicit expressions for $e_1, e_2, \cdots, e_k$, and show that these vectors are linearly independent.

When λ_1 is k-tuple and $\operatorname{rank}(A - \lambda_1 I) = n - m$, there are m proper and $k - m$ generalized eigenvectors which belong to λ_1. In the extreme case when rank

$(A-\lambda_1 I)=0$, $A=\lambda_1 I$ and any n linearly independent vectors of n-space are proper eigenvectors.

We shall first describe a procedure for computing eigenvectors which is a generalization of the method used for the companion matrix [1].

2. Computation of Eigenvectors. The eigenvalues λ_i of a matrix A are the zeros of the function

$$f(\lambda) = \det(A - \lambda I). \tag{3}$$

If λ is not an eigenvalue, $f(\lambda)\neq 0$ and the equation

$$(A - \lambda I)e(\lambda) = [f(\lambda), 0, \cdots, 0]^{\mathsf{T}} \tag{4.1}$$

has a unique column vector $e(\lambda)$ as solution whose components are *polynomials* in λ. We shall omit the transpose symbol T and denote a column by a row in square brackets. If the eigenvalues are all simple, a single eigenvector belongs to each eigenvalue and is given by $e(\lambda_i)$ provided $e(\lambda_i)\neq 0$; for when $\lambda=\lambda_i$, equation (4.1) reduces to (1).

If $f(\lambda)$ is chosen as the jth component (instead of the first) the equation is numbered as (4.j). All of the n equations (4.1) to (4.n) will yield the same eigenvector $e(\lambda_i)$ when λ_i is a simple zero provided $e(\lambda_i)\neq 0$. The computation is shown in Example 1.

If λ_1 is a k-tuple zero of $f(\lambda)$ and rank $(A-\lambda_1 I)=n-1$, then there is one proper eigenvector e_1 and $k-1$ generalized eigenvectors $e_2, e_3, \cdots, e_k$. These satisfy equation (4.1) and the $k-1$ equations obtained from it by differentiating $k-1$ times with respect to λ and then putting $\lambda=\lambda_i$. From (4.1) we have

$$\begin{aligned}
(A - \lambda I)e'(\lambda) - e(\lambda) &= [f'(\lambda), 0, \cdots, 0] \\
(A - \lambda I)e''(\lambda) - 2e'(\lambda) &= [f''(\lambda), 0, \cdots, 0] \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(A - \lambda I)e^{k-1}(\lambda) - (k-1)e^{k-2}(\lambda) &= [f^{(k-1)}(\lambda), 0, \cdots, 0].
\end{aligned} \tag{5}$$

Since λ_i is a k-tuple zero of $f(\lambda)$, all vectors on the right vanish when $\lambda=\lambda_i$ and yield the system

$$\begin{aligned}
(A - \lambda_i I)e'(\lambda_i) &= e(\lambda_i) \\
(A - \lambda_i I)\tfrac{1}{2}e''(\lambda_i) &= e'(\lambda_i) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(A - \lambda_i I)\frac{1}{(k-1)!}e^{k-1}(\lambda_i) &= \frac{1}{(k-2)!}e^{k-2}(\lambda_i).
\end{aligned} \tag{6}$$

A comparison with equation (2) now reveals that a set of generalized eigenvectors belonging to λ_i is given by

$$e_j = \frac{1}{(j-1)!}e^{j-1}(\lambda_i), \qquad j = 2, 3, \cdots, k. \tag{7}$$

The existence of this solution shows that equations (2) are consistent. The computation is illustrated in Example 2.

The linear independence of the set of eigenvectors may be shown by assuming the linear relation

$$c_1e_1 + c_2e_2 + \cdots + c_ke_k = 0$$

and premultiplying it $k-1$ times by the matrix $(A-\lambda_iI)$. We then find in turn that

$$\begin{aligned} &c_2e_1 + c_3e_2 + \cdots + c_ke_{k-1} = 0 \\ &\cdots\cdots\cdots\cdots\cdots \\ &c_{k-1}e_1 + c_ke_2 = 0 \\ &c_ke_1 = 0. \end{aligned}$$

Since $e_1 \neq 0$, $c_k = 0$; and on passing upward through these equations we find that all the constants $c_i = 0$.

When $\operatorname{rank}(A-\lambda_1 I) = n-m (m>1)$, the computation of m proper and $k-m$ generalized eigenvectors for k-tuple λ_1 is illustrated in Examples 2 and 4.

The author believes that the foregoing procedure yields all the eigenvectors, proper and generalized, but realizes that a strict proof cannot rest on examples, however diverse.

Example 1. For the symmetric matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad f(\lambda) = -(\lambda-1)(\lambda+1)(\lambda-2)$$

and 1, -1, 2 are the eigenvalues. Equation (4.1) becomes

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} e(\lambda) = \begin{bmatrix} f(\lambda) \\ 0 \\ 0 \end{bmatrix};$$

it has the unique solution

$$e(\lambda) = [\lambda^2 - \lambda - 1, \lambda - 1, 1]$$

and the mutually orthogonal eigenvectors are

$$e(1) = [-1, 0, 1], \qquad e(-1) = [1, -2, 1], \qquad e(2) = [1, 1, 1].$$

Equation (4.3) yields $e(\lambda) = (1, \lambda-1, \lambda^2-\lambda-1)$ and gives the same results. But (4.2) gives

$$e(\lambda) = (\lambda-1)[1, \lambda-1, 1],$$

$$e(1) = [0, 0, 0], \qquad e(-1) = [-2, 4, -2], \qquad e(2) = [1, 1, 1].$$

We cannot divide out $\lambda-1$ and then put $\lambda=1$ for $f(\lambda)/(\lambda-1)\neq 0$ when $\lambda=1$. See Example 3, in which this division is permissible.

Example 2. For the matrix

$$A=\begin{bmatrix} 3 & 2 & -1 \\ -3 & -2 & 2 \\ -1 & -1 & 2 \end{bmatrix}, \qquad f(\lambda) = -(\lambda-1)^3$$

and 1 is a triple eigenvalue. Equation (4.1),

$$\begin{bmatrix} 3-\lambda & 2 & -1 \\ -3 & -2-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{bmatrix} e(\lambda) = \begin{bmatrix} f(\lambda) \\ 0 \\ 0 \end{bmatrix}$$

has the unique solution $e(\lambda)=[\lambda^2-2,\ 4-3\lambda,\ 1-\lambda]$. Since $\operatorname{rank}(A-I)=2$, this yields the only proper eigenvector $e_1=e(1)=(-1,\ 1,\ 0)$. The two generalized eigenvectors are

$$e_2 = e'(1) = (2, -3, -1),$$
$$e_3 = \tfrac{1}{2}e''(1) = (1, 0, 0).$$

Example 3. For the matrix

$$A=\begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}, \qquad f(\lambda) = -(\lambda+1)(\lambda-1)^2,$$

and -1, 1, 1 are the eigenvalues. Equation (4.1)

$$\begin{bmatrix} -1-\lambda & 2 & -2 \\ 4 & -3-\lambda & 4 \\ 4 & -4 & 5-\lambda \end{bmatrix} e(\lambda) = \begin{bmatrix} f(\lambda) \\ 0 \\ 0 \end{bmatrix}$$

has the unique solution

$$e(\lambda) = (\lambda-1)[\lambda-1, 4, 4].$$

Moreover $E(\lambda)=e(\lambda)/(\lambda-1)=[\lambda-1,\ 4,\ 4]$ satisfies

$$(A-\lambda I)E(\lambda) = [1-\lambda^2, 0, 0]$$

and yields the proper eigenvectors

$$E(1) = 4[0, 1, 1], \qquad E(-1) = 2[-1, 2, 2].$$

Since $\operatorname{rank}(A-I)=1$, a 2-space of proper eigenvectors belongs to 1. One is given by equation (4.2), namely

$$E(\lambda) = \frac{e(\lambda)}{\lambda-1} = [2, \lambda-3, -4];$$

hence

$$E(1) = -2[-1, 1, 2], \qquad E(-1) = -2[-1, 2, 2].$$

All vectors in the space spanned by $[0, 1, 1]$ and $[-1, 1, 2]$ are eigenvectors for $\lambda = 1$. Another such vector is given by equation (4.3):

$$E(\lambda) = \frac{e(\lambda)}{\lambda - 1} = [-2, 4, \lambda + 5];$$

$$E(1) = 2[-1, 2, 3], \qquad E(-1) = 2[-1, 2, 2].$$

Example 4. For the matrix

$$A = \begin{bmatrix} 9 & 2 & -6 \\ 4 & 2 & -3 \\ 12 & 3 & -8 \end{bmatrix}, \qquad f(\lambda) = (1 - \lambda)^3,$$

and 1 is a triple eigenvalue. Equations (4.1), (4.2) and (4.3) yield respectively

$$e_1(\lambda) = (\lambda - 1)[\lambda + 7, 4, 12], \qquad E_1(\lambda) = [\lambda + 7, 4, 12];$$
$$e_2(\lambda) = (\lambda - 1)[2, \lambda, 3], \qquad E_2(\lambda) = [2, \lambda, 3];$$
$$e_3(\lambda) = (\lambda - 1)[6, 3, 10 - \lambda], \qquad E_3(\lambda) = [6, 3, 10 - \lambda].$$

When $\lambda = 1$, all give the same proper eigenvector:

$$E_1(1) = 4[2, 1, 3], \qquad E_2(1) = [2, 1, 3], \qquad E_3(1) = 3[2, 1, 3].$$

But since rank $(A - I) = 1$, there *must* be a second proper eigenvector e_3 and a generalized eigenvector $e_2 = e_1'$ corresponding to $e_1 = [2, 1, 3]$ which is given by

$$\tfrac{1}{4}E_1' = [\tfrac{1}{4}, 0, 0] = e_{21},$$

or $E_2' = [0, 1, 0] = e_{22}$, or $\tfrac{1}{3}E_3' = [0, 0, -\tfrac{1}{3}] = e_{23}$.

Since a generalized eigenvector satisfies (2), we have

$$(A - \lambda_1 I)e_{21} = e_1, \qquad (A - \lambda_1 I)e_{22} = e_1, \qquad (A - \lambda_1 I)(e_{21} - e_{22}) = 0,$$

that is, $e_{21} - e_{22}$ is a proper eigenvector. Thus we have three new proper eigenvectors

$$e_{21} - e_{22} = [\tfrac{1}{4}, -1, 0] = \tfrac{1}{4}[1, -4, 0],$$

$$e_{21} - e_{23} = [\tfrac{1}{4}, 0, \tfrac{1}{3}] = \tfrac{1}{12}[3, 0, 4],$$

$$e_{22} - e_{23} = [0, 1, \tfrac{1}{3}] = \tfrac{1}{3}[0, 3, 1].$$

Thus any of the vectors $[1, -4, 0]$, $[3, 0, 4]$, $[0, 3, 1]$ may be chosen for e_3 and all three lie in a plane with e_1.

The generalized eigenvector corresponding to $e_1 = [2, 1, 3]$ is most simply chosen as $e_2 = [0, 1, 0]$.

Example 5. When $A = I$, the unit matrix of order 3, $f(\lambda) = (1-\lambda)^3$ and equations (4.1), (4.2), (4.3) yield in turn, after rejecting the factor $(1-\lambda)^2$ since $1-\lambda$ still remains on the right,

$$e_1 = [1, 0, 0], \qquad e_2 = [0, 1, 0], \qquad e_3 = [0, 0, 1].$$

Since $\operatorname{rank}(A - I) = 0$, all vectors in the 3-space spanned by e_1, e_2, e_3 are proper eigenvectors.

3. Reduction to a Jordan Matrix. From the eigenvectors of a matrix, arranged in groups according to the distinct eigenvalues, we form the matrix

$$E = (e_1 | e_2 | \cdots | e_n) \tag{8}$$

having e_i as *columns*. Since the vectors e_i are linearly independent, they admit a reciprocal set $e^1, e^2, \cdots, e^n$ for which

$$e_i \cdot e^j = \delta_i^j, \qquad i, j = 1, 2, \cdots, n. \tag{9}$$

These equations show that the matrix having e^j as *rows* is the inverse of E:

$$E^{-1} = \begin{pmatrix} e^1 \\ \hline e^2 \\ \hline \cdot \\ \cdot \\ \hline e^n \end{pmatrix} \tag{10}$$

Then the product

$$E^{-1}AE = J \tag{11}$$

is the Jordan matrix for A corresponding to the grouping of eigenvectors in (8). The proof is immediate and is a direct consequence of equations (1), (2) and (9).

Thus in Example 1 ($\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$)

$$E^{-1}AE = \begin{pmatrix} e^1 \\ \hline e^2 \\ \hline e^3 \end{pmatrix} (e_1 | -e_2 | 2e_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In Example 2 ($\lambda_1 = \lambda_2 = \lambda_3 = 1$)

$$E^{-1}AE = \begin{pmatrix} e^1 \\ \hline e^2 \\ \hline e^3 \end{pmatrix} (e_1 | e_2 + e_1 | e_3 + e_2) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In Example 3 ($\lambda_1=-1$, $\lambda_2=\lambda_3=1$)

$$E^{-1}AE=\left(\begin{array}{c}e^1\\ \hline e^2\\ \hline e^3\end{array}\right)(-e_1\,|\,e_2\,|\,e_3+e_2)=\begin{pmatrix}-1&0&0\\0&1&1\\0&0&1\end{pmatrix}.$$

In Example 4 ($\lambda_1=\lambda_2=\lambda_3=1$) when e_3 is referred to e_1

$$E^{-1}AE=\left(\begin{array}{c}e^1\\ \hline e^2\\ \hline e^3\end{array}\right)(e_1\,|\,e_2+e_1\,|\,e_3)=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix};$$

and when e_3 is referred to e_2

$$E^{-1}AE=\left(\begin{array}{c}e^1\\ \hline e^2\\ \hline e^3\end{array}\right)(e_1\,|\,e_2\,|\,e_3+e_2)=\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}.$$

4. Matrix with specified eigenvalues and eigenvectors. Let an arbitrary set of linearly independent vectors e_i be chosen and assigned to arbitrary eigenvalues λ_i singly or in groups. In each group specify which vectors are to be proper and which generalized eigenvectors, and refer each generalized eigenvector to its proper companion. For each specific order and grouping we can construct a matrix A which has the eigenvalues and eigenvectors thus specified. This is rather surprising since the generalized eigenvectors must conform to equations (2). To achieve this result, let J be the Jordan matrix which corresponds to the order and grouping of the eigenvalues with due regard to the reference of generalized to proper eigenvectors. Then the required matrix A may be obtained at once from equation (11):

$$A = EJE^{-1}, \tag{12}$$

where E has its columns formed by the specified eigenvectors in the order corresponding to J. We note again that each generalized eigenvector must be referred to its proper companion. To be specific, if the proper e_1 has the generalized e_2 and e_3 referred to it, we write $e_2=e_1'$, $e_3=e_1''$.

Problem. Find a matrix A with the eigenvalues and eigenvectors

$$\begin{aligned}\lambda_1&=1, & e_1&=(2,0,-1,1)\\ \lambda_2&=1, & e_2&=(-1,1,1,1)=e_1'\\ \lambda_3&=1, & e_3&=(1,1,1,1)\\ \lambda_4&=2, & e_4&=(2,0,1,0),\end{aligned}$$

corresponding to this order.

We have

$$E = \begin{pmatrix} 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \qquad E^{-1} = \frac{1}{2}\begin{pmatrix} 0 & -2 & 0 & 2 \\ -1 & -5 & 2 & 4 \\ 1 & 7 & -2 & -4 \\ 0 & -4 & 2 & 2 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \qquad A = EJE^{-1} = \frac{1}{2}\begin{pmatrix} 0 & -18 & 8 & 12 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 2 & -2 \\ -1 & -5 & 2 & 6 \end{pmatrix}.$$

If we require a matrix B in which e_2 is proper and $e_3 = e_2'$ is referred to e_2,

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} 1 & -15 & 6 & 8 \\ 1 & 9 & -2 & -4 \\ 1 & 3 & 2 & -2 \\ 1 & 7 & -2 & -2 \end{pmatrix}.$$

5. The inverse matrix. When the company *Matrices to Order* is organized its promotion literature might contain this incentive: "If cash is sent with order we will supply the inverse of the desired matrix free of charge." Indeed this may be done with but slight additional labor; for from (12) we have

$$A^{-1} = EJ^{-1}E^{-1}. \tag{13}$$

Since J is a quasi-diagonal matrix, J^{-1} is also quasi-diagonal with the reciprocals of the original Jordan blocks in its main diagonal. For each block such as

$$J_i = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad J_i^{-1} = \begin{pmatrix} \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} & -\lambda^{-4} \\ 0 & \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} \\ 0 & 0 & \lambda^{-1} & -\lambda^{-2} \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

Thus for the foregoing problem giving A,

$$J^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad A^{-1} = \frac{1}{2}\begin{pmatrix} 4 & 14 & -6 & -10 \\ 0 & 2 & 0 & 0 \\ -1 & 3 & 3 & 3 \\ 1 & 5 & -2 & -2 \end{pmatrix}.$$

6. An Application. The beautiful theorem of Gantmacher-Krein [2] states that a *totally positive* $n \times n$ matrix (whose minors of *all* orders are positive) has always n distinct positive eigenvalues

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0,$$

and that the associated eigenvectors $e_1, e_2, \cdots, e_n$ have components with exactly $0, 1, 2, \cdots, n-1$ variations in sign.

Is it necessary for a matrix to be totally positive to have this property?

Let us construct a matrix A with

$$\lambda_1 = 3, \quad e_1 = (1, 1, 1)$$
$$\lambda_2 = 2, \quad e_2 = (1, -1, -1); \qquad \text{then } E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$
$$\lambda_3 = 1, \quad e_3 = (1, -1, 1)$$

and

$$A = EJE^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5 & 2 & -1 \\ 1 & 4 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

which is not totally positive. This sufficient condition for the stated property is therefore not necessary.

This paper was presented in part at the annual meeting of the Texas section of the MAA at Fort Worth, April 9, 1965.

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THE MAXIMAL THEOREMS OF HARDY AND LITTLEWOOD

KEITH PHILLIPS, University of Washington and California Institute of Technology

1. Introduction. In this paper we give a unified treatment of the famous maximal theorems of Hardy and Littlewood. Roughly speaking, these theorems give estimates of the maximal value of the difference quotient of an indefinite integral. They were originally proved by Hardy and Littlewood in 1930 [5]. They have had numerous applications, and in fact were proved in order to solve certain problems in complex function theory. Other applications appear in [9] (e.g., to conjugate functions, or Hilbert transforms), [1] and [8] (to ergodic theory), [7] (to harmonic functions), and [2] (to singular integrals). We are not concerned in this paper with applications, but with presenting the main facts and methods associated with maximal functions. Subsection (3.6) is of secondary interest, and can be skipped. Section 4 is not required for Section 5.

We begin by establishing notation. First, f will denote a nonnegative, extended real-valued, Lebesgue measurable function on R (the real numbers) such that $\int_F f d\lambda < \infty$ for all compact sets F; λ is Lebesgue measure for R. (Later, in sect. 5, f will be defined on R^m). Define three "maximal functions" of f as follows:

$$M_r f(x) = \sup\left\{\frac{1}{\lambda(I)}\int_I f d\lambda\colon I = [x, u],\ x < u < \infty\right\}$$

$$M_l f(x) = \sup\left\{\frac{1}{\lambda(I)}\int_I f d\lambda\colon I = [u, x],\ -\infty < u < x\right\} \tag{1.1}$$

$$Mf(x) = \sup\left\{\frac{1}{\lambda(I)}\int_I f d\lambda\colon I \text{ is a closed interval containing } x\right\}.$$

Thus $M_r f(x)$, $M_l f(x)$, and $Mf(x)$ are three different maximal averages of the function f. All of the maximal theorems are inequalities giving bounds for the integral of one of the maximal functions composed with a monotonic function. As a result, the mappings M_r, M_l, and M carry certain function spaces to others. Notice that the suprema defining our maximal functions are over finite intervals of all possible lengths. By the continuity of the indefinite integrals involved, each of the maximal functions is lower semicontinuous (and hence Borel measurable). The inequalities $f \leqq M_j f \leqq Mf (j = r, l)$ are immediate, but much more is true.

(1.2) THEOREM. *The equality* $Mf = \mathrm{Max}(M_r f, M_l f)$ *holds.*

Proof. Fix x. For $u < x < t$, let

$$A(t) = (t - x)^{-1}\int_x^t f d\lambda \quad \text{and} \quad B(u) = (x - u)^{-1}\int_u^x f d\lambda; \qquad A(x) = B(x) = 0.$$

Let $p_1(u, t) = (t-x)(t-u)^{-1}$, $p_2(u, t) = (x-u)(t-u)^{-1}$, and $C = p_1 A + p_2 B$, for $u \leqq x \leqq t$ and $u \neq t$; then

$$Mf(x) = \sup\{C(u, t)\colon u \leqq x \leqq t,\ t \neq u\}.$$

Let $(u_n, t_n)_{n=1}^{\infty}$ be such that $\lim_{n\to\infty} C(u_n, t_n) = Mf(x)$. The sequence $(p_1(u_n, t_n), p_2(u_n, t_n))_{n=1}^{\infty}$ is contained in the compact subset $[0, 1] \times [0, 1]$ of the plane, and so has a convergent subsequence. Hence there is a sequence $(u_k, t_k)_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} C(u_k, t_k) = Mf(x) \quad \text{and} \quad \lim_{k\to\infty} (p_1(u_k, t_k), p_2(u_k, t_k)) = (a, b),$$

where $a + b = 1$. For each k, the inequality

$$p_1(u_k, t_k)\, A(t_k) + p_2(u_k, t_k) B(u_k) \leqq p_1(u_k, t_k) M_r f(x) + p_2(u_k, t_k) M_l f(x)$$

holds; hence the inequality

$$Mf(x) \leqq a M_r f(x) + b M_l f(x)$$

holds. Since $a + b = 1$, the inequality $Mf(x) \leqq \mathrm{Max}(M_r f(x), M_l f(x))$ follows. The reverse inequality is immediate from the definitions (1.1). ∎

2. Two fundamental lemmas. The heart of our proofs of the maximal theorems is in the following two lemmas. The lemmas themselves give precise infor-

mation about the size of the maximal functions. For a positive real number t and an extended real-valued nonnegative function g on R, we let

$$E_t[g] = \{x \in R: g(x) > t\}.$$

(2.1) LEMMA. *The equalities*

$$\text{(i)} \qquad \lambda(E_t[M_j f]) = \frac{1}{t}\int_{E_t[M_j f]} f d\lambda \qquad (j = r, l)$$

and the inequality

$$\text{(ii)} \qquad \lambda(E_t[Mf]) \leqq \frac{2}{t}\int_{E_t[Mf]} f d\lambda$$

hold for every $t>0$.

Proof. We prove (i) with $j=r$, the case $j=l$ being almost the same. It is easy to see that $E_t[M_r f]$ is open. Let $\{]\beta_k, \gamma_k[\}_{k=1}^{\infty}$ be the unique pairwise disjoint intervals with union $E_t[M_r f]$. Consider an interval $]\beta_k, \gamma_k[$ (which may of course be infinite). For each $x \in]\beta_k, \gamma_k[$, the set

$$N_x = \left\{s: \int_x^s f d\lambda > t(s-x),\ s \in]x, \gamma_k] \cap R\right\}$$

is nonvoid. This is trivial if $\gamma_k = \infty$. If $\gamma_k < \infty$ and $N_x = \varnothing$, there would be a $w > \gamma_k$ such that $\int_x^w f d\lambda > t(w-x)$. We would have

$$\int_{\gamma_k}^{w} f d\lambda = \int_x^w f d\lambda - \int_x^{\gamma_k} f d\lambda > t(w - \gamma_k),$$

a contradiction since $\gamma_k \notin E_t[M_r f]$. Let $\gamma = \sup N_x$. If $\gamma < \gamma_k$, then the equality $\int_x^y f d\lambda = t(\gamma - x)$ must hold; an obvious argument proves this. Therefore (since N_γ is nonvoid) there is a $y \in]\gamma, \gamma_k] \cap R$ such that $\int_\gamma^y f d\lambda > t(y-\gamma)$. It follows that $\int_x^\gamma f d\lambda > t(y-x)$, a contradiction since $y > \gamma$. We thus have $\gamma_k = \sup N_x$, and hence the inequality $\int_x^{\gamma_k} f d\lambda \geqq t(\gamma_k - x)$ holds for all $x \in]\beta_k, \gamma_k[$; letting $x \to \beta_k$, we get

$$\text{(1)} \qquad \int_{\beta_k}^{\gamma_k} f d\lambda \geqq t(\gamma_k - \beta_k).$$

If $]\beta_k, \gamma_k[$ is infinite, the equality (i) follows. If $]\beta_k, \gamma_k[$ is finite, then the reverse inequality in (1) is immediate ($\beta_k \notin E_t[M_r f]$). Hence in all cases equality holds in (1), and so (i) is proved for $j=r$.

To prove (ii), note that Theorem (1.2) implies that $E_t[Mf] = E_t[M_l f] \cup E_t[M_r f]$. Hence, using (i), we have

$$\lambda(E_t[Mf]) \leqq \lambda(E_t[M_r f]) + \lambda(E_t[M_l f]) \leqq \frac{2}{t}\int_{E_t[Mf]} f d\lambda. \quad \blacksquare$$

(2.2) LEMMA. *For every* $k\in]0, 1[$ *and every* $t>0$, *the following inequalities hold*:

(i) $$\lambda(E_t[M_jf]) \leqq \frac{1}{(1-k)t}\int_{E_{kt}[f]} f d\lambda \qquad (j = r, l);$$

(ii) $$\lambda(E_t[Mf]) \leqq \frac{2}{(1-k)t}\int_{E_{kt}[f]} f d\lambda.$$

Proof. Let

$$g(x) = \begin{cases} f(x) & \text{if } f(x) > kt \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $M_rf \leqq M_rg + kt$, and $E_t[M_rf] \subset E_{(1-k)t}[M_rg]$. Applying (2.1.i), we obtain

(1) $$\lambda(E_t[M_rf]) \leqq \frac{1}{(1-k)t}\int_{E_{(1-k)t}[M_rf]} g d\lambda.$$

Since $g=0$ on $E_{kt}[f]'$, the integral in (1) is less than $\int_{E_{kt}[f]} f d\lambda$.

It is clear how to obtain (i) for $j=l$, and also how to use (2.1.ii) to prove (ii). ∎

Our proof of (2.1.i) is a variant of one of F. Riesz [6]. Lemma (2.2) appears in [4]; the method of proof is due to Wiener [8].

3. $\mathfrak{L}_p$ maximal theorems. For a Lebesgue measurable function g (defined on R) and a positive real number p, we let $\|g\|_p = [\int_R |g|^p d\lambda]^{1/p}$. The function space $\mathfrak{L}_p$ is all g's such that $\|g\|_p < \infty$. The following lemma, an easy application of Fubini's theorem, will be useful. We use the symbol ξ for characteristic functions.

(3.1) LEMMA. *Let* χ *be a function on* $[0, \infty[$ *that is absolutely continuous on every finite interval, and satisfies* $\chi(0)=0$. *For every nonnegative Lebesgue measurable function* g *on* R *and every Lebesgue measurable set* E, *the equality*

$$\int_E (\chi \circ g) d\lambda = \int_0^\infty \chi'(t)\lambda(E \cap E_t[g]) dt$$

holds.

Proof. Write

$$\int_E \chi \circ g \, d\lambda = \int_R \xi_E(x) \int_0^{g(x)} \chi'(t) dt dx$$

$$= \int_0^\infty \int_R \xi_{E \cap E_t[g]}(x) \chi'(t) dx dt = \int_0^\infty \chi'(t)\lambda(E \cap E_t[g]) dt;$$

the use of Fubini's theorem is justified since the set $\{(x, t): x \in E \text{ and } g(x) > t\}$ is product measurable. ∎

(3.2) THEOREM. *For $p > 1$ and $f \in \mathfrak{L}_p$, the inequalities*

(i) $$\|M_j f\|_p \leqq \frac{p}{p-1}\|f\|_p \qquad (j = r, l)$$

and

(ii) $$\|Mf\|_p \leqq \frac{2^{1/p}p}{p-1}\|f\|_p$$

hold.

Proof. The following calculation (using (3.1), (2.2.i), and Fubini's theorem) shows that $M_j f \in \mathfrak{L}_p$:

$$\begin{aligned}\int_R (M_j f)^p d\lambda &= p\int_0^\infty \lambda(E_t[M_j f])t^{p-1}dt \leqq \frac{p}{1-k}\int_0^\infty t^{p-2}\int_{E_{kt}[f]} f d\lambda dt \\ &= \frac{p}{1-k}\int_R f(x)\int_0^\infty \xi_{E_{kt}[f]}(x)t^{p-2}dtdx \\ &= \frac{p}{(p-1)(1-k)k^{p-1}}\int_R f(x)^p dx.\end{aligned}$$

The constant k satisfies $0 < k < 1$. The application of Fubini's theorem is justified because the set $\{(x, t): f(x) > kt\}$ is product measurable. To obtain (i), we use (2.1.i), Fubini's theorem, and Hölder's inequality to calculate as follows:

$$\begin{aligned}\int_R (M_j f)^p d\lambda &= p\int_0^\infty t^{p-2}\int_{E_t[M_j f]} f(x)dxdt = p\int_R\int_0^\infty \xi_{E_t[M_j f]}(x)f(x)t^{p-2}dtdx \\ &= p\int_R f(x)\frac{(M_j f(x))^{p-1}}{p-1}dx \\ &\leqq \frac{p}{p-1}\left[\int_R f^p d\lambda\right]^{1/p}\left[\int_R (M_j f)^{p'(p-1)}d\lambda\right]^{1/p'}.\end{aligned}$$

Since $p'(p-1) = p$ and $1 - 1/p' = 1/p$, the inequality (i) follows. The use of Fubini's theorem is justified because the set $\{(x, t): M_j f(x) > t\}$ is product measurable.

To prove (ii), let $A = \{x: M_l f(x) \geqq M_r f(x)\}$. Then we have, by (1.2),

$$\int_R (Mf)^p d\lambda = \int_A (M_l f)^p d\lambda + \int_{A'} (M_r f)^p d\lambda \leqq 2\left(\frac{p}{p-1}\right)^p \int_R f^p d\lambda;$$

the inequality (ii) follows. ∎

(3.3) REMARKS. (a) The calculation giving (3.2.i) in the above proof is a modification of one given by Flett [4].

(b) The inequalities of Theorem (3.2) are false if $p=1$. In fact, if $f\in\mathfrak{L}_1(R)$ and $0<b<x$, then the inequality $M_l f(x)\geqq x^{-1}\int_0^b f d\lambda$ holds. Thus, except for trivial f's, $M_l f$ is not in $\mathfrak{L}_1(R)$. The fact that we allow arbitrary intervals in our definition of the maximal functions is not, however, the only difficulty in the $\mathfrak{L}_1$ case. If $f(x)=\xi_{[0,1/2]}(x)/[x(\log x)^2]$, then $f\in\mathfrak{L}_1(R)$; but $(M_l f)\xi_{[0,1/2]}\notin\mathfrak{L}_1(R)$. In fact, we have $M_l f(x)\geqq(x|\log x|)^{-1}$ if $x\in]0,\frac{1}{2}[$. (This example appears in [9], p. 33, but with the equality $M_l f(x)=(x|\log x|)^{-1}$. Strict inequality holds, however, for $x\in]1/e, 1/2[$.) In the case $p=1$, there are two replacement results for Theorem (3.2). We state and prove them below.

(3.4) THEOREM. *Let E be a Lebesgue measurable set, and suppose $k\in]0, 1[$. The inequalities*

$$\text{(i)}\qquad \int_E (M_j f)d\lambda \leqq \frac{1}{k}\lambda(E)+\frac{1}{1-k}\int_R f(x)\log^+ f(x)dx \qquad (j=r, l)$$

and

$$\text{(ii)}\qquad \int_E (Mf)d\lambda \leqq \frac{1}{k}\lambda(E)+\frac{2}{1-k}\int_R f(x)\log^+ f(x)dx$$

hold. For $0<p<1$, the inequalities

$$\text{(iii)}\qquad \int_E (M_j f)^p d\lambda \leqq \frac{\lambda(E)^{1-p}}{1-p}\left(\int_R f d\lambda\right)^p \qquad (j=r, l)$$

and

$$\text{(iv)}\qquad \int_E (Mf)^p d\lambda \leqq \frac{2^p\lambda(E)^{1-p}}{1-p}\left(\int_R f d\lambda\right)^p$$

hold.

Our proof of (i) and (ii) (below) is essentially as in [4]. Our proof of (iii) and (iv), based also on (2.2), uses a technique appearing in [3] and [2] (but not in [4]). The definition of $\log^+$ is $\log^+ t=\operatorname{Max}\{\log t, 0\}$. If g is Lebesgue measurable and $\lambda(E)=0$, then $\int_E g d\lambda=0$, even if $\lambda(\{x: g(x)=\pm\infty\})>0$.

Proof. For $j=r, l$, we have

$$\int_E (M_j f)d\lambda = \int_0^\infty \lambda(E_t[M_j f]\cap E)dt = \int_0^{1/k} + \int_{1/k}^\infty$$

$$\leqq \frac{1}{k}\lambda(E)+\frac{1}{1-k}\int_{1/k}^\infty \frac{1}{t}\int_{E_{kt}[f]} f(x)dxdt$$

$$= \frac{\lambda(E)}{k} + \frac{1}{1-k}\int_R f(x)\left\{\int_{1/k}^{\infty} \xi_{E_{kt}[f]}(x)\,\frac{1}{t}\,dt\right\}dx$$

$$= \frac{\lambda(E)}{k} + \frac{1}{1-k}\int_R f(x)\log^+ f(x)\,dx;$$

and so (i) is established. Clearly (ii) can be proved similarly, using (2.2.ii).

To prove (iii), suppose that $\lambda(E)>0$ and $\int_R f d\lambda < \infty$. Let α be any positive real number. We write

$$\int_E (M_j f)^p d\lambda = p\int_0^{\infty} t^{p-1}\lambda(E_t[M_j f]\cap E)\,dt = p\left(\int_0^{\alpha/k} + \int_{\alpha/k}^{\infty}\right)$$

(1)

$$\leqq \frac{\alpha^p}{k^p}\lambda(E) + \frac{p}{1-k}\int_R f(x)\left\{\int_{\alpha/k}^{\infty} t^{p-2}\xi_{E_{kt}[f]}(x)\,dt\right\}dx.$$

The inner integral above is $[f(x)^{p-1}-\alpha^{p-1}]/k^{p-1}(p-1)$ if $f(x)>\alpha$ and is 0 otherwise; it therefore does not exceed $\alpha^{p-1}/k^{p-1}(1-p)$. Thus we get the estimate

$$\int_E (M_j f)^p d\lambda \leqq \frac{\lambda(E)}{k^p}\alpha^p + \frac{p\alpha^{p-1}}{(1-p)k^{p-1}(1-k)}\int_R f d\lambda. \tag{2}$$

Taking $\alpha = (k/(1-k))(\lambda(E))^{-1}\int_R f d\lambda$ (this value minimizes the right side of (2)), we obtain

$$\int_E (M_j f)^p d\lambda \leqq \frac{1}{(1-k)^p}\,\frac{\lambda(E)^{1-p}}{1-p}\left[\int_R f d\lambda\right]^p.$$

Letting $k\to 0$, we obtain (iii). Obvious modifications in the proof yield (iv). ∎

(3.5) REMARK. All of the inequalities in (3.4) are, of course, interesting only if $\lambda(E)<\infty$. Consider the function $\xi=\xi_{[-1,1]}$. We have $M\xi(x)=2/(1+|x|)$ if $|x|\geqq 1$ (see 3.6). Thus ξ and $\xi\log^+\xi$ are in $\mathfrak{L}_1(R)$, but $M\xi$ is not in $\mathfrak{L}_1(R)\cup\mathfrak{L}_p(R)$, $p<1$. This example shows that M does not carry $\mathfrak{L}_1\log^+\mathfrak{L}_1$ into $\mathfrak{L}_1$ or $\mathfrak{L}_1$ into $\mathfrak{L}_p$, $p<1$; and so we cannot expect replacement results for (3.4) when $\lambda(E)=\infty$.

(3.6) BEST CONSTANTS. In various places in the preceding sections, the constants obtained can be shown to be the best possible. First, let $\xi_\delta=\xi_{[-\delta,\delta]}$, $0<\delta<1$. An elementary calculation shows that

$$M\xi_\delta(x) = \begin{cases} 1 & \text{if } |x|\leqq\delta \\ 2\delta/(\delta+|x|) & \text{if } |x|>\delta. \end{cases}$$

We have $\lambda(E_t[M\xi_\delta])=2$ and $t^{-1}\int_{-1}^{1}\xi_\delta d\lambda=\delta+1$, if $t=2\delta(\delta+1)^{-1}$; therefore, 2 in (2.1.ii) is best possible.

Let $p>1$ and $0<\alpha<1$, and put $f_\alpha(x)=x^{-\alpha/p}\xi_{[0,1]}(x)$. The equality $\lim_{\alpha\to 1}\|M_j f_\alpha\|_p\|f_\alpha\|_p^{-1}=p/(p-1)$ shows that the constant in (3.2.i) is the best possible. The example is due to Flett, [4]. Let $0<p<1$, $0<\alpha<1$, and $f_\alpha(x)=x^{-\alpha}\xi_{[0,1]}(x)$. The inequality

$$\left[\int_0^1 (M_1 f_\alpha)^p d\lambda\right]\left[\int_R f_\alpha d\lambda\right]^{-p} \geqq (1-\alpha p)^{-1}$$

shows that the constant in (3.4.iii) is the best possible.

Finally, the constant $(1-k)^{-1}$ in (3.4.i) is the best possible in the following sense. If B_1 and B_2 are constants independent of f and E such that

$$\int_E (M_1 f)\, d\lambda \leqq B_2 \lambda(E) + B_1 \int_R f(x) \log^+ f(x)\, dx,$$

then we have $B_1 \geqq 1$. The following example is due to Flett [4]. Let $E=[0, b]$ and let $f(x)=x^{-1}[\log 1/x]^{-(2+\epsilon)}\xi_E(x)$. Suppose that b is small enough so that f is decreasing on $]0, b]$ and so that $0<x\leqq b$ implies that $\log(\log 1/x)\geqq 0$. An elementary calculation shows that

$$M_1 f(x) = \frac{1}{1+\epsilon}\,\frac{1}{x}\left[\log\frac{1}{x}\right]^{-(1+\epsilon)}, \qquad 0 < x \leqq b.$$

Hence for any constant B_2 we have

$$\left[\int_E M_1 f\, d\lambda - B_2\lambda(E)\right]\left[\int_R f\log^+ f\right]^{-1} \geqq (1+\epsilon)^{-1}\left[\int_0^b \left(\frac{1}{x}\log\frac{1}{x}\right)^{-(1+\epsilon)} dx - B_2 b\right] \times\left[\int_0^b \frac{1}{x}\left(\log\frac{1}{x}\right)^{-(1+\epsilon)} dx\right]^{-1}.$$

Letting $b\to 0$, we obtain $(1+\epsilon)^{-1}$ on the right.

4. The Hardy-Littlewood maximal theorem. Versions of the inequalities (3.2.i), (3.2.ii), and (3.4.i) appear in Hardy and Littlewood's original paper [5]. Sections 2 and 3 combine some of the refinements made in subsequent years, and some related results. As is apparent from the previous references, important contributions have been made by several authors: F. Riesz, Wiener, Flett, and Edwards and Hewitt, to mention those who bear the most influence on our treatment. Hardy and Littlewood based their proofs on a more general maximal theorem. This theorem, which we now prove by the methods of Section 3, is usually called the Hardy-Littlewood maximal theorem.

Notation is as in Sections 1 and 2. We also let f^* denote a decreasing function on $]0, \infty[$ that is equimeasurable with f; that is, $\lambda(E_t[f^*])=\lambda(E_t[f])$ for all $t>0$. (Such functions exist; e.g., f^* can be taken as the inverse of the decreasing function $y\to\lambda(E_y[f])$, with suitable modifications when this function is not strictly decreasing.) The equality

$$\int_R f d\lambda = \int_0^\infty f^* d\lambda = \int_0^\infty \lambda(E_t[f])\, dt \tag{4.1}$$

is a result of (3.1).

(4.2) LEMMA. *The inequality*

$$\int_B f d\lambda \leqq \int_0^{\lambda(B)} f^* d\lambda$$

holds for every Lebesgue measurable set B.

Proof. Clearly we have $\lambda(E_0[f\xi_B]) \leqq \lambda(B)$. Thus $(f\xi_B)^* = 0$ on $]\lambda(B), \infty[$, and we have

$$\int_B f d\lambda = \int_0^{\lambda(B)} (f\xi_B)^* d\lambda \leqq \int_0^{\lambda(B)} f^* d\lambda.$$

We will prove the inequality $(f\xi_B)^* \leqq f^*$(a.e.) used here. In fact, if $g \leqq f$ and $g^*(t_0) > f^*(t_0)$ for some point t_0 of continuity of g^*, then $g^* > f^*$ on some interval $[t_0, t_1]$. Thus we have

$$\lambda(E_{f^*(t_0)}[f^*]) \leqq t_0 < t_1 \leqq \lambda(E_{f^*(t_0)}[g^*]),$$

a contradiction since the extremes are equal to $\lambda(E_{f^*(t_0)}[f])$ and $\lambda(E_{f^*(t_0)}[g])$. Since g^* is continuous a.e., we have $g^* \leqq f^*$ a.e. ∎

The above lemma is well known; it appears, e.g., in [9], p. 31.

We extend f^* by letting $f^*(t) = 0$ if $t \leqq 0$.

(4.3) HARDY-LITTLEWOOD MAXIMAL THEOREM. *Let χ be an increasing non-negative function on $[0, \infty[$, and suppose $f \in \mathfrak{L}_1(R)$. The following inequalities hold:*

(i) $$\int_R \chi \circ (M_j f) d\lambda \leqq \int_R \chi \circ (M_1 f^*) d\lambda \qquad (j = r, l);$$

(ii) $$\int_R \chi \circ (Mf) d\lambda \leqq 2 \int_R \chi \circ (M_1 f^*) d\lambda.$$

Proof. It is easy to see that $M_1 f^*(x) = (1/x)\int_0^x f^* d\lambda$, since f^* is decreasing. It follows from this equality that $M_1 f^*$ is decreasing. By (2.1.i) and (4.2), we have

(1) $$\lambda(E_t[M_j f]) \leqq \frac{1}{t} \int_0^{\lambda(E_t[M_j f])} f^* d\lambda \qquad (t > 0);$$

and this inequality implies that $M_1 f^*(\lambda(E_t[M_j f])) \geqq t$, if $0 < \lambda(E_t[M_j f]) < \infty$. Since $M_1 f^*$ is decreasing, we infer that

(2) $$\lambda(E_t[M_j f]) \leqq \lambda(E_t[M_1 f^*]),$$

if $\lambda(E_t[M_j f]) < \infty$. The equality $\lambda(E_t[M_j f]) = \infty$ would imply, by (1), that $f^*, f \notin \mathfrak{L}_1(R)$; hence, (2) holds for all $t > 0$. Let $F_t[g] = \{x : g(x) \geqq t\}$ $(t > 0,\ g \geqq 0)$. An application of Fatou's lemma, using (2), shows that

(3) $$\lambda(F_t[M_j f]) \leqq \lambda(F_t[M_1 f^*]).$$

Next, let $S=\{x: 0\leqq x_1<x<x_2\Rightarrow\chi(x_1)<\chi(x)<\chi(x_2)\}$. Then χ is strictly increasing on S. On S, we define α as the inverse of $\chi:\chi(\alpha(t))=t$ if $t\in\chi(S)$. If $t\in\chi([0, \infty[)'$, let $\beta=\sup[\{x: \chi(x)<t\}\cup\{0\}]$ and $\gamma=\inf[\{x: \chi(x)>t\}\cup\{\infty\}]$. Then we have $]\beta, \gamma[\subset\chi([0, \infty[)'$, $t\in[\beta, \gamma]$, and β or $\gamma\in\chi([0, \infty[)$. If $\gamma\in\chi([0, \infty[)$, let $\alpha(t)=\inf\{x: \chi(x)=\gamma\}$; if $\beta\in\chi([0, \infty[)$ and $\gamma\notin\chi([0, \infty[)$, let $\alpha(t)=\sup\{x: \chi(x)=\beta\}$. Then $\chi(\alpha(t))$ is either β or γ, and is unequal to t. The function α is thus defined a.e. on $[0, \infty[$ (it is not defined on $\chi(S')$). Let $A=\{t: \chi(\alpha(t))\leqq t\}$ and $B=\{t: \chi(\alpha(t))>t\}$. Then the equalities

$$(4)\qquad \begin{aligned} E_t[\chi\circ g] &= E_{\alpha(t)}[g], \qquad t\in A \\ E_t[\chi\circ g] &= F_{\alpha(t)}[g], \qquad t\in B \end{aligned}$$

hold for any nonnegative function g. Using (2), (3), (4), and our previous results, we calculate as follows:

$$\begin{aligned} \int_R \chi\circ(Mf)d\lambda &= \int_0^\infty \lambda(E_t[\chi\circ Mf])dt \\ &= \int_A \lambda(E_{\alpha(t)}[Mf])dt + \int_B \lambda(F_{\alpha(t)}[Mf])dt \\ &\leqq \int_0^\infty \lambda(E_t[\chi\circ M_1 f^*])dt = \int_R \chi\circ(M_1 f^*)d\lambda. \end{aligned}$$

To prove (ii), we write

$$\begin{aligned} \int_R \chi\circ(Mf)d\lambda &= \int_R \operatorname{Max}\{\chi\circ(M_r f), \chi\circ(M_1 f)\}d\lambda \\ &\leqq \int_R \{\chi\circ(M_r f)+\chi\circ(M_1 f)\}d\lambda \leqq 2\int_R \chi\circ(M_1 f^*)d\lambda. \quad \blacksquare \end{aligned}$$

The constant 2 in (4.3.ii) is the best possible; for, an equimeasurable function for ξ_1 (see Section 3.6) is $\xi_1^*=\xi_{[0,2]}$, and we have

$$\lim_{\substack{p\to 1\\ p>1}}\left[\int_R (M\xi_1)^p d\lambda\right]\left[\int_R (M_1\xi_1^*)^p d\lambda\right]^{-1} = 2.$$

5. Generalizations for R^m. The maximal function Mf defined in Section 1 can be written as

$$Mf(x) = \sup\left\{\lambda(I)^{-1}\int_{x+I} f d\lambda : I \text{ is a finite interval containing } 0\right\}.$$

We see that the mapping M depends on the measure λ and the collection of finite intervals containing 0; averages are taken over intervals containing x. Given a measure space $(X, \mathfrak{A}, \mu)$, what is needed to define formally an operator

M on the positive measurable functions is a family $\mathfrak{B}$ of subsets of X over which to take averages. Maximal functions have been successfully defined and heavily used by several mathematicians by obtaining suitable families $\mathfrak{B}$ on certain measure spaces. Examples are: Calderon [1], to ergodic theorems on locally compact groups; Calderon and Zygmund [2], to singular integrals in R^m; Edwards and Hewitt [3], to pointwise convergence of convolutions on locally compact groups; Smith [7], to harmonic functions in R^m. For the presentation of the maximal theorems in sections 2, 3, and 4, the crucial result is really Lemma (2.1). The other results are obtained from it with no significant use of specific properties of the underlying space R. In this section we prove an analogue of (2.1) for Euclidean m-space R^m, using open spheres in place of intervals. The precise arguments of the proof of (2.1) are not possible if $m \geqq 2$, for the open sets in R^m are not unique disjoint countable unions of spheres. Instead, we need the following covering theorem. All writers dealing with generalized maximal theorems use covering arguments of roughly the same type. The one given below is patterned after Theorem (2.2) of [3]. For $r > 0$, let $S_r = \{x \in R^m : \|x\| < r\}$; and for $n \in Z$ (the integers), let $B_n = S_{2^{-n}}$. Notice that $B_n + B_n = B_{n-1}$. Let $\mathfrak{B} = \{x + B_n : x \in R^m, n \in Z\}$.

(5.1) A COVERING THEOREM. *Let $\mathfrak{B}^\dagger \subset \mathfrak{B}$ and suppose $E \subset R^m$ satisfies*

(i) $\lambda(E + B_n) < \infty$ *for all* $n \in Z$;

(ii) *for each* $x \in E$, *there is some* n *such that* $x + B_n \in \mathfrak{B}^\dagger$;

(iii) $\{n : x + B_n \in \mathfrak{B}^\dagger$ *for some* $x \in E\}$ *is bounded below. Then there are (possibly finite) sequences* $(x_k)_{k=1}^{\infty}$ *and* $(n_k)_{k=1}^{\infty}$ *such that* $x_k \in E$, $(x_k + B_{n_k})_{k=1}^{\infty}$ *is a pairwise disjoint sequence in* $B^\dagger$, *and* $\lambda(E) \leqq 2 \sum_{k=1}^{\infty} \lambda(B_{n_k})$.

Proof. Let $n_1 = \operatorname{Min}\{n : x + B_n \in \mathfrak{B}^\dagger$ for some $x \in E\}$, and choose $x_1 \in E$ such that $x_1 + B_{n_1} \in \mathfrak{B}^\dagger$. Sequences (x_k) and (n_k) satisfying

(1) $\{x_k + B_{n_k}\}_{k=1}^{p}$ is a pairwise disjoint sequence in $\mathfrak{B}^\dagger$ for all p;

(2) $n_k = \operatorname{Min}\{n : x + B_n \in \mathfrak{B}^\dagger$ and $(x + B_n) \subset [\bigcup_{i=1}^{k-1}(x_i + B_{n_i})]'$, some $x \in E\}$;

(3) $x_{p+1} \in [\bigcup_{i=1}^{p}(x_k + B_{n_{k-1}})]'$ for all p

will be defined by induction. Thus suppose that $(x_k)_{k=1}^{p}$ and $(n_k)_{k=1}^{p}$ satisfy (1), (2), (3). If $E \subset \bigcup_{k=1}^{p}[x_k + B_{n_{k-1}}]$, we obtain a finite sequence satisfying (1) and (2) and $E \subset \bigcup_{k=1}^{p}(x_k + B_{n_{k-1}})$. Otherwise, let $x \in E \cap [\bigcup_{k=1}^{p}(x_k + B_{n_{k-1}})]'$ and let j be the smallest integer such that $x + B_j \in \mathfrak{B}^\dagger$. We will show that $x + B_j \subset [\bigcup_{k=1}^{p}(x_k + B_{n_k})]'$. If $(x + B_j) \cap (x_1 + B_{n_1}) \neq \varnothing$, we have $x \in x_1 + (B_{n_1} + B_j) \subset x_1 + B_{n_1 - 1}$; a contradiction. If there are k's such that $(x + B_j) \cap (x_k + B_{n_k}) \neq \varnothing$, let q be the least among them $(1 < q \leqq p)$. Then the inclusion $x + B_j \subset [\bigcup_{i=1}^{q-1}(x_i + B_{n_i})]'$ holds. Therefore, the inequality $j \geqq q$ holds, and $x \in x_q + B_{n_q - 1}$; a contradiction. We can thus define $n_{p+1} = \operatorname{Min}\{j : x + B_j \in \mathfrak{B}^\dagger$ and $(x + B_j) \subset [\bigcup_{i=1}^{p}(x_i + B_{n_i})]'$, some $x \in E\}$, and select x_{p+1} such that $x_{p+1} + B_{n_{p+1}} \in \mathfrak{B}^\dagger$.

We claim that $E \subset \bigcup_{k=1}^{\infty}(x_k + B_{n_k - 1})$ (this is already established if the sequences are finite). We have $\lim_{k \to \infty} n_k = \infty$; proof:

$$\sum_{k=1}^{\infty} \lambda(B_{n_k}) = \sum_{k=1}^{\infty} \lambda(x_k + B_{n_k}) < \lambda(E + B_{n_1}) < \infty.$$

Hence, for given $x \in E$ and $p \in Z$, we must have $(x+B_p) \cap (x+B_{n_k}) \neq \varnothing$ for some k; if not, x and p would have been selected for x_j and n_j as soon as n_j exceeded p. If q is the least integer such that $(x+B_p) \cap (x_q+B_{n_q}) \neq \varnothing$, then we argue as before to obtain $x \in x_q + B_{n_q-1}$.

Finally, we have

$$\lambda(E) \leqq \sum_{k=1}^{\infty} \lambda(B_{n_k-1}) \leqq 2 \sum_{k=1}^{\infty} \lambda(B_{n_k}). \quad \blacksquare$$

Let λ denote m-dimensional Lebesgue measure.

(5.2) THEOREM. *Let $p \geqq 1$, and suppose $f \in \mathfrak{L}_p(R^m)$. Define*

$$Mf(x) = \sup_{r>0} \lambda(S_r)^{-1} \int_{x+S_r} f d\lambda.$$

There is a constant α such that $\lambda(E_t[Mf]) \leqq (2/t) \int_{E_{\alpha t}[Mf]} f d\lambda$ for all $t > 0$.

Proof. With notation as above, let

$$M'f(x) = \sup_{n \in Z} \lambda(B_n)^{-1} \int_{x+B_n} f d\lambda.$$

Small changes in x yield small changes in $\int_{x+S_r} f d\lambda$ for fixed r; it follows easily from this fact that Mf and $M'f$ are Borel measurable, in fact lower semi-continuous. Define $\mathfrak{B}^\dagger = \{x+B_n : \lambda(B_n)^{-1} \int_{x+B_n} f d\lambda > t\}$. Then $E_t[M'f]$ satisfies (5.1.ii) for this $\mathfrak{B}^\dagger$. Condition (5.1.iii) is also satisfied, for there cannot be sequences $(x_i)_{i=1}^{\infty}$ and $(n_i)_{i=1}^{\infty}$ such that $\lim_{i \to \infty} n_i = -\infty$ and $\int_{x+B_{n_i}} f > t\lambda(B_{n_i})$. This last assertion is obvious if $p = 1$ and is an easy application of Hölder's inequality if $p > 1$. Condition (5.1.i) is satisfied for each set $E_j = B_j \cap E_t[M'f]$, $j \in Z$; and clearly (ii) and (iii) are also satisfied for the E_j's. For fixed j, then, let $(x_k+B_{n_k})_{k=1}^{\infty}$ be the sequence guaranteed by (5.1). The inequality

$$\lambda(E_j) \leqq 2t^{-1} \int_U f d\lambda \left(U = \bigcup_{k=1}^{\infty} (x_k + B_{n_k}) \right)$$

thus holds. If $x \in U$ and $x \in x_k + B_{n_k}$, then we have

$$\lambda(B_{n_k-1})^{-1} \int_{x+Bn_k-1} f d\lambda = 2^{-m} \lambda(B_{n_k})^{-1} \int_{x+B_{nk-1}} f d\lambda > 2^{-m} t;$$

therefore, $U \subset E_{2^{-m}t}[M'f]$. We conclude, letting $j \to -\infty$, that

$$\lambda(E_t[M'f] \leqq \frac{2}{t} \int_{E_{\beta t}[M'f]} f d\lambda \quad (\beta = 2^{-m}).$$

For a given $r>0$, let n be the unique integer such that $B_n \subseteq S_r \subset B_{n-1}$. Then $\lambda(S_r)^{-1}\int_{x+S_r} f d\lambda \leqq 2^m \lambda(B_{n-1})^{-1}\int_{x+B_{n-1}} f d\lambda$ for any $x \in R^m$. Hence we have $Mf \leqq 2^m M'f$ and $E_t[Mf] \subset E_{t2^{-m}}[M'f]$. The inequality

$$\lambda(E_t[Mf]) \leqq \frac{2}{t}\int_{E_{\alpha t}[Mf]} f d\lambda, \quad \alpha = 2^{-2m}$$

follows. ∎

The proofs of the R^m analogues of (2.2), (3.2), (3.4), and (4.3) are almost identical to those given in Sections 2, 3, 4 for $m=1$; (5.2) is used in place of (2.1). Of course, the constants change. Lemmas (3.1) and (4.1) offer no difficulty in R^m.

The family $\{x+S_r: x \in R^m, r>0\}$ over which the averages are taken in (5.2) can be replaced by other families. For example, m-dimensional cubes symmetric about the origin would do. In fact, m-dimensional rectangles symmetric about O and satisfying the condition that the ratio of the maximal side length to the minimal side length does not exceed a fixed constant could replace the S_r's. Sufficient conditions (on the sets over which averages are taken) to yield maximal theorems are given in each of [2], [3], and [7].

Hardy and Littlewood devoted the main part of their proofs to the discrete analogues of the theorems in Sections 3 and 4. It is instructive to formulate these analogues (for Z, e.g.). The facts in these cases are corollaries of the results in Sections 2, 3, and 4. Similarly, there are results for sums of multiple sequences (maximal functions for Z^m). These follow from the R^m theorems.

The author would like to thank Professor Edwin Hewitt for the original suggestion that a paper of this type be written, and for helpful suggestions during its preparation.

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ON QUALITATIVE PROBABILITY

C. VILLEGAS, Instituto de Matemática y Estadística, Montevideo

1. Introduction. The rapid development of mathematical probability over the past thirty years was based on the firm axiomatic foundation laid out by Kolmogorov in 1933 [6]. Following Kolmogorov, almost all mathematical papers on probability which have been published since then, have considered probability as a (countably additive) measure P defined on a σ-algebra of sets of a space Ω, satisfying $P(\Omega)=1$. Although this formulation is entirely satisfactory from a mathematical point of view, problems still remained in the applications in several fields, suggesting the desirability of some changes in the general axiomatic theory. Thus, Glivenko [4], Koopman [7] and Halmos [5] considered the events as undefined terms, subject to the usual operations of union, intersection and complementation, satisfying the axioms of a Boolean algebra [11]. It has been pointed out also that a probability measure satisfying $P(\Omega)=1$ cannot be applied to some problems in which unbounded measures naturally occur. We cannot speak for instance of a probability distribution which is uniform on the whole real axis, or on the whole plane, or on all integers. One objection that usually arises when unbounded measures are proposed is that probability is used to build models in which it plays the role of an idealized frequency, and therefore its value can never exceed one. Trying to solve this difficulty without altering the frequency interpretation of probability, Rényi [9] presented a modified axiomatic theory, in which conditional probability, instead of absolute probability, is the fundamental concept. He shows then that under general conditions a (possibly unbounded) measure can be defined, in terms of which the conditional probability is given by $P(A\,|B)=\mu(A\cap B)/\mu(B)$.

There is however an increasing number of statisticians who believe that, by adhering exclusively to an objective interpretation of probability, the field of application of probability theory has been unnecessarily narrowed. They argue that there is a need for using also personal or subjective probabilities, and that such probabilities have an important role to play in various fields of application, notably in statistical inference and decision theory. Motivated in part by this subjective interpretation of probability there have been various attempts to build an axiomatic theory in which qualitative probability is the fundamental concept, and the numerical probabilities are derived from it as the only compatible probability measure. Clearly, qualitative statements such as "the event A is less probable than B" are closer to our intuitive representation of the situation considered than similar statements concerning numerical probabilities, like "the probability of A is equal to P," even when a frequency interpretation is possible.

An axiomatic theory based on qualitative probability was given by Bernstein in 1917 [1]. Slightly improved axioms were proposed later by de Finetti ([2], Sec. 13; [3]). Bernstein, de Finetti and Koopman derived an additive probability measure from a qualitative probability under the simplifying assumption

that, for any integer n, there are n mutually exclusive, equally probable events. L. J. Savage [10] has shown that this strong assumption may be relaxed, thereby gaining in generality at the expense of a corresponding loss in simplicity. Kraft, Pratt and Seidenberg [8] considered the case of finite Boolean algebras, showing, for example, that there are qualitative probabilities on finite Boolean algebras without any compatible measure. Villegas [13] proved that, if an atomless and monotonely continuous qualitative probability is defined in a Boolean σ-algebra of sets, then there is one and only one compatible additive probability measure, and it is countably additive.

Although the present writer subscribes to the opinion that the interpretation given to the word probability may be different in different contexts, and even in different parts of the same problem, in this paper we shall not enter into the controversy over the objective vs. subjective interpretation of probability. We simply remark that a qualitative probability, as a numerical one, may be interpreted in either way, and therefore the following axiomatic theory is compatible with both interpretations of probability.

2. Events. We shall consider the *events* as undefined elements of a *Boolean ring* $\mathfrak{R}$. The two ring operations, addition and multiplication, are also called *symmetric difference* $+$ and *intersection* $\cap$. As is well known, the *union* $\cup$ of two events and the Boolean *inclusion* $\subset$ may be defined in terms of the ring operations. The Boolean inclusion is a partial order with a first element, O, which is the zero of the ring, and is called the *impossible event.* Two events, A, B are *incompatible* or *mutually exclusive* if $A \cap B \equiv O$ (in this paper the identity of events will be denoted by the sign $\equiv$, leaving the sign $=$ for the relation "equally probable" to be introduced later). A *Boolean algebra* $\mathfrak{A}$ of events is a Boolean ring with a unit element, which is called the *sure event*, and is usually denoted by Ω. Unless the contrary is specified, we shall not assume the existence of a sure event.

A fundamental result in the theory of Boolean rings is that every Boolean ring is isomorphic to a ring of sets of a space. This result is due to Stone [12] and his construction has the following simple interpretation. A *possible realization* of the ring of events $\mathfrak{R}$ is a classification of the elements of $\mathfrak{R}$ into two classes: the events which *occur* and the events which *do not occur*. This classification is such that:

(i) the class of the events which occur is non-empty;

(ii) $A+B$ occurs if and only if only one of the events A, B occurs;

(iii) $A \cap B$ occurs if and only if both events A, B occur.

It can be easily shown that the impossible event O never occurs. If A is an event different from O, then it can be proved (using Zorn's lemma or an equivalent) that there is one realization in which A occurs (theorem of existence of realizations). The set Ω of all possible realizations is the *possibility space* of the ring $\mathfrak{R}$. Given an event A, let $\overline{A}$ be the set of all realizations of $\mathfrak{R}$ in which A does occur. Using the theorem of existence of realizations it can be proved easily that the natural correspondence $A \to \overline{A}$ is a Boolean isomorphism between $\mathfrak{R}$ and the

Boolean ring of sets $\overline{\mathfrak{R}} = \{\overline{A}: A \in \mathfrak{R}\}$. We may identify, therefore, the events with the sets of $\overline{\mathfrak{R}}$, that is, we can think of the events as sets of realizations.

We shall say that an event A is the *intersection* $\cap_{i=1}^{\infty} A_i$ *of a denumerable family of events* $\{A_i: i=1, 2, \cdots\}$ if from $B \subset A_i$, $i=1, 2, \cdots$ it follows that $B \subset A$ (the words set, family and collection will be used as synonyms in this paper). The *union of a denumerable family of events* is defined in a similar way. A *Boolean σ-ring* is a Boolean ring in which every denumerable collection of elements has a union and an intersection. We shall say that a Boolean ring is a *Boolean δ-ring* if every denumerable collection of events has an intersection. In general, Boolean δ-rings (or σ-rings) are not isomorphic to Boolean δ-rings (or σ-rings) of sets of a space Ω.

3. Qualitative probability. A *total pre-order* in the Boolean ring of events $\mathfrak{R}$ is a relation $\leqq$ such that the following conditions obtain:

Comparability (TPO_1): If A and B are events, then either $A \leqq B$ or $B \leqq A$;

Transitivity (TPO_2): If $A \leqq B$ and $B \leqq C$, then $A \leqq C$.

Obviously, from (TPO_1) it follows immediately that for any event A, we have $A \leqq A$. If $A \leqq B$ holds but $B \leqq A$ does not, then we shall write, as usual, $A < B$. A *qualitative probability* in the Boolean ring of events $\mathfrak{R}$ is a total pre-order satisfying the following conditions:

Monotony (QP_1): If $B_1 \cap B_2 \equiv O$, then from $A_1 \leqq B_1$ and $A_2 \leqq B_2$ it follows that

$$A_1 \cup A_2 \leqq B_1 \cup B_2,$$

and if, in one of the first two inequalities, the sign $\leqq$ is replaced by $<$, then the last one holds with the sign $\leqq$ replaced by $<$.

(QP_2): For any event A, $O \leqq A$.

The above set of axioms is not independent. In fact, as was pointed out to the author by J. J. Schäffer, it can be shown that (TPO_2) is a consequence of (TPO_1) and (QP_1). If $\leqq$ is a qualitative probability and $A \leqq B$, then we shall say that A is less probable than B. As usual, instead of $A \leqq B$ we shall write also $B \geqq A$ and we shall say that B is more probable than A. If $A < B$ (or equivalently $B > A$), then we shall say that A is strictly less probable than B (or that B is strictly more probable than A). If $A \leqq B$ and $B \leqq A$, then we shall write $A = B$ and we shall say that A and B are equally probable events. The pair $(\mathfrak{R}, \leqq)$ in which $\mathfrak{R}$ is a Boolean ring and $\leqq$ is a qualitative probability defined in $\mathfrak{R}$ will be called a *qualitative probability ring*. A *qualitative probability algebra* is a pair $(\mathfrak{A}, \leqq)$ in which $\mathfrak{A}$ is a Boolean algebra, and $\leqq$ is a qualitative probability defined on $\mathfrak{A}$ and satisfying

(QPA): $O < \Omega$.

If $A = O$, we shall say that A is a *null* or an *almost impossible event*. In a qualitative probability algebra, we shall say that an event A is an *almost sure event* if $A = \Omega$. Two events A, B will be called *almost incompatible* if $A \cap B = O$. The following propositions are left as exercises for the reader.

(1) *Sharpened transitivity.* If $A \leqq B$ and $B < C$, then $A < C$.

(2) *Compatibility with the inclusion.* If $A \subset B$, then $A \leqq B$.

(3) If $A_2 \subset A_1$ and $B_2 \subset B_1$, then from $A_1 \leqq B_1$ and $A_2 \geqq B_2$ it follows that

$$A_1 - A_2 \leqq B_1 - B_2.$$

(4) *Duality.* In a qualitative probability algebra, if $A \leqq B$, then $B^c \geqq A^c$, where the superscript c denotes complementation.

A uniform partition of order n of an event $A > O$ is a collection of n equally probable, almost incompatible nonnull events, the union of which is A.

LEMMA. *If $\{A_i: i=1, \cdots, m\}$ is a uniform partition of order m of an event $E > O$, and $\{B_j: j=1, \cdots, n\}$ is a uniform partition of order $n \leqq m$ of the same event E, then $A_i \leqq B_j$ for every i, j.*

Proof. Assume, on the contrary, that $B_j < A_i$ for some i, j. Then the same relationship must hold for all i, j because the partitions are uniform. Hence, by monotony we have

$$\bigcup_{j=1}^{n} B_j < \bigcup_{i=1}^{n} A_i,$$

and therefore

$$E < \bigcup_{i=1}^{n} A_i,$$

contrary to the assumption that the A_i's constitute a uniform partition of E.

COROLLARY 1. *If $\{A_i\}$, $\{B_j\}$ are two uniform partitions of the same order, of an event $E > O$, then $A_i = B_j$ for all i, j.*

COROLLARY 2. *If $\{A_i\}$ is a uniform partition of order m of an event $E > O$, and $\{B_j\}$ is a uniform partition of order $n = m\,k$ of the same event, then, for any i, A_i and the union of k events B_j are equally probable events.*

4. Compatible additive measures. Let $\mathfrak{R}$ be a Boolean ring, and let $\mathfrak{F}$ be a subset of $\mathfrak{R}$, which contains the impossible event O. An *additive functional* on $\mathfrak{F}$ is a real valued function ϕ defined on $\mathfrak{F}$ such that

$$\phi(A \cup B) = \phi(A) + \phi(B) \tag{4.1}$$

whenever A, B, $A \cup B$ belong to $\mathfrak{F}$ and $A \cap B \equiv O$. It follows immediately that $\phi(O) = 0$. An additive functional on $\mathfrak{F}$ is *compatible* with a qualitative probability $\leqq$ defined on $\mathfrak{R}$, if $A \leqq B$ is equivalent to $\phi(A) \leqq \phi(B)$ for all pairs of events A, B belonging to $\mathfrak{F}$. Thus, if an additive functional ϕ defined on $\mathfrak{F}$ is compatible with a qualitative probability $\leqq$, the knowledge of ϕ implies the knowledge of the qualitative probability within $\mathfrak{F}$. An *additive measure* on $\mathfrak{F}$ is a nonnegative additive functional on $\mathfrak{F}$, i.e., is an additive functional μ defined on $\mathfrak{F}$ such that $\mu(A) \geqq 0$ for all $A \in \mathfrak{F}$. Since $A \geqq O$ for all events A, it follows that, if an additive functional is compatible with a qualitative probability, then it is an additive

measure. An *additive probability measure* P is an additive measure defined in a Boolean algebra such that $P(\Omega)=1$. We shall say that a qualitative probability ring is *fissionable* if every event can be partitioned in two equally probable events, and that it is reducible if it is fissionable and for every pair of events A, B such that $O<A<B$, there is a uniform partition $\mathcal{E}$ of B such that $E<A \leqq B-E$ for every $E\in\mathcal{E}$. Clearly, if $(\mathfrak{R}, \leqq)$ is fissionable, then every nonnull event has uniform partitions of order $n=2^p$ for any positive integer p. Given an event $E\in\mathfrak{R}$, the *section of* $\mathfrak{R}$ *determined by* E is the collection of all events $A\in\mathfrak{R}$ such that $A\leqq E$.

THEOREM 4.1. *If* $(\mathfrak{R}, \leqq)$ *is a reducible qualitative probability ring, then there is one and only one (up to an arbitrary scale factor) compatible additive measure on any section of* $\mathfrak{R}$.

Proof. We assume that there are nonnull events, since otherwise the theorem is trivial. The proof will be divided into several steps.

(a) *Uniqueness.* Let μ be a compatible additive measure defined on the section of $\mathfrak{R}$ determined by the event $E>O$. Let $\{E_i: i=1, \cdots, n\}$ be a uniform partition of E of order $n=2^p$. Then we have

$$\mu(E) = \sum_{i=1}^{n} \mu(E_i).$$

But, since the events E_i are equally probable, we have $\mu(E_1)= \cdots =\mu(E_n)$ and therefore, for $m\leqq n$, $\mu(E_1\cup \cdots \cup E_m)=\mu(E)m/n$. By compatibility $\mu(E)>0$, and consequently

$$\frac{\mu(E_1 \cup \cdots \cup E_m)}{\mu(E)} = \frac{m}{n}.$$

Consider now an arbitrary event $A\leqq E$. Let

$$(4.2) \qquad \begin{aligned} r &= \max\left\{m: \bigcup_{i=1}^{m} E_i \leqq A, m = 0, \cdots, n\right\} \\ s &= \min\left\{m: A \leqq \bigcup_{i=1}^{m} E_i, m = 0, \cdots, n\right\}. \end{aligned}$$

Note that, by convention, $\bigcup_{i=1}^{0} E_i \equiv O$. By compatibility we have then

$$\frac{r}{n} \leqq \frac{\mu(A)}{\mu(E)} \leqq \frac{s}{n}.$$

The uniqueness of $\mu(A)/\mu(E)$ follows immediately from the fact that n may be made arbitrarily large and $s-r\leqq 1$.

(b) *Definition of* μ. In order to prove that the above procedure gives a consistent definition for $\mu(A)/\mu(E)$, it is sufficient to show that, if $\mathcal{E}'=\{E_i': i=1, \cdots, n'\}$ and $\mathcal{E}''=\{E_i'': i=1, \cdots, n''\}$ are two uniform partitions of E of

orders $n'=2^{p'}$, $n''=2^{p''}$, and if r', s'; r'', s'' are the corresponding numbers defined as in (4.2) then

$$\frac{r'}{n'} \leqq \frac{s''}{n''}. \tag{4.3}$$

Let $\mathcal{E}=\{E_i: i=1, \cdots, n\}$ be a uniform partition of order $n=n'n''$. Since any element of $\mathcal{E}'$ and the union of n'' elements of $\mathcal{E}$ are equally probable events, from $\bigcup_{i=1}^{r'} E_i' \leqq A$ it follows that $\bigcup_{i=1}^{r'n''} E_i \leqq A$ and therefore $r'n'' \leqq r$, or, equivalently, $r'/n' \leqq r/n$. Similarly, $s/n \leqq s''/n''$, and from the last two inequalities (4.3) follows immediately.

(c) *Additivity.* Let A, B be two disjoint events, and let $C \equiv A \cup B$. We assume that A, B and C belong to the section of $\mathfrak{R}$ determined by E. Let $\{E_i: i=1, \cdots, n\}$ be a uniform partition of E of order $n=2^p$ and let $r(A)$, $s(A)$; $r(B)$, $s(B)$; $r(C)$, $s(C)$ be the corresponding numbers defined as in (4.2). Then we have

$$\bigcup_{i=1}^{r(C)} E_i \leqq C \leqq \bigcup_{i=1}^{s(C)} E_i, \qquad \bigcup_{i=1}^{s(A)} E_i \geqq A \geqq \bigcup_{i=1}^{r(A)} E_i.$$

By subtraction we have

$$\bigcup_{i=s(A)+1}^{r(C)} E_i \leqq B \leqq \bigcup_{i=r(A)+1}^{s(C)} E_i. \tag{4.4}$$

These inequalities are also valid when the unions are empty; i.e., when $r(A)=s(C)$ or when $r(C) \leqq s(A)$, in which case, by convention, the empty unions are equal to the impossible event O. From (4.4) it follows that

$$r(C) - s(A) \leqq r(B) \leqq s(B) \leqq s(C) - r(A).$$

Multiplying by $\mu(E)/n$ and letting $n=2^p \to \infty$, we get

$$\mu(C) - \mu(A) \leqq \mu(B) \leqq \mu(C) - \mu(A).$$

Since the first and last members are equal, (4.1) follows immediately.

(d) *Compatibility.* Assume that $A < B \leqq E$. Since $(\mathfrak{R}, \leqq)$ is reducible there is an event $C \subset B$ such that $A \leqq C < B$. Let $\{E_i: i=1, \cdots n\}$ be a uniform partition of E of order $n=2^p$ such that $E_1 \cup E_2 \leqq B-C$. Let $s(A)$, $r(A)$, $s(B)$, $r(B)$ be defined as in (4.2). Then by the axiom of monotony, from

$$\bigcup_{i=1}^{s(A)-1} E_i \leqq C \quad \text{and} \quad E_{s(A)} \cup E_{s(A)+1} \leqq B - C$$

it follows that

$$\bigcup_{i=1}^{s(A)+1} E_i \leqq B.$$

Hence $s(A)+1 \leqq r(B)$ and therefore $\mu(A) < \mu(B)$. If, on the other hand, $A=B$, then the definition of μ shows that $\mu(A)=\mu(B)$.

THEOREM 4.2 *If* $(\mathfrak{A}, \leqq)$ *is a reducible qualitative probability algebra, then there is one and only one additive probability measure which is compatible with the qualitative probability* $\leqq$.

Proof. This theorem follows immediately from the previous one.

We shall say that a qualitative probability ring $(\mathfrak{R}, \leqq)$ is *σ-bounded* if there is a sequence of events $E_1 \leqq E_2 \leqq \cdots$ such that, if $A \in \mathfrak{R}$, then there is an integer n for which $A \leqq E_n$.

THEOREM 4.3. *If* $(\mathfrak{R}, \leqq)$ *is a reducible and σ-bounded qualitative probability ring, then there is one and only one (up to an arbitrary scale factor) compatible additive measure.*

Proof. Let $O < E_1 \leqq E_2 \leqq \cdots$ be a sequence of events such that $A \leqq E_n$ for any event A and some integer n, depending on A. Let $\mathfrak{F}_n$ be the family of all events which are less probable than E_n. Let μ_n be the additive measure defined on $\mathfrak{F}_n$ which is compatible with $\leqq$ and is such that $\mu_n(E_n) = 1$. Obviously, if $m \leqq n$ we will have $\mu_n(E_m) > 0$. Define $\mu(E_1) > 0$ arbitrarily, and then define $\mu(E_n)$ by

$$\mu_n(E_1) = \frac{\mu(E_1)}{\mu(E_n)} \cdot \tag{4.5}$$

Then for $A \in \mathfrak{F}_n$ define

$$\mu(A) = \mu_n(A)\mu(E_n). \tag{4.6}$$

It is clear that this is a compatible additive measure; however, we must prove that this definition is consistent, or, in other words, we have to prove that, if $A \in \mathfrak{F}_n$, $A \in \mathfrak{F}_m$, then

$$\mu_n(A)\mu(E_n) = \mu_m(A)\mu(E_m). \tag{4.7}$$

Assume that $m < n$. Then for any $A \in \mathfrak{F}_m$,

$$\mu_n(A)/\mu_n(E_m) = \mu_m(A) \tag{4.8}$$

because the left hand member of this equality is a compatible additive measure defined in $\mathfrak{F}_m$ which gives to E_m the value 1. Since $E_1 \in \mathfrak{F}_m$, from (4.8) we have

$$\mu_n(E_1)/\mu_n(E_m) = \mu_m(E_1). \tag{4.9}$$

Then from (4.5) and (4.9) we get

$$\mu_n(E_m)\mu(E_n) = \mu(E_m). \tag{4.10}$$

By multiplying of (4.8) and (4.10), we get equation (4.7) immediately.

5. Monotone continuity. We shall say that a sequence of events $\{A_n : n = 1, 2, \cdots\}$ *converges from above to* A and we shall write $A_n \searrow A$, if it is a monotone decreasing sequence which converges to A, that is, if $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} A_n \equiv A$. A qualitative probability $\leqq$ defined on a Boolean δ-ring is monotonely continuous if from $A_n \searrow A$ and $A_n \geqq B$ for all n, it follows that $A \geqq B$.

If $\mathfrak{R}$ is a Boolean δ-ring, and $\leqq$ is a monotonely continuous qualitative probability defined on $\mathfrak{R}$, then the pair $(\mathfrak{R}, \leqq)$ will be called a *qualitative probability δ-ring.*

THEOREM 5.1. *Let $\mathfrak{R}$ be a Boolean δ-ring, let $\leqq$ be a qualitative probability defined on $\mathfrak{R}$, and let μ be a compatible additive measure. Then μ is countably additive if and only if $\leqq$ is monotonely continuous.*

Proof. The proof is omitted since it is similar to the corresponding proof for Boolean algebras given in [13].

THEOREM 5.2. *Any atomless qualitative probability δ-ring is reducible.*

Proof. By Theorem 3.4 of [13], the qualitative probability δ-ring is fissionable, and therefore, given two events A, B such that $O<A<B$, there is a sequence of uniform partitions of B, $\mathcal{E}_n = \{E_1^{(n)}, \cdots, E_{2^n}^{(n)}\}$, such that

$$E_m^{(n-1)} \equiv E_{2m-1}^{(n)} \cup E_{2m}^{(n)}.$$

That is, the elements of the partition $\mathcal{E}_n$ are obtained by partitioning each element of $\mathcal{E}_{n-1}$ in two equally probable events. By Lemma 3.2 of [13], the limit of the monotone sequence $\{E_1^{(n)}\}$ is a null event, and therefore there is one n such that $E_1^{(n)}<A$. On the other hand, the sequence $\{C_n\}$ defined by $C_n \equiv B - E_1^{(n)}$ is a monotone increasing sequence which converges to an event which differs from B at most by a null event; therefore there is one n such that $A \leqq C_n < B$.

THEOREM 5.3. *Let $(\mathfrak{R}, \leqq)$ be a σ-bounded, atomless qualitative probability δ-ring. Then there is one and only one (up to an arbitrary scale factor) compatible additive measure, and it is countably additive.*

Proof. This is an immediate consequence of Theorems 5.1, 5.2 and 4.3.

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A THEOREM ON REPEATING DECIMALS

W. G. LEAVITT, University of Nebraska

It is well known that a real number is rational if and only if its decimal expansion is a repeating decimal. For example, $2/7 = .285714285714 \cdots$. Many students also know that if n/m is a rational number reduced to lowest terms (that is, n and m relatively prime), then the number of repeated digits (we call this the *length of period*) depends only on m. Thus all fractions with denominator 7 have length of period 6. A sharp-eyed student may also notice that when the period (that is, the repeating digits) for 2/7 is split into its two half-periods 285 and 714, then the sum $285+714=999$ is a string of nines. A little experimentation makes it appear likely that this is always true for a fraction with the denominator 7, as well as for fractions with denominators 11, 13, or 17. A natural conjecture is that all primes with even length of period (note that many primes, such as 3 and 31, have odd length of period) will have a similar property. This conjecture is, in fact, true but it is unfortunately not a criterion for primeness, since many composite numbers (such as 77) also have the property. The relevant theorem appears not to be well known, although it was discovered many years ago. (L. E. Dickson [see 1, p. 163] attributes the result to E. Midy, Nantes, 1836). The proof of the theorem is simple and elegant, and since it also provides a nice example of the usefulness of the concept of the order of an element of a group, it deserves to be better known.

In the following, we will develop from the beginning the theory of repeating decimals. This is to provide the necessary machinery for the proof of Midy's theorem, as well as for completeness.

Write (n, m) for the G.C.D. of n and m. Assuming n/m is a fraction reduced to lowest terms is thus equivalent to supposing $(n, m)=1$. (Note: we are interested only in positive fractions, so we restrict ourselves to $m>1$ and $n>0$.) Without loss of generality, we may also assume $(10, m)=1$, for if m is divisible by 5 or 2 we could multiply numerator and denominator of n/m by the appropriate power of 2 or 5 to obtain $n/m=10^{-h}n'/m'$ (for some integer $h\geqq 1$), where we still have $(n', m')=1$, and m' is not divisible by either 2 or 5. Since dividing by 10^h simply moves the decimal place, it is clear that the repeating digits of n/m are the same as those for n'/m'.

Write $n/m=c+h_1/m$ where $0<h_1<m$ and $c\geqq 0$ is an integer. Since $n=cm+h_1$, it is clear that we still have $(h_1, m)=1$. Also, since c may be any integer, the

repeating digits we are after will be those of h_1/m. Now we are interested only in the digits obtained upon division by m (not in the position of the decimal point). Thus when in the division process we bring down the first zero, we are actually dividing $10h_1$ by m. Let a_1 be the first digit in the quotient ($0 \leqq a_1 \leqq 9$) and let h_2 be the next remainder, then

$$h_2 = 10h_1 - a_1m.$$

Repeating the division process to obtain the next digit a_2 and the next remainder h_3,

$$h_3 = 10h_2 - a_2m.$$

Thus in general for any $t \geqq 2$,

$$h_t = 10h_{t-1} - a_{t-1}m. \tag{1}$$

Substituting for h_{t-1} from the preceding relation of (1) yields

$$h_t = 10^2h_{t-2} - (10a_{t-2} + a_{t-1})m,$$

and in general for any s (where $t > s \geqq 1$),

$$h_t = 10^sh_{t-s} - (10^{s-1}a_{t-s} + 10^{s-2}a_{t-s+1} + \cdots + 10a_{t-2} + a_{t-1})m. \tag{2}$$

Note that since $(m, 10h_1) = 1$ it follows from (1) that m and h_2 have no common factors, so $(m, 10h_2) = 1$. It is thus clear (inductively) that for all the remainders h_i we have $(m, 10h_i) = 1$. But the total number of integers less than m and prime to m is finite. (Remark that this number is the so-called "Euler ϕ-function" $\phi(m)$). Accordingly there must exist integers k and $k+r$ such that $h_k = h_{k+r}$. Then from (2)

$$10^rh_k - h_{k+r} = (10^{r-1}a_k + \cdots)m,$$

so that m divides $(10^r - 1)h_k$. But $(m, h_k) = 1$, and so m divides $10^r - 1$ (write this $m \mid 10^r - 1$). Thus $10^r \equiv 1 \pmod{m}$, and we assume that r is the least such integer. Another way of saying this is: r is the order of 10 in the multiplicative group (modulo m) of all integers $< m$ and relatively prime to m. As we mentioned above, the order of this group is $\phi(m)$, so by the Lagrange theorem, $r \mid \phi(m)$.

Writing, as we have indicated, the decimal expansion of h_1/m as $.a_1a_2 \cdots$, we can prove:

THEOREM 1. *If r is the order of 10 modulo m, then $a_{r+i} = a_i$ for all $i \geqq 1$. Conversely, if there exist positive integers k and u such that $a_{u+i} = a_i$ for all $i \geqq k$, then $r \mid u$.*

Proof. From (2) we have $h_{r+i} \equiv 10^rh_i \pmod{m}$, and since $10^r \equiv 1 \pmod{m}$, it follows that $h_{r+i} \equiv h_i \pmod{m}$. But all remainders $h_i < m$, so this implies $h_{r+i} = h_i$. Then from (1)

$$0 = h_{r+i+1} - h_{i+1} = (a_{r+i} - a_i)m,$$

whence

$$a_{r+i} = a_i. \tag{3}$$

Conversely, suppose $a_{u+i}=a_i$ for all $i \geqq k$. If we take w large enough so that $wr \geqq k$, then this says that $a_{u+wr+i}=a_{wr+i}$ for all $i \geqq 1$, so by (3), $a_{u+i}=a_i$. Now let $d=(u, r)$ and take advantage of the fact that there exist integers s and t such that $su+tr=d$. We know that either s or t must be positive, so suppose $s>0$ (same proof if $t>0$). Then $a_{su+i}=a_i$ and by using (3), $a_{su+tr+i}=a_i$. Thus $a_{d+i}=a_i$. Once again, using (1), it is clear that

$$10(h_{d+i} - h_i) = h_{d+i+1} - h_{i+1}, \tag{4}$$

and $10(h_{d+i+1}-h_{i+1})=h_{d+i+2}-h_{i+2}$. Multiplying (4) by 10 and adding, yields

$$10^2(h_{d+i} - h_i) = h_{d+i+2} - h_{i+2}.$$

Thus (inductively), for any $q \geqq 1$,

$$10^q(h_{d+i} - h_i) = h_{d+i+q} - h_{i+q}.$$

But since all $h_i<m$, the right hand side (in absolute value) is less than m. If we choose q large enough so that $10^q>m$, this relation could be true only if $h_{d+i}=h_i$. As in the discussion of above, this implies $10^d \equiv 1 \pmod{m}$. But r is the order of 10 modulo m and so from group theory we know that $r \mid d$. Since $d=(u, r)$, it follows that $r=d$ and so $r \mid u$.

From this theorem it is clear that the digits repeat if and only if we moved ahead r digits (or some multiple of r digits). That is to say, all fractions h_1/m have periods of length precisely r, where r is the order of 10 modulo m.

Note that when m is prime, $\phi(m)=m-1$. Thus when m is prime, $r \mid m-1$.

We now discuss the case of even period r.

LEMMA 1. *If there exists a positive integer v such that $10^v \equiv -1 \pmod{m}$, then r is even, where r is the order of 10 modulo m.*

Proof. Since $10^{2v} \equiv 1 \pmod{m}$ and r is the period of 10 modulo m, we know that $r \mid 2v$. If r were odd then $r \mid v$ and thus $10^v \equiv 1 \pmod{m}$. But from $10^v \equiv -1 \pmod{m}$ we would then get $0 \equiv 2 \pmod{m}$, which is impossible (since we are assuming $(m, 10)=1$).

LEMMA 2. *If m is prime and r is even, then $10^{r/2} \equiv -1 \pmod{m}$.*

Proof. Write $r=2w$, then since $10^r \equiv 1 \pmod{m}$ we have $(10^w-1)(10^w+1) \equiv 0 \pmod{m}$. Since m is prime it must divide one or the other of the terms on the left-hand side. But r is the order of 10 and so $10^w \not\equiv 1 \pmod{m}$. Thus m can only divide the second of the two factors, that is $10^w \equiv -1 \pmod{m}$.

We will continue to assume that $r=2w$ and that the period is $a_1 \cdots a_w a_{w+1} \cdots a_{2w}$. Let c and d be the two half-periods, so that $c=a_1 \cdots a_w$ and $d=a_{w+1} \cdots a_{2w}$ (or writing in terms of powers of 10, $c=a_1 10^{w-1}+a_2 10^{w-2} + \cdots + a_w$ and $d=a_{w+1}10^{w-1}+a_{w+2}10^{w-2}+ \cdots + a_{2w}$).

We can now prove Midy's theorem in a very simple way.

THEOREM 2. *If* $r=2w$ *and* $10^w \equiv -1 \pmod{m}$, *then* $c+d=10^w-1=99\cdots 9$.

Proof. From (2) we have $h_{w+1}=10^w h_1-cm$, $h_{2w+1}=10^w h_{w+1}-dm$. But $r=2w$ is the period, so $h_{2w+1}=h_1$ and thus

$$(h_1 + h_{w+1})(10^w - 1) = (c + d)m. \tag{5}$$

But by hypothesis $10^w-1\equiv -2 \pmod{m}$, so 10^w-1 and m are relatively prime. From (5) we therefore have that $m \mid h_1+h_{w+1}$. But each $h_i<m$ so $h_1+h_{w+1}<2m$. It can therefore be divisible by m only if it equals m, that is, $h_1+h_{w+1}=m$. Hence $c+d=10^w-1$.

Notice that, once again, this theorem depends only on the denominator m of the fraction. Let us say that an integer m which satisfies Theorem 2 has the *nines-property*. From Lemma 2, it immediately follows that:

COROLLARY 1. *Every prime with even period has the nines-property.*

However, it is also true, as the following corollary shows, that there are many composite numbers with the nines-property.

COROLLARY 2. *If* $m \mid (10^p+1)$ *where* p *is prime, then* m *has the nines-property.*

Proof. By hypothesis, $10^p\equiv -1 \pmod{m}$ and so $10^{2p}\equiv 1 \pmod{m}$. Since r is the order of 10 modulo m, it follows that $r \mid 2p$. If $r \mid 2$ then $m \mid 99$. But we always have $10^p\equiv 1 \pmod 3$ and so $10^p \not\equiv -1 \pmod 3$. Since $10^p\equiv -1 \pmod{m}$, it follows that m and 3 are relatively prime. Thus in this case we could only have $m=11$, and it is easily verified directly that $m=11$ satisfies Theorem 2.

The remaining cases are $r=p$, which (as we saw in the proof of Lemma 1) contradicts $10^p\equiv -1 \pmod{m}$ or $r=2p$. In the latter case, Theorem 2 is evidently satisfied.

From this corollary, it follows that there are many composite nines-numbers. For example $10^3+1=7\cdot 11\cdot 13$ so that in addition to primes 7, 11, and 13 we also have composites 77, 91, and 143 (as well as 1001 itself) as nines-numbers.

COROLLARY 3. *If* m *is prime and* $10^{(m-1)/2}\equiv -1 \pmod{m}$ *then* m *is a nines-number.*

Proof. By Lemma 1, r is even and so by Corollary 1, m is a nines-number.

This corollary, together with the theory of quadratic residues, shows that any prime of form $40k \pm 7, \pm 11, \pm 17$, or ± 19 is a nines-number. (For a discussion of quadratic residues see, for example, [2, Chapter 5]. The condition $10^{(m-1)/2}\equiv -1 \pmod{m}$ is the so-called Euler criterion that 10 should be a quadratic nonresidue of the prime m [2, p. 46]. A determination of the form given for the primes for which 10 is a quadratic non-residue is found, as an example, in [2, p. 74].)

On the other hand, there are many primes m with odd periods. If, for example, $(m-1)/2$ is odd and $10^{(m-1)/2}\equiv 1 \pmod{m}$, then clearly r is odd, since it

must divide $(m-1)/2$. Again from the theory of quadratic residues, this class will contain all primes of form $40k+3$, -13, -9, or -1.

Note that Corollary 3 does not give all prime nines-numbers. One such prime which escapes Corollary 3 is 13.

Another familiar class of numbers, the Fermat numbers $m=2^{2^n}+1$, are covered by the following:

COROLLARY 4. *Every prime Fermat number* >5 *is a nines-number.*

Proof. This can be established directly, or one can use a simple induction to show that when $n \geqq 2$, $2^{2^n} \equiv 16 \pmod{40}$. The result is thus a special case of the remark following Corollary 3. This class is, of course, somewhat restricted since the only Fermat primes known which are >5 are given by $n=2$, 3, or 4.

This paper was presented at the May, 1965 meeting of the Nebraska Section of the Mathematical Association of America.

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AN INTEGRATED MODEL FOR EUCLIDEAN AND NON-EUCLIDEAN GEOMETRIES

L. G. NOVOA, University of Alabama

1. Introduction. The usual geometric models for non-Euclidean geometries are specifically designed either for hyperbolic geometry (Poincaré, Cayley-Klein, etc.) or for elliptic geometry (bundle and spherical models).

In this paper, we present models which are constructed by using essentially the same idea for all three types of classical two dimensional geometries: hyperbolic, elliptic and Euclidean.

We consider first a set Γ whose elements are certain ellipses in three dimensional Euclidean space, introduce a suitable definition of distance, and show that in a class of subspaces the metric gives rise, quite naturally, to the different types of geometries.

2. The metric space (Γ, δ). Consider the system of quadrics in E^3 with cartesian coordinates X, Y, Z:

$$X^2 + Y^2 - kZ^2 = 1 \quad \text{for all real values of } k. \tag{1}$$

Let these surfaces be intersected by planes which pass through the origin but do not contain the axis OZ. We shall be interested only in those intersections which are ellipses.

To the family Γ of all such ellipses we give a metric structure in the following way.

Let γ be an ellipse of Γ. We project it on the cylinder C: $X^2+Y^2=1$ axially with respect to OZ. This means that to every point P of γ we assign the intersection P' of the line perpendicular to OZ and passing through P, with the cylinder. This gives a closed curve γ' on the cylinder. Let Γ' be the family of all γ'.

In cylindrical coordinates (r, θ, Z) the equation of the cylinder is $r=1$, and θ, Z can be considered as coordinates of a point on the cylinder.

Let the ellipse γ be the intersection of the quadric of parameter k with the plane $Z=mX\cos\theta_0+mY\sin\theta_0$. Then the equation of the curve γ' on the cylinder is:

$$Z=\frac{m\cos(\theta-\theta_0)}{\sqrt{[1-km^2\cos^2(\theta-\theta_0)]}},\qquad \left(\text{if } k>0, \text{ then } |m|<\frac{1}{\sqrt{k}}\right).$$

Two different ellipses of Γ have different projections on C, as can be easily checked. To each system of values for k, m, θ_0 there correspond a definite γ and, also, a definite γ'. If k, m, θ_0 are fixed, then Z in (1) becomes a function of θ: $\phi(\theta)$. Let $\phi_1(\theta)$ and $\phi_2(\theta)$ represent the functions corresponding to two ellipses γ_1, γ_2 of Γ. We define the distance $\delta(\gamma_1, \gamma_2)$ by:

$$\delta(\gamma_1,\gamma_2)=\frac{1}{2}\int_0^{\pi}|\phi_1(\theta)-\phi_2(\theta)|\,d\theta. \tag{2}$$

It is well known from the theory of function spaces that this definition gives a metric structure to the set Γ' and hence to Γ. We designate by (Γ, δ) the metric space thus generated.

3. The subspaces for constant k. We now study the two dimensional subspaces of (Γ, δ) obtained by giving a fixed value to k. There are three different types, depending on whether $k>0$, $k<0$, or $k=0$. The point Q of Γ for which $m=0$ belongs to all of them. We call Q the origin of Γ and it is the circle common to all quadrics of (1).

Straightforward computation gives for the integral: $I=\int_\alpha^{\alpha+\pi}\phi(\theta)\,d\theta$, where $\phi(\theta)=[m\cos(\theta-\theta_0)]/\sqrt{[1-km^2\cos^2(\theta-\theta_0)]}$, the values:

$$\begin{cases} I=-\dfrac{2}{\sqrt{k}}\sinh^{-1}\dfrac{m\sin(\alpha-\theta_0)}{\sqrt{[k^{-1}-m^2]}} & \text{for } k>0,\\[2ex] I=-\dfrac{2}{\sqrt{-k}}\sin^{-1}\dfrac{m\sin(\alpha-\theta_0)}{\sqrt{[m^2-k^{-1}]}} & \text{for } k<0,\\[2ex] I=-2m\sin(\alpha-\theta_0) & \text{for } k=0. \end{cases} \tag{3}$$

To compute the distance between two points (m_1, θ_1) and (m_2, θ_2), for some

subspace, we use the periodic properties of ϕ and put:

$$2\delta(\gamma_1, \gamma_2) = \int_0^{\pi} |\phi_1 - \phi_2| \, d\theta = \left| \int_{\alpha_0}^{\alpha_0+\pi} (\phi_1 - \phi_2) d\theta \right| ,$$

where α_0 is the value of θ for one of the common points of γ_1' and γ_2' on the cylinder i.e.: $\phi_1(\alpha_0) = \phi_2(\alpha_0)$. This implies that:

$$m_1 \cos(\alpha_0 - \theta_1) = m_2 \cos(\alpha_0 - \theta_2). \tag{4}$$

If $k > 0$ we use (3) to obtain

$$\delta(\gamma_1, \gamma_2) = \frac{1}{\sqrt{k}} \left| \sinh^{-1} \frac{m_1 \sin(\alpha_0 - \theta_1)}{\sqrt{[k^{-1} - m_1^2]}} - \sinh^{-1} \frac{m_2 \sin(\alpha_0 - \theta_2)}{\sqrt{[k^{-1} - m_2^2]}} \right| ;$$

calling $R = 1/\sqrt{|k|}$, taking the hyperbolic cosine of both sides, and using (4), we get

$$\cosh \frac{\delta}{R} = \frac{R^2 - m_1 m_2 \cos(\theta_1 - \theta_2)}{\sqrt{[R^2 - m_1^2]}\sqrt{[R^2 - m_2^2]}} . \tag{5}$$

The distance ρ from γ to the origin is: $\cosh \rho/R = R/\sqrt{[R^2 - m^2]}$. If we take the number ρ as a coordinate (instead of m) formula (5) becomes:

$$\cosh \frac{\delta}{R} = \cosh \frac{\rho_1}{R} \cosh \frac{\rho_2}{R} - \sinh \frac{\rho_1}{R} \sinh \frac{\rho_2}{R} \cos(\theta_1 - \theta_2), \tag{6}$$

which is the standard formula for the distance in hyperbolic geometry when we take hyperbolic polar coordinates (ρ, θ).

A similar analysis for the case $k < 0$ gives for the distance:

$$\cos \frac{\delta}{R} = \cos \frac{\rho_1}{R} \cos \frac{\rho_2}{R} + \sin \frac{\rho_1}{R} \sin \frac{\rho_2}{R} \cos(\theta_1 - \theta_2) \tag{7}$$

which again is the well-known formula for the distance in spherical geometry when we take polar coordinates (ρ, θ).

Finally if $k = 0$ then $\delta = m_1 \sin(\alpha_0 - \theta_1) - m_2 \sin(\alpha_0 - \theta_2)$, from which, squaring and using (4), we get: $\delta^2 = m_1^2 + m_2^2 - 2m_1 m_2 \cos(\theta_1 - \theta_2)$. Since m represents, in this case, the distance ρ to the origin we have:

$$\delta^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2)$$

which is the distance in polar coordinates in the Euclidean plane.

In summary, the subspaces with $k > 0$ are hyperbolic planes, those with $k < 0$ are spherical regions (in fact half-spheres) and for $k = 0$ we get a Euclidean plane.

It is interesting to note that the intersections which are not ellipses, and which exist if $k \geqq 0$, can be identified in a rather natural way with the ideal and ultra-ideal points of the euclidean and hyperbolic planes.

4. The case $k<0$. In this case the quadric of (1) is the ellipsoid σ:

$$X^2 + Y^2 + \frac{Z^2}{R^2} = 1, \qquad \text{where } R = \frac{1}{\sqrt{|k|}}$$

and ρ, as defined above, is properly bounded by $(\pi/2)R$. Hence the corresponding 2-dimensional subspace S is isometric to an open half-sphere with the metric defined by (7). To pass from S to an elliptic plane we have to add the intersections γ^* of σ with the planes through OZ, which were omitted at the beginning, and redefine the distance in the usual way:

$$\delta' = \text{Min } (\delta, \pi R - \delta). \tag{8}$$

This process will add to the half-sphere mentioned above its boundary: $\rho=(\pi/2)R$ and at the same time will identify antipodal points, if we agree that $\delta'(P, Q)=0$ means $P=Q$.

Since the axial projections of "vertical" ellipses (those which pass through OZ) on the cylinder are not well defined, we agree to associate to each such ellipse γ^*, determined by the plane $\theta=\theta_0$, the function ϕ_{γ^*} defined as follows:

$$\phi_{\gamma^*} = -\frac{\pi}{2}R \quad \text{for } 0 \leqq \theta \leqq \theta_0$$

and

$$\phi_{\gamma^*} = \frac{\pi}{2}R \quad \text{for } \theta_0 < \theta \leqq \pi.$$

Call S' the set of vertical ellipses. The set $P=S\cup S'$ is a metric space as soon as we extend the definition of distance of paragraph 2 to the new elements γ^* and the associated functions ϕ_{γ^*}.

Finally, the redefinition (8) will make P into an elliptic plane. We leave the details to the reader, since they are similar to the usual procedure to pass from the metric 2-sphere to the elliptic plane.

When $k=-1$, the geometric meaning of definition (2) becomes particularly clear. In this case, the quadric becomes a sphere, and the axial projection of this sphere on the cylinder is area-preserving. Hence, the integral of (2) is proportional to one of the areas intercepted, on the sphere, by two great circles. The convention (8) means that δ' is proportional to the smaller of the two angles formed by the great circles. In other words, the metric δ' is equivalent to the angular metric of spherical geometry, which is, in turn, equivalent to the metric of the elliptic plane.

5. Geodesics. It is not difficult to see that, for a fixed k, the set of intersections of a family of planes which contain a fixed line l through O, with the corresponding quadric, constitutes a set of points of Γ, which is isometric to a circle if $k<0$ or to a straight line if $k\geqq 0$, and represents, therefore, a geodesic of Γ.

To every such geodesic we can associate the pair of points L_1, L_2, in which l meets the quadric, and which are common to all the intersections. Conversely, all the geodesics of the subspace $k=$constant can be obtained in this way.

This means that the lines (geodesics) of the Euclidean or non-Euclidean planes, considered in the preceding sections, can be identified with pairs of points, symmetric with respect to O, and contained in the corresponding quadric.

A BARRIER FUNCTION FOR THE ORIGIN OF A CONCAVE n-DIMENSIONAL HYPERSPHERICAL SECTOR

R. A. FRISCH, General Cable Corporation's Research Laboratories, Bayonne, N.J.

In the study of partial differential equations in an n-dimensional space ($n\geqq 2$, integer), the concept of a barrier function has been widely used to guarantee continuity of the solution of a partial differential equation on the boundary of the domain in which the equation is defined [1, pp. 168 *et seq.*]. A barrier function can be defined as follows:

Let the partial differential equation be defined in a domain G with boundary G' and closure $\overline{G}$. Let $Q\in G'$ be a point of the boundary, let $N_Q\in\overline{G}$ be a neighborhood of Q and let $P\in N_Q$ ($P\neq Q$) be a generic point of N_Q. A barrier function B for the point Q will have the following properties:

(i) B is defined, continuous, and singlevalued in N_Q and on its boundary;

(ii) $B(Q)=0$;

(iii) $B(P)>0$;

(iv) B is superharmonic in N_Q.

The existence of a barrier at a point Q guarantees the continuity of the solution f of the given partial differential equation at Q [1, 184 *et seq.*], and therefore the existence of a barrier function at every point of G' guarantees the continuity of f at the boundary.

The barrier function for the origin of a convex n-dimensional hyperspherical sector of radius R is well known [1, 187]. For the origin of a concave sector, however, proof of existence of a barrier for $n\geqq 3$ has rested on the exclusion from the sector of another origin-centered sector of radius r, $r<R$, and on letting r tend to zero [1, 187]. In the following a barrier is constructed for this origin, thus avoiding the use of a limiting process.

In an n-dimensional space with Cartesian coordinates x_i, $i=1, 2, \cdots, n$, $n\geqq 3$, let there be defined the two functions

$$r^2=\sum_{i=1}^{n}x_i^2 \qquad r\geqq 0, \tag{1}$$

$$\phi=\arccos(x_n/r) \qquad 0\leqq\phi\leqq\pi. \tag{2}$$

In this space, an origin-centered hyperspherical concave sector is defined by

$$0 < r < R \qquad R \text{ finite}, \tag{3}$$

$$0 \leqq \phi < \theta \qquad \pi/2 < \theta < \pi. \tag{4}$$

This sector, which we call S, is a domain. Let S' be the boundary of S and let $\overline{S} = S \cup S'$.

Consider the function

$$F = r^{1-\alpha} \cos \phi/k. \tag{5}$$

If $k>1$ is finite and such that $\phi/k < \pi/2$, and if $\alpha<1$, then F is obviously zero at the origin, positive everywhere else in $\overline{S}$ and continuous on S'.

The Laplacian of F is

$$\Delta F = \sum_{i=1}^{n} \frac{\partial^2 F}{\partial x_i^2} = k^{-2} r^{-\alpha-1} \left\{ \cos \frac{\phi}{k} \left[k^2(n-1-\alpha)(1-\alpha) - 1 \right] - k \sin \frac{\phi}{k} \operatorname{ctg} \phi \right\} . \tag{6}$$

Since $k^{-2}r^{-\alpha-1}>0$ for $k>1$ and finite, $\Delta F \leqq 0$ in S if and when

$$\cos \frac{\phi}{k} \left[1 - k^2(n-1-\alpha)(1-\alpha) \right] + k \sin \frac{\phi}{k} \operatorname{ctg} \phi \geqq 0. \tag{7}$$

Since $\sin \phi/k \geqq 0$ and $\cos \phi/k > 0$ everywhere in S, and since for $\pi/2 < \phi < \pi$, $\operatorname{ctg} \phi < 0$, it follows that

$$1 - k^2(n-1-\alpha)(1-\alpha) > 0 \tag{8}$$

is a necessary condition for the fulfillment of (7). Solution of (8) together with the formerly established condition $\alpha<1$ yields

$$0 < \tfrac{1}{2}\left[n - \sqrt{n^2 - 4(n-1-k^{-2})} \right] < \alpha < 1. \tag{9}$$

It is clear that if (8) is satisfied, (7) will be satisfied for $\phi=0$, for any $k>1$. For other values of ϕ, (7) can be written

$$G(\phi) = \eta\lambda \operatorname{ctg} \lambda\phi + \operatorname{ctg} \phi \geqq 0, \tag{10}$$

$$0 < \eta < 1, \qquad \eta = 1 - k^2(n-1-\alpha)(1-\alpha), \tag{11}$$

$$0 < \lambda < 1, \qquad \lambda = 1/k. \tag{12}$$

Two immediate properties of $G(\phi)$ are: $1-G \geqq 0$ for $0 < \phi \leqq \pi/2$, for any λ and η; $2 - dG/d\phi < 0$ for any λ and η for $0 < \phi < \pi$.

Therefore if for θ of (4), $G(\theta) \geqq 0$, then $G(\phi) > 0$ for $0 < \phi < \theta$.

$G(\theta)$ can be written, using Riemann's Zeta-function, as

$$(13)\qquad \eta\lambda\left[\frac{1}{\lambda\theta}-\sum_{n=1}^{\infty}\frac{2\zeta(2n)}{\pi^{2n}}(\lambda\theta)^{2n-1}\right]>\frac{1}{\pi-\theta}-\sum_{n=1}^{\infty}\frac{2\zeta(2n)}{\pi^{2n}}(\pi-\theta)^{2n-1}.$$

Introducing λ into the bracket, multiplying by $\theta(\pi-\theta)>0$ due to (4), we obtain (13) in the form

$$(14)\qquad \eta(\pi-\theta)-\theta\geqq\sum_{n=1}^{\infty}2\zeta(2n)\left[\eta(\pi-\theta)\left(\frac{\lambda\theta}{\pi}\right)^{2n}-\theta\left(\frac{\pi-\theta}{\pi}\right)^{2n}\right].$$

Since the zeta-function of a real argument larger than one is monotonically decreasing, (14) will be satisfied if

$$(15)\qquad \eta(\pi-\theta)-\theta\geqq A\left[\eta(\pi-\theta)\sum_{n=1}^{\infty}\left(\frac{\lambda\theta}{\pi}\right)^{2n}-\theta\sum_{n=1}^{\infty}\left(\frac{\pi-\theta}{\pi}\right)^{2n}\right],$$

where $A=2\zeta(2)$. The inequality (15) can be written as

$$(16)\qquad \frac{\eta(\pi-\theta)-\theta}{A}+\theta\zeta\left[\left(\frac{\pi}{\pi-\theta}\right)^{2}\right]\geqq\eta(\pi-\theta)\zeta\left[\left(\frac{\pi}{\lambda\theta}\right)^{2}\right]$$

and, since the zeta-function of a real argument larger than one is always larger than one, (16) will be satisfied if

$$(17)\qquad \frac{\eta(\pi-\theta)-\theta}{A}+\theta\geqq\eta(\pi-\theta)\zeta\left[\left(\frac{\pi}{\lambda\theta}\right)^{2}\right].$$

Putting $K\geqq\zeta[(\pi/\lambda\theta)^2]>1$, we see that (17) is satisfied if

$$(18)\qquad 2A\eta K\theta+2A\theta-\theta-\eta\theta\geqq 2A\eta K\pi-\eta\pi.$$

Inequality (18) will be satisfied if

$$(19)\qquad 2A\eta K\theta+2A\theta-\theta-\eta K\theta\geqq 2A\eta K\pi-\eta\pi,$$

or if

$$(20)\qquad \theta\geqq\pi\frac{\eta(2AK-1)}{(2A-1)(1+\eta K)}.$$

Inequality (20) will hold for $\theta\geqq\pi/2$ if

$$(21)\qquad K\leqq\frac{2A-1+2\eta}{\eta(2A+1)}.$$

The right side of (21) being always larger than one for $0<\eta^*<1$, it will always be possible to choose $\epsilon^*>0$ such that $K\geqq 1+\epsilon^*$.

Since $\zeta(x)\to 1$ if $x\to\infty$, there will exist a value $\lambda^*<1$ such that $\zeta[(\pi/\lambda^*\theta)^2]\leqq 1+\epsilon^*$. Thus (11) can be solved for α in terms of η^* and $k^*=1/\lambda^*$. Introducing the solution $\alpha^*<1$, together with k^*, into (7), we see that this inequality will

always be satisfied in S. Therefore $\Delta F \leqq 0$ everywhere in S. Consequently F is also superharmonic in S and constitutes a barrier function for the origin of the hyperspherical sector defined by (3) and (4).

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MATHEMATICAL NOTES

A NOTE ON LARGE MEASURABLE SETS

D. M. Topping, Tulane University

1. Purpose. A well-known result in the classical theory of Lebesgue measure describes a measurable set as the disjoint union of a Borel set and a null set. In the Bourbaki theory of integration on locally compact spaces, the question of finding a suitable counterpart to this result for non-σ-finite spaces does not appear to have been adequately answered in the literature.

We shall furnish a simple and natural generalization of the classical situation which, in a sense, is the best possible. The technique used is a slight variation of the old "exhaustion" method in modern transfinite disguise.

2. The setting. We begin with a locally compact Hausdorff space X and a positive integral (or Radon "measure") I, defined on the space of real continuous functions having compact supports. The theory proceeds by extending I to an "upper integral" $\bar{I}$ defined on *all* nonnegative functions on X with values in the interval $[0, +\infty]$. One then obtains a Carathéodory outer measure $\bar{\mu}$ by setting $\bar{\mu}(A) = \bar{I}(\chi_A)$ for each subset A of X, where χ_A is the characteristic function of the set A. The measurable sets can then be defined by the classical inequality of Carathéodory: A is *measurable* if $\bar{\mu}(S) \geqq \bar{\mu}(S \cap A) + \bar{\mu}(S \cap A')$, for all subsets S of X. Tracing through the theory, we find that the measurable sets thus defined are just the sets which are "locally measurable" [1] with respect to the ring of *summable* sets (sets whose characteristic functions are summable—these are precisely the measurable sets A with $\bar{\mu}(A) < +\infty$). For a set to be measurable, it is enough that it intersect every compact set in a summable set. An extremely lucid account of the theory appears in [2] Chapter 3, to which we refer for further details.

By a *Borel* set we shall mean an element of the σ-algebra generated by the open sets of X. Another notion of "Borel set" is in current usage (these are the σ-bounded Borel sets), and Berberian [1] has described very nicely the connections between the two. A *locally Borel* set is a subset of X which intersects each compact set in a Borel set. It is easily seen that a subset of X is locally Borel if and only if it meets each σ-bounded Borel set in a Borel set. A *locally*

null set is a subset N of X such that $\mu(N\cap K)=0$ for each compact subset K of X [2] (esp. pp. 124–130, 136 and 228). It is clear that locally Borel and locally null sets are measurable.

3. The decomposition. In the setting described above, our observation can now be stated. The σ-finite case is, of course, classical.

THEOREM. *Let A be measurable (and non-σ-finite). Then there is a family $\mathfrak{A}$ of pairwise disjoint compact sets, each contained in A, such that:*

(1) $\mu(U\cap K)>0$ *if U is open, $K\in\mathfrak{A}$ and $U\cap K\neq\varnothing$.*

(2) $B=\bigcup\{K:K\in\mathfrak{A}\}$ *is locally Borel.*

(3) $N=A-B$ *is locally null.*

(4) *Each open set of finite positive outer measure meets only countably many of the sets in $\mathfrak{A}$.*

Proof. The empty family satisfies (1) vacuously and Zorn's lemma shows that there is a maximal family $\mathfrak{A}$ of pairwise disjoint compact subsets of X, each contained in A, satisfying (1). Property (4) follows immediately from (1).

If C is a compact subset of X, then C meets only countably many of the sets in $\mathfrak{A}$, say $\{K_n:n=1,\ 2,\ \cdots\}$. Now $B\cap C=\bigcup\{K_n\cap C:n=1,\ 2,\ \cdots\}$ which is manifestly a Borel set. Thus B is locally Borel, and it follows that N is measurable. The proof that N is locally null proceeds exactly as in [2] ((11.39), p. 129).

REMARK. The set B in the theorem might be called a "locally Borel kernel" of A. Applying the construction to $X-A$, one obtains a "locally Borel cover," i.e., a locally Borel set $E\supset A$ with $E-A$ locally null.

4. An example. It is natural to ask if the set B in part (2) of the theorem can be taken to be a Borel set (clearly Borel sets are locally Borel). In special cases (apart from the classical σ-compact situation), the locally Borel sets will coincide with the Borel sets, e.g., when X is uncountable and discrete.

Perhaps the best "large" locally compact spaces for integration theory are the paracompact spaces, which include, for example, locally compact groups (such spaces are known to be the disjoint union of open σ-compact subspaces). Even here, however, locally Borel and Borel sets need not coincide, as shown by a simple example, essentially due to Kuratowski ([3] bottom of p. 268). Take X to be the product of a discrete real line with the usual real line. (Kuratowski uses $[0,\ 1]\times\Omega$, with Ω discrete, but this is an unimportant modification.) Kuratowski's argument shows that there are locally Borel sets in X which are not Borel sets, (this example was discovered independently by R. G. Douglas in conversation with the author). Moreover, X is a metrisable abelian group (hence paracompact), and with Haar measure (see [2] (11.33), p. 127) it is easily seen that our theorem gives the sharpest description of measurable sets that general circumstances will permit.

In case our space is a locally compact group, a sharper analysis can be made. The result below was noted in conversation with M. A. Moskowitz. For sim-

plicity, we assume that our group G is a (not necessarily connected) Lie group, but a standard approximation argument can be used to extend the theorem to the locally compact case.

THEOREM 2. *Locally Borel sets in G are Borel sets if and only if G is either σ-compact or zero-dimensional (i.e. discrete).*

Proof. If U is a symmetric neighborhood of the identity with compact closure, then $G_0=\bigcup_{n=1}^{\infty}\overline{U}^n$ is a σ-compact subgroup, which we may assume to be connected.

Assume now that G is neither σ-compact nor zero-dimensional. Since G_0 is locally Euclidean and not zero-dimensional, it contains a Borel set B_α of each class $\alpha<\Omega$. Well-order the cosets of G_0. Then, by assumption, there are *at least* Ω ordinals used in the well-ordering (otherwise G would be σ-compact). Choose a Borel set B_α of class $\alpha<\Omega$ from the α-th coset of G_0. Then as in Kuratowski's example, $\bigcup_{\alpha<\Omega}B_\alpha$ is locally a Borel set, but not a Borel set. The reverse implication is clear.

Berberian has recently communicated a related result (to appear in the MONTHLY): Haar measure on the Borel sets (as defined in section 2 above) of a locally compact group G is simultaneously inner and outer regular if and only if G is either σ-compact or discrete.

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A NONLINEAR GENERALIZATION OF A THEOREM ON FUNCTION ALGEBRAS

S. CATER, Portland State College

In this paper X will be a compact Hausdorff space, $C(X)$ will denote the algebra of all continuous real valued functions on X, and $D(X)$ will denote the family of all continuous functions from X to the interval $[0, 1]$. From the work of R. I. Jewett [1] we infer immediately the following variation of the Stone-Weierstrass Theorem.

PROPOSITION 1. *Let L be a nonvoid uniformly closed point separating subset of $D(X)$ satisfying*

(i) *if $f, g\in L$, then $fg\in L$,*

(ii) *if $f\in L$, then $1-f\in L$.*

Then there exists a (possibly void) totally disconnected closed subset E of X such that L consists of all the functions $f\in D(X)$ for which $f(E)$ is a subset of the doubleton $\{0, 1\}$.

This work inspires a nonlinear generalization of some well-known results on function algebras (for example, see [2], pp. 245–6) which we hope is of some intrinsic interest. We present

THEOREM 1. *Let L be as in Proposition* 1 *and let u be a mapping of L into the interval* $[0, 1]$ *satisfying*

(1) $u(fg)=u(f)u(g)$ *for each* $f, g\in L$,

(2) $u(1-f)=1-u(f)$ *for each* $f\in L$.

Then there exists a unique point $x_0\in X$ *such that* $u(f)=f(x_0)$ *for each* $f\in L$.

From this we can establish immediately

THEOREM 2. *Let L be as above and let Y be another compact Hausdorff space. Suppose ϕ is a mapping of L into $D(Y)$ satisfying* (1) *and* (2). *Then there exists a continuous mapping p of Y into X such that $\phi(f)(y)=f(py)$ for each $f\in L$, $y\in Y$.*

In [3] Milgram provides a result related to Theorem 2 which describes all the isomorphisms ϕ of the multiplicative semi-group $C(X)$ onto $C(Y)$; given ϕ there exists a homeomorphism T of Y onto X and for each $x\in X$ there exists a multiplicative mapping p_x of the real line onto itself for which

$$\phi(f)(y) = p_{Ty}[f(Ty)] \quad \text{for each } f \in C(X),\ y \in Y.$$

In [3] it is essential, however, that ϕ be a 1-1 mapping of $C(X)$ onto $C(Y)$, and functions in $C(X)-D(X)$ are required in the arguments (see Example 1). For more on multiplicative transformations see Bourgin [4] and [5].

Proof of Theorem 1. Either $u(L)$ is the doubleton $\{0, 1\}$ or else $u(L)$ contains a number $\neq 0$ or 1. We begin with the trivial case in which u takes only the values 0 and 1.

Let $f\in L$ such that $u(f)=1$. Then $1\in f(X)$; for otherwise there would exist an integer n such that $f^n<\frac{1}{2}<1-f^n$, $u(f^n)=u(f)^n=1$ and $u(1-f^n)=0$. But $f^n(1-f^n)^{-1}\in L$ and $u(f^n)=u(1-f^n)u[f^n(1-f^n)^{-1}]=0$, which is impossible.

Now suppose that $f_1 \cdots, f_m$ are finitely many functions in L which satisfy $u(f_i)=1$ for each i. Then $u(f_1 \cdots f_m)=u(f_1) \cdots u(f_m)=1$ and it follows that there exists a nonvoid compact set K such that $(f_1 \cdots f_m)(K)=f_i(K)=1$ for each i. Sets of the form $X(f=1)$, where $u(f)=1$, have the finite intersection property. By the compactness of X, there exists a nonvoid subset S of X such that $f(S)=1$ for each $f\in L$ satisfying $u(f)=1$. By (2) we have $f(S)=0$ for each $f\in L$ satisfying $u(f)=0$. Since L separates points in X, S must consist of just one point x_0. Hence $u(f)=f(x_0)$ for each $f\in L$ and the proof is complete if u takes only the values 0 and 1.

Now assume that there exists an $f\in L$ such that $0<u(f)<1$. By (1) we have $f^h\in L$ and $u(f^h)=u(f)^h$ for any positive rational number h. If h and j are positive rational numbers and k is a positive real number such that $j<k<h$, then $f^{k-j}\in L$, $f^{h-k}\in L$, $f^k\in L$, and

$$u(f)^h = u(f^h) = u(f^{h-k})u(f^k) \leqq u(f^k) = u(f^{k-j})u(f^j) \leqq u(f^j) = u(f)^j.$$

It follows that $u(f^k)=u(f)^k$ for any positive number k.

Let $f_1, \cdots, f_n$ be finitely many functions in L satisfying $0<u(f_i)<1$ for each i. Let k_i be the positive number for which $u(f_i)^{k_i}=\frac{1}{2}$ for each i, and let k be the positive number for which $3^k=\frac{1}{2}(4^k)$. Set

$$f=[1-f_1^{k_1}(1-f_1^{k_1})]^{k/n}[1-f_2^{k_2}(1-f_2^{k_2})]^{k/n}\cdots[1-f_n^{k_n}(1-f_n^{k_n})]^{k/n}.$$

From the preceding paragraph we get $f\in L$ and $u(f)=\frac{1}{2}$. Furthermore, $(1-r(1-r))^k\geqq\frac{1}{2}$ for any real number r, and equality holds if and only if $r=\frac{1}{2}$. Hence $f\geqq\frac{1}{2}\geqq 1-f$. We claim that the strict inequality $f>1-f$ cannot hold. For, if it did, there would be a number $j>1$ such that $f^j>1-f$ and $u(f^j)=u(f)^j=(\frac{1}{2})^j<\frac{1}{2}=u(1-f)$; but $f^{-j}(1-f)\in L$, and $u(1-f)=u(f^j)u[f^{-j}(1-f)]\leqq u(f^j)$, which is impossible. Hence there is a nonvoid compact subset C of X such that $f(C)=\frac{1}{2}$. It follows that $f_i^{k_i}(C)=\frac{1}{2}=u(f_i)^{k_i}$ and $f_i(C)=u(f_i)$ for each i. The compact subsets of the form $X(f=u(f))$, where $f\in L$ and $0<u(f)<1$, have the finite intersection property. By the compactness of X, there exists a nonvoid set S such that $f(S)=u(f)$ for each $f\in L$ satisfying $0<u(f)<1$.

We claim that $f(S)=u(f)$ for any $f\in L$. Suppose that $u(f)=1$. Select $g\in L$ such that $0<u(g)<1$. Then $0<u(fg)=u(g)<1$, and we have $(fg)(S)=u(fg)=u(g)=g(S)$. Hence $f(S)=1$. By (2) we also get $f(S)=0$ if $u(f)=0$. Of course, the set S consists of just one point because L separates points in X. This completes the proof of Theorem 1.

Proof of Theorem 2. Select $y\in Y$ and define $u(f)=\phi(f)(y)$ for each $f\in L$. Then u is a mapping of L into $[0, 1]$ satisfying (1) and (2). By Theorem 1 there is a point $y^*\in X$ such that $u(f)=f(y^*)$ for each $f\in L$. We need only make $py=y^*$. To show that p is continuous is routine; note that if U is a neighborhood of py in X, then there exists an $f\in L$ such that $f(py)=1$, $f(X-U)=0$, and $p[Y(\phi(f)>0)]\subset U$, $y\in Y(\phi(f)>0)$.

We now present examples which show that neither (1) nor (2) can be omitted from Theorem 2.

Example 1. Let N be the set of positive integers under the discrete topology and let X be the Stone-Čech compactification of N. Note that any bounded function on N extends to a unique continuous function on X. For each $f\in D(X)$, let $\phi(f)$ be the function in $D(X)$ for which $\phi(f)(n)=[f(n)]^n$ for $n\in N$. Then ϕ is obviously a 1-1 multiplicative mapping of $D(X)$ onto $D(X)$. But $\phi(\frac{1}{2})=0$ on $X-N$ and $\phi^{-1}(\frac{1}{2})=1$ on $X-N$. Since $\phi(0)=0$ and $\phi(1)=1$, it is clear that neither ϕ nor ϕ^{-1} can be implemented by a continuous mapping of X into X. Hence (2) cannot be omitted from Theorem 2.

Example 2. Let N and X be the same as in Example 1. For each positive integer n let p_n be the mapping of the unit interval into itself defined in the following way; $p_n(r)=2^n r^{n+1}$ for $0\leqq r\leqq\frac{1}{2}$, and $p_n(r)=1-p_n(1-r)$ for $\frac{1}{2}<r\leqq 1$. For each $f\in D(X)$, let $\phi(f)$ be the function in $D(X)$ for which $\phi(f)(n)=p_n[f(n)]$, each $n\in N$. Then ϕ is a 1-1 mapping of $D(X)$ onto $D(X)$ which satisfies (2). But $\phi(\frac{1}{4})=0$ on $X-N$ and $\phi^{-1}(\frac{1}{4})=\frac{1}{2}$ on $X-N$. Since $\phi(0)=0$ and $\phi^{-1}(\frac{1}{2})=\frac{1}{2}$ it

is clear that neither ϕ nor ϕ^{-1} can be implemented by a continuous mapping of X into X. Hence (1) cannot be omitted from Theorem 2.

We conclude with one further comment. Milgram [3] showed that the multiplicative semi-group $C(X)$ characterizes a compact space X. By replacing $C(X)$ with $D(X)$ in Theorem 3.1 and in the preceding work in [3] we see that the multiplicative semi-group $D(X)$ also characterizes a compact space X. Functions exceeding 1 are not needed in this portion of Milgram's arguments.

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ON SIMPLE RINGS WITH UNIFORM RIGHT IDEALS

KWANGIL KOH, North Carolina State University

If R is a ring, a right ideal I of R is said to be *a complement right ideal* of R provided that there exists a right ideal K in R such that $I \wedge K = (0)$ and if J is a right ideal such that $J > I$ then $J \wedge K \neq (0)$. A right ideal U of R is said to be *uniform* if any pair of nonzero right ideals of R, which are contained in R, has a nonzero intersection. A ring R is *a right Ore domain* if R is a uniform right ideal and R contains no divisors of zero. In [1], C. Faith proved that if R is a simple ring with identity which contains a minimal complement right ideal, then R is the endomorphism ring of a unital torsion-free module over an integral domain. In proving the theorem cited above, he used a theorem of Utumi [5], viz., the maximal right quotient ring S of R is a full ring of linear transformations in a right vector space over a field D.

The purpose of this note is to give an elementary proof of Faith's theorem without using Utumi's result.

LEMMA. *Let R be a prime ring with 1 such that the right singular ideal of R is zero. If R contains a uniform right ideal U, then for each $u \in U$, $u \neq 0$, $\mathrm{Hom}_R(uR, uR)$ is a right Ore domain.*

Proof. Let $(u)^r = \{r \in R \mid ur = 0\}$. Then $(u)^r$ is a maximal annihilator right ideal of R by [4; 4.11]. Define $N = \{r \in R \mid r(u)^r \leqq (u)^r\}$ and $K = N/(u)^r$. By [2; Theorem 1, p. 25], $K \cong \mathrm{Hom}_R(uR, uR)$. Let $k_1, k_2 \in K$ such that $k_i = n_i + (u)^r$ for some $n_i \in N$, $n_i \notin (u)^r$, $i = 1, 2$. If $k_1 k_2 = 0$ then $n_1 n_2 \in (u)^r$ and $(un_1)^r > (u)^r$ since $n_i(u)^r \leqq (u)^r$ and $n_2 \in (un_1)^r$ but $n_2 \notin (u)^r$. This is impossible since $(u)^r$ is a maximal annihilator right ideal. Hence K is an integral domain. Let $J = (n_1R + (u)^r) \wedge (n_2R + (u)^r)$. Then $J > (u)^r$ since $uR \cong R/(u)^r$ and uR is a uni-

form right ideal of R. Since $(uJ)(uJ)\neq(0)$, we can choose $j\in J$ such that $uju\neq 0$. $ju\in N\wedge J$ and $ju\notin(u)^r$ since $u(ju)(u)^r=(0)$ and $uju\neq 0$. There exist r_1, $r_2\in R$ with r_1, $r_2\notin(u)^r$ and a_1, $a_2\in(u)^r$ such that $ju=n_1r_1+a_1=n_2r_2+a_2$. Hence $n_ir_i=ju-a_i\in N$ for $i=1$, 2. Now $n_ir_i(u)^r\leqq(u)^r$ for all $i=1$, 2, hence if r_1 or $r_2\notin N$, say $r_1\notin N$ then $(un_1)^r>(u)^r$. This follows from the fact that $r_1(u)^r$ is not a subset of $(u)^r$ and $u(n_1r_1)(u)^r=(0)$. This is impossible since $un_1\neq 0$ and $(u)^r$ is a maximal annihilator right ideal of R. This proves that K is a right Ore domain.

THEOREM. *If R is a simple ring with identity which contains a uniform right ideal U, then there exists a left ideal L in R and a right Ore-domain K such that L is a unital torsion-free right K-module and $R\cong\mathrm{Hom}_K(L, L)$.*

Proof. By lemma, $K=\mathrm{N}/(u)^r\cong\mathrm{Hom}_R(uR, uR)$ is a right Ore-domain. Let $L=\{r\in R\,|\,r(u)^r=(0)\}$. Define $l\cdot(n+(u)^r=ln$ for all $l\in L$ and $n+(u)^r\in K$, where $n\in N$. Since $ln(u)^r=(0)$, L becomes a right K-module under this definition. If $x\in L$ and $x\neq 0$ then $(x)^r=(u)^r$ since $x(u)^r=(0)$ and $(u)^r$ is a maximal annihilator right ideal of R. Hence, if $xn=0$ for some $n\in N$ then $n\in(u)^r$. Thus L is a torsion-free K-module. Define a mapping f from R into $\mathrm{Hom}_K(L, L)$ by $f(r)(l)=rl$ for all $l\in L$. It is easy to see that f is a ring homorphism of R into $\mathrm{Hom}_K(L, L)$. The kernel of f is zero since $L^l=\{r\in R\,|\,rL=(0)\}=(0)$. Let $t\in\mathrm{Hom}_K(L, L)$ and $l\in L$. Since $1\in R=LR$, $1=\sum_{i=1}^n l_ir_i$ for some $l_i\in L$, $r_i\in R$ and some positive integer n.

$$t(l)=t(1l)=t\left(\sum_{i=1}^n l_ir_il\right)=\sum_{i=1}^n t(l_i)r_il$$

since t is a K-homomorphism and $r_il\in N$ for all $1\leqq i\leqq n$. Let $\bar{l}_i=t(l_i)$, then

$$t(l)=\left(\sum_{i=1}^n \bar{l}_ir_i\right)l \quad\text{and}\quad t=f\left(\sum_{i=1}^n \bar{l}_ir_i\right).$$

This proves that $R\cong\mathrm{Hom}_K(L, L)$.

REMARK. If R is a simple ring with 1 which contains a uniform right ideal U then $\hat{R}$, the maximal right quotient ring of R in the sense of [3], is a simple ring with a minimal right ideal with 1. By [2; Structure Theorem, p. 75], $\hat{R}$ is isomorphic to the set of all linear transformations of finite rank of a vector space M over a division ring Δ and since $1\in\hat{R}$, $\hat{R}$ is isomorphic to the ring of all linear transformations of the finite dimensional vector space M over Δ. Hence $\hat{R}$ is a simple ring with minimum conditions on one-sided ideals. It is easy to check that $\mathrm{Hom}_R(uR, uR)$ is isomorphic to a subring of $\mathrm{Hom}_{\hat{R}}(u\hat{R}, u\hat{R})=D$ which is a division ring. Furthermore, if we let $(\bar{u})^r=\{x\in\hat{R}\,|\,ux=0\}$ and $\bar{L}=\{x\in\hat{R}\,|\,x(\bar{u})^r=(0)\}$ then $\hat{R}\cong\mathrm{Hom}_D(\bar{L}, \bar{L})$.

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ON A SEMI-PRIME SELF INJECTIVE RING

KWANGIL KOH, North Carolina State University

1. The purpose of this note is to prove that if a ring R with 1 is *semi-prime* [2] and *right self injective* [6] then every maximal annihilator right ideal (if it exists) is a maximal right ideal which is generated by an idempotent in R. If $a\in R$ such that $(a)^r=\{r\in R\,|\,ar=0\}$ has a nonzero intersection with each nonzero right ideal of R then a is called a right singular element of R. The set of all right singular elements in R will be denoted by R_r^Δ. R_r^Δ is a two sided ideal of R and it is called the *right singular ideal* of R [4]. In the sequel, R will denote a semi-prime and right self injective ring with 1 which contains a maximal annihilator right ideal.

2. LEMMA 2.1. *If I is a maximal annihilator right ideal of R then there exists an idempotent e in R such that $I=(e)^r$.*

Proof. Let I be a maximal annihilator right ideal of R. Then there exists $a\in R$ such that $I=(a)^r$. Let $L=\{r\in R\,|\,rI=0\}$ and let $S=L\wedge R_r^\Delta$. If $x\in S$ and $x^2\neq 0$ then $(x)^r=(x^2)^r$, since $(x)^r\leqq(x^2)^r$ and $(x)^r=I$. Let $y\in(x)^r\wedge xR$. Then $y=xr$ for some $r\in R$ and $xy=x(xr)=0=x^2r=xr=y$. This implies that $x\notin R_r^\Delta$. Thus if $x\in S$ then $x^2=0$. Hence, if $y\in S$ then $xy=0$ since $(x)^r=(y)^r$ or $y=0$. This implies that $S=(0)$ since R is semi-prime. Hence there exists a nonzero right ideal J in R such that $J\wedge I=(0)$. Let f be an R-homomorphism of $J\oplus I$ into R such that $f(j)=j$ for all $j\in J$ and $f(i)=0$ for all $i\in I$. Since R is right self injective, there must exist a nonzero element $e\in R$ such that $f(j)=ej=j$ for all $j\in J$ and $f(i)=ei=0$ for all $i\in I$. Since $(e^2-e)^r\geqq J\oplus I>I$ and I is a maximal annihilator right ideal, $e^2=e$. Since $ei=0$ for all $i\in I$, $I=(e)^r$.

LEMMA 2.2. *If I is a maximal annihilator right ideal of R then the module $R-I$ is isomorphic to eR for some idempotent e in R and $\mathrm{Hom}_R(eR, eR)$ is a division ring.*

Proof. By Lemma 2.1, $I=(e)^r$ for some idempotent $e\in R$. Since $x=(1-e)x$ for any $x\in(e)^r$, $(e)^r=(1-e)R$ and since $R=eR\oplus(1-e)R$, $R-I\cong eR$. Let $f\in\mathrm{Hom}_R(eR, eR)$ and $f\neq 0$. Since R is right self injective, there is $u\in R$ such that $f(er)=uer$ for all $er\in eR$. Define a relation g between ueR and eR by $g(uer)=er$ for all $uer\in ueR$. Then g is a mapping since $uer=uer'$ implies that $r-r'\in(ue)^r=(e)^r$. Clearly g is an R-homomorphism of ueR onto eR. Since eR is a direct summand of a right self injective ring R, eR must be injective as R-module. Hence there is an extension $\bar{g}$ of g to eR such that $\bar{g}\in\mathrm{Hom}_R(eR, eR)$ and $\bar{g}\cdot f$ is an identity mapping. Thus $\mathrm{Hom}_R(eR, eR)$ is a division ring.

THEOREM. *If R is a semi-prime and a right self injective ring with* 1, *then any maximal annihilator right ideal is a maximal right ideal which is generated by an idempotent in R.*

Proof. Let I be a maximal annihilator right ideal of R. Then, by Lemma 2.1, $I=(e)^r$ for some idempotent e in R. Hence $I=(1-e)R$. By [3; Proposition 3, p. 51] and by Lemma 2.2, $\mathrm{Hom}_R(eR, eR)=eRe$ is a division ring. Hence, by [3; Proposition 1, p. 65], eR must be a minimal right ideal of R. Thus $I=(e)^r$ is a maximal right ideal of R.

COROLLARY. *If R is a semi-prime and a right self injective ring with* 1 *which contains a maximal annihilator right ideal, then R contains a minimal right ideal and a minimal left ideal of R.*

REMARK. If we assume a maximum condition on annihilator right ideals in R, in addition, then by Lemma 2.1 $R_r^{\Delta}=0$. Since any complement right ideal of a right self injective ring is an annihilator right ideal (See [5] Lemma 2.4), a maximum condition on annihilator right ideals in R implies that a maximum condition on complement right ideals in R. Thus by a theorem of Goldie [2] and by [1; Th. 1], R is semi-simple Artinian.

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LOCATION OF THE ZEROS OF A POLYNOMIAL

MOHAN LAL TIKOO, University of Jammu and Kashmir, India

That the zeros of the polynomial $P_n(z)=c_0+c_1z+\cdots+c_nz^n$, $c_n\neq 0$, are the eigenvalues of its companion matrix leads to easier methods in obtaining the bounds for all the zeros of a polynomial. Howard E. Bell [1], by making use of the Gershgorin's theorem [2], has obtained some well-known theorems as a direct consequence of the above fact. In this note we apply the Gershgorin's theorem to columns and obtain some simple bounds for the zeros of $P_n(z)$.

THEOREM (a). (Gershgorin). *Let $M\equiv[c_{ij}]$ be an (n, n) complex matrix, and let $C^{(i)}$ be the sum of the moduli of the off-diagonal elements in the i-th column. Then each eigenvalue of M lies in the union of the circles*

$$|z - c_{ii}| \leqq C^{(i)} \quad (i = 1, 2, 3, \cdots, n). \tag{1}$$

The analogous result holds if the rows of M are considered.

Like Bell, we take the larger circles $|z| \leqq |c_{ii}| + C^{(i)}$, $i = 1, 2, \cdots, n$.

THEOREM 1. *All the zeros of the polynomial $P_n(z) = c_0 + c_1 z + \cdots + c_n z^n$, $c_n \neq 0$, lie in the circle*

$$|z| \leqq \max\left(\frac{K_i}{K_{i+1}}, \sum_{i=0}^{n-1} \frac{|c_i|}{|c_n| K_{i+1}}\right) \quad (i = 0, 1, 2, \cdots, n-1) \tag{2}$$

where K_i are positive numbers with $K_0 = 0$ and $K_n = 1$.

Proof. The companion matrix $C(P_n)$ of the polynomial $P_n(z)$ is

$$C(P_n) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0/c_n \\ 1 & 0 & \cdots & 0 & -c_1/c_n \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & -c_{n-1}/c_n \end{bmatrix}.$$

Let $D = \operatorname{diag}(K_1, K_2, \cdots, K_n)$, where K_i are arbitrary positive numbers and $K_0 = 0$, $K_n = 1$, then

$$D^{-1}C(P_n)D = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0/c_n K_1 \\ K_1/K_2 & 0 & \cdots & 0 & -c_1/c_n K_2 \\ & & \cdots & & \\ 0 & 0 & \cdots & K_{n-1}/K_n & -c_{n-1}/c_n K_n \end{bmatrix}$$

whence, by Theorem (a), the eigenvalues of $D^{-1}C(P_n)D$ lie in

$$|z| \leqq \max\left(\frac{K_i}{K_{i+1}}, \sum_{i=0}^{n-1} \frac{|c_i|}{|c_n| K_{i+1}}\right) \quad (i = 0, 1, 2, \cdots, n-1).$$

Since $D^{-1}C(P_n)D$ is similar to $C(P_n)$, the theorem follows.

COROLLARY 1. *All the zeros of $P_n(z)$ lie in the circle*

$$|z| \leqq \max\left(1, \sum_{i=0}^{n-1} |c_i| / |c_n|\right). \tag{3}$$

Proof. In (2) take $K_i = 1$ for all $i = 1, 2, \cdots, n$.

REMARK 1. The bound in (3) is best possible. If $P_n(z) = -nz^n + z^{n-1} + \cdots + z + 1$, then all the zeros of this polynomial lie in $|z| \leqq 1$. Obviously, $\sum_{i=0}^{n-1} |c_i|/|c_n| = 1$, and hence (3) gives the actual bound for the zeros of the considered polynomial.

REMARK 2. Inequality (3) also gives a traditional result that if $|c_0| + |c_1| + \cdots + |c_{n-1}| \leqq |c_n|$ then the zeros of $\sum_{i=0}^{n} c_i z^i$ lie in $|z| \leqq 1$.

COROLLARY 2. *Let* $P_n(z)=c_0+c_1z+c_2z^2+\cdots+c_nz^n$, *where* $c_i\neq 0$ *for all* $i=1, 2, \cdots, n$ *then all the zeros of* $P_n(z)$ *lie in*

$$|z| \leqq \max\left(|c_{i-1}|/|c_i|\, t, \sum_{i=1}^{n} 1/|c_n|\, t^i\right), \tag{4}$$

where $t>0$ *is arbitrary.*

Proof. In (2) put $K_i=|c_{i-1}|\, t^i$.

REMARK 3. The result obtained in (4) is best possible and can easily be verified if we take $P_n(z)=-nz^n+z^{n-1}+\cdots+z+1$, and $t=1$.

The referee has pointed out that the result of Corollary 1 is known. He has attributed this result to Parodi.

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GENERALIZED BASES AND DIGITAL SUMS

J. R. TROLLOPE, University of Alberta

Many authors have considered problems dealing with digital sums of integers. For example, if we let $\alpha(n, r)$ denote the sum of the digits of n when n is represented in base r, then Legendre's theorem [1] states that

$$\alpha(n, r) = n + \sum_{i=1}^{\infty} (1-r)\left[\frac{n}{r^i}\right], \tag{1}$$

the square bracket denoting the integral part of its argument. Also, the asymptotic relation

$$A(n, r) \equiv \sum_{k<n} \alpha(k, r) = \frac{r-1}{2}\,\frac{n \log n}{\log r} + O(n), \tag{2}$$

has been discussed by various authors ([2]–[5]) and an explicit formula for $A(n, 10)$ has been given by d'Ocagne [6]. Lately there has been a renewed interest in generalized bases ([7], [8]) and the object of this paper is to consider digital sums relative to a particular type of generalized base.

Let $\{\xi_i | i=0, 1, 2, \cdots\}$ be a sequence of positive integers with $\xi_0=1$ and $\xi_i>1$ when $i\geqq 1$. Define the integers r_k by

$$r_k = \prod_{i\leq k} \xi_i. \tag{3}$$

It can be easily shown that every positive integer n has a unique representation of the form

now show that $p|C: C \to p(C)$ is injective. Let $x, y \in C$ be such that $p(x) = p(y) = b$. Since C is pathwise connected there exists a path $\beta: I \to C$ such that $\beta(0) = x$ and $\beta(1) = y$. Thus, $\alpha = p \circ \beta$ is a closed path in U with initial point b. Thus there exists a contracting homotopy of α in B. Since (X, p, B) has the contractible lifting property, there exists a contracting homotopy F of α in B which has a lifting F' such that $F'(t, 0) = \beta(t)$. Define $f: I \to X$ by $f(s) = F'(0, s)$. Since $p \circ f(s) = p \circ F'(0, s) = b$, f is a path in $p^{-1}(b)$ and since $p^{-1}(b)$ is totally pathwise disconnected, f is constant. Since $f(0) = F'(0, 0) = \beta(0) = x$, $f(s) = x$ for all $s \in I$. Thus, in particular $F'(0, 1) = x$. Define $g: I \to X$ by $g(s) = F'(1, s)$. A similar argument shows that g is constant and since $g(0) = F'(1, 0) = \beta(1) = y$, $g(s) = y$ for all $s \in I$. Thus, in particular, $F'(1, 1) = y$. Define $h: I \to X$ by $h(t) = F'(t, 1)$. The same argument shows that h is constant. But $h(0) = F'(0, 1) = x$ and $h(1) = F'(1, 1) = y$. Hence $x = y$. It follows that $p|C$ is injective and hence is a homeomorphism. Hence (X, p, B) is a sheaf.

Easy examples show that the triple (X, p, B) of the above theorem is in general not a covering space. Thus, under the hypotheses of the above theorem and the theorem of Browder, we have completed the diagram below

$$\begin{array}{ccc} \text{WEAK FIBRE SPACE} & \Rightarrow & \text{SEMI-WEAK FIBRE SPACE} \\ \Downarrow & & \Downarrow \\ \text{COVERING SPACE} & \Rightarrow & \text{SHEAF.} \end{array}$$

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NOTE ON THE SINGLE VARIABLE BELL POLYNOMIALS

JOHN BRILLHART, University of California, Berkeley

In a recent paper [1, p. 22] L. Carlitz has shown that the single variable Bell polynomials $A_n(x)$ have no repeated roots by relating them to the cyclotomic polynomials mod 2. In this note we establish this result by two other mod 2 arguments.

The polynomials $A_n(x)$ can be defined by

$$A_n(x) = \sum_{k=0}^{n} S(n, k)x^k, \; n \geqq 0, \quad \text{where} \quad S(n, k) = \frac{1}{k!}\sum_{r=0}^{k} (-1)^{k-r}\binom{k}{r} r^n$$

are the Stirling numbers of the second kind. The first few of these polynomials are:

$$A_0(x) = 1, \quad A_1(x) = x, \quad A_2(x) = x^2 + x,$$
$$A_3(x) = x^3 + 3x^2 + x, \quad A_4(x) = x^4 + 6x^3 + 7x^2 + x.$$

The $A_n(x)$ are most easily generated by the recursion formula

$$A_{n+1}(x) = x[A_n(x) + A_n'(x)], \qquad n \geqq 0. \tag{1}$$

For simplicity we write $A_n = A_n(x)$.

THEOREM. *A_n has no multiple roots.*

Proof 1. According to a result of Touchard [2], we have $A_{n+p} \equiv A_{n+1} + x^p A_n \pmod{p}$ for p a prime, $n \geqq 0$. Putting $p = 2$ and shifting the index, we can write

$$A_n \equiv A_{n+1} + x^2 A_{n-1} \pmod 2, \qquad n \geqq 1. \tag{2}$$

If (2) is combined with (1), we obtain

$$(x + 1)A_n + xA_n' \equiv x^2 A_{n-1} \pmod 2. \tag{3}$$

Now let $d(x)$ be the G.C.D. of A_n and A_n'. Then $d(x)$ will be monic with integral coefficients, since A_n has these properties. From (3) we see that $d(x) \pmod 2$ will divide $x^2 A_{n-1}$, inasmuch as it divides the left side of the identity. But the constant term of A_n' is 1, so that $x \nmid d(x) \pmod 2$. Hence, $d(x) \mid A_{n-1} \pmod 2$. It now follows from repeated use of (2) that $d(x)$ will in turn divide A_{n-2}, A_{n-3}, $\cdots$, $A_0 = 1 \pmod 2$. Thus, $d(x) \equiv 1 \pmod 2$. But $d(x)$ is monic. Therefore, $d(x) = 1$, which proves the theorem.

Proof 2: Put $A_n = xB_n$. Then B_n satisfies the comparable modular identities $B_n \equiv B_{n-1} + x^2 B_{n-2} \pmod 2$ and $B_n' \equiv B_n + xB_{n-1} \pmod 2$. We now show that the discriminant D of B_n is odd, and hence not 0. For our purposes then, it will suffice to compute $D(B_n) \pmod 2$. We do this by appealing to both the well-known relations between $D(B_n)$ and the resultant $R(B_n, B_n')$ and the properties of R. (See Uspensky [3], Swan [4, 1104].) Thus,

$$\begin{aligned} D(B_n) &\equiv R(B_n, B_n') \equiv R(B_n, B_n + xB_{n-1}) = R(B_n, xB_{n-1}) \\ &= R(B_n, x)R(B_n, B_{n-1}) = R(B_n, B_{n-1}) = R(B_{n-1}, B_n) \\ &\equiv R(B_{n-1}, B_{n-1} + x^2 B_{n-2}) = R(B_{n-1}, x^2 B_{n-2}) \\ &= R(B_{n-1}, B_{n-2}) = \cdots = R(B_2, B_1) = 1 \pmod 2. \end{aligned}$$

In general, if F and G are two polynomials, then $D(FG) = D(F)D(G)\,[R(F,G)]^2$. Hence, $D(A_n) = D(xB_n) = D(B_n) \equiv 1 \pmod 2$.

REMARK: It is clear from the above proofs that A_n also has no multiple factors mod 2.

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MEDIANS OF A SIMPLEX

ARUN SANYAL, Indian Institute of Technology, Kharagpur, India

The purpose of this note is to establish a theorem on medians of a simplex in n-dimensional Euclidean space $E[n]$.

Let $A_0A_1 \cdots A_n$ be a simplex $\mathcal{S}$ in $E[n]$. Any r vertices of the simplex form an $(r-1)$-simplex and the rest of the vertices form an $(n-r)$-simplex. These two opposite simplexes will be denoted respectively by $\mathcal{S}_{r-1}$ and $\mathcal{S}_{n-r}$. There are $\binom{n+1}{r}$ pairs of opposite simplexes. We shall take $r \leqq n+1-r$.

The line joining the centroids of two opposite $\mathcal{S}_{r-1}$ and $\mathcal{S}_{n-r}$ will be called (cf. [2]) an r-median of $\mathcal{S}$. If $r=n+1-r$, an r-median will be called a bimedian. There are then $\frac{1}{2}\binom{2r}{r}$ bimedians of an $\mathcal{S}$ in $E[2r-1]$. A 1-median will be called a median.

In the following theorem, we shall denote by A_i the coordinate vector of a point A_i with respect to an origin, and the scalar product $A_i \cdot A_i$ by A_i^2.

THEOREM. *The sum of the squares of* $\binom{n+1}{r}$ *r-medians of a simplex* $\mathcal{S}$ *in* $E[n]$ is $[(n+1)!]/[nr(n-r+1)(n-r+1)!r!]$ *times the sum of the squares of its edges if* $r \neq n+1-r$.

Proof. Let d be the sum of the squares of the edges. Then

$$d = \sum_{i<j} (A_i - A_j)^2 \qquad (i, j = 0, 1, \cdots, n).$$

Since n edges pass through each vertex,

$$d = n \sum_{i=0}^{n} A_i^2 - 2 \sum_{i<j} A_i \cdot A_j.$$

Let us denote the sum of the squares of the r-median by D. The square of the length of one r-median is

$$\left(\frac{A_0 + A_1 + \cdots + A_{r-1}}{r} - \frac{A_r + A_{r+1} + \cdots + A_n}{n-r+1} \right)^2.$$

Now, (a) there are

$$\binom{n}{r-1} \mathcal{S}_{r-1} \quad \text{and} \quad \binom{n}{n-r} \mathcal{S}_{n-r}$$

which have the vertex A_i. So the coefficient of A_i^2 in D is

$$\frac{1}{r^2}\binom{n}{r-1} + \frac{1}{(n-r+1)^2}\binom{n}{n-r} = \frac{(n+1)!}{r(n-r+1)(n-r+1)!r!}$$

and, (b) there are

$$\left[\binom{n-1}{r-2}\right] \mathcal{S}_{r-1} \quad \text{and} \quad \left[\binom{n-1}{n-r-1}\right] \mathcal{S}_{n-r}$$

which have both the vertices A_i, A_j. Also, there are

$$\left[\binom{n}{r-1}-\binom{n-1}{r-2}\right]\mathfrak{S}_{r-1} \quad \text{and} \quad \left[\binom{n-1}{r-2}-\binom{n-1}{n-r-1}\right]\mathfrak{S}_{n-r}$$

which have the vertex A_i or A_j. So the coefficient of $A_i \cdot A_j$ in D is given by the following expression:

$$2\left[\frac{(n-1)!}{r^2(r-2)!(n-r+1)!}+\frac{(n-1)!}{(n-r+1)^2 r!(n-r-1)!}\right.$$
$$\left.-\frac{2(n-1)!}{r(n-r+1)(n-r)!r!}\right]=\frac{-2(n+1)!}{nr(n-r-1)(n-r+1)!r!}.$$

Hence

$$D=\frac{(n+1)!}{nr(n-r+1)(n-r+1)!r!}\left[n\sum_{i=0}^{n}A_i^2-2\sum_{i<j}A_i\cdot A_j\right]$$
$$=\frac{(n+1)!}{nr(n-r+1)(n-r+1)!r!}d.$$

Putting $r=1$, in the result of the theorem, we get

$$D=\frac{n+1}{n^2}d,$$

which can be written in the following way: The sum of the squares of the medians of a simplex is $(n+1)/n^2$ times the sum of the squares of the edges. The result for $n=2$ can be found in books of school geometry, (cf. [3] pp. 231, Ex 9) and for $n=3$ we get it in [1, p. 65].

If $n=2r-1$ there are

$$\frac{1}{2}\binom{2r}{r}$$

bimedians and

$$D=\frac{(2r)!}{2r^2(2r-1)r!r!}d.$$

For $n=3$ this reduces to the result given in [1, p. 64] and [4, p. 67].

Thanks are due to Dr. Sahib Ram Mandan for his guidance in preparation of this note.

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THE INVOLUTORY ANTIAUTOMORPHISMS OF THE QUATERNION ALGEBRA

RABE VON RANDOW, Otago University, Dunedin, New Zealand

Let H denote the algebra of quaternions and A^* the set of automorphisms and antiautomorphisms of H. Clearly A^* is a group and, since the square of any antiautomorphism is an automorphism, the automorphisms of H form a subgroup A of index 2 in A^*. Furthermore, as H is a vector space of dimension 4 over the field R of real numbers, A and A^* are clearly subgroups of $GL(4, R)$. The object of this note is to determine the involutory antiautomorphisms of H. If E is an involutory antiautomorphism of H and $P \in A^*$, then obviously $P^{-1}EP$ is again an involutory antiautomorphism of H, and we will say that E and $P^{-1}EP$ are equivalent modulo A^*. We will show that, up to equivalence mod A^*, there are only two involutory antiautomorphisms of H, namely

$$\begin{aligned} &a + bi + cj + dk \to a - bi - cj - dk, \\ &a + bi + cj + dk \to a - bi + cj + dk. \end{aligned} \tag{1}$$

The first of these is of course the usual conjugation of quaternions. In [1] we consider Hermitian forms over H, utilising the second involutory antiautomorphism. This gives rise to a theory of Hermitian forms completely analogous to that of quadratic forms over C, whereas the usual theory of Hermitian forms over H is analogous to that of quadratic forms over R or Hermitian forms over C.

Determination of A and A^*. Suppose that $S \in A^*$, $S \notin A$, and express S as a real 4×4 matrix with respect to the usual base of H. Then, as

$$\begin{aligned} &S(1) = 1, \qquad (S(i))^2 = (S(j))^2 = (S(k))^2 = -1, \\ &S(j)\cdot S(i) = S(k), \quad S(k)\cdot S(j) = S(i), \quad S(i)\cdot S(k) = S(j), \end{aligned} \tag{2}$$

an easy calculation shows that S is of the form

$$S = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix},$$

where T is orthogonal of determinant -1.

Similarly, suppose that $S \in A$. Then the relations (2) and $S(i)\cdot S(j) = S(k)$, $S(j)\cdot S(k) = S(i)$, $S(k)\cdot S(i) = S(j)$ show that S is of the form

$$S = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix},$$

where T is orthogonal of determinant 1.

Determination of the involutory antiautomorphisms of H. Let E be an involutory antiautomorphism of H. Then E has the form

$$E = \begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix},$$

where F is orthogonal of determinant -1 and involutory, hence symmetric and orthogonal of determinant -1. The eigenvalues of E can therefore only be 1, 1, 1, -1, and 1, -1, -1, -1. Hence E has, with respect to some base in H, one of the following standard forms:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The first three are equivalent mod A^*, however, and not equivalent to the last one, as they have differing eigenvalues. This leaves the two involutory anti-automorphisms (1), which are clearly the only ones mod A^*, as equivalence mod A^* is nothing but equivalence under orthogonal reduction.

The involutory automorphisms of H are obtained similarly.

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IF $R[X]$ IS NOETHERIAN, R CONTAINS AN IDENTITY

R. W. GILMER, JR., Florida State University

The well-known Hilbert basis theorem shows that if R is a commutative ring with identity and if R is Noetherian, then $R[X]$, the polynomial ring in one indeterminate over R, is also Noetherian. In this note we establish the converse of this result.

THEOREM. *If R is a commutative ring such that $R[X]$ is Noetherian, then R contains an identity.*

Proof. The mapping $f(X) \to f(0)$ is a homomorphism of $R[X]$ onto R. Hence R is Noetherian. If now $r \in R$, then the chain of ideals $(r) \subseteq (r, rX) \subseteq \cdots \subseteq (r, rX, \cdots, rX^n) \subseteq \cdots$ contains only finitely many distinct ideals. Thus, for some n, $rX^{n+1} \in (r, rX, \cdots, rX^n)$, say

$$(*) \qquad rX^{n+1} = \sum_{i=0}^{n} f_i(X) \cdot rX^i + \sum_{i=0}^{n} n_i \cdot rX^i,$$

where $f_i(X) = \sum_{j=0}^{m_i} f_j^{(i)} X^j$, $f_j^{(i)} \in R$, and each n_i is an integer. Equating coefficients of X^{n+1} in (*) we have $r = \sum_{i=0}^{n} (f_{n-i}^{(i)} \cdot r) = g \cdot r$ where $g = \sum_{i=0}^{n} f_{n-i}^{(i)} \in R$. We then note that for any elements $r_1, r_2, g_1, g_2 \in R$ such that $r_1 = g_1 r_1$ and $r_2 = g_2 r_2$, we have $r_1 = (g_1 + g_2 - g_1 g_2) r_1$ and $r_2 = (g_1 + g_2 - g_1 g_2) s$. By induction it follows that if $\{r_i\}_{i=1}^{k}$ is a finite set of elements of R then there exists an element $u \in R$ such that $r_i = ur_i$ for $i = 1, \cdots, k$. Since R is Noetherian, there is a finite subset T of R such that T generates R as an ideal of R. If $e \in R$ is such that $et = t$ for each $t \in T$, then it follows that e is an identity of R.

RELATING PROPER AND NORMAL CONES

M. D. LEVIN, University of Iowa

In [1], Namioka states that in pseudo-normed spaces, normal cones are proper. The statement is true only in normed spaces. Moreover, in normed euclidean spaces cones are normal if and only if their closures are proper. An example is given to demonstrate that this fact is not necessarily true in other normed spaces.

Notation. The closed unit ball in a pseudo-normed space will be denoted S.

DEFINITIONS. *Let E be a real linear space.* (1) *A subset C of E is a cone in E if, whenever x, $y\in C$ and $t\geqq 0$, both tx and $x+y\in C$.* (2) *A cone C in E is proper in case $x\in C$ and $-x\in C$ imply $x=0$.* (3) *If p is a pseudo-norm on E, a cone C in E is normal if $(S+C)\cap(S-C)$ is p-bounded.* (4) *If x is a nonzero element of E, $Rx=\{tx: t\in R\}$ is called a line.*

Notes. (1) If C is a cone in a topological vector space, then $\overline{C}$ is a cone. (2) A cone is proper if and only if it contains no line.

PROPOSITION 1. *If C is a normal cone in a normed space, then C is proper.*

Proof. If C is not proper, then there is a line L in C. Then $L\subset(S+C)\cap(S-C)$. Hence, $(S+C)\cap(S-C)$ is not bounded.

Example. A normal cone in a pseudo-normed space need not be proper. Let p be the pseudo-norm on R^2 given by $p((x, y))=|x|$. Then $S=\{(x, y): |x|\leqq 1\}$. Let $C=\{(x, y): x\geqq 0\}$. $(S+C)\cap(S-C)=S$, so C is normal. But $(0, 1)$ and $-(0, 1)$ belong to C, so C is not proper.

PROPOSITION 2. *Let C be a normal cone in a pseudo-normed space. Then $\overline{C}$ is normal.*

Proof. $\overline{C}\subset C+S=C-S$. Hence, $S+\overline{C}\subset 2S+C=2(S+C)$ and $S-\overline{C}\subset 2(S-C)$. It follows that

$$(S+\overline{C})\cap(S-\overline{C})\subset 2[(S+C)\cap(S-C)].$$

Since $(S+C)\cap(S-C)$ is bounded, so is $(S+\overline{C})\cap(S-\overline{C})$.

PROPOSITION 3. *Let p be a norm on R^n. Let C be a closed proper cone in R^n. Then C is normal.*

Proof. Suppose that $(S+C)\cap(S-C)$ is not p-bounded. Then there exists a sequence $\{x_j\}$ in $(S+C)\cap(S-C)$ such that $p(x_j)\to\infty$. Since S is compact, one can assume that $x_j/p(x_j)\to y$. Because $p(y)=1$, $y\neq 0$. Write $x_j=u_j+v_j$, where $u_j\in S$ and $v_j\in C$. Then $v_j/p(x_j)=x_j/p(x_j)-u_j/p(x_j)$. The right hand member of this equation converges to y. Since $v_j/p(x_j)\in C$ and C is closed, $y\in C$. Similarly $-y\in C$. Then C is not proper.

PROPOSITION 4. *Let p be a norm on R^n. Let C be a cone in R^n. Then C is normal if and only if $\overline{C}$ is proper.*

Proof. If $\overline{C}$ is proper then, by proposition 3, $(S+\overline{C})\cap(S-\overline{C})$ is p-bounded. But $(S+C)\cap(S-C)\subset(S+\overline{C})\cap(S-\overline{C})$. Then C is normal. On the other hand, if C is normal, then by proposition 2, $\overline{C}$ is normal, and by proposition 1, $\overline{C}$ is proper.

THEOREM. *A cone C in a pseudo-normed linear space (E, p) is normal if and only if there exists $e>0$ such that whenever $x, y\in C$ and $p(x)=p(y)=1$, it is true that $p(x+y)\geqq e$.*

Proof. Namioka [1], page 30.

Example. A closed proper cone in a non-euclidean normed space need not be normal. Let $\|\ \|$ be the usual norm on R^2. Let E be the set of all sequences $\{x_n\}$ with values in R^2 such that $\sum_{n=1}^{\infty}\|x_n\|^2$ is a convergent series. Define norm p on E by $p(\{x_n\})=(\sum_{n=1}^{\infty}\|x_n\|^2)^{1/2}$. E is actually a Hilbert space. Let $C_n=\{(y, z)\in R^2: yn^2-|z|\geqq 0\}$. Then C_n is $\|\ \|$-closed in R^2. Let $C=\{\{x_n\}\in E: x_n\in C_n$ for each $n\}$. Then C is a closed complete proper cone in (E, p) but C is not normal.

Proof. Suppose that C is normal. Then, by the theorem, there exists $e>0$ such that if $u, v\in C$ and $p(u)=p(v)=1$, then $p(u+v)\geqq e$. Let N be an integer such that $N>2/e$. Let δ_{ij} be the Kronecker delta. Let $u=\{u_n\}$ be the element of E such that $u_n=\delta_{nN}\,(1/N,\ (N^2-1)^{1/2}/N)$. Let $v=\{v_n\}$ be the element of E such that $v_n=\delta_{nN}(1/N,\ -(N^2-1)^{1/2}/N)$. $u, v\in C$; $p(u)=p(v)=1$; $p(u+v)=2/N<e$. Contradiction.

The author thanks the referee for suggesting simplifications in most of the original proofs.

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COMPACT CONVEX SETS IN INNER PRODUCT SPACES

S. K. BERBERIAN, University of Iowa

G. H. Orland recently proved that a compact convex set of complex numbers is generated by its bare points [3, p. 78, Lemma 3]. In this note we show that Orland's arguments extend to any inner product space; the Krein-Milman theorem for inner product spaces follows as a corollary.

For the rest of the paper E denotes a nonzero inner product space, with inner product (x, y); for definiteness we assume that E is a complex inner product space, though the arguments are also valid for real inner product spaces, with the simplification $\Re(x, y)=(x, y)$ ($\Re$ denotes real part). Our convention is that convex sets are nonempty.

DEFINITION. *Let C be a convex subset of the inner product space E. A point x of C will be called a bare point of C if there exists a point x_0 in E and a real number $r>0$ such that $\|x-x_0\|=r$ and $\|y-x_0\|\leqq r$ for all y in C; informally, there exists a closed ball containing C, with x on the surface (i.e., there exists a sphere through x which encloses C).*

It is a well-known property of inner product spaces that every point of the sphere $\|x-x_0\|=r$ is an extremal point of the closed ball $\|x-x_0\| \leqq r$ (see [1] p. 32, Exercises 4 and 5); it follows at once that every bare point of a convex set C in E is an extremal point of C.

THEOREM. *A compact convex set in an inner product space is the closed convex hull of its bare points.*

Before giving the proof, which follows Orland's model faithfully, we note an immediate corollary: a compact convex set in an inner product space is the closed convex hull of its extremal points; this is the Krein-Milman theorem for inner product spaces (cf. [2] p. 84). This result carries over at once to any topological vector space which is normable by an inner product—for instance, any finite-dimensional separated topological vector space [2, p. 27, Théorème 2].

The following lemmas are "geometrically obvious" and surely well known; we suppress most of the details.

LEMMA 1. *Let A be a convex set in E, and let x_0 be a point of E not in A. Suppose that there exists a point y_0 in A such that*

$$\|x_0 - y_0\| \leqq \|x_0 - y\| \qquad \text{for all } y \text{ in } A.$$

Then $\Re(x_0-y_0, y) \leqq \Re(x_0-y_0, y_0)$ for all y in A.

Proof. Consider first the case that $y_0=0$, and fix a point y in A. For any α with $0<\alpha<1$ we have $\alpha y \in A$, and so $\|x_0\|^2 \leqq \|x_0-\alpha y\|^2$; thus $2\Re(x_0, y) \leqq \alpha\|y\|^2$, and $\Re(x_0, y) \leqq 0$ results on letting α tend to zero. The general case follows at once by translation.

LEMMA 2. *With notations as in Lemma 1, assume moreover that A is bounded. Then there exists a positive integer n such that A is contained in the closed ball with center $w_0 = -n(x_0-y_0)+y_0$ and radius $r_0=(n+1)\|x_0-y_0\|$.*

Proof. By translation one may assume that $y_0=0$. Suppose, on the contrary, that for each n there exists a point y_n in A such that

$$\|y_n - (-nx_0)\| > (n+1)\|x_0\|;$$

by squaring both sides and invoking Lemma 1, one arrives at

$$\|x_0\|^2 < \|y_n\|^2(1+2n)^{-1},$$

and since $\|y_n\|$ is bounded this implies $x_0=0$, a contradiction (recall that $x_0 \notin A$). With notation as in the statement of Lemma 2, we remark that x_0 lies on the surface of the indicated closed ball, that is, $\|x_0-w_0\|=r_0$.

LEMMA 3. *With notations as in Lemma 1, let $u_0=(x_0+y_0)/2$. Then $\|u_0-y_0\| \leqq \|u_0-y\|$ for all y in A.*

Proof. For all y in A,

$$\|x_0 - y\| \geqq \|x_0 - y_0\| = \|x_0 - u_0\| + \|u_0 - y_0\|,$$

whence

$$\|u_0 - y_0\| \leqq \|x_0 - y\| - \|x_0 - u_0\|;$$

the lemma then follows at once from the inequality

$$|\,\|x_0 - y\| - \|x_0 - u_0\|\,| \leqq \|u_0 - y\|.$$

Proof of the theorem. Assume that C is a compact convex subset of the inner product space E; in particular, C is complete and bounded. Let S be the set of all bare points of C; one argues exactly as in [3] that S is nonempty (an endpoint of a segment in C of maximal length is a bare point). Let D be the closed convex hull of S; thus $D \subset C$, and D is compact and convex. It is to be shown that $D = C$.

Assume, on the contrary, that there is a point z_0 of C which does not belong to D. Let y_0 be the unique point of D such that

$$\|z_0 - y_0\| \leqq \|z_0 - y\| \qquad \text{for all } y \text{ in } D$$

(cf. [1] p. 68, Theorem 1); setting $x_0 = (z_0 + y_0)/2$, we have

$$\|x_0 - y_0\| \leqq \|x_0 - y\| \qquad \text{for all } y \text{ in } D$$

by Lemma 3. Observe that $x_0 \notin D$, since

$$\|z_0 - x_0\| = \tfrac{1}{2}\|z_0 - y_0\| < \|z_0 - y_0\|.$$

By Lemma 2 there exists a positive integer n such that D is contained in the closed ball with center $w_0 = -n(x_0 - y_0) + y_0$ and radius $r_0 = (n+1)\|x_0 - y_0\|$. Thus

$$\|y - w_0\| \leqq r_0 \qquad \text{for all } y \text{ in } D, \tag{1}$$

and, as noted earlier,

$$\|x_0 - w_0\| = r_0. \tag{2}$$

On the other hand the point z_0 does not belong to this closed ball, since by (2)

$$\|z_0 - w_0\| = \|z_0 - x_0\| + \|x_0 - w_0\| > \|x_0 - w_0\| = r_0;$$

thus

$$\|z_0 - w_0\| > r_0. \tag{3}$$

Let $r = \sup\{\|x - w_0\| : x \in C\}$; since $z_0 \in C$ we have

$$r \geqq \|z_0 - w_0\| > r_0 \tag{4}$$

by (3). By compactness there is a point u of C such that $\|u - w_0\| = r$; since

$$\|x - w_0\| \leqq r \qquad \text{for all } x \text{ in } C,$$

by the definition of r, it follows that u is a bare point of C. Then $u \in S \subset D$ and therefore $\|u - w_0\| \leqq r_0$ by (1); that is, $r \leqq r_0$, contrary to (4).

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NONPERIODICITY OF SOLUTIONS OF A THIRD ORDER EQUATION

W. R. Utz, University of Missouri

In Problem 5195, (this Monthly, 72 (1965) p. 433), A. C. Lazer shows that for the differential equation

$$y''' - p(x)y = 0, \tag{1}$$

wherein $p(x)$ is positive and continuous in an interval I, no solution can be tangent to the line $y=0$ more than once in I. That solutions of (1) may have one tangency is clear from the fact that initial conditions

$$y(a) = y'(a) = 0, \qquad y''(a) \neq 0$$

may be assumed.

Suppose that $p(x)$ is positive and continuous for all real x, then if $y''(a)$ is selected to be positive, it is known (M.Švec, Czechoslovak Math. J., (7) 82 (1957) pp. 450–462) that for this solution

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} y'(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} y''(x)$$

exists but may be infinite. In case, for this one solution, $\lim_{x\to\infty} y''(x) < \infty$, no solution of the equation oscillates (i.e., has positive maxima and negative minima for arbitrarily large x).

Since $-y(x)$ satisfies (1) if $y(x)$ does, a similar result is available at once if $y''(a) < 0$.

The purpose of this note is to point out that by using the same argument the theorem of Lazer is valid for

$$y''' - p(x)f(y) = 0, \tag{2}$$

where $f(y)$ is an odd function, and, moreover, solutions are not periodic.

Theorem 1. *If $p(x)$ is continuous and positive in the interval I: $[a, b]$ and if for all real y,*

$$yf(y) > 0 \qquad \textit{for } y \neq 0,$$

then no solution $y \not\equiv 0$ of (2) in (a, b) can be tangent to $y=0$ more than once.

Proof. Let $y(x) \not\equiv 0$ be a solution of (2), valid in (a, b) and consider

$$v(x) = y''y - \tfrac{1}{2}y'^2. \tag{3}$$

Since $v'(x)=y'''(x)y(x)=p(x)y(x)f(y(x))>0$ for $y\neq 0$, $v(x)$ is a continuous strictly increasing function. Hence $v(x)$ has at most one zero. Since $y(x)=y'(x)=0$ implies that $v(x)=0$, the theorem is proved.

THEOREM 2. *Under the hypotheses of the previous theorem, no solution of* (2) *is periodic.*

Proof. If $y(x)$ is a periodic solution of (2) of period $t>0$, then

$$v(x) = y''(x)y(x) - (y'(x))^2/2$$

is continuous and periodic of period t. As in the proof of Theorem 1, however,

$$v'(x) > 0$$

for $y(x)\neq 0$; hence $v(x)$ never decreases and can't be periodic.

NOTE ON ABSTRACT COMMUTATIVE IDEAL THEORY

P. J. MCCARTHY, University of Kansas

Let L be a multiplicative lattice that is commutative and residuated. We denote the join and meet of two elements a, $b\in L$ by $a\cup b$ and $a\cap b$, respectively. There is a multiplication defined on L such that $ab\leqq a\cap b$, $ab=ba$, and $a(b\cup c)=ab\cup ac$ for all a, b, $c\in L$. Further, for each pair of elements a, $b\in L$ there is an element $a:b\in L$ characterized by the fact that if $c\in L$ then $bc\leqq a$ if and only if $c\leqq a:b$. The element $a:b$ is called the *residual* of a by b. Various properties of the operations of join, meet, multiplication, and residuation are given in [3] and in [1, pp. 482–483].

An element $m\in L$ is called *join principal* if $(a\cup bm):m=a:m\cup b$ for all a, $b\in L$, *meet principal* if $(a\cap b:m)m=am\cap b$ for all a, $b\in L$, and *principal* if it is both join principal and meet principal. This definition of principal element is that given by Dilworth [1]: it is, in general, different from the definition of principal element used by Ward and Dilworth in [3]. On page 497 of [1], Dilworth points out that every element of the lattice $L(R)$ of ideals of a ring R is principal when R is a Dedekind domain. This prompts the following question: for which rings R (commutative and with unity) is every element of $L(R)$ principal? We emphasize that the principal elements of $L(R)$ are not the same as the principal ideals of R. Every principal ideal of R is a principal element of $L(R)$ [1, p. 481], but there may be principal elements of $L(R)$ which are not principal ideals of R, as the remark about Dedekind domains shows. In this note we shall answer the question we have asked, at least for Noetherian rings.

We shall call the lattice L an *M-lattice* if it satisfies the following condition: if a, $b\in L$ and if $a\leqq b$ then there is an element $c\in L$ such that $a=bc$.

THEOREM 1. *Let L be a commutative and residuated multiplicative lattice satisfying* (1) *the ascending chain condition, and* (2) *every element of L is a join of principal elements. Then, every element of L is principal if and only if L is an M-lattice.*

Proof. Suppose that L is an M-lattice. Then, by [3, Theorem 13.2], L is a distributive lattice. (We caution the reader that the use of the word principal in Theorem 13.2 of [3] is not the same as in this note.) It is a consequence of (1) and (2) that every element of L is a join of a finite number of principal elements. Therefore, to show that every element of L is principal, it is sufficient to show that the join of two join principal elements is join principal and that the join of two meet principal elements is meet principal.

Let m be a join principal element of L and suppose $m \leqq d$ for some $d \in L$. Then $m = cd$ for some $c \in L$, and since m is join principal, $(a \cup bcd) : cd = a : m \cup b$ for all $a, b \in L$. Hence $((a \cup bcd) : c) : d = a : m \cup b$. However, $(a \cup bd)c = ac \cup bcd \leqq a \cup bcd$, and consequently $a \cup bd \leqq (a \cup bcd) : c$. Therefore, $(a \cup bd) : d \leqq a : m \cup b$ for all $a, b \in L$. Thus, if m and n are join principal elements of L, we have for all $a, b \in L$,

$$\begin{aligned}(a \cup b(m \cup n)) : (m \cup n) &\leqq (a : m \cup b) \cap (a : n \cup b) \\ &= (a : m \cap a : n) \cup b = a : (m \cup n) \cup b.\end{aligned}$$

Since the reverse inequality always holds, $m \cup n$ is join principal.

Now, let m be a meet principal element of L and suppose that $m \leqq d$ for some $d \in L$. Then $m = cd$ for some $c \in L$, and since m is meet principal, $(a \cap b : cd)cd = am \cap b$ for all $a, b \in L$. Then, since $(b : cd)c = ((b : d) : c)c \leqq b : d$, we have $am \cap b \leqq (ac \cap (b : cd)c)d \leqq (a \cap b : d)d$ for all $a, b \in L$. Thus, if m and n are meet principal elements of L then we have, for all $a, b \in L$,

$$\begin{aligned}(a \cap b : (m \cup n))(m \cup n) &\geqq (am \cap b) \cup (an \cap b) \\ &= (am \cup an) \cap b = a(m \cup n) \cap b.\end{aligned}$$

Since the reverse inequality always holds, $m \cup n$ is meet principal.

If every element of the lattice L is meet principal (and so, if every element of L is principal) then for every $a, b \in L$ we have $a \cap b = (a : b)b$ [1, Corollary 3.1]. Hence, if $a \leqq b$ then $a = bc$ where $c = a : b$. Therefore, L is an M-lattice. This completes the proof of Theorem 1. Note that we have shown that if every element of L is meet principal then every element of L is join principal, and so principal.

A commutative ring R with unity is called a *multiplication ring* if $L(R)$ is an M-lattice. These rings have been the object of considerable study (for example, see [2] and the references in that paper). From Theorem 1 we obtain the following result, which answers the question asked earlier, at least for Noetherian rings.

THEOREM 2. *Let R be a Noetherian commutative ring with unity. Then every element of $L(R)$ is principal if and only if R is a multiplication ring.*

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CLASSROOM NOTES

Material for this department should be sent to George Raney, Department of Mathematics, University of Connecticut, Storrs, CT 06268.

THE FIRST SYMMETRIC DERIVATIVE

C. E. AULL, Virginia Polytechnic Institute

1. Introduction. This note is concerned with the symmetric derivative

$$f^s(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

and its relation with the ordinary derivative and the Lipschitz condition for a function of one variable. The symmetric derivative has applications to Fourier series and the Poisson integral. For a discussion of these, see A. Zygmund, [5, p. 100]. Throughout the paper "$G(x)$ exists" will mean "$G(x)$ exists and is finite."

It may happen that $f^s(a)$ exists even when $f(a)$ does not. The function $f(x) = 1/x^2$ has a symmetric derivative even at 0, $f^s(0)$ being 0.

The Dirichlet function [$f(x) = 1$ for x rational and $f(x) = 0$ for x irrational] has a symmetric derivative for rational points but not for irrational points.

2. Relation to the ordinary derivative. It is well known [5, p. 22] that if the ordinary derivative exists at a point, then the symmetric derivative does also. This result follows from the limit of the sum theorem and the following expression

$$\frac{f(x+h) - f(x-h)}{2h} = \frac{f(x+h) - f(x)}{2h} + \frac{f(x) - f(x-h)}{2h}.$$

But even the continuity of $f(x)$ and the existence of $f^s(x)$ does not guarantee the existence of $f'(x)$; for example, consider $f(x) = |x|$ at 0.

It will be shown here by means of a sequence of theorems that the continuity of $f^s(x)$ at a point "a" and the continuity of $f(x)$ in a neighborhood of "a" suffice for the existence of $f'(a)$.

LEMMA 1. *Let $f(x)$ be continuous on $a \leqq x \leqq b$ and let $f^s(x)$ exist on $a < x < b$. Let $f(b) > (<) f(a)$. Then there exists a point c, $a < c < b$, such that $f^s(c) \geqq (\leqq) 0$.*

Proof. Let k be such that $f(a) < k < f(b)$. The set $\{x : f(x) > k,\ a < x < b\}$ is bounded below by a so it has a g.l.b., c, distinct from a and b by the continuity of $f(x)$. Furthermore $f^s(c) \geqq 0$, since there are points $x > c$ in every neighborhood of c such that $f(x) > k$ and $f(x) \leqq k$ for $a \leqq x \leqq c$. The proof with the inequalities reversed is similar.

LEMMA 2. *There is a nondenumerable number of points such that $f^s(x) \geqq (\leqq) 0$ for $f(x)$ fulfilling the conditions of Lemma 1.*

LEMMA 3. (Quasi-Rolle's Theorem) *Let $f(x)$ be continuous on $a \leqq x \leqq b$ and let*

$f^s(x)$ exist on $a<x<b$, and let $f(a)=f(b)=0$. Then there exists a point x_1, $a<x_1<b$, such that $f^s(x_1)\geqq 0$ and a point x_2, $a<x_2<b$, such that $f^s(x_2)\leqq 0$.

Proof. If $f(x)\equiv 0$, the conclusion is immediate. If $f(x)\not\equiv 0$, there exists either c such that $f(c)>0$ or d such that $f(d)<0$ or both. By Lemma 1, there exist points x_1 and x_2, $a<x_1<c<x_2<b$ or $a<x_2<d<x_1<b$, such that $f^s(x_1)\geqq 0$ and $f^s(x_2)\leqq 0$.

THEOREM 1. (Quasi-Mean Value Theorem) *Let $f(x)$ be continuous on $a\leqq x_2\leqq b$, and let $f^s(x)$ exist on $a<x<b$. Then there exist points x_1 and x_2, $a<x_1<b$, $a<x_2<b$, such that*

$$f^s(x_2)\leqq\frac{f(b)-f(a)}{b-a}\leqq f^s(x_1).$$

Proof. This follows from Lemma 3 in a manner analogous to the proof of the Mean Value Theorem from Rolle's theorem.

THEOREM 2. *Let $f(x)$ and $f^s(x)$ be continuous on $a<x<b$. Then $f'(x)$ exists and $f'(x)=f^s(x)$.*

Proof. For h sufficiently small ($a<x+h<b$), by the previous theorem, there exist x_1 and x_2 strictly between x and $x+h$ such that

$$f^s(x_2)\leqq\frac{f(x+h)-f(x)}{h}\leqq f^s(x_1).$$

If strict inequalities hold, we apply the intermediate value theorem for continuous functions. In any case there exists x_3 strictly between x and $x+h$ such that

$$f^s(x_3)=\frac{f(x+h)-f(x)}{h}.$$

Taking the limit of both sides as $h\to 0$, knowing that the limit of the left side exists, one obtains $f^s(x)=f'(x)$. The following even stronger theorem can be proven.

THEOREM 3. *Let $f^s(x)$ be continuous at a point a and let $f(x)$ be continuous in a neighborhood of a; then $f'(a)$ exists and $f'(a)=f^s(a)$.*

Proof. For $\epsilon>0$, there exists N_a such that if $x\in N_a$, $f(x)$ is continuous and $f^s(a)-\epsilon<f^s(x)<f^s(a)+\epsilon$. By Theorem 1, there exist x_1 and x_2 strictly between a and $a+h$ such that

$$f^s(x_2)\leqq\frac{f(a+h)-f(a)}{h}\leqq f^s(x_1)$$

for N_a such that $(a+h)\in N_a$, $h\neq 0$. Hence

$$f^s(a)-\epsilon<\frac{f(a+h)-f(a)}{h}<f^s(a)+\epsilon.$$

The existence of $f'(a)$ follows and $f'(a)=f^s(a)$.

It is of interest that if the first symmetric derivative exists almost everywhere, then the ordinary derivative exists almost everywhere. For proof see Khintchine [3, p. 217]. For continuous functions the ordinary derivative exists everywhere the symmetric derivative exists, except, perhaps, on a set of first category. See Bruckner and Goffman [2, p. 511].

3. Relation to the Lipschitz condition. It is well known that the existence of a bounded derivative for $f(x)$ on $a<x<b$ implies that $f(x)$ satisfies a Lipschitz condition. The following theorem is the analogous theorem for the symmetric derivative.

THEOREM 4. *Let $f^s(x)$ be bounded on $a<x<b$, and let $f(x)$ be continuous on $a<x<b$. Then $f(x)$ satisfies a Lipschitz condition on $a<x<b$. That is, for x, t, $a<x<b$, $a<t<b$, there exists a constant A such that $|f(x)-f(t)| \leqq A|x-t|$, where A is independent of x and t.*

Proof. For some A, $|f^s(x)| \leqq A$, $a<x<b$. By Theorem 1,

$$-A \leqq \frac{f(x)-f(t)}{(x-t)} \leqq A, \quad (a<x<b,\ a<t<b)$$

and $|f(x)-f(t)| \leqq A|x-t|$, $(a<x<b,\ a<t<b)$.

The converse is not true, as shown by the following

Example. Let $f(x)=0$, $(-1 \leqq x \leqq 0)$; for $x=1/2^{2n}$, $x \leqq 1$ let $f(x)=0$; for $x=1/2^{2n-1}$, $x \leqq 1$, let $f(x)=1/2^{2n}$. Connect successive points $(1/2^n, f(1/2^n))$ by straight lines to define the function at other points on $0 \leqq x \leqq 1$.

For all x and t in the interval $(-1, +1)$, $|f(x)-f(t)| \leqq |x-t|$. Clearly, $f^s(0)$ does not exist.

In the above example $f(x)$ crosses the line $y=x/4$ an infinite number of times.

This example suggests the following definition and theorem.

DEFINITION: *$f(x)$ is said to satisfy condition (F) at a point c if $f(x)$ crosses every straight line m through $(c, f(c))$ at most a finite number of times in some neighborhood N_c^m.*

THEOREM 5. *Let $f(x)$ satisfy Lipschitz condition in $a<x<b$ and let condition (F) be satisfied for each x. Then $f^s(x)$ exists and is bounded for $a<x<b$.*

Proof. We will show that if $f^s(c)$ does not exist for some c, $a<c<b$, then there is a line through $(c, f(c))$ such that $f(x)$ crosses this line an infinite number of times in every N_c. If $f^s(c)$ does not exist, either the left hand or right hand derivative does not exist [5, p. 22]. Consider the second case, the first case being similar. The Lipschitz condition implies that the Dini derivatives $f^+(c)$ and $f_+(c)$ are bounded. See Boas [1, p. 113]. Set $f^+(c)=M$ and $f_+(c)=m$. There exists $\epsilon>0$ such that $M-m>4\epsilon$. In every neighborhood of c, there are points s, $(s>c)$ and t, $(t>c)$ such that

(1) $f(s) > (M-\epsilon)(s-c)+f(c)$ and (2) $f(t) < (m+\epsilon)(t-c)+f(c)$.

Let $s_1(c<s_1<b)$ be one such point satisfying (1) and let $t_1(c<t_1<s_1)$ satisfy (2), followed by $s_2(c<s_2<t_1)$ satisfying (1) and then by $t_2(c<t_2<s_2)$ satisfying (2). This process can be continued indefinitely. By an application of the intermediate value theorem for continuous functions we obtain an infinite number of crossing points of

$$y = (m + 2\epsilon)(x - c) + f(c).$$

From the Lipschitz condition, it follows that $f^s(x)$ is bounded in $a<x<b$ and the theorem is proved.

Condition (F) is not a necessary condition. Consider $G(x)=f(x)-f(-x)$ where $f(x)$ is the function defined in the previous example.

4. Some further remarks. At a relative maximum within an open interval $f^s(x)$ need not be 0, consider

$$f(x) = x - |2x|; \qquad f^s(0) = 1,$$

and this is a relative maximum at 0.

For a further discussion of Dini derivatives including a quasi-mean value theorem see [1]. For a discussion of conditions weaker than the Lipschitz condition but stronger than continuity see [4].

Finally, I would like to thank F. Brooks, R. Haines, and F. Leetch for their valuable suggestions in regard to this paper.

Most of the above research was done at Kent State University.

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ON A MEAN VALUE PROPERTY FOR SOLUTIONS OF A WAVE EQUATION

S. C. CHU, Bellcomm, Washington, D. C. (Now at University of Delaware)

In [1] Epstein showed that if D is "a simply connected plane domain of finite area and t a point of D such that, for every function u harmonic in D and integrable over D, the mean value of u over the area of D equals $u(t)$, then D is a disc and t is its center." This result completes a triple of properties for harmonic functions; the other two being that if $u(x, y)$ is harmonic then it has the mean value property over a disc, and if $u(x, y)$ is continuous and has the mean value property over a disc, then it is harmonic in the disc.

There is an analogous triplet for functions which are solutions of the one-dimensional wave equation. It is well known that (see [2]) a function $u(x, t)$

which is, say, twice continuously differentiable in a domain Ω, satisfies the wave equation

$$u_{xx} = u_{tt}, \tag{1}$$

in Ω, if and only if u satisfies the difference equation

$$u(A) + u(C) = u(B) + u(D), \tag{2}$$

where A, B, C, and D are the vertices of any rectangle contained in Ω, whose sides are characteristics (i.e., lines with equations $x+t=$constant or $x-t=$constant).

The purpose of this note is to give an analogue of Epstein's theorem, and thereby complete the triplet.

THEOREM. *Let Q be a quadrilateral in a plane domain Ω, and let Q have vertices $A=(x_1, t_1)$, $B=(x_2, t_2)$, $C=(x_3, t_3)$, and $D=(x_4, t_4)$. If every function $u(x, t)$ which satisfies the wave equation* (1) *in Ω also satisfies the difference equation* (2), *at the vertices of Q, then Q is a rectangle with sides which are characteristic.*

Proof. It can be assumed without loss that $D=(x_4, t_4)=(0, 0)$. The functions $x+t$ and $x-t$ are certainly solutions of (1). Substitution of these two functions into equation (2), and addition and subtraction of the two resulting equations give

$$x_3 = x_2 - x_1, \tag{3}$$

$$t_3 = t_2 - t_1. \tag{4}$$

Furthermore, the functions $(x+t)^2$ and $(x-t)^2$ are also solutions of (1). Substitution of these functions into equation (2) yields

$$(x_1 + t_1)^2 + (x_3 + t_3)^2 - (x_2 + t_2)^2 = 0,$$
$$(x_1 - t_1)^2 + (x_3 - t_3)^2 - (x_2 - t_2)^2 = 0.$$

By applying the relations (3) and (4) to the last pair of equations, one obtains

$$(x_1 + t_1)(x_1 + t_1 - x_2 - t_2) = 0, \tag{5}$$

$$(x_1 - t_1)(x_1 - t_1 - x_2 + t_2) = 0. \tag{6}$$

Equation (5) implies that either

$$x_1 = -t_1, \tag{7}$$

or

$$x_1 - x_2 = t_2 - t_1. \tag{8}$$

If (7) is true, then $x_1 \neq t_1$ (otherwise, $x_1=t_1=0$, and Q degenerates to a triangle). Therefore, (6) gives $x_1-t_1-x_2+t_2=0$, or

$$t_2 = x_2 - 2x_1. \tag{9}$$

If (8) is true, then $x_2-x_1 \neq t_2-t_1$ (otherwise $x_2=x_1$, $t_2=t_1$). Therefore, (6) gives

(10) $$t_1 = x_1$$

and hence

(11) $$t_2 = 2x_1 - x_2.$$

One has, therefore, from equations (3), (4), (7), (9), (10), and (11) either

$$t_1 = -x_1, \quad t_2 = x_2 - 2x_1, \quad x_3 = x_2 - x_1, \quad t_3 = x_2 - x_1,$$

or

$$t_1 = x_1, \quad t_2 = 2x_1 - x_2, \quad x_3 = x_2 - x_1, \quad t_3 = x_1 - x_2.$$

This means that the four vertices of Q are either

$$(0, 0), \quad (x_1, -x_1), \quad (x_2, x_2 - 2x_1), \quad (x_2 - x_1, x_2 - x_1),$$

or

$$(0, 0), \quad (x_1, x_1), \quad (x_2, 2x_1 - x_2), \quad (x_2 - x_1, x_1 - x_2).$$

In both cases, each adjacent pair of vertices lies on a line with equation $x+t=$constant or $x-t=$constant. This is the desired result.

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SOME ESTIMATES FOR $\sigma(n)$

R. L. DUNCAN, Lock Haven State College

Let $H(n)=1+\frac{1}{2}+\cdots+1/n$. Using the estimate $H(n)=O(\log n)$, Cohen [1] has shown that

(1) $$\sigma(n) = O(n \log n),$$

where $\sigma(n)$ is the sum of the divisors of n. His proof consists of the observation that

(2) $$\sigma(n) = \sum_{d/n} d = n \sum_{d/n} 1/d \leqq nH(n).$$

Our purpose here is to refine this observation to obtain improvements in (1). To do this we use the explicit estimate

(3) $$H(n) \leqq 1 + \log n.$$

The following proof of (3) requires only (see [2])

(4) $$\left(1 + \frac{1}{n}\right)^{n+1} \geqq e.$$

If we assume (3), then

$$H(n+1) = H(n) + \frac{1}{n+1} \leqq 1 + \log n + \frac{1}{n+1}$$
$$= 1 + \log (n+1) + \frac{1}{n+1}\left[1 - \log\left(1 + \frac{1}{n}\right)^{n+1}\right]$$
$$\leqq 1 + \log (n+1) \quad \text{by} \quad (4).$$

Since $H(1) \leqq 1 + \log 1$, the desired result (3) is true for every positive integer n by mathematical induction. The results (2) and (3) imply the explicit estimate

$$\sigma(n) \leqq n \log n + n \tag{5}$$

which implies (1).

We now refine (2) to obtain sharper estimates for $\sigma(n)$. It is obvious that

$$\sigma(n) = n \sum_{d/n} \frac{1}{d} \leqq n \sum_{k=1}^{d(n)} \frac{1}{k} = nH(d(n)), \tag{6}$$

where $d(n)$ is the number of positive divisors of n. It follows from (3) and (6) that

$$\sigma(n) \leqq n \log d(n) + n \tag{7}$$

and this improves (5). Since $\log d(n) = o(\log n)$ [3, Theorem 315], (7) implies that

$$\sigma(n) = o(n \log n) \tag{8}$$

and this improves (1).

We now establish several further explicit estimates for $\sigma(n)$. If p^α is the highest power of p which divides n, then

$$\sigma(n) = n \prod_{p/n}\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^\alpha}\right) < n \prod_{p/n}\left(1 - \frac{1}{p}\right)^{-1}$$
$$= n \prod_{p/n}\left(1 + \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right)^{-1}.$$

Since $\zeta(2) = \prod_p (1 - (1/p^2))^{-1}$, we have

$$\sigma(n) < \zeta(2) n \prod_{p/n}\left(1 + \frac{1}{p}\right). \tag{9}$$

If we let n be the rth power of the product of the first r primes and let $r \to \infty$, a similar argument shows that

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n} \prod_{p/n}\left(1 + \frac{1}{p}\right)^{-1} = \zeta(2).$$

It should be observed in conjunction with (9) that

$$\sigma(n) \geqq n \prod_{p/n}\left(1+\frac{1}{p}\right) \tag{10}$$

with equality holding if and only if n is square-free. Thus the constants in (9) and (10) are the best possible. By (9) we have

$$\sigma(n) < \zeta(2)n \sum_{d/n} \frac{|\mu(d)|}{d} \leqq \zeta(2)nH(2^{\omega(n)}),$$

where $\omega(n)$ is the number of distinct prime divisors of n. It now follows from (3) that

$$\sigma(n) < \zeta(2)n[1 + \omega(n) \log 2]. \tag{11}$$

Also, since $\zeta(2) = \pi^2/6 < 10/6$ and $\log 2 < 7/10$, we have

$$\sigma(n) < \frac{n}{6}[7\omega(n) + 10] \tag{12}$$

and this is virtually as good as (11).

We conclude with a simple proof that

$$\sigma(n) = O(n \log \log n). \tag{13}$$

If $r = \omega(n)$ and p_r is the rth prime, then

$$\log \prod_{p/n}\left(1+\frac{1}{p}\right) = \sum_{p/n} \log\left(1+\frac{1}{p}\right) \leqq \sum_{p/n} \frac{1}{p}$$

$$\leqq \sum_{p \leqq p_r} \frac{1}{p} = \log \log p_r + O(1).$$

Hence

$$\prod_{p/n}\left(1+\frac{1}{p}\right) = O(\log p_r) = O(\log r) = O(\log \log n).$$

The desired result now follows from (9). The above proof relies on Theorems 9 and 427 of Hardy and Wright [3] and the trivial fact that $\omega(n) = O(\log n)$.

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A COUNTEREXAMPLE ON INDEPENDENT EVENTS

E. L. Crow, Environmental Science Services Administration, Boulder, Colorado

In probability theory the events A, B, and C are said to be independent (or statistically independent) if

$$(1)\quad P[AB] = P[A]P[B], \qquad P[AC] = P[A]P[C], \qquad P[BC] = P[B]P[C],$$

$$(2)\quad P[ABC] = P[A]P[B]P[C].$$

Examples are given in the texts by Cramér [1], Fisz [2], and Parzen [5] which show that (1) does not imply (2). Cramér's example, attributed to S. Bernstein, is a three-dimensional distribution of random variables (X, Y, Z) in which each of the four points (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) carries the probability $\frac{1}{4}$. It is then easily verified that

$$\begin{aligned} P[X = 1] &= P[Y = 1] = P[Z = 1] = \tfrac{1}{2}, \\ P[X = 1, Y = 1] &= \tfrac{1}{4} = P[X = 1]P[Y = 1], \\ P[X = 1, Z = 1] &= \tfrac{1}{4} = P[X = 1]P[Z = 1], \\ P[Y = 1, Z = 1] &= \tfrac{1}{4} = P[Y = 1]P[Z = 1], \end{aligned}$$

but $P[X=1, Y=1, Z=1] = \frac{1}{4} \neq P[X=1]P[Y=1]P[Z=1]$.

As the fullness of the definition suggests, it is also true that (2) does not imply (1). This is not specifically pointed out in any of the texts I have examined [6], and it seems desirable to give an example demonstrating this proposition. Let (X, Y, Z) have the locations (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), and (1, 1, 1) with the probabilities $\frac{1}{8}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{2}{8}$, and $\frac{1}{8}$ respectively. Then

$$\begin{aligned} P[X = 1] &= P[Y = 1] = P[Z = 1] = \tfrac{1}{2}, \\ P[X = 1, Y = 1, Z = 1] &= \tfrac{1}{8} = P[X = 1]P[Y = 1]P[Z = 1], \end{aligned}$$

but

$$\begin{aligned} P[X = 1, Y = 1] &= \tfrac{3}{8} \neq P[X = 1]P[Y = 1], \\ P[X = 1, Z = 1] &= \tfrac{1}{8} \neq P[X = 1]P[Z = 1], \\ P[Y = 1, Z = 1] &= \tfrac{1}{8} \neq P[Y = 1]P[Z = 1]. \end{aligned}$$

Hence (2) does not imply (1).

Fisz and Parzen express Bernstein's example in terms of urn models; this has the advantage of eliminating reference to random variables. The latter example above can also be so expressed. Thus, we consider drawing one ball from an urn with 1 red ball, 1 blue, 3 yellow, 2 purple, and 1 white. We consider all balls to be equally probable. Let A be the event of drawing a red or a purple or a white ball, B the event of drawing a blue or a purple or a white ball, and C the event of drawing a yellow or a white ball. Then (2) holds but not (1).

Lancaster [4] recently generalized Bernstein's example to any number of

points and any number of dimensions. Geisser and Mantel [3] gave a useful continuous example.

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6. Added in proof: Exercise 11 on page 44 of Introduction to Probability and Statistics from a Bayesian viewpoint, Part 1, Probability, by D. V. Lindley (Cambridge Univ. Press, 1965) asks for the construction of such an example.

THE ALGEBRAIC DERIVATIVE OF MIKUSIŃSKI

R. G. Buschman, State University of New York at Buffalo

Jan Mikusiński has introduced into his operational calculus [2] an *algebraic derivative* which corresponds to differentiation with respect to the operator s. The operator D is defined initially by

$$Da = \{-ta(t)\};$$

that is, for locally integrable functions it corresponds to multiplication of the functional values by $-t$. In the field of operators, D is shown to produce the usual differentiation formulas for sums, differences products, and quotients. Further, if the operator is a rational function of s, then D corresponds to d/ds.

In order to introduce meaning for algebraic derivatives of other than integral order, it is convenient to reconsider the form, using $-D$ as the operator; thus

$$(-D)^n a = \{t^n a(t)\}.$$

This corresponds to the situation for Laplace transforms, where we have

$$\{t^n F(t)\} = (-d/ds)^n f(s),$$

$$\{t^{-n} F(t)\} = \int_s^\infty \cdots \int_s^\infty f(u)(du)^n.$$

Further, the correspondence is preserved if we introduce the fractional integral operator:

$$K^\alpha f(s) = [1/\Gamma(\alpha)] \int_s^\infty (u - s)^{\alpha-1} f(u)\,du, \qquad (\alpha > 0),$$

then $(-D) = K^{-1}$ and $(-D)^\beta = (-D)^n K^{n-\beta}$, where n is chosen such that $n - \beta > 0$. This allows the introduction of algebraic fractional integrals and derivatives

consistent with the algebraic derivative. Thus, if we take

$$K^{\alpha}a = \{t^{-\alpha}a(t)\},$$

we have an analog to the operational calculus relation $s^{-\alpha}a = \{I^{\alpha}a(t)\}$, where

$$I^{\alpha}a(t) = [1/\Gamma(\alpha)]\int_0^t (t-u)^{\alpha-1}a(u)du.$$

Tables of fractional integrals are available [1].

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PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

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All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P. O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before October 31, 1967. ***Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.***

E 1995. *Proposed by L. J. Mordell, St. Johns College, Cambridge, England*

$ABCD$ is a tetrahedron T, and P is any point of T. The circumsphere of T has center O and radius R, and a concentric sphere of radius $R_1 \geqq \frac{1}{2}R$, cuts OA in A_1, etc. Let $|P|$ be the least of the distances PA_1, PB_1, PC_1, PD_1. Prove that $|P| \leqq R_1$, with equality only if $R_1 = \frac{1}{2}R$ and P is distant $\frac{1}{2}R$ from A_1, B_1, C_1 or D_1.

A similar result holds for a simplex and then, when $R_1 = R$, it is due to M. J. Godwin, J. London Math. Soc., 40(1965)699.

E 1996. *Proposed by Erwin Just, Bronx Community College*

Let G be an infinite group in which no element other than the identity has finite order. If $(xy)^2=(yx)^2$ for all $x, y\in G$, must G be abelian?

E 1997. *Proposed by D. Rameshwar Rao, Osmania University, India*

If P is a product of n distinct primes, then there exist distinct positive integers $a_i (i=1, 2, \cdots, n)$ such that

$$P-1=\sum a_1+\sum a_1a_2+\cdots+\sum a_1a_2\cdots a_{n-1}+a_1a_2\cdots a_n.$$

The set of n a's is unique.

E 1998. *Proposed by C. F. Moppert, Melbourne University, Australia*

Characterize the rational primes p for which

$$(p+1)\phi(p-1)=\phi(p^2-1) \quad \text{if } p\equiv -1 \pmod 4,$$
$$(p-1)\phi(p-1)=\phi(p^2-1) \quad \text{if } p\equiv 1 \pmod 4,$$

where ϕ is Euler's totient function.

E 1999. *Proposed by L. Carlitz, Duke University*

I. Put

$$S_n=\sum_{i+j+k=n}\binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{k}$$

where the summation is over all nonnegative i, j, k such that $i+j+k=n$. Show that $S_n-S_{n-1}=\binom{2n}{n}$. A combinatorial proof would be preferred.

II. Put

$$R_{m,n}=\sum_{i+j+k\leq \min(m,n)}\binom{i+j}{i}\binom{j+k}{k}\binom{m-i-j}{k}\binom{n-j-k}{i}.$$

Show that $R_{m,n}-2R_{m-1,n-1}=\binom{mn}{m}$.

E 2000. *Proposed by Howard Kleiman, London, England*

Let q be any prime number >2, and let m be any positive integer. Then show that

$$\sum_{n=1}^{q} x^{q-n}\Bigg|\left\{\sum_{\substack{m,n=1\\ m+n=q}}^{q-1} x^{mn}+q-1\right\}.$$

E 2001. *Proposed by Harry Lass, California Institute of Technology*

Show from elementary probability considerations that

$$\int_0^1 x^k(1-x)^l dx=\frac{k!l!}{(k+l+1)!}.$$

E 2002. *Proposed by F. Leuenberger, Feldmeilen, Switzerland*

Construct a triangle ABC, having given the two sides AB and AC, and knowing that the line joining the circumcenter and the incenter is parallel to the interior angle bisector of angle A.

E 2003. *Proposed by Robert Abilock, Brandeis University*

N riflemen are distributed at random points on a plane. At a signal, each one shoots at and kills his nearest neighbor. What is the expected number of riflemen who are left alive?

Solve the problem in three dimensions (where there are interesting physical applications).

E 2004. *Proposed by C. C. Lindner, Coker College, Hartsville, S. C.*

Show that the following quotient is integral for all integers $n>0$ and $m>1$,

$$\left\{\prod_{i=0}^{n-1}(m^n - m^i)\right\}/n!.$$

E 2005. *Proposed by W. A. McWorter, University of British Columbia*

Let $p_1, \cdots, p_t$ be distinct primes and n a positive integer, and let $k=p_1 \cdots p_t$. Show that there exist nonnegative integers $a_1, \cdots, a_t$ such that

$$\sum_{i=1}^{t} a_i p_i = \binom{kn-1}{k-1} - 1.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

Inscribed Regular Polygon of 13 Sides

E 1835 [1965, 1128]. *Proposed by Tudor Zamfirescu, Bucharest, Rumania*

A circumference is given in the plane. Show that the construction of an inscribed regular polygon with 13 sides is possible using a straightedge and an angle trisector.

Solution by Michael Goldberg, Washington, D.C. It is known that all Euclidean constructions can be performed with the straight edge alone, if a circumference and its center are given. Since the center is not given, it can be found by means of the trisector. Taking any two points of the circumference, trisect an angle inscribed in this arc. Then the intersections of the joins of the original two points with the trisection points on the arc determine a diametral line. Another diametral line, determined similarly, locates the center of the circumference.

The problem requires the construction of the 12 imaginary 13th roots of unity. The method is suggested in Problem 3, p. 40, of Dickson, *Theory of Equa-*

tions. Let these roots be $R, R^2, \cdots, R^{12}$. If we take the sixth power of a root to obtain the next root, the 12 roots have the sequence $R, R^6, R^{10}, R^8, R^9, R^2, R^{12}, R^7, R^3, R^5, R^4, R^{11}$. Taking every third term, let

$$z_1 = R + R^8 + R^{12} + R^5, \qquad z_2 = R^6 + R^9 + R^7 + R^4,$$
$$z_3 = R^{10} + R^2 + R^3 + R^{11}.$$

Then

$$z_1 + z_2 + z_3 = -1,$$
$$z_2z_1 + z_3z_2 + z_1z_3 = 4(R + R^2 + \cdots + R^{12}) = -4,$$
$$z_1z_2z_3 = 4 + 5(R + R^2 + \cdots + R^{12}) = 4 - 5 = -1.$$

Then z_1, z_2, z_3 are the roots of the cubic equation $z^3+z^2-4z+1=0$. The roots of this cubic are obtained graphically by means of the Cardan formulas (Dickson, Theory of Equations, pp. 46–48) by employing Euclidean constructions in conjunction with the angle trisector.

Hence R, R^8, R^{12}, R^5 are the roots of the quartic equation $w^4-z_1w^3+z_2w^2-z_3w+1=0$. The roots of this equation are obtained by use of the Descartes formulas (pp. 52–54). Hence R and the other roots are constructible.

See also Jacob Steiner, *Geometrical Constructions with a Ruler, Given a Fixed Circle with its Center*, translated from the German by Marion E. Stark and R. C. Archibald, 1950.

A Limit Which Is Not the Derivative

E 1852 [1966, 83]. *Proposed by Stephen Haines and L. F. Leetch, Bowling Green State University, Ohio*

If $F^*(x) = \lim_{h\to 0}\{F(x+h) - F(x-h)\}/2h$, prove or disprove the following statement: If F and F^* are continuous at x, then F' exists at x and $F'(x) = F^*(x)$.

Solution by Freshman Honors Students, University of Connecticut. As a counterexample we present a function F, such that F and F^* are continuous at $x=0$, but F' does not exist at $x=0$. Define F as follows: $F(\pm 1/n) = 1/n$ for all positive integers n, $F(x)=0$ for all other real numbers x. It is immediately seen that F^* is continuous and in fact $F^*(x)\equiv 0$ for all real x, F is continuous at $x=0$, yet F' does not exist at $x=0$.

Also solved by R. D. Berlin, C. R. Conniff, Erhard Heil (W. Germany), Elaine Rakshys, Jürg Rätz (Switzerland), Simeon Reich (Israel), Francis Siwiec, Al Somayajulu, and the proposers.

Solution of Simultaneous Quadratic Equations

E 1855 [1966, 198]. *Proposed by L. M. Young, San Fernando Valley State College*

Let λ be a given (real or complex) number. Solve the system:

$$x_{n+1} = x_n^2 - \lambda \quad (n = 1, 2, 3, 4),$$

where $x_5 = x_1$, subject to the condition $x_3 \neq x_1$.

Solution by R. C. Lyness, Blackpool, England. Let x_1 and x_3 be the roots of $x^2 - px + q = 0$ and put $s_r = x_1^r + x_3^r$. From the two equalities

$$x_3 = x_1^4 - 2\lambda x_1^2 + \lambda^2 - \lambda, \qquad x_1 = x_3^4 - 2\lambda x_3^2 + \lambda^2 - \lambda,$$

we have (adding) $s_1 = s_4 - 2\lambda s_2 + 2\lambda(\lambda - 1)$, and (subtracting and dividing by $(x_1 - x_3)$), $-1 = s_1 s_2 - 2\lambda s_1$. Substitution for s_r in terms of p and q gives $q = (p^3 - 2\lambda p + 1)/2p$ and, upon eliminating q,

$$p^6 - 4\lambda p^4 + 4p^3 + 4\lambda p^2 - 1 = 0.$$

This is a cubic in $y = p - 1/p$.

We therefore solve $y^3 - (4\lambda - 3)y + 4 = 0$ for y, then $p^2 - yp - 1 = 0$ for p and finally $x^2 - px + (p^3 - 2\lambda p + 1)/2p = 0$ for x_1 and x_3.

Also solved by L. E. Ward, and by W. B. Tuel, Jr.

Roots of a Transcendental Equation

E 1857 [1966, 198]. *Proposed by Steven Rosencrans and James Potter, Massachusetts Institute of Technology*

Show that the solutions λ of $\tan \lambda = a\lambda/(\lambda^2 + c)$, where a is real and c is positive, are either real or (pure) imaginary.

I. *Solution by W. G. Tuel, Jr., San Jose, California.* Let $\lambda = s + it$, where s and t are real. Then the given equation may be broken into

$$\frac{\tan s \operatorname{sech}^2 t}{1 + \tan^2 s \tanh^2 t} = \frac{as^2(s^2 + t^2 + c)}{(s^2 - t^2 + c)^2 + 4s^2t^2}, \tag{1}$$

$$\frac{\tanh t \sec^2 s}{1 + \tan^2 s \tanh^2 t} = \frac{at(c - s^2 - t^2)}{(s^2 - t^2 + c)^2 + 4s^2t^2}, \tag{2}$$

by equating real and imaginary parts. Assume $st \neq 0$. Then upon division (1) and (2) give

$$\frac{\tan s \operatorname{sech}^2 t}{\tanh t \sec^2 s} = \frac{s}{t}\left[\frac{s^2 + t^2 + c}{c - s^2 - t^2}\right],$$

$$\left(\frac{\sin 2s}{2s}\right) \Big/ \left(\frac{\sinh 2t}{2t}\right) = \frac{c + (s^2 + t^2)}{c - (s^2 + t^2)}. \tag{3}$$

But, for all $s, t \neq 0$, the left member of (3) is less than 1, while the right member is greater than 1. This contradiction proves $st = 0$, which implies the statement of the problem.

II. *Solution by Martin Ziegler, Technische Hochschule, Clausthal-Zellerfeld, Germany.* Suppose a, c to be real. Then every solution of $\tan\lambda = a\lambda/(\lambda^2+c)$ with λ^2 nonreal must be a zero of $F(\lambda) \equiv (\lambda^2+c)\sin\lambda - a\lambda\cos\lambda$. We shall show that, with the condition $c+a^2/4 \geqq 0$, $F(\lambda)$ has only zeros with λ^2 real. Consider the eigenvalue problem for $y = y(x;\lambda)$ in $0 \leqq x \leqq 1$:

$$(1)\quad y'' + \lambda^2 y = 0, \qquad (2)\quad U_0[y] \equiv y'(0) - b_0 y(0) = 0,$$

$$(3)\quad U_1[y] \equiv y'(1) + b_1 y(1) = 0,$$

with real b_0, b_1. It is well known that (1) – (3) is a self-adjoint and normal problem with only real eigenvalues λ^2. (Cf. L. Collatz, *Eigenwertaufgaben mit technischen Anwendungen*, Leipzig 1949, p. 54.) On the other hand, direct calculation shows for the solutions of (1), $y(x;\lambda) = \alpha\cos\lambda x + \beta\sin\lambda x$, that (2), (3) is non-trivially solvable if and only if

$$G(\lambda) \equiv (\lambda^2 - b_0 b_1)\sin\lambda - (b_0 + b_1)\lambda\cos\lambda = 0.$$

A solution $F(\lambda) \equiv G(\lambda)$ exists if b_0, b_1 are the two solutions of $b^2 - ab - c = 0$, and b_0, b_1 are real provided $c+a^2/4 \geqq 0$, which gives some extension of the conditions, $c > 0$, a real, in the problem.

Also solved by G. Di Antonio, and by Louise S. Grinstein.

M. S. Klamkin notes that the present problem is a slight extension of E 1295 [1958, 450], where the first solution is similar to I above. (Note a slight misprint where the denominator on the right of the first equation should be $[1+m^2(a^2-b^2)]^2+4m^4a^2b^2$.)

System with Restricted Commutativity of Addition

E 1858 [1966, 198]. *Proposed by S. E. Dickson, University of Nebraska*

Assume the usual properties of the real numbers except for commutativity of addition. Given that the equation $x+y = y+x$ holds for real numbers x and y less than 10^{-26} apart, show that this latter restriction can be removed.

II. *Solution by J. L. Brown, Jr., Pennsylvania State University.* Let x', y' be arbitrary *distinct* real numbers and let $a = 10^{30}(x'-y')$. Since $a \neq 0$, the equations $ax = x'$ and $ay = y'$ have unique real solutions x and y respectively. Moreover, $a(x-y) = ax - ay = x' - y'$, so that $x - y = 10^{-30} < 10^{-26}$. Therefore

$$x' + y' = ax + ay = a(x+y) = a(y+x) = ay + ax = y' + x'.$$

Since commutativity is obvious when $y' = x'$, we have that addition is commutative for any pair x', y' of real numbers.

Also solved by J. C. Abad, I. K. Abruob, Merrill Barneby, S. I. Brown, J. P. Celenza, C. W. Dodge, A. H. Durfee, G. F. Feissner & V. W. Goodman, Arne Garness & G. A. Heuer, E. D. Gingerich, Anton Glaser, J. E. Homer, Jr., H. D. Junghenn, Erwin Just, E. S. Langford, Arthur Lieberman, D. C. B. Marsh, Liselotte Miller, Norman Miller, D. E. Nixon & J. P. Thomas, Howard Penn, Robert Prener, Stanley Rabinowitz, Azriel Rosenfeld, G. F. Schumm, D. L. Silverman, Robin Spital, L. A. Steen, E. J. Wegman, John Wessner, P. C. Yang, and K. L. Yocom, several of whom note that the result is already contained in E 1812 [1967, 199].

Characteristic Roots of a Matrix

E 1859 [1966, 198]. *Proposed by Judith H. Hallman, University of North Carolina and Western Electric Company*

Let $k_1, k_2, \cdots, k_n$ be any real numbers different from zero, and let $A=(a_{ij})$ be the matrix $a_{ij}=k_i/k_j$. Find the n characteristic roots of A.

III. *Solution by Judith Richman, Drexel Institute of Technology.* The rows of the matrix A are proportional to each other, so that its rank is 1. It follows that $n-1$ of the characteristic roots are zeros. Since the diagonal elements of A are unity, the trace is n, and the sum of the characteristic roots is n. Thus the roots are 0 of multiplicity $n-1$, and n.

Also solved by H. D. Abramson, W. R. Allaway, D. A. Blaeuer, J. L. Brenner, Martin Brown, R. B. S. Brooks & E. M. Scheuer, J. F. Burke, John Burslem, J. P. Celenza, P. L. Claypool, Red Cougar, R. W. Feldman, R. O. Fisher, A. V. Geramita, M. G. Greening (Australia), Eldon Hansen, J. E. Homer, Jr., Edward Hook, M. S. Klamkin, J. D. E. Konhauser, J. A. Lambert (Australia), E. S. Langford, T. J. Lee, M. W. Legg, Arthur Lieberman, D. C. B. Marsh, C. P. McCarty, Lieselotte Miller, M. G. Murdeshwar, Dave Nixon, Stanton Philipp, Stanley Rabinowitz, Steven Russ M.S.R.K. Sastry, Perry Scheinok, A. J. Schneider, R. Sivaramakrishnan (India), J. B. Skinner, Sidney Spital, Rory Thompson, W. G. Tuel, Jr., John Wessner, J. E. Wilkins, Jr., H. H. Wong, and K. L. Yocom.

Four Trigonometric Inequalities

E 1861 [1966, 199]. *Proposed by T. R. Curry, Bay Shore, N. Y.*

Given a triangle ABC with the usual notation for sides, angles, area, circumradius and inradius, prove the following inequalities, with equality holding if and only if the triangle is equilateral:

$$(1)\qquad 4\Delta\sqrt{3} \leqq \frac{9abc}{a+b+c}, \qquad (2)\qquad 2\Delta\sqrt{3} \leqq 9rR,$$

$$(3)\qquad \cot A + \cot B + \cot C \geqq \frac{(a^2+b^2+c^2)(a+b+c)\sqrt{3}}{9abc}.$$

$$(4)\qquad \csc A + \csc B + \csc C \geqq 2\sqrt{3}.$$

Solution by Leon Bankoff, Los Angeles, California. In any triangle, as is well known, $\sum \sin A$ is a maximum when $A=B=C$, so that $\sum \sin A \leqq 3\sqrt{3}/2$ or

$$(*)\qquad a+b+c \leqq 3R\sqrt{3}.$$

From (*) and $R=abc/4\Delta$ we have at once (1). Again, (*) and $2\Delta=r(a+b+c)$ give (2).

We have $2R\sum ab\cos C = abc\sum \cot A$ in which the left member may be replaced, by the law of cosines, to give $R(a^2+b^2+c^2)=abc\sum \cot A$, which with (*), proves (3).

Since $(a+b+c)(1/a+1/b+1/c) \geqq 9$, and $\sum \csc A = 2R(1/a+1/b+1/c)$, (4) follows at once, again using (*).

Also solved by A. N. Aheart, Louise S. Grinstein, M. G. Greening (Australia), M. H. Hayamizu, M. S. Klamkin, D. C. B. Marsh, Norman Miller, Stanley Rabinowitz, Stanton Philipp, Simeon Reich (Israel), M. S. R. K. Sastry, R. Sivaramakrishnan (India), Stephen Spindler, Sidney Spital, J. L. Standig, P. D. Thomas, and J. Wesley.

Radius of Curvature

E 1863 [1966, 199]. *Proposed by R. G. Albert, West Newton, Mass.*

The size and shape of a conic with positive eccentricity are determined by two parameters, the distance d from a focus to the nearest directrix and the eccentricity e. Evaluate the radius of the osculating circle at a vertex as a function of d and e.

Solution by P. D. Thomas, Naval Research Laboratory, Washington, D.C. It is known that the radius of curvature of a conic at a vertex is equal to the length of the semilatus rectum. But the length of the latus rectum is $2de$, whence the desired radius equals de.

Otherwise: the polar equation of the conic in terms of d and e is $r=de/(1+e\cos\theta)$. The radius of curvature is given by

$$R = (r^2 + r'^2)^{3/2}/(r^2 + 2r'^2 - rr'') = de(1 + 2e\cos\theta + e^2)^{3/2}/(1 + e\cos\theta)^3$$

which becomes, for a vertex ($\theta=0, \pi$), $R=de$.

Also solved by Michael Goldberg, P. F. Hultquist, J. D. E. Konhauser, R. C. Lyness (England), D. C. B. Marsh, Liselotte Miller, C. S. Ogilvy, Simeon Reich (Israel), Robin Robinson, Perry Scheinok, Sidney Spital, W. G. Tuel, Jr., W. K. Viertel, and Charles Wexler.

Group Whose Elements Satisfy $A^2 = I$

E 1864 [1966, 200]. *Proposed by R. E. Mikhel, Tri-State College, Angola, Ind.*

If G_i is an abelian group of involutoric matrices ($A^2=I$) prove that the order of G_i is of the form 2^n.

Solution by Douglas Lind, University of Virginia. Suppose a prime p is a divisor of the order of G_i. By Cauchy's theorem for abelian groups, there is an $A \in G_i$ whose order is p. But $A^2=I$, so that $p \leqq 2$, implying the result.

Also solved by I. K. Abruob, D. P. Ambrose, Joe Beckely, D. C. Eckert, R. C. Freiwald, L. D. Haugh, G. A. Heuer, J. E. Homer, Jr., Roman Kaluzniacki, M. S. Klamkin, D. C. B. Marsh, J. M. Petro, Simeon Reich (Israel), Stephen Rhodes, Michael Schulz, R. Sivaramakrishnan (India), and Dalton Tarwater.

Notes. Without appropriate restrictions the statement is false, for if F is an infinite field of characteristic 2, then

$$\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F\right\}$$

is an infinite abelian group of involutoric matrices. Perhaps the notation G_i implies a finite group.—*Petro.*

The assumption that G is abelian is superfluous; it follows from $(AB)^2 = I = A^2B^2$—*Abruob.*

The problem is stated and solved in V. I. Smirnov, *Linear Algebra and Group Theory*, ex. 17, p. 377. It also is a special case ($p=2$) of a theorem on p. 129 of Paley and Weichsel, *A First Course in Abstract Algebra—Homer.*

Smallest Circle Containing n Given Points

E 1866 [1966, 309]. *Proposed by Michael Fried, University of Michigan*

Given n points in the Euclidean plane. Find a ruler and compass construction for the smallest circle containing these n points.

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa. If a circle contains the n points and does not pass through any of them, it can be replaced by a smaller concentric circle containing the n points and passing through at least one of them; call the desired circle O and this point A. If a circle passes through A, contains the n points and does not pass through another of them, it can be replaced by a smaller circle containing the n and tangent to the first circle at A; thus O passes through two of the n points; call them A and B. If a circle passes through A and B, contains the n points and does not pass through any other of them, it can be replaced by a smaller circle containing the n points and passing through A and B, *unless* A and B are on the same diameter of the first circle. Therefore, either O passes through three of the n points or O has two of them at opposite ends of a diameter. Ruler and compass constructions of the circles determined by these conditions are routine. The trick is to do it efficiently.

Also solved by D. J. G. Baron, Michael Goldberg, Andrew Jarosak, Norman Miller, W. L. Mrozek, and Oswald Wyler.

Mrozek notes that it is shown in Rademacher and Toeplitz, *The Enjoyment of Mathematics*, 1957, pp. 103–110, that the desired circle is unique, has the properties described above, and has radius no greater than $(d/3)\sqrt{3}$, where d is the largest distance between any two of the points of the set.

An Inequality for the Inverse Tangent

E 1867 [1966, 309]. *Proposed by R. E. Shafer, Lawrence Radiation Laboratory*

Show that for $x>0$,

$$\tan^{-1} x > \frac{3x}{1 + 2\sqrt{1 + x^2}}.$$

I. *Solution by Louise S. Grinstein, New York, N.Y.* Direct computation gives

$$\frac{dF}{dx} = \frac{(\sqrt{1 + x^2} - 1)^2}{(1 + x^2)(1 + 2\sqrt{1 + x^2})^2},$$

where $F(x) = \tan^{-1}x - 3x/(1+2\sqrt{1+x^2})$. Now dF/dx is positive for all $x \neq 0$,

whence F is an increasing function. Since $F(0)=0$, it follows that $F(x)>0$ for $x>0$.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* It follows from $(\cos\phi-1)^2\geqq 0$ that

$$1 \geqq \frac{3+6\cos\phi}{(\cos\phi+2)^2}.$$

The desired result is obtained directly upon inetgration of the latter inequality with respect to ϕ from 0 to $\tan^{-1}x$ $(x>0)$.

III. *Solution by J. D. E. Konhauser, University of Minnesota.* The substitution $x=\tan y$ transforms the given inequality into $y>3\sin y/(\cos y+2)$, which is a special case of an inequality discussed on pp. 105–6 of D. S. Mitrinovic, *Elementary Inequalities.*

Also solved by V. S. Ananthachar (India), A. Behr, G. Di Antonio, W. G. Dotson, Jr., E. P. Emerson, Marco Ettrick & Stanley Rabinowitz, H. J. Fletcher, R. E. Giudici, P. L. Gore, H. A. Greenbaum, Robert Groover III, G. A. Heuer, A. S. Howard, J. E. Joseph & O. G. Smith, D. G. Kabe, Geoffrey Kandall, B. W. King, Kenneth Kramer, Arthur Lieberman, Beatriz Margolis, Charles McCracken, Renate McLaughlin, Lieselotte Miller, Norman Miller, N. B. Murray Larry Ortloff & John Knox, Stanton Philipp, Dante Pingenot, Joel Pitcairn, George Purdy (England), Simeon Reich (Israel), M. S. R. K. Sastry, Perry Scheinok, J. G. Slagle, Sidney Spital, J. L. Standig, L. A. Steen, Franz Streit (Switzerland), J. A. Tierney, Gregory Wulczyn, H. L. Walton, Charles Wexler, J. E. Wilkins, Jr., J. C. Williams, and K. L. Yocom.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before December 31, 1967. Any contributor wishing acknowledgment of his solution should include a self-addressed stamped postcard. (Solvers outside the United States omit the stamp.)

5500. *Proposed by W. A. McWorter, University of British Columbia*

Let H be a nontrivial finite group. Then there exists a finite group G in which H is properly embedded as its own normalizer in G.

5501. *Proposed by Harsh Pittie, Princeton University*

Let S be a σ-finite measure space. Prove that $L^p(S)$ is τ sequentially complete, where τ is any locally convex Hausdorff topology such that the conjugate of $L^p(S)$ under τ is L^q, $(q=p/(p-1))$.

5502. *Proposed by Roy O. Davies, The University, Leicester, England*

Is it possible for the Cauchy product $\sum_{n=1}^{\infty}c_n$ of two conditionally convergent series $\sum_{n=1}^{\infty}a_n$, $\sum_{n=1}^{\infty}b_n$ to be absolutely convergent? $(c_n=a_1b_n+a_2b_{n-1}+\cdots+a_nb_1.)$

5503. *Proposed by M. T. L. Bizley, London, England*

Prove or disprove the following conjecture. The number of lattice paths from $(0, 0)$ to (km, kn) where m, n are coprime integers and k is any integer, having just r lattice points below the line $y/x = n/m$ (not counting the endpoints) is:

$$\psi_k + \psi_{k-1}\phi_1 + \psi_{k-2}\phi_2 + \cdots + \psi_p\phi_{k-p}, \quad \text{for } (p-1)(m+n) \leqq r < p(m+n),$$

$(p = 1, 2, \cdots, k)$, where ϕ_k, ψ_k are respectively the coefficients of x^k in $\phi \equiv \exp(F_1x + F_2x^2 + \cdots) - 1$ and $\psi = 1 - \exp(-F_1x - F_2x^2 - \cdots)$, and where

$$F_j = \frac{1}{j(m+n)}\binom{jm+jn}{jm}.$$

5504. *Proposed by Erwin Just, Bronx Community College*

Let f be a function defined on $(0, 1)$ such that $f'(x) = 1$ at each point of an infinite subset of $(0, 1)$. Must there exist a point at which $|f|$ is differentiable?

5505. *Proposed by Erwin Just, Bronx Community College*

If p is a prime and $p > m^{1/2}$, prove that any group of order p^2m is not simple.

5506. *Proposed by J. D. Aczel, University of Waterloo, Canada*

Find all real functions satisfying

$$f(xy) = f(x)f(y) \qquad \text{and} \qquad f(x + k) = f(x) + f(k)$$

for all real x, y and for one real $k \neq 0$.

5507. *Proposed by B. S. Yadav, Sardar Vallabhbhai Vidyapeeth, India*

If $\{\delta_n\}$ is a sequence of real numbers such that

$$\sum_{\nu=1}^{\infty} \frac{\log \nu}{\nu^2}\left\{\sum_{k=1}^{\nu} k^2\delta_k^2\right\}^{1/2} + \sum_{\nu=1}^{\infty} \frac{\log \nu}{\nu}\left\{\sum_{k=\nu+1}^{\infty} \delta_k^2\right\}^{1/2} < \infty.$$

Prove or disprove that $\sum_{n=2}^{\infty}\delta_n^2 \log n < \infty$.

5508. *Proposed by Albert Wilansky, Lehigh University*

Let X be a topological space which has a one-point compactification Y and a two-point compactification Z such that Y and Z are homeomorphic. Show that for every natural number n, X has an n-point compactification homeomorphic with Y; and give an example of a space X with this property, where Y is Hausdorff.

5509. *Proposed by A. K. Dewdney, University of Michigan*

Prove or disprove the following statement: C is a continuous curve from the point $(0, 0)$ to the point $(n, 0)$ in R^2. Denote by $T: R^2 \to R^2$ the translation $(x, y) \to (x+1, y)$ of the plane. Then $C \cap T(C) \neq \varnothing$, i.e., the translated curve intersects the original curve at some point.

SOLUTIONS OF ADVANCED PROBLEMS

The Coefficients of $(\Sigma c_i x^i)^n$

5325 [1965, 913; 1966, 901; 1967, 329]. *Proposed by D. J. Newman, Yeshiva University, and W. E. Weissblum, AVCO Research*

Suppose $0 \leqq c_i \leqq 1$ and $\sum c_i = 2$. Prove that all the coefficients of $(\sum c_i x^i)^n$ are $\leqq \binom{n}{q}$, where $q = [n/2]$.

III. *Completion of Solution I by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa.* The final inequality in Solution I [1966, 901–2] is correct only for n odd, when it is an equality. For the induction from n even to $n+1$ odd, we let $n = 2m$ and put

$$\left(\sum c_i x^i\right)^m = \sum d_i x^i$$

with

$$d_i \leqq \binom{m}{q} \quad \text{and} \quad q = [m/2].$$

$(\sum c_i x^i)^{n+1} = \sum e_i x^i$ then yields

$$e_N = \sum_{i+j+k=N} d_i d_j c_k \leqq \binom{m}{q}^2 \sum c_k = 2\binom{m}{q}^2 < \binom{2m+1}{m} = \binom{n+1}{q'}$$

with $q' = [(n+1)/2]$. To establish the last inequality consider the cases m even and m odd separately. We use the identity

$$\binom{a}{0}\binom{b}{c} + \binom{a}{1}\binom{b}{c-1} + \cdots + \binom{a}{c}\binom{b}{0} = \binom{a+b}{c}.$$

(See, e.g., Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, equation (12.9).) For m odd,

$$2\binom{m}{q}^2 = \binom{m}{q}\binom{m}{q+1} + \binom{m}{q+1}\binom{m}{q} \leqq \binom{2m}{m} < \binom{2m+1}{m},$$

where we have used $a = b = c = m$. For m even

$$\binom{m}{q}^2 < \binom{m+1}{q}\binom{m}{q}$$

and

$$\binom{m}{q}^2 \leqq \binom{m+1}{q-1}\binom{m}{q+1} + \binom{m+1}{q+1}\binom{m}{q-1}$$

hold and thus

$$2\binom{m}{q}^2 < \binom{2m+1}{m},$$

where we have used $a=m+1$ and $b=c=m$. Since Solution I gives the induction from odd to even, starting at $n=1$, we do not need to prove the case $n=2$. However, it is simple to do so:

$$e_N = \sum_{i+j=N} c_i c_j \leqq \sum c_j = 2 = \binom{2}{1}.$$

The Sequence Function $x_n=f(x_{n+1}, x_{n+2}, \cdots)$

5348 [1965, 1136; 1966, 1131]. *Proposed by Fred Galvin, St. Paul, Minnesota*

Show that there is a function f such that for any sequence $(x_1, x_2, \cdots)$ we have $x_n=f(x_{n+1}, x_{n+2}, \cdots)$ for all but finitely many n.

Editorial Note. It has been pointed out by the proposer that the "stronger result" claimed in the printed solution is incorrect. B. L. D. Thorpe cites the following counterexample to the argument as presented. Select a well ordering in which the first element is the sequence $(2, 1, 0, 1, 0, 1, 0, \cdots)$. Consider the sequence $(x_1, x_2, \cdots)=(1, 0, 1, 0, 1, 0, \cdots)$. Then $f(x_3, x_4, \cdots)=f(1, 0, 1, 0, \cdots)=2\neq x_2$, and $f(x_5, x_6, \cdots)=f(1, 0, 1, 0, \cdots)=2\neq x_4$, etc. Clearly $f(x_{n+1}, x_{n+2}, \cdots)$ fails to equal x_n for infinitely many values of n.

Dr. Silverman has, accordingly, withdrawn his submission with apologies. We offer instead:

II. *Solution by B. L. D. Thorp, York University, Toronto.* This problem can be generalized as follows: Let S denote the set of all real sequences $x=(x_1, x_2, \cdots)$ and let ϕ be a strictly increasing mapping of the positive integers N into itself. If the mapping T on S is defined by

$$Tx = (x_{\phi(1)}, x_{\phi(2)}, x_{\phi(3)}, \cdots),$$

characterize those functions ϕ for which there exists a function f on S satisfying

$$x_n = f(T^n x) \quad (x \in S)$$

for all but finitely many n.

It is shown that f exists if and only if the complement of $\phi(N)$ in N consists of exactly one point. Suppose that $N\backslash\phi(N)$ is empty. Then T is the identity mapping and no f exists. If $N\backslash\phi(N)$ is infinite, the sequence

$$x_i = \begin{cases} 0 & i \in \phi(N) \\ 1 & i \notin \phi(N) \end{cases}$$

fails to satisfy the condition for any function f. Suppose that $N\backslash\phi(N)$ consists of the points $k_1<k_2<\cdots<k_p$, where $p>1$. The sequence

$$x_i = \begin{cases} 1 & \text{for } i \geqq k_p \text{ and } i - k_p \equiv 0 \pmod{p} \\ 0 & \text{otherwise} \end{cases}$$

fails to satisfy the condition for any function f on S. The necessity of the condition is established.

Suppose that $\{k\} = N \backslash \phi(N)$. Define an equivalence relation on S by writing $x \sim y$ if there exist $p, q \in N$ such that $T^p x = T^q y$. Clearly $x \sim y$ if and only if there exist positive integers p, q such that $x_{p+i} = x_{q+i}$ for $i \geqq k$ and $x_i = y_t$ for $1 \leqq t \leqq k-1$. Suppose that the equivalence class A contains an element which is eventually periodic. An element x in A can be chosen and $r \geqq k$ determined such that $x_i = x_{i+r}$ for $i \geqq k$. Define f on A by

$$f(y) = \begin{cases} x_{r+n} & \text{if } y = T^n x \quad (n = 0, 1, 2, \cdots) \\ 0 & \text{otherwise.} \end{cases}$$

f is well-defined on A and satisfies the required condition. If B is an equivalence class which does not contain an element which is eventually periodic, choose an element x of B and define f on B by

$$f(y) = \begin{cases} 0 & \text{if } y = x \\ x_n & \text{if } y = T^n x \quad (n = 1, 2, \cdots) \\ 0 & \text{otherwise.} \end{cases}$$

Sums and Composition of Legendre Symbols

5388 [1966, 421]. *Proposed by L. Carlitz, Duke University*

Let p be an odd prime and put

$$K(a, b) = \sum_{x=1}^{p-1} e(ax + bx') \qquad (e(x) = e^{2\pi x i/p}), \quad xx' \equiv 1 \pmod{p},$$

$$J(a) = \sum_{x=1}^{p-1} \psi(x)\psi(x^2 - a),$$

where $\psi(x) = (x/p)$, the Legendre symbol. Show that, for $p \equiv 1 \pmod 4$,

$$\sum_{a=1}^{p-1} K(a, b)J(4a) = \begin{cases} 0 & (\psi(b) = -1) \\ 2pG(c) & (b = c^2, \quad c \neq 0), \end{cases}$$

where $G(c) = \sum_{x=1}^{p-1} \psi(x)e(cx)$.

I. *Solution by R. A. Smith, University of Toronto.* We prove the following more general result: Let p be an odd prime and let $\psi(x) = (x/p)$, the Legendre symbol. For any integer z co-prime to p, define z' by $zz' \equiv 1 \pmod p$. Put

$$K(a, b) = \sum_{x=1}^{p-1} e(ax + bx'), \qquad (b, p) = 1,$$

(since $\psi(-1)=1$). We have therefore, for $b=c^2$, $c\neq 0$,

$$\sum_{a=1}^{p-1} K(a, b)J(a) = 2pG(c).$$

Also solved by M. G. Greening (Australia), and K. S. Williams.

Nonisomorphic Ordered Fields

5400 [1966, 673]. *Proposed by W. C. Waterhouse, Harvard University*

Find, or disprove the existence of, two (totally) ordered fields which are isomorphic as fields and isomorphic as ordered sets, but not isomorphic as ordered fields.

Solution by G. A. Heuer, University of California at Berkeley. We exhibit three ordered fields, two of which are Archimedean ordered and the third non-Archimedean ordered, all three of which are isomorphic as fields, all three isomorphic as ordered sets, and no two of which are isomorphic as ordered fields.

Let Q be the field of rationals, and $F_1=Q(\alpha)$, $F_2=Q(\beta)$, and $F_3=Q(t)$, where t is an indeterminate. (α and β are algebraically independent transcendentals over the rationals.) F_1 and F_2 are ordered as subfields of the ordered field R of real numbers. In F_3 a polynomial is positive if and only if its leading coefficient is positive, and a quotient of polynomials is positive iff numerator and denominator are both positive or both negative. F_3 is a well-known example of a non-Archimedean ordered field; F_1 and F_2, as subfields of R, are Archimedean. That all three are isomorphic fields is familiar; that all are isomorphic as ordered sets follows from the fact that any two denumerably infinite dense ordered sets without first or last elements are order isomorphic. That F_3 is not isomorphic to F_1 or F_2 as ordered fields is already clear; that F_1 and F_2 are not isomorphic ordered fields is seen as follows. The only field isomorphisms of F_1 and F_2 are those generated by $\alpha\to a\beta+b$, where a and b are rational. Let $\beta'=a\beta+b$, and let m and n be integers such that m/n lies between α and β', and $n>0$. Then $m-n\alpha$ and $m-n\beta'$ lie on opposite sides of 0, so no field isomorphism of F_1 onto F_2 maps positives to positives.

Various generalizations to larger sets of such fields are more or less immediately noticed. In particular, one may have an infinite sequence $\{K_0, K_1, \cdots\}$ of such fields, with each $K_i\subset K_{i+1}$ and K_{i+1} non-Archimedean over K_i (i.e., possessing infinitely large elements over K_i). Let $\{a_1, a_2, \cdots\}$ be an algebraically independent, denumerably infinite set of transcendental real numbers, and $\{t_1, t_2, \cdots\}$ a denumerably infinite set of indeterminates. The fields $K_0=Q(a_1, a_2, \cdots)$, $K_n=K_{n-1}(t_n)$ for $n=1, 2, \cdots$, where the order in K_0 is induced by that in R, and K_n gets its order from K_{n-1} as F_3 did from Q, are all isomorphic as fields and as ordered sets, but no two are isomorphic as ordered fields.

Also solved by D. A. Hejhal, J. O. Kiltinen, W. M. Lambert, Jr., J. G. Mauldon (England), D. J. Newman, W. R. Scott, William Smythe, and the proposer.

Matrices of Rank 2

5401 [1966, 674]. *Proposed by Charles Wells, Duke University*

Let ϵ be a primitive nth root of unity. Define a matrix $A=(a_{ij})$ by $a_{rs}=\epsilon^r-\epsilon^s$ for $r, s=1, 2, \cdots, n$. Find the rank of A.

Solution by Dennis A. Hejhal, Student, University of Chicago. For $n\geqq 2$, the rank of A is 2. More generally, this is also true of $a_{rs}=x_r-y_s$ if $\{x_r\}$, $\{y_s\}$ are nonconstant. To see this, note that if the first column is subtracted from the kth column, the result is the constant column y_1-y_k; this implies that the rank of A cannot exceed 2. The identity

$$\begin{vmatrix} x_j - y_t & x_j - y_u \\ x_k - y_t & x_k - y_u \end{vmatrix} = (x_j - x_k)(y_t - y_u)$$

and the nonconstancy of $\{x_r\}$, $\{y_s\}$ now imply that the rank of A is precisely 2.

Also solved by A. S. Adikesavan (India), P. N. Bajaj, D. L. Barnett & P. J. Nikolai, W. J. Blundon, J. L. Brenner, Robert Breusch, Red Cougar, M. S. Demos, M. A. Fechter, G. J. Foschini, P. M. Gibson & J. A. Marlin, Eldon Hansen, W. E. Hoff, Thomas Jefferson, Jr.,GeoffreyKandall, Daniel Marcus, J. G. Mauldon (England), M. D. Mavinkurve (India), Ka Menehune, Norman Miller, J. P. Muskat, Dave Nixon, Vishnu Parkash (India), Alfred Pietrowski, J. R. Purdy, Sister Marion Beiter & Robert DeCarli, Sidney Spital, J. E. Wilkens, Jr., and the proposer.

A Pair of Oscillatory Functions

5402 [1966, 674]. *Proposed by David Shelupsky, The City College, New York*

For $N>-1$ let the functions f and g be defined by the expansions

$$f(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+N+1}}{(2m+N+1)!(2m+1)!},$$

$$g(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+N}}{(2m+N)!(2m)!},$$

which somewhat resemble the expansions of the trigonometric functions. Prove that each of these functions has, in common with the trigonometric functions, an infinite number of real zeros.

I. *Solution by H. C. Howard, University of Wisconsin at Milwaukee.* A simple calculation yields the following equations for $f(x)$, $g(x)$:

$$xf''(x) - (N-1)f'(x) = g(x)$$
$$xg''(x) - (N-1)g'(x) = -f(x).$$

From these it follows that both $f(x)$ and $g(x)$ are solutions of

$$[x^{-N-2}y''(x)]'' + x^{-N}y(x) = 0.$$

Thus f and g are solutions of an ordinary differential equation of the form $(r(x)y''(x))''+p(x)y(x)=0$, with $r(x)$ and $p(x)$ positive as $x\to\infty$. Oscillatory properties of solutions of such equations have been studied by Leighton and Nehari in their paper, *On the oscillatory solutions of self-adjoint linear differential equations of the fourth order*, Trans. A. M. S., v. 89 (2), (1958) 325–377. The solution of the present problem is contained in

THEOREM 11.1: *If there exists a real number* α *such that* $\limsup_{x\to\infty}x^{-2-\alpha}r(x)<1$, *and* $\liminf_{x\to\infty}4x^{2-\alpha}p(x)>\alpha^2$, *then every nontrivial solution of the equation* $(ry'')''+py=0$ *possesses an infinite number of distinct real zeros.*

Thus, choosing $\alpha=-(N-1)$ for example, and noting $\limsup_{x\to\infty}x^{-2+N-1}x^{-N+2}=0$, and $\liminf_{x\to\infty}4x^{2+N-1}x^{-N}=\infty$, an application of this theorem yields the desired result.

II. *Solution by L. E. Ward, Sr., Escondido, Cal.* The identity

$$e^{-N\pi i/4}x^{N/2}J_N(2\sqrt{ix}) = g(x) - if(x)$$

yields, for large positive x, the asymptotic relations

$$f(x) \sim p(x)\cdot\sin(\sqrt{2x} - N\pi/4 - \pi/8),$$
$$g(x) \sim p(x)\cdot\cos(\sqrt{2x} - N\pi/4 - \pi/8),$$

where $p(x)=e^{\sqrt{2x}}x^{N/2-1/4}/2\sqrt{\pi}$. The result of the problem follows from these relations. It is to be noted that the restriction $N>-1$ is not needed. The functions f, g are closely related to Thompson's functions; see, e.g., Watson, *Bessel Functions*, rev. ed., section 3.8.

III. *Solution by A. Jakimovski, University of Tel-Aviv, Israel.* We shall prove that $f(x)$ and $g(x)$ have an infinite number of zeros and that all the zeros are real. The functions $z^{-N-1}f(z)$ and $z^{-N}g(z)$ are entire functions of order $1/2$. Therefore they have an infinite number of real or complex zeros. (See Titchmarsch, *The Theory of Functions* (Oxford, 1939), Section 8.26, p. 252.) By applying the theorem of Section 8.62 in Titchmarsch (which is true for $a\geqq 0$) to the pair of functions

$$f_1(z) = \sum_{n=0}^{\infty}\frac{z^n}{(2n)!} = \cos i\sqrt{z} = \prod_{n=0}^{\infty}\left(1+\frac{z}{(n-1/2)^2\pi^2}\right),$$

$$\phi_1(z) = \frac{1}{\Gamma(2z+N+1)}, \quad (N>-1),$$

it follows that all the zeros of the function

$$\sum_{n=0}^{\infty} \frac{z^n}{(2n+N)!(2n)!}$$

are real and negative. Hence all the zeros of $g(z)$ are real. Similarly, by applying the same theorem to the pair of functions

$$f_2(z) = \sum_{m=0}^{\infty} \frac{z^m}{(2m+1)!} = \frac{\sin i\sqrt{z}}{i\sqrt{z}} = \prod_{n=0}^{\infty}\left(1 + \frac{z}{\pi^2 n^2}\right),$$

$$g_2(z) = \frac{1}{\Gamma(2z+N+2)}, \quad (N > -2),$$

it follows that all the zeros of $f(z)$ are real.

IV. *Solution by F. W. Steutel, Technische Hogeschool Twente, Netherlands.* The Laplace transforms ϕ and γ of f and g (restricted to $x \geqq 0$) are

$$\phi(\tau) = \tau^{-N-1} \sin\frac{1}{\tau}, \qquad \gamma(\tau) = \tau^{-N-1} \cos\frac{1}{\tau}.$$

If $f(x)$ has no zeros on (x_0, ∞) then in

$$\phi(\tau) = \int_0^{x_0} e^{-\tau x} f(x)\,dx + \int_{x_0}^{\infty} e^{-\tau x} f(x)\,dx$$

the first integral remains bounded if $\tau \downarrow 0$ while the second is monotone. It would follow that $\phi(\tau)$ either has a limit (finite or infinite) for $\tau \downarrow 0$ or remains bounded. As this clearly contradicts the facts, $f(x)$ (and in the same way, $g(x)$) has zeros exceeding x_0 for every x_0. Therefore $f(x)$ and $g(x)$ have infinitely many zeros.

Also solved by M. D. Mavinkurve (India), J. J. Roseman, and the proposer.

Primitive Simple Rings

5403 [1966, 674]. *Proposed by Kwangil Koh, North Carolina State University*

It is known that a simple ring is not necessarily primitive. Prove that if R is a simple ring such that for each right (or left) ideal I of R, $IR = I (RI = I)$, then R is primitive.

Solution by Ram N. Gupta, Ramjas College, Delhi, India. It is sufficient to show that R has a maximal right ideal. Let $x \neq 0$ be in R. Let $\bar{x}$ be the right ideal of R generated by x. Then $\bar{x}R = \bar{x}$, so $xR = \bar{x}$. $xe = x$ for some $e \in R$. $ey - y \in x^r$ for every $y \in R$ and $e \notin x^r$. Use Zorn's lemma to show that there exists a maximal member J of the family $\{I: I$ right ideal of R, $e \notin I$, $I \supseteq x^r\}$. J will in fact be a maximal right ideal of R.

REMARK. The above argument can be used to prove that a simple ring for which there exists M, a strictly cyclic ($xR = M$ for some $x \neq 0$ in M) right R module, is primitive.

Also solved by R. A. Howland, and by the proposer.

Almost-Pythagorean Triples

5404 [1966, 674]. *Proposed by George Grossman, Board of Education, New York City*

By an almost-Pythagorean triangle we mean one whose sides are natural numbers satisfying $a^2+b^2=c^2\pm 1$. Derive formulas for determining a, b, c. In particular, show that if (a_1, b_1, c_1) is a solution, then there are infinitely many solutions (a, b, c) with $a/c=a_1/c_1$ and infinitely many with $a/b=a_1/b_1$.

I. *Solution by Charles Wexler, Arizona State University.* Given

$$a^2 \pm 1 = (c + b)(c - b), \tag{1}$$

we assume there are integral solutions, and put $r=c+b$, $s=c-b$, so that $c=\frac{1}{2}(r+s)$, $b=\frac{1}{2}(r-s)$. We note that

$$rs = a^2 \pm 1, \qquad r \geqq s, \qquad r \equiv s \pmod 2. \tag{2}$$

Conversely, if r, s satisfy (2) for some choice of a, then $b=\frac{1}{2}(r-s)$, $c=\frac{1}{2}(r+s)$ are integers satisfying (1). Now, in case $a^2-1=rs$, we see that for every $a\geqq 2$ there always exist natural numbers r, s, given by (2) (excluding the trivial case $a+1=r=s+2$), which produce satisfactory values of b, c. In case $a^2+1=rs$, $r\equiv s \pmod 2$ implies a is even. Hence for every even a there exist r, s, and, hence, b, c.

If (a_1, b_1, c_1) is a solution, we seek integers k, b such that (ka_1, b, kc_1) is a solution of (1). Substitution in (1) gives $k^2a_1^2-1=k^2c_1^2-b^2$ or $b^2-k^2(b_1^2-1)=1$. This is a Pell equation of the form $x^2-Dy^2=1$, where $D=b_1^2-1$ is nonsquare, which is known to have infinitely many solutions in integers b, k.

If we wish solutions (a, b, c) with $k=a/a_1=b/b_1$, we obtain similarly $c^2-(c_1^2+1)k^2=-1$. Again c_1^2+1 is nonsquare. This type of Pell equation does not always have solutions, but when it has one solution, as in our case, it also has infinitely many, as required.

II. *Solution by Michael Goldberg, Washington, D. C.* A solution for $a^2+b^2=c^2+1$ is obtained as a special case of the equation $a^2+b^2=c^2+d^2$, for which the general solution is given in L. E. Dickson, *History of the Theory of Numbers*, v. 2, p. 254. It is

$$2a = uv + xy, \qquad 2b = ux - vy, \qquad 2c = ux + vy, \qquad 2d = uv - xy.$$

If we choose arbitrary sets u, v, x, y subject to the condition that $2d=uv-xy=2$, then all solutions of the special case may be obtained.

Solutions for $a^2+b^2=c^2-1$ are obtained as a special case of the equation $a^2+b^2+d^2=c^2$. The general solution of this equation is given (loc. cit. p. 269). It is

$$a = 2(ux + vy), \qquad b = 2(uy - vx), \qquad c = u^2 + v^2 + x^2 + y^2,$$

$$d = u^2 + v^2 - x^2 - y^2.$$

If we choose arbitrary sets u, v, x, y subject to the condition

$$d = (u + x)(u - x) - (y + v)(y - v) = 1,$$

then all the solutions of the special case may be obtained.

Since, in both cases there are four variables connected by the relation $d=1$, the imposition of the condition $a/c=k$, or $a/b=k$, still leaves two variables in a Diophantine equation, and this permits an infinity of solutions.

Also solved by V. S. Ananthachar (India), Walter Bluger, Julia Caspino, M. A. Ettrick, M. G. Greening (Australia), K. E. Hirst (England), Norman Miller, C. Stanley Ogilvy, Vishnu Parkash (India), and A. Thyagaraja (India).

Editorial Note. Thyagaraja, in his solution, shows that $(2mn+1, mn^2-m+n, mn^2+m+n)$ is a solution of $a^2+b^2=c^2+1$.

Bluger shows that if (a, b, c) is a solution of $a^2+b^2=c^2-1$, then so is $(2ac, 2bc, 2c^2-1)$, establishing infinitely many solutions with constant ratio a/b. Likewise if (a, b, c) is a solution of $a^2+b^2=c^2+1$, then so is $(2ab, 2b^2-1, 2bc)$.

Ogilvy referred to another recent appearance of the problem. See *The New York State Mathematics Teachers Journal*, 16 (1966) 19.

Solution I above meets the stated requirements of the problem but is not quite satisfactory as a complete solution of (1). Factorization of $a^2\pm1$ is required, and it may be entirely impossible to find all possible values of r and s when a is large. In solution II where we must solve $uv-xy=2$, we can start with arbitrary values of u and x, with $(u, x)|2$, and solve the resulting linear Diophantine equation for v, y, of which there are infinitely many sets. A similar comment applies to the condition on d in the second case.

Normal Subgroups in a Finite Group of Even Order

5406 [1966, 674]. *Proposed by C. C. Lindner, Coker College, Hartsville, S. C.*

Prove that a finite group of order $2n$ which contains an element of order n has at least $\tau(n)+1$ normal subgroups. $\tau(n)$ is the number of divisors of n.

Solution by Gomer Thomas, University of Illinois. Let G have order $2n$, let x be an element in G of order n, and let $\langle x\rangle$ be the cyclic group generated by x. Then $|G:\langle x\rangle| = 2$, so $\langle x\rangle$ is normal in G. For every integer m dividing n there is exactly one subgroup of $\langle x\rangle$ of order m, namely the subgroup generated by $x^{n/m}$. Thus $\langle x\rangle$ has exactly $\tau(n)$ subgroups, each of which is characteristic in $\langle x\rangle$ and hence normal in G, so the number of normal subgroups of G (including G itself) is at least $\tau(n)+1$.

Also solved by W. O. Alltop, J. L. Berggren, J. L. Brenner, P. R. Chernoff, Eleanor G. Dawley, J. F. Dillon, Annabel Fong, N. Ganesan (India), Kenneth Gehner, H. A. Gindler, Myron Goldberg & Gautam Pandya, Agatha Himmelfarb, E. C. Hook, Erwin Just, R. S. Kaluzniacki, Geoffrey Kandall, Betty Kvarda, Arthur Lieberman, A. F. Long, Jr., M. D. Mavinkurve (India), Jagan M. Mital (India), Wanda J. Mourat, Vishnu Parkash (India), Donald Quiring, W. R. Scott, K. N. Sigmon, C. H. Toll, Qazi Zameeruddin (India), and the proposer.

Note. As observed by several solvers, the critical aspect of this problem is contained in I. N. Herstein, *Topics in Algebra*, p. 45, ex. 13.

A Nondecreasing Indicator Function

5407 [1966, 674]. *Proposed by S. W. Golomb, University of Southern California*

Let $f(n)$ be a function with both range and domain the positive integers, and define $g(n)$ to be the number of values of m such that $f(m)=n$. If $f(n)$ is monotone nondecreasing and satisfies $f(n)=g(n)$ for all n, then $f(n)$ is uniquely determined. Find an asymptotic expression for $f(n)$ as $n\to\infty$.

I. *Solution by Daniel Marcus, Student, Adelphi University.* First notice that the sequence defined by $a_n=f(n)$ is 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, $\cdots$. It follows from the properties of f that the kth block of consecutive equal entries (all equal to k) has length a_k. Now suppose that h is a continuous, differentiable function defined for all positive x, such that h agrees with f at the terminal point of each block, that is, $h(n)=f(n)$ for n satisfying $n=\sum_{k=1}^{f(n)}f(k)$. Then by the mean value theorem, there must exist, for each n, ξ_n in the open interval $(\sum_{k=1}^{f(n)-1}f(k),\ \sum_{k=1}^{f(n)}f(k))$ such that $h'(\xi_n)=1/f(f(n))$. Allowing $h'(\xi_n)\approx h'(n)$, this leads us to suspect that an asymptote F of f satisfies the differential equation $F'(x)=1/F(F(x))$. One solution is easily obtainable. Setting $F(x)=ax^r$, we find that $a=(\phi-1)^{-1/(\phi+1)}$ and $r=\phi-1$, where $\phi=\frac{1}{2}(1+\sqrt{5})$. Therefore as $n\to\infty$, $f(n)$ is approximated by $((\sqrt{5}+1)/2)^{(3-\sqrt{5})/2}n^{(\sqrt{5}-1)/2}$.

II. *Solution by N. J. Fine, Pennsylvania State University.* We shall prove that $f(n)\sim\delta^{\delta-1}n^{\delta}$, where $\delta=(\sqrt{5}-1)/2$.

Let $k(0)=0$ and for $M\geqq 1$, let $k(M)$ be the greatest j for which $f(j)=M$. Thus

$$k(M-1)<n\leqq k(M)\Leftrightarrow f(n)=M\quad(M\geqq 1). \tag{1}$$

Since the number of n's for which (1) holds is $f(M)$, we have

$$f(M)=k(M)-k(M-1)\quad(M\geqq 1). \tag{2}$$

Therefore

$$k(M)=\sum_{n=1}^{M}f(n). \tag{3}$$

We need some preliminary estimates. Since $f(n)\geqq 1$, we have $k(M)\geqq M$, so $M=f(k(M))\geqq f(M)$. By (3),

$$k(M)\leqq\sum_{n=1}^{M}n=\frac{M^2+M}{2}\leqq M^2.$$

Hence, for all $n\geqq 1$, by (1), $n\leqq k(f(n))\leqq f^2(n)$. That is

$$f(n)\geqq\sqrt{n}\quad(n\geqq 1). \tag{4}$$

From (3) again,

$$k(M)\geqq\sum_{n=1}^{M}\sqrt{n}\geqq\int_0^M x^{1/2}dx=\tfrac{2}{3}M^{3/2}\quad(M\geqq 1). \tag{5}$$

Now suppose that N is an integer >1 and that A and B are positive numbers such that

$$Aj^{\delta} \leqq f(j) \leqq Bj^{\delta} \quad (j \leqq N). \tag{6}$$

Then, for all $M \leqq N$,

$$k(M) = \sum_{j=1}^{M} f(j) \leqq B \sum_{j=1}^{M} j^{\delta} \leqq B \int^{M+1} x^{\delta} dx = \frac{B}{1+\delta} (M+1)^{1+\delta}$$

and

$$k(M) \geqq A \sum_{j=1}^{M} j^{\delta} \geqq A \int_{0}^{M} x^{\delta} dx = \frac{A}{1+\delta} M^{1+\delta}.$$

Using the fact that $1+\delta = 1/\delta$, we have

$$A\delta M^{1/\delta} \leqq k(M) \leqq B\delta(M+1)^{1/\delta} \quad (M \leqq N). \tag{7}$$

If $n \leqq k(N)$, then $f(n) \leqq N$, so we can put $M = f(n)$ and $M = f(n)-1$ in (7) to get $k(f(n)) \leqq B\delta(f(n)+1)^{1/\delta}$, $k(f(n)-1) \geqq A\delta(f(n)-1)^{1/\delta}$. By (1), $k(f(n)-1) < n \leqq k(f(n))$, so

$$A\delta(f(n)-1)^{1/\delta} < n \leqq B\delta(f(n)+1)^{1/\delta}, \quad (n \leqq k(N)). \tag{8}$$

Now if $n > N$, we have by (4), $1/f(n) \leqq 1/f(N) \leqq 1/\sqrt{N}$, and

$$f(n) + 1 = f(n)\left(1 + \frac{1}{f(n)}\right) \leqq f(n)\left(1 + \frac{1}{\sqrt{N}}\right),$$

$$f(n) - 1 = f(n)\left(1 - \frac{1}{f(n)}\right) \geqq f(n)\left(1 - \frac{1}{\sqrt{N}}\right).$$

Hence, from (8),

$$A\delta\left(1 - \frac{1}{\sqrt{N}}\right)^{1/\delta} f(n)^{1/\delta} < n \leqq B\delta\left(1 + \frac{1}{\sqrt{N}}\right)^{1/\delta} f(n)^{1/\delta}, \quad (N < n \leqq k(N)).$$

Therefore

$$(B\delta)^{-\delta}\left(1 + \frac{1}{\sqrt{N}}\right)^{-1} n^{\delta} \leqq f(n) \leqq (A\delta)^{-\delta}\left(1 - \frac{1}{\sqrt{N}}\right)^{-1} n^{\delta}, \quad (N < n \leqq k(N)).$$

Since $(1+1/\sqrt{N})^{-1} > (1-1/\sqrt{N})$, we have

$$(B\delta)^{-\delta}\left(1 - \frac{1}{\sqrt{N}}\right) n^{\delta} \leqq f(n) \leqq (A\delta)^{-\delta}\left(1 - \frac{1}{\sqrt{N}}\right)^{-1} n^{\delta}, \quad (N < n \leqq k(N)).$$

For $x > 0$, define $\psi(x) = cx^{-\delta}$, where $c = \delta^{-\delta}$. Also write $\lambda(N) = 1 - 1/\sqrt{N}$. Then

$$\psi(B)\lambda(N)n^{\delta} \leqq f(n) \leqq \psi(A)\lambda^{-1}(N)n^{\delta} \quad (N < n \leqq k(N)). \tag{9}$$

Let us say that (A, B) is a *stable pair* if

$$0 < A \leqq \psi(B), \qquad \psi(A) \leqq B. \tag{10}$$

If (A, B) in (6) is a stable pair, then from (9) and (10),

$$A\lambda(N)n^{\delta} \leqq f(n) \leqq B\lambda^{-1}(N)n^{\delta} \quad (N < n \leqq k(N)). \tag{11}$$

Since $\lambda(N) < 1$, $\lambda^{-1}(N) > 1$, so from (6) and (11),

$$A'n^{\delta} \leqq f(n) \leqq B'n^{\delta} \quad (n \leqq k(N)), \tag{12}$$

where

$$A' = A\lambda(N), \qquad B' = B\lambda^{-1}(N). \tag{13}$$

Now, observe that if (A, B) is a stable pair and if $0 < \lambda \leqq 1$, then $(\lambda A, \lambda^{-1}B)$ is also a stable pair. For

$$\psi(\lambda^{-1}B) = \lambda^{\delta}\psi(B) \geqq \lambda\psi(B) \geqq \lambda A > 0,$$
$$\psi(\lambda A) = \lambda^{-\delta}\psi(A) \leqq \lambda^{-1}\psi(A) \leqq \lambda^{-1}B.$$

Thus we have the basis for an induction. If we define $N_0 = N$, $N_{r+1} = k(N_r)$, $(r \geqq 0)$, and $\lambda_{r+1} = \lambda(N_r)$, $(r \geqq 0)$, the hypothesis that (A, B) is a stable pair and that (6) holds leads to

$$A\lambda_1\lambda_2 \cdots \lambda_r n^{\delta} \leqq f(n) \leqq B(\lambda_1\lambda_2 \cdots \lambda_r)^{-1}n^{\delta}, \quad (n \leqq N_r) \tag{14}$$

for every $r \geqq 1$.

To begin the induction, we take $N_0 = 2$, $A = 1$, $B = \delta^{-1}$. It is easily verified that (6) holds and that (A, B) is a stable pair, so (14) holds. Now we use (5) to get a lower bound for N_r. For $M \geqq 3$, $\frac{2}{3}M^{3/2} > M^{1.1}$. Since $N_1 = k(2) = 3$, we have $N_2 > 3^{1.1}$, $N_3 > 3^{(1.1)^2}$, and for all $r \geqq 1$,

$$N_r > 3^{(1.1)^{r-1}}.$$

Hence in (14) the products $\lambda_1\lambda_2 \cdots \lambda_r$ are bounded away from 0, and there are positive constants A_0, B_0 such that

$$A_0 n^{\delta} \leqq f(n) \leqq B_0 n^{\delta} \quad (n \geqq 1). \tag{15}$$

Thus, if we define

$$\alpha = \liminf f(n)n^{-\delta}, \qquad \beta = \limsup f(n)n^{-\delta},$$

we have $0 < \alpha \leqq \beta < \infty$. It remains to show that $\alpha = \beta$ and to evaluate this constant. Now

$$\limsup N^{-1/\delta}k(N) = \limsup \frac{1}{N^{1+\delta}} \sum_{j=1}^{N} f(j) \leqq \frac{1}{1+\delta} \limsup f(j)j^{-\delta} = \beta\delta.$$

Since $n \leqq k(f(n))$,

$$\limsup f(n)^{-1/\delta}n \leqq \beta\delta, \qquad \limsup f(n)^{-1}n^{\delta} \leqq (\beta\delta)^{\delta},$$

so

$$\alpha^{-1} = (\liminf f(n)n^{-\delta})^{-1} \leqq (\beta\delta)^{\delta},$$

that is,

$$\alpha\beta^{\delta} \geqq \delta^{-\delta}. \tag{16}$$

Similarly, we find $\liminf N^{-1/\delta}k(N) \geqq \alpha\delta$. Since $n > k(f(n)-1)$,

$$\liminf (f(n) - 1)^{-1/\delta}n \geqq \alpha\delta.$$

Therefore $\beta^{-1} = (\limsup (f(n)-1)n^{-\delta})^{-1} = \liminf (f(n)-1)^{-1}n^{\delta} \geqq (\alpha\delta)^{\delta}$, that is,

$$\delta^{-\delta} \geqq \beta\alpha^{\delta}. \tag{17}$$

From (16) and (17) we see that

$$\left(\frac{\alpha}{\beta}\right)^{1-\delta} = \frac{\alpha\beta^{\delta}}{\beta\alpha^{\delta}} \geqq 1,$$

so $\alpha \geqq \beta$. Since $\alpha \leqq \beta$, equality holds. Putting $\beta = \alpha$ in (16) and (17), we get $\alpha^{1/\delta} = \alpha^{1+\delta} = \delta^{-\delta}$, so $\alpha = \delta^{-\delta^2} = \delta^{\delta-1}$. Hence finally, as promised,

$$\lim_{n\to\infty} f(n)n^{-\delta} = \alpha.$$

Also solved by J. H. van Lint, and by the proposer.

Collections of Sets of Positive Measure

5408 [1966, 674]. *Proposed by Thaddeus Dankel, Princeton University*

Let C be any infinite collection of distinct Lebesgue-measurable subsets of $[0, 1]$ each having measure $\geqq \epsilon$, where ϵ is a fixed positive number <1. Show that there exists an infinite subcollection $C_0 \subset C$ of distinct sets E such that $m(\bigcap_{E\in C_0} E) > 0$.

I. *Solution by P. R. Chernoff and W. C. Waterhouse, Harvard University.* The stated result is false no matter how close ϵ is to 1. Indeed, let S_n be the set of numbers in $(0, 1)$ whose nth digit in the b-ary expansion is not 0. The intersection of any k of these has measure $((b-1)/b)^k$. Hence the intersection of any infinite subcollection has measure zero.

On the other hand, if 1 is an accumulation point of $\{m(E) : E \in C\}$ there is an infinite subset $C_0 \subset C$ such that $\sum_{E\in C_0}(1-m(E)) < 1$. Hence $m(\bigcap_{E\in C_0} E) > 0$.

II. *Comment by J. Gillis, The Weizmann Institute of Science, Israel.* If C is nondenumerable then the result is true. Indeed it is then known that for any $\eta > 0$ we can find an infinite subcollection of C whose intersection has measure

$>\epsilon-\eta$ [cf. J. Gillis, Quart. J. Math., 7 (1936) 191–198]. Without this additional condition it has been shown that for any $\eta>0$ and natural number r we can always find r distinct sets E whose intersection has measure $>\epsilon^r-\eta$ and that this result is best possible [cf. J. Gillis, J. London Math. Soc., 11(1936)139–141].

Also solved by T. V. Fossum, Frank Gilfeather, A. Jakimovski (Israel), J. G. Mauldon (England), David Preiss (Czechoslovakia), W. R. Scott, Yam-Tung Siu, and L. J. Wallen.

Fossum shows that there exists an infinite subcollection C_0 such that $\cap_{E\in C_0} E$ is nonempty, under the hypothesis of the problem as stated.

Subrings of Direct Sums of Copies of I_k

5409 [1966, 675]. *Proposed by R. A. Melter, University of Massachusetts*

Let k be a square-free positive integer greater than 1 and let R be a subring of the complete direct sum of an arbitrary number of copies of I_k, the ring of integers modulo k. Show that R is the direct sum of a finite number of p-rings. (A p-ring is a ring in which the identities $px=0$ and $x^p=x$ are satisfied for all x and for some positive prime p.)

Solution by G. A. Heuer, University of California at Berkeley. Let $k=p_1\ p_2 \cdots p_r$, where the p_i are distinct and prime. Let P_i be the subring of I_k generated by k/p_i. P_i is isomorphic to the field I_{p_i}, which is clearly a p_i-ring. Moreover, it is readily seen that I_k is the internal direct sum of the P_i.

Let S be the complete direct sum of a family of copies of I_k (indexed by A, say). If S_i is the complete direct sum of a family of copies of P_i indexed by A, then each S_i is a p_i-ring, and S is isomorphic to $S_1\oplus \cdots \oplus S_r$. R is a subring of S; if we identify S with $S_1\oplus \cdots \oplus S_r$, and put $R_i=R\cap S_i$, we have $R=R_1\oplus \cdots \oplus R_r$, where each R_i, being a subring of a p-ring, is a p-ring.

Also solved by N. J. Fine, Wanda J. Mourant, and the proposer.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto, 5, Canada. Correspondence about reviews is welcomed.

Algèbre linéaire et géométrie élémentaire. By Jean Dieudonné. Hermann, Paris, 1964. 233 pp. 36 F.

It would be an exciting experiment to have a couple of living mathematicians each write a book on elementary linear algebra. I guess that it would not be difficult to find criteria to compare the results systematically. The most striking feature and perhaps a means to arrange them in a linear order would probably be the amount of regard or disregard of the author for the reader. There are textbook writers who believe that a mathematical exposition without logical gaps explains itself to the presumed reader who faith-

fully follows the author step by step, and that the secret about what the whole thing means need not to be told by the author explicitly, but has to be grasped by the reader from the details he has swallowed. There are other guides who prefer to take the guided ones into confidence, to tell them where the road goes and why this one is better than another. The main stream textbook writer sails between Scylla and Charybdis. As a mathematician he is possessed by systematism, and as a teacher he is plagued by didactic scruples. Indeed, systematism is a nondidactical attitude which often turns antididactical. A common way out is to give rein to one's passion for systematism in the main part and to give full credit to didactic goals in the problems set after every chapter. This way out is as old as Euclid, who during centuries puzzled his readers by the seemingly crazy, because unmotivated, arrangement of his theorems and proofs, while, on the other hand, he used to introduce his so-called constructions by motivations called analysis.This Euclidean spirit still prevails in many textbooks, and so it does in Dieudonné's.

The inventor of the slogan "A bas Euclide" a follower of Euclid? It looks odd, but it does so simply because few slogans have been misunderstood as badly as this one. Partly, it was Dieudonné's own fault. When he cried "A bas Euclide," he actually meant "A bas 'Euclide'," (viz. the Euclid of French lycée textbooks), but in oral discussions it is a hard thing to pronounce quotation marks.

To come back to our experimental couple of textbook writers, one can easily imagine many respects in which their productions would diverge from each other, from the dreary examples of French typography to some highlights of international snobism. As a typographical product Dieudonné's book, though still difficult to read, is much superior to the average French production. As to the content proper and its arrangement the couple of authors would not probably disagree very much. Their interpretations of what elementary linear algebra means, would be as identical as they can be. So there is some reason to ask why Dieudonné wrote a book which could have been written by many other people. Of course Dieudonné performed much better than other people would have done, but probably not so much better that it would justify his preoccupation with this subject.

Dieudonné has tried to answer this question in the preface, which faithfully reflects Dieudonné's sympathetic image as a forceful defender of all possible inconsistencies. According to the second paragraph of the preface this book is a teacher's textbook, and to prove this pretention, he hammers at the educational merits of its subject matter and the educational flaws of any other one, only to finish with the anticlimatical last paragraph where suddenly he declares that he has no personal interest in problems of secondary education. "I simply wished to add to the files of the future historian an example of what one could do in this matter if one would try to act in a rational way." I do not understand this emphatic utterance, and I doubt whether anybody, Dieudonné included, can explain it. Does it mean that the teacher's textbook being written, Dieudonné retires into an ivory tower to have teachers who do not agree to accept it as their textbook, judged by history?

However, it is by no means clear in which sense Dieudonné has meant this book as a teacher's textbook. He emphasizes that it is the solid mental basis on which the teacher should develop his teaching, but he does not tell to whom this book addresses itself, whether to the teacher who does not know anything about linear algebra and who has to learn it from this book, or to the more advanced one who knows the subject perfectly well and will use the book to brush up his knowledge or to check it on some details. The level of this book is in between both, though even the experienced reader may be puzzled by some detail on some page n, which on page $n+100$ may be justified as a necessity of the design or as conditioned by a whim of the author.

For instance, the field of reals is introduced as an ordered field, in which real square and cube roots exist. The author admits and stresses that stronger axioms have been

omitted because they will not be needed. This is perfectly legal. However, it is not the whole truth. In the Introduction the author said that he curtailed the axiomatic system of reals in order to make it *impossible* that angles in the usual intuitive sense could be defined on the strength of the axioms. Even if the reader is able to connect a strange feature of Dieudonné's axiomatic system to that remark in the introduction, he has to wait till Appendix I to understand things clearly. But that is still not yet the whole story. Even an experienced reader may have difficulties with the secret why Dieudonné just allows the solutions of quadratic and cubic equations but of no higher ones. Faithful to the euclidean spirit, he reveals the secret only implicitly, in the last chapter where eigenvalues of 3-3-symmetric matrices are needed.

Another example: Some teacher might wonder what linear algebra can tell about areas and volumes. He will not find these words in the index or anywhere in the book. Of course Dieudonné knows what alternating bilinear forms and Gram's determinants have to do with these intuitive notions. Did he trust in some magic link by which this very essential connection would be transferred from the unconscious of the writer to that of the reader? What is the reason, a principle or a whim?

I feel embarrassed that I have to bother a mathematician I agree with in all principles of mathematical education, with such silly questions. I believe that Dieudonné's basic ideas on teaching mathematics are sound and healthier than the classical ones (if there are any) and than most of the current attempts at modernizing. It is a great pity that they are often uttered in a bizarre, preposterous and misleading way. It is more than a pity, it is a danger to base good ideas on bad arguments, particularly if it is done with Dieudonné's authority. There are people who pick up his wrong arguments to use them for bad ideas, and there are other ones who reject all good ideas which look like Dieudonné's, because Dieudonné's were based on bad arguments.

Dieudonné is completely right when attacking a kind of plane geometry instruction which can be characterized by such terms as Stewart's theorem, nine-point-circle, construction of triangles from three fanciful data, though I cannot believe that there are many class rooms in the world where such geometry is still taught. He is definitely wrong if he uses such exaggerations and degenerations, which have been the final result of a bad educational system and a wrong educational philosophy, in a way that can be understood as an argument against classical plane geometry (though it is not meant so). If one considers how in a much shorter time the same kind of excesses have developed in secondary school text books on set theory, one is inclined to conclude that it is not the subject that is wrong in our mathematics teaching but something else.

Dieudonné admits that the student has to learn geometrical constructions though he warns against the *peste* of what he calls "le plus gigantesque canular" of traditional instruction, the postulate of construction with ruler and compasses. This is a boomerang argument. The classical construction postulate means algebraically restricting to square root adjunctions. Is it much more liberal, is it an essentially less gigantic "canular" to admit both square and cube roots only as did Dieudonné, particularly if he did so with the sole aim to block the measure of angles? Of course, neither is a canular, but both are expressions of the same dogmatic spirit, which often turns antididactical, the spirit of methodical purity and systematism. There is some difference too; the classical construction postulate, though symptom of a bad philosophy, has been a source of beautiful results, while Dieudonné's is no more than a whim, even—I believe—in his own eyes.

Sometimes Dieudonné lends his ear to critical remarks. His much too complicated treatment of plane rotations is the same as given in many lectures, but finally he has yielded to criticism and at least added a paragraph on complex numbers, which was badly needed though obviously it sins gravely against methodical purity. But mostly he pretends to be blind and deaf though I firmly believe he is not. The worst example is angles.

It is a fact that in traditional instruction the word angle covers quite different notions (as the word "point" does in many different deductive systems). First, there is Euclid's notion of angle, restricted to the interval between 0 and 180 degrees, but nevertheless added up to $(n-2)\cdot 180°$ for the angles of an n-gon. There are the oriented or non-oriented angles of straight lines in "analytic" geometry which are magnitudes mod 180° and those of vectors in trigonometry which are counted mod 360°, and finally there are a few others in the geometry of space. It is a necessity that the student learns to analyze these notions and to understand their interrelations, and of course the teacher should be prepared to guide the student to this analysis. (For simplicity's sake I use designations as 180° and 360° though the foregoing has nothing to do with measuring angles.) *Measuring* angles is a quite different problem within a deductive system of geometry. If the measure is to be multiplicative, it arises easily by algebraic means from any algebraization of geometry. The usual measure, however, is additive and has to be obtained from the multiplicative one by taking logarithms. It is quite improbable that analysis (or topology) as far as needed for logarithms exceeds secondary school level, though certainly it is a heavy blow for systematism and methodical purity. So additive angle measures are banned by people who believe in these principles.

However one thinks about this point, measuring angles has nothing to do with the problems tackled in the last paragraph. After so many oral discussions and a few written explanations it is a sad thing to have them again mixed up in Dieudonné's introduction, though it is unthinkable that he should not know what is wrong in this confusion. Adding up angles beyond 360° has nothing to do with any angle measure. It simply means passing over to the universal covering group of the cyclically ordered group of angles; to construct the universal covering one needs no topology but only the cyclic order. This is as clear as it can be and it justifies in the most elementary way the usual operations on angles exorcized by Dieudonné. It is hard to understand how after all Dieudonné can propagate again and again such a confusion.

Dieudonné's activity, and particularly the present book, adds a tragedy to the challenge of modernizing mathematical education. I call it a tragedy, because Dieudonné's educational principles are the soundest of all, but instead of elaborating them he dwells on the brink of the field, where some curiosities have drawn his attention. He knows and he stresses that geometry cannot be taught by linear algebra alone, whereas many modernizers believe they can do without intuitively based geometry. Likewise Dieudonné firmly opposes those who refuse to teach more mathematics at secondary level than can be taught by the most exacting mathematical methods, which means a small amount of calculus or even no calculus at all. Instead he is realistic enough to insist on teaching as much mathematics as is needed in applications, which means, more calculus, and at an earlier age than hitherto. It is a pity that he calls the methods by which this should be done "experimental" or "physical." He still does not acknowledge that psychologically there are different levels of exactness in mathematical reasoning, and that assigning a lower bound to what may be called mathematical exactness is an artificial procedure in education. For instance, today quite a few people would not be satisfied with Dieudonné's loose use of bound variables as constants (such as ϵ in the field axioms: R7, there is an $\epsilon \neq \omega$ such that $\epsilon\zeta = \zeta$ for all ζ, and R8, for any $\zeta \neq \omega$ there is a ζ'' such that $\zeta\zeta'' = \epsilon$).

Without doubt Dieudonné would be a most competent and authoritative mathematician to tell which kind of geometry besides linear algebra and which kind of calculus should be taught and in which way it should be done. Instead he always comes back to teaching linear algebra or rather a system of linear algebra featuring mainly some oddities. He even never explains what linear algebra surrounded by all the other mathematics taught in an "experimental" way actually means. One would expect that he conceives it as an example of strictly deductive mathematics with a certain formal educa-

tional value. But the only argument Dieudonné gives, is the dogmatic statement that this is the minimum of knowledge a "bachelier ès-sciences" is expected to have when he enters the "classes propédeutiques de l'enseignement supérieur." Such statements only add to the tragic embarrassment of people who fight for modernizing mathematical instruction.

H. FREUDENTHAL, Utrecht University

Rational Mechanics. By C. W. Kilmister and J. E. Reeve. American Elsevier, New York, 1966. xi+354 pp. $9.75.

Two generations ago, every mathematician was familiar with classical mechanics, the higher parts of the subject usually were taught by mathematics departments, and some of the finest mathematicians published textbooks on analytical dynamics. Mathematicians trained since the war may well have had no contact with mechanics past a chapter or two in a required freshman physics course, and often they suffer from fantastic misconceptions regarding the contents and objectives of mathematical theories in which force, rather than mere geometry or probability, plays a part. The authors of the book under review lay blame on the textbooks, with their "completely nineteenth-century emphasis on techniques for the solution of special problems, . . . a collection of tricks with no great intellectual value. . . . "

While mechanics has almost totally disappeared from mathematics teaching and training, it has been revived as a field of modern research. This new work, some of it at a high level of mathematical abstraction, is often described by the revived term "rational mechanics," which seems to have been used little in English between the publication of Gibbs' *Statistical Mechanics* in 1901 and the foundation of the *Journal of Rational Mechanics and Analysis* in 1952. The authors' title suggests, and their preface encourages us to expect, that they will present mechanics "as the fascinating and strikingly beautiful mathematical theory that it is, . . . persuading the reader that dynamics is very much a living subject. . . . "

In fact, while some areas such as existence theory receive fuller and more scrupulous treatment than in most books, the authors do not give evidence of any contact with recent research. Even within the almost completely standard scope they have chosen, they themselves admit the logical weaknesses that any critical reader will see at once. On p. 129 they define a rigid body as a certain kind of finite system of mass points; on p. 130 they give a second definition, allowing the number of particles to be nondenumerable, and they then state that the difficulty of passing from the former theory, in which they have proved some theorems, to the latter, which most of the rest of their book concerns, is "more or less insuperable," and the matter is not made clearer when they discuss strings and chains at the end. While it comes as a welcome relief when authors of a textbook reject Newton's laws as "vague and obscure, and expositions which appear to be based on Newton's laws are really based on the unstated assumptions made in interpreting the laws," in the present case the unpromising choice of Mach as a guide for a mathematical book seems responsible for some statements vaguer and obscurer than most traditional writing, e.g.:

"Axiom 2. There exists a finite set of N particles, $N>1$, and these particles can influence each other's motion. A *particle* is an entity whose motion is completely specified by a continuous point."

There are not many mathematicians who will find this new book clearer than the standard ones, e.g. Osgood's *Mechanics* for beginners and Whittaker's *Analytical Dynamics* for the advanced. In fact, the authors make the difficulties more difficult than need be. If we can teach set theory to kindergarteners, abstract algebra to freshmen,

and functional analysis to seniors, why should the "satisfactory treatment . . . based on the theory of continuous media" be "outside the scope of the normal undergraduate syllabus" (p. 130)?

There are many special problems compactly stated, set up, and solved at various points. The main departure from older courses seems to lie in the frank discussions of some difficulties concerning constraints and friction, and in the brief treatment of the theory of impulses as a partly independent discipline. Despite the authors' intention to the contrary, the book remains a loosely connected set of interesting special cases, tastefully selected, very British and closely tied to examinations for an "honours course." It will be a welcome addition to many private and institutional libraries.

C. TRUESDELL, The Johns Hopkins University

MAA Studies in Mathematics, Vol. 3. Studies in real and complex analysis. Edited by I. I. Hirschman, Jr. Prentice-Hall, Englewood Cliffs, N. J., 1965. 213 pp. $4.00.

Volume 3 of this outstanding series of mathematical expositions more than meets the high standard set by the previous volumes.

The contents are: 1. Several Complex Variables, by H. J. Bremermann; 2. Nonlinear Mappings Between Banach Spaces, by L. M. Graves; 3. What is a Semi-group? by Einar Hille; 4. The Laplace Transform, the Stieltjes Transform and their Generalizations, by I. I. Hirschman and D. V. Widder; 5. A Brief Introduction to the Lebesgue-Stieltjes Integral, by H. H. Schaefer; 6. Harmonic Analysis, by Guido Weiss; 7. Toeplitz Matrices, by Harold Widom.

The articles are written by highly competent contributors to current mathematical developments, who have apparently given a great deal of time and thought to expressing their subjects. The incipient mathematician is fortunate, indeed, in having available this richness of material in such readable form. Moreover, the book is highly recommended reading for the professional mathematician, who will find it enjoyable and informative.

The paper by Weiss on harmonic analysis is a remarkable exposition of one of the vital areas of mathematics. It takes the reader from the beginning to an introduction to some ideas that are at the forefront today. The papers by Schaefer and Graves are beautiful presentations, quite self-contained, of important but smaller mathematical topics. The papers by Widom and by Hirschman and Widder, although narrower in scope, give the reader a feeling for the mathematical heartbeat. Those of Bremerman and Hille are elegantly written surveys of larger topics.

Finally, thanks are due to Richard V. Andree without whose persevering efforts this series would never have been started.

CASPER GOFFMAN, Purdue University

Inverse Functions. By William K. Smith. Macmillan, New York, 1966. 98 pp. $2.95.

Although the author achieves his stated purpose: "to make available to students (primarily of calculus) a reasonably complete, self-contained treatment of inverse functions, a subject which is discussed in most calculus books with distressing brevity," his book will have little appeal to the audience for whom it is intended. The fault, I think, lies in the choice of topic: whereas Professor Smith's earlier book, *Limits and Continuity*, helped clarify a concept that universally worries calculus students, few beginners are aware of gaps in the standard textbook treatment of inverse functions.

The material is uneven. The first twenty pages overexplain elementary properties of sets and functions. A student in need of such hand-holding would find the later sections too abstract. Sections 3–7, constituting the heart of the discussion, treat inverse functions both analytically and geometrically. The final section (pp. 60–80) is devoted to

trigonometric functions and the nonexistence of f^{-1}. The author's style makes his arguments easy to follow. Problems are plentiful, as are clearly-marked diagrams. Certainly a beginner in analysis would benefit from the practice provided on manipulation of functions and functional notation and from additional exposure to concepts such as monotonicity, symmetry and continuity.

WINIFRED ASPREY, Vassar College

Discourse on Fourier Series. By Cornelius Lanczos. Hafner, New York, 1966. viii+255 pp $9.85.

The subject of Fourier series is germinal. To an unusual extent, the interplay between theory and practical use has produced a growing body of knowledge with wide appeal. This discourse (and it is not a treatise nor a handbook nor a textbook but a discourse) prepared for students of mathematics, physics, and engineering on the half-graduate and graduate level is a mathematically careful history of this exciting interplay.

Especially in the beginning, very close attention is paid to elementary concepts such as 'limit,' 'uniform convergence,' and 'function.' For example, he considers in what senses 0.33333 . . . is a sum and a limit and a number. This careful keeping to the foundations pays off later. Examples that seem mere formula substitution with some authors become in his hands fresh and intuitive.

The first hundred pages include simple Sturm-Liouville problems, Fejér means, the Gibbs phenomenon, local smoothing of a series, an interesting comparison of Fourier series and Taylor series, and the usual elementary matters known to anyone who has worked with the classical vibrating string problem.

The next fifty pages include curve fitting problems, the discovery by Fourier analysis of 'hidden' properties of data, and methods of improving convergence.

The last hundred pages are on the Fourier integral and the Fourier and Laplace transforms. This last section is more difficult and the reader could profitably bring to it stronger tools such as a knowledge of asymptotic series, uniform convergence of infinite integrals, and the Cauchy integral theorem.

This good book will be of most value to those many students who know a little about Fourier series and want to know what has been happening in other aspects of the field. In attitude and content, the closest book to it is Langer's 1947 Slaught Memorial Paper, "Fourier's Series; The Genesis and Evolution of a Theory," now, lamentably, out of print. Langer has a fuller discussion of the very early history but Lanczos has a more extensive volume and includes many interesting problems. Neither book considers Lebesgue integrals or trigonometric series fitted into the setting of function spaces.

DAVID DICKINSON, University of Massachusetts

Introduction to Matrices and Linear Transformations. By Daniel T. Finkbeiner. Freeman, San Francisco and London, 1966. xi+297 pp. $7.75.

The first edition of this book has been used with great success by many graduate students as a primary reference for linear algebra. These same students may have at one time felt that the book was too simple, but continued use has shown them the depth of material in it. The second edition follows the essential format of the first, with some very useful additions.

In choosing a text for a linear algebra course, an important consideration is the background of the students. Those with limited experience in abstract mathematics should probably be introduced to linear algebra by means of familiar techniques, such as solving simultaneous linear equations. However, many present undergraduates have been exposed to "new math" and hence should be able to take a formal, axiomatic approach in stride. This text does utilize the axiomatic approach, which is particularly desirable for

those students with a serious interest in mathematics. The book is primarily a development of the *theory* of linear algebra, and not of techniques of application.

Nonetheless, both in the text and in the exercises, the applications of the theory are illustrated much more frequently than is usually found in essentially theoretical texts. For example, matrix applications in quantum mechanics, special relativity, Markov processes, linear equations, and differential equations are mentioned. The applications of linear algebra to linear programming are given the most attention. In particular, one expanded section of the second edition is on combinatorial equivalence and its role in the theory of linear programming.

Other important changes in this edition are the expansion of the discussion of metric concepts and the inclusion of some of the theory of normal matrices and transformations. And a new appendix, treating the notation appropriate to right and left hand operations, seems particularly valuable, in view of the fact that the student will likely encounter both notational systems.

K. R. REBMAN, University of Michigan

Number Theory. By Z. I. Borevich and I. R. Shafarevich. Translated by Newcomb Greenleaf for Scripta Technica. Academic, New York, 1966. x+435 pp. $12.95.

This is quite a distinctive and valuable book because it overlaps very little the other available texts on the subject in English. Written with verve and sophistication, it offers a high-level second course on the subject. (The translator's preface to the book takes a somewhat different view. "While some results from elementary number theory are occasionally used, a previous course in number theory is certainly not necessary, although the reader without such a course may have a few occasions for consulting a more elementary text.")

It is not possible to list the subject matter of the book in a brief way because of the many techniques, primarily algebraic, that are developed at great length to attack the purely number-theoretic questions. Aside from the final chapter, the topics discussed come under the headings of congruences, divisibility, and Diophantine equations, especially representation by forms. It should be added at once that these are treated in a broad framework going far beyond the discussion of integers or rational numbers. For example consider a polynomial $F(x_1, x_2, \ldots, x_n)$ with integral coefficients. At the outset of the book it is pointed out that if $F=0$ has a solution in integers, then $F\equiv 0 \pmod{m}$ is solvable for every value of the modulus m, but that the converse holds for some polynomials F but not others. To study this converse proposition the writers use such methods as the embedding of the base field of rational numbers in p-adic number fields, in algebraic fields, or in the real field, and then examining the question in the wider context. Such methods, while far from new in the literature, are not given much attention in many of the standard books on number theory.

Although algebraic techniques are central, geometric methods are also used, and in the last of the five lengthy chapters, analytic methods. The principal topic in the final chapter is the class number problem in algebraic fields.

There is an algebraic supplement of 32 pages, partly with proofs and partly without. There are many problems throughout the book, even in the algebraic supplement, most of which amount to nontrivial extensions of the theory.

IVAN NIVEN, University of Oregon

Mathematical Analysis. By L. A. Lyusternik and A. R. Yanpol'skii. Pergamon Press, Long Island City, N. Y., 1965. 404 pp. $10.00.

According to the editors, the book has been designed for the reader acquainted with the fundamentals of mathematical analysis and it will serve both as a source of information in new branches of the subject and as a digest of those with which he is already familiar. This is a correct assertion. A person who has some background in analysis

should find this a refreshing book to read, in that it develops many important topics in a precise and understandable manner. There is a clear development of ideas along with many helpful examples from which one can learn without going through the logical details of long proofs.

In the development, precise definitions are given and theorems are stated without proof. The examples given to illustrate the ideas are usually good, but a few more in crucial places would improve the book. A skillful and knowledgeable teacher could use the book effectively as a text in a course in which the techniques and methods of mathematical analysis are taught. There are no exercises.

In each chapter there is a blend of classical and modern mathematics, bringing together ideas that are not found in any one volume. Chapter I, "The Arithmetical Linear Continuum and Functions Defined There," deals in a rather sophisticated way with properties of real numbers and convergence of sequences of real numbers and of functions. Chapter II, "n-Dimensional Spaces and Functions Defined There," gives much of the material necessary for a study of functional analysis. Chapter III, "Series," on both numerical series and series of functions, gives various methods of summing series. Chapter IV, "Orthogonal Series and Orthogonal Systems," brings together many important results and includes a discussion of the classical orthogonal systems. Chapter V is an especially useful chapter on continued fractions which includes important results involving matrix methods. The last chapter is entitled, "Some Special Constants and Functions."

This volume should prove to be popular as a reference book for the mathematics teacher and for the physicist and engineer who uses the techniques of mathematical analysis.

W. G. JOHNSON, Hiram College

Some basic problems of the mathematical theory of elasticity. By N. I. Muskhelishvili Noordhoff, Groningen, Holland, 1963. xxxi+718 pp. Dfl. 57, 50.

This is a translation, by J. R. M. Radok, of the fourth Russian edition published in 1954. The first English version of the original work appeared in 1953. This edition contains numerous additions and changes, most of which are small, but at times essential. We shall list a few of the more important additions.

1. Section 41, entitled "Reduction of the fundamental problems to problems of complex function theory" has been completely rewritten and now contains an additional subsection clarifying the results for multiply connected regions.

2. In section 42, proofs of the uniqueness of regular solutions have been added.

3. Section 84, entitled "General solution of the fundamental problems for regions, mapped on to the circle by help of polynomials," contains an additional note describing how one can obtain an existence theorem in this case.

4. Section 139, some general formulas on the torsion of compound bars, now contains a proof of the fact that when the torsion function satisfies the boundary conditions, the resultant forces on the ends are zero.

This translation is admirably done and assists considerably in spreading the methods and results of Muskhelishvili outside the Soviet Union. There are, however, some annoying printers' errors in this second addition. In particular, (at least in the reviewer's copy), pages 326–327, 330–331, 334–335, and 372–373 are completely blurred. Only one part of section 138a, page 606 appears, while page 607 is exactly the same as page 620 in section 140a. Also, throughout the text various pages (i.e. 139, 460) appear in faint print. It is hoped that these errors are easily remedied.

MORTON LOWENGRUB, Wesleyan University

Introduction to functional analysis for scientists and technologists. By B. Z. Vulikh. Addison-Wesley and Pergamon Press, Reading, Mass., 1963. 397 pp. $10.00.

This book is a translation of the Russian book entitled "*Vredeniye v funktsional'nyi analiz*" published in Moscow by Fizmatgy in 1958. The title of the book is a bit misleading. There are no examples drawn from science or technology. The text appears to lack any motivational material necessary to arouse the interest of potential scientists and engineers in North American universities. Barring some rather serious translational difficulties, the text is far more suitable for the mathematics major than for a student of science or technology, but one must be extremely careful in assigning this book for students to read. There is serious misuse of the definite articles "the" and "a" throughout the book. A typical example is in the following definition found on page 165,

"DEFINITION: The complete orthonormal system of elements is said to be *the orthonormal basis* of the Hilbert space."

Chapter 6 is particularly poor in this respect.

There are many additional mistranslations or awkward translations of mathematical terms. An example of this is found in the third sentence of page 21. Also, one finds such statements as "functions of an interval" being used for a function defined over an interval, (see page 177).

In the reviewer's opinion, the book is not a total loss. There is a very nice section on the application of functional analysis to the solution of infinite dimensional linear equations, Fredholm integral equations, and Sturm-Liouville systems of differential equations.

The book is divided into thirteen chapters covering the following material: finite and infinite dimensional Euclidean spaces, metric spaces, normed linear spaces, Hilbert space, L^2 spaces, linear operators and functionals, adjoint and self-adjoint operators in Hilbert space, completely continuous operators, approximate solution of functional equations, and partially ordered normed spaces. There is a short but impressive bibliography.

MORTON LOWENGRUB, Wesleyan University

Partial Differential Equations of Mathematical Physics. By S. L. Sobolev. Translated from the third Russian edition by E. R. Dawson. English translation edited by T.A.A. Broadbent. ADIWES International Series in Mathematics. Pergamon Press, Oxford, 1964. x+427 pp. $14.00.

This is an excellent exposition of classical methods in partial differential equations. The presentation is clear, and in many instances the author goes to great length to prepare the unfamiliar reader. Only a knowledge of advanced calculus is assumed. Topics such as multiple integrals, Lebesgue integration, integrals depending on a parameter, and integral equations are developed as needed. The concept of generalized solutions is explained and motivated.

The translation is faithful; errors are few. Under present standards the book can only be recommended as an undergraduate text. It is unfortunate that it was not translated many years ago when there was a great need for books on this topic in English. In spite of the fact that the original Russian editions were published so long ago, the book remains a fine introduction to the subject.

MARTIN SCHECHTER, Institute for Advanced Study

General Theory of Functions and Integration. By Angus E. Taylor. Blaisdell, Waltham, Mass., 1965. 437 pp. $12.50.

Since there are many texts on the theory of functions and integration, another one must answer a few questions. Is it sufficient in scope? Are the examples and problems adequate? Finally and perhaps most important: in what mathematical or pedagogical way does this book differ from its predecessors?

The scope of Professor Taylor's book is certainly adequate. The basic tools of analysis are well covered. One might regret the omission of Borel structures, measures on arbitrary products, and measurability of the composition of a Borel function of k variables with a k-vector of measureable functions.

A forte of this book is its selection of examples and problems to teach and challenge. In many cases when new material is introduced, the solutions of the problems are outlined and references to the literature are given.

There are several pedagogical novelties. The theory of integration is developed twice, once through outer measures and once through the Daniell approach. Then the complete relationship between these two methods is shown. There are two chapters on general topology which would make an excellent outline for a brief undergraduate course. Chapter 9 on functions of one variable includes such things as Helly's theorem and Lebesgue's theorem on the differentiation of functions of bounded variation. It can be read independently and serve as a bridge between an elementary and more advanced course.

If a student were to choose one book from which to study integration, this book would be a good choice. He should be warned that, as with all books, the work appears of uniform color and that the short paragraph devoted to Fatou's lemma will be as important as many of the complex theorems he will encounter.

FRANK HAHN, Yale University

A Course of Higher Mathematics, Vol. V (Integration and Functional Analysis). By V. I Smirnov. Translated by D. E. Brown. Translation edited by I. N. Sneddon. Addison-Wesley (distributor for Pergamon Press), Reading, Mass., 1964. xiv+635 pp. $17.50.

The main headings in the table of contents are the following: Chapter I: The Stieltjes integral. Chapter II: Set functions and the Lebesgue integral (section 1: Set functions and the theory of measure; section 2: Measurable functions; section 3: The Lebesgue integral). Chapter III: Set functions; absolute continuity; generalization of the integral. Chapter IV: Metric and normed spaces. Chapter V: Hilbert space (section 1: The theory of bounded operators; section 2: Spaces l_2 and L_2; section 3: Unbounded operators).

The chapter on Hilbert space is by far the longest. Starting from scratch with the axioms of Hilbert space, this chapter includes a treatment of a Fredholm theorems for completely continuous (compact) operators, the spectral theorem for self-adjoint operators (bounded first, unbounded later), the corresponding operational calculus, the extension of closed symmetric operators via the Cayley transform, and Friedrichs extensions of semi-bounded operators. The treatment is classical in spirit, detailed, and illustrated with excellent, nontrivial examples. Among works covering roughly the same ground, the chapter is unusual in that it makes use of Hellinger integrals (in discussing the continuous spectrum) and generalized derivatives (in discussing examples of unbounded operators).

If the translation were improved, the exposition for the most part would be clear, detailed, and easy to follow; nevertheless, by withholding examples and explanations at strategic places, the author often does not let the reader know what is really going on.

The translation and editing are bad. Among other things, the style is wooden, many meanings are not conveyed accurately, many terms are mistranslated, many statements are mangled, typographical errors abound, the index is poor. The book is not a pleasure to read.

There are no exercises.

R. R. CHRISTIAN, The University of British Columbia

Differential Geometry on Complex and Almost Complex Spaces. By K. Yano. Pergamon Press, Macmillan, New York, 1965. xii+326 pp. $12.50.

This book is intended to serve as an introduction to differential geometry on complex and almost complex spaces, and in all respects this intention has been fully realized. Stylistically it resembles the author's earlier book on Lie derivatives, but contextually they are quite different. The notation used is fairly standard, and throughout the book free use is made of the notion of Lie derivatives.

Chapter I is a concise review (not an introduction) of the basic elements of Riemannian geometry; chapter II includes a discussion of harmonic Killing, projective Killing, and conformal Killing vectors, along with the theorem of Green. In chapter III complex spaces are introduced; the structure tensor and linear connections are defined, then a metric, leading to Hermite and Kähler spaces. Chapter IV is devoted to Kähler spaces, including a discussion of the various types of vectors and a theorem of Matsushima on the decomposition of a contravariant analytic vector, followed by a generalization by Lichnerowicz. Almost complex spaces, in particular almost Hermite and almost Kähler spaces, are introduced in chapter V, while chapter VI is concerned with linear connections on the various types of spaces. Chapters VII, VIII, and IX are devoted, respectively, to almost Kähler, almost Tachibana, and almost Hermite spaces together with the various vectors on these spaces. Chapter X deals with locally product spaces where the square of the structure tensor is $+1$ rather than -1 as in the complex case; chapter XI covers almost product spaces. The final chapter is a study of holomorphically projective transformations. Here F-connections, semisymmetric and half-symmetric connections are discussed, and other concepts such as holomorphic projective flatness and curvature are examined in this context, with applications to almost Kähler and Kähler-Einstein spaces.

The only defect that mars an otherwise excellent presentation is the number of typographical errors that occur, especially in the early chapters where the average rate is about 1 every 10 pages. Since the book is intended primarily as a reference, there are no problems or exercises included. The bibliography is extensive.

A. L. Hilt, Temple University and General Electric Co.

Linear Geometry. By Rafael Artzy. Addison-Wesley, Reading, Mass., 1965. ix+273 pp. $9.75.

This is a fine introduction to linear geometry, suitable at the advanced undergraduate and graduate levels. It is "modern" in several ways. The vocabulary and the notations are often of 20th century origin; a certain amount of algebra and algebraic methods developed in the last fifty years are presented; and the treatment has most of the time an axiomatic, hence in some sense an "abstract" and "most general" character.

The first three chapters contain an introduction to projective, affine, euclidean, elliptic, hyperbolic geometries. Coordinates over given fields are used and transformation groups play the central role in the spirit of Klein's Erlangen program. The fourth and last chapter develops geometries from geometric axioms in the spirit of Hilbert's classic "Foundations of Geometry." Of course, algebra is quickly used again. Ternary rings and loops appear. The connections between algebraic laws and geometric theorems (Desargues, Pappus) are explained. This amounts to a rather concentrated introductory text into the theory of rudimentary planes and the algebra of ternary rings. Many references are given, and I hope the interested student will be eager to use them after the study of Artzy's stimulating text.

The limitations of the book are obvious. The bulk of the work is done in two dimen-

sions. Analysis exists (the reals are introduced) but does not flourish. The geometry is linear, in some places bilinear, and the conics serve principally as sets of points which are infinitely far away, hence Pascal, Brianchon and Steiner have nothing to say. These restrictions, however, are well justified. The author offers a book of manageable size to be used as a text in a course on linear geometry treated in the algebraic way.

Finally, I cannot help but add a remark on a point which struck me and which might be of general interest apart from this particular book on linear geometry. The author refers in the text to "the German mathematician C. F. Gauss (p. 255), "the British mathematician Arthur Cayley" (p. 174), "the French mathematician Henri Poincaré" (p. 55), etc. A man's nationality is mentioned in about thirty cases, and you begin to be interested in the score (the Germans lead with 11 quotations, happily followed by 7 Americans). Should a man's nationality be mentioned in a text book of this type? In my opinion, in most cases it is not necessary and I would like to suggest not doing it at all. The repeated use of national labels might generate micro-gaullistic feelings of all sorts.

ALFRED AEPPLI, University of Minnesota

Elementary Methods in Analytic Number Theory. By A. O. Gelfond and Yu. V. Linnik. Revised and edited by L. J. Mordell. Rand McNally, Chicago, 1965. xii+242 pp. $8.75.

As the authors state in the preface, many problems of analytic number theory can be treated today by elementary methods. The object of the book is to present elementary solutions of a number of famous problems. The wealth of material treated is best indicated by the following summary. Chapter 1. Additive properties. Schnirelman's method. Theorems of Mann and Erdös. Chapter 2. Elementary solution of Waring's problem and the Hilbert-Kamke problem. Chapter 3. Distribution of primes in arithmetic progressions. The prime number theorem. Chapter 4. Distribution of Gaussian primes. Chapter 5, 6. Methods of V. Brun and A. Selberg. Chapter 7. Distribution of the fractional parts of numerical sequences. Chapter 8. Lattice point problems. Chapter 9. Distribution of power residues. Chapter 10. Yu. I Manin's proof of Hasse's theorem on the number of solutions of the congruence $y^2 \equiv x^3+ax+b \pmod{p}$. Chapter 11. Siegel's theorem concerning the number of classes of binary quadratic form with negative discriminant. Chapter 12. Transcendence of e^{ω} and α^{β}. Chapters 5 and 6 were written by A. I. Vinogradov; Chapter 10 was written by Yu. I. Manin.

It will be evident from the above that the book will be of great value to all who are interested in number theory. It is also pleasant to mention that the book is very attractive in appearance.

L. CARLITZ, Duke University

Recherches sur la Théorie Générale des Systèmes Formels. By J. Porte. Collection de Logique Mathématique, Serie A, N° 18, Gauthier-Villars, Paris, and E. Nauwelaerts, Louvain, 1965. 146 pp. 30 F.

This book starts off as a study of formal systems: deductional systems (based upon a deducibility relation), logistic systems (axioms and rules of inference), and thetic systems (theorems), in order of increasing generality. The topics covered at this level include various notions of equivalence of systems, derivability and independence of rules and axioms, properties roughly equivalent to the validity of *modus ponens* and the deduction theorem, extensions of systems, and congruence relations in the set of formulas. Then the rest (well over half) of the book is devoted to propositional calculi. Here the abstract notions of the first part are studied in concrete situations and are related to the semantic notion of a model: a set B (truth values), a subset of designated truth values, and a set of truth functions on B (corresponding to connectives of formal

systems). A chapter is given over to classical propositional calculi, especially the Łukasiewicz-Tarski system, and there is also brief mention of intuitionistic, implicational, and modal systems. Quantification theory and other more advanced topics are not treated at all.

For the most part, only simple and straightforward results are proved. For the many difficult problems which suggest themselves and for solutions of some of these problems, the reader is supplied with copious references to the literature. Nevertheless, the author's labor of definition and clarification serves a useful purpose; there are many places in logic (for example, equivalence of propositional systems involving different connectives) where a little carelessness leads to unexpected errors.

ELLIOTT MENDELSON, Queens College, Flushing, N. Y.

A Treatise on Trigonometric Series. By N. K. Bary. Translated from the Russian by Margaret F. Mullins. Macmillan, New York, 1965. Volume I, xx+553 pp. $12.00; Volume II, xv+508 pp. $15.00.

This treatise on trigonometric series appeared in Russian in 1961 and was preceded in 1959, by the two-volume second edition of Zygmund's "Treatise on Trigonometric Series." The reviewer does not propose to make any major comparison of these two standard works.

The book opens with an introductory chapter followed by an exhaustive Chapter I on the basic concepts. The material here is the standard introductory one and the 165 pages comprising it should perhaps be read by all mathematicians. The next chapter is on the nature and properties of Fourier coefficients of functions which are of bounded variation or of a Lipschitz class and includes such standard results as the Hausdorff-Young Theorem, generalized Parseval equation in L^2, and necessary conditions for a sequence to be one of Fourier coefficients. Chapter III gives, in detail, various tests for the convergence of a Fourier series at a point, while Chapter IV continues the study of the convergence of Fourier series of continuous functions and gives constructions of continuous functions whose Fourier series diverge at a set of points of cardinality $\aleph_0$ or c. Chapter V is devoted primarily to the various standard results, mainly those of Kolmogorov and Plessner, Beurling and Salem and Zygmund, and gives also examples due to Kolmogorov (respectively, Marcinkiewicz) for the existence of functions whose Fourier series are a.e. unbounded (bounded) and diverge a.e.

Chapter VI merits special attention since it gives information, not found in other books, on the following main problem: given a measurable function f and an $\epsilon>0$, is it possible to alter the function f over a set of measure $<\epsilon$ so that the new function g will have a Fourier series which converges a.e. or everywhere or uniformly? The extensive work of Menshov and his associates is summarized there.

Chapters VII thru XIII, again, contain detailed surveys of known results on the summability or strong summability, by standard summation processes, of Fourier series and conjugate trigonometric series and on the absolute convergence of Fourier series, general properties of sine and cosine series with decreasing coefficients, general trigonometric series and their convergence, divergence, etc. and lacunary series are also discussed in those chapters.

Chapters XIV and XV are again worthy of special mention. They are of a very special nature and contain several new concepts and results, many of which are those of the author herself. Chapter XIV is devoted to the problem of uniqueness of the expansion of functions by trigonometric series and the Fourier series of that sum function. A collection of open problems on this general topic is also presented at the end of the chapter. The problem of "representing" a function by trigonometric series is the theme of Chapter XV and extensive details of the works of Menshov and his associates (including the author) are found.

The first five chapters are within the reach of a specially interested first year graduate student; the value of the book to such a student is enhanced by the inclusion, at the end of these chapters, of selected problems. The remainder of the book is meant for a research worker. The principal virtues of this comprehensive work are the systematic, though slow, development of the topics discussed and the clear manner in which they are presented. The book is exceptionally well written and is self-contained. An extensive bibliography is one of its major assets. Contrasted with Zygmund's book, this work contains no major notes at the end of any chapter. As one may expect, both works have a great deal in common; however, much of the material in chapters VI, XIV and XV seem to be found only in Bary's.

Miss Bary's treatise will remain long a monumental job excellently executed by the author and gratefully received by the workers in that field.

M. S. RAMANUJAN, The University of Michigan

Elementary Differential Equations and Boundary Value Problems. By W. E. Boyce and R. C. DiPrima. Wiley, New York, 1965. 485 pp. $8.95.

Elementary Differential Equations. By W. E. Boyce and R. C. DiPrima. Wiley, New York, 1965. 361 pp. $7.75.

The second book, consisting of the first eight chapters of the first, omits the chapters on boundary value problems, Fourier series, and partial differential equations. Therefore, this will be a review of the first book.

In this subject it is not easy to tread the middle of the road between rigor and practicality. In this case, the authors have been quite successful in doing so. To be sure, not all results are proved, but all theorems are carefully stated and in almost every case some indication of the proof is given. This should give the student a sense of security with the subject. He will appreciate the basis for most results and at least know where the difficulties lie otherwise. After this treatment there should be no reason to have to brainwash any student before he goes on to advanced topics in ordinary differential equations.

The chapter on first order equations has the standard techniques and applications. There is an appendix which derives the equation of motion of a body with variable mass, an interesting problem which has applications in rocketry. This chapter also has a proof of existence and uniqueness for the initial value problem for a single first order equation based on a condition slightly less general than a Lipschitz condition. The chapter on second order linear equations begins with a nice discussion of how solutions can be obtained from linearly independent solutions of the homogeneous equation. It also contains the usual discussions of constant coefficients, variation of parameters, and undetermined coefficients. There is a separate chapter on series solutions near ordinary and regular singular points, unfortunately without general proofs of convergence. The three different cases for the roots of the indicial equation are carefully delineated and illustrated using the Bessel equation. The next three chapters are on higher order linear equations, the Laplace transform, and systems of first order equations. Enough linear algebra is introduced to make the discussion of systems quite coherent, but the general existence theorem for systems is not proved. The method of the Laplace transform is illustrated for systems. The chapter on numerical methods is better than one finds in most books on elementary differential equations. The authors carefully distinguish between local formula error, accumulated formula error, and round-off error, and they compare the efficiency of several methods in terms of local formula error. The last three chapters are on boundary value problems, Fourier series, and the separation of variables method of solving certain second order partial differential equations.

Probably because of the lack of discussion of various special functions, there is a heavy emphasis in the last three chapters on problems which lead to trigonometric

eigenfunctions. Perhaps this is inevitable in an elementary course, but one wonders if it would be possible to introduce some additional problems based on the Bessel functions or the Legendre functions. Also, the concept of the Green's function seems so important that it should not be relegated to the problems. These are just minor criticisms of an otherwise well conceived and well executed book. The authors are to be commended for helping to upgrade the literature in an area where often in the past "cookbooks" have been all too common.

J. W. DETTMAN, Oakland University

Topics in Algebra. By I. N. Herstein. Blaisdell, Waltham, Mass., 1964. 350 pp. $8.50.

This is an excellent text designed for an honors section in algebra, but the manner in which it is written makes it possible to teach a course at several different levels. By amplifying the proofs, going over them thoroughly in class, and assigning the problems intended to illustrate the theory, one can gear the course to the average student. The above average student could be encouraged to work on the "not too obvious problems." Another good exercise is to have the students puzzle out the proofs for themselves.

The book has many attractive features. The problems are excellent, varying from the immediate to those one had best not assign unless he had a few weeks spare time to work on them himself. Many theorems are proved in two or three different ways and in varying degrees of generality depending upon the context in which they occur. Most important, the author demands a lot of the student; and, if the instructor supports the author in this demand, the student will learn a great deal from the book.

Most important of all, I enjoyed teaching a course with this text.

ROBERT ELLIS, Wesleyan University

Number Theory. By J. Hunter. Oliver and Boyd, Edinburgh and London, 1964. 149 pp. $2.75.

This elegant little book is one of the "University Mathematical Texts." In the introductory chapter there is an excellent, if brief, informal discussion of the systems of positive integers, all integers, rational numbers, equivalence relations, groups, rings and fields. There are six more chapters. They deal with division and factorization properties of the integers, congruences, algebraic congruences and primitive roots, quadratic residues, representation of integers by binary quadratic forms and some diophantine equations. An attempt is made in various parts of the book to exhibit the close relation between certain parts of number theory and properties of finite abelian groups and commutative rings.

The book developed out of lecture courses in number theory given to students of mathematics at Glasgow University at various stages of an honors course. A strong student who has had a course in modern algebra can read the little book on his own with profit. The author is not long-winded. His proofs on occasion are extremely terse. In my opinion the writing is too condensed for the usual number theory course given in this country, and there are not enough problems. Each chapter has only one problem list appearing at the end of the chapter.

SOLOMON HURWITZ, City College, City University of New York

Einführung in die Wahrscheinlichkeitsrechnung und mathematische Statistik. By D. Morgenstern. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1964. 224 pp. DM 34.50.

This book has many virtues; among them are a clear and effective presentation of the standard material of mathematical statistics, numerous examples, a wealth of good exercises, and a method of presentation which stresses motivation and intuition.

It is organized in two parts, the first dealing with discrete distributions and the second

with continuous distributions. This leads to a certain amount of logical redundancy, but pays handsome dividends in readability and in keeping statistical motivation in the foreground. For example, the central limit theorem is proved for a number of special cases in part 1 and in a general form in part 2.

Anyone using this book, either as a textbook or for self study, might profitably supplement it with one of the standard texts on probability theory. The author does not usually state theorems separately, so they must be extracted from the body of the text; in fact, some of the hypotheses turn up in the footnotes.

The book contains more than the average number of misprints and inconsistencies of notation, and some of the references are rather incomplete. In spite of these difficulties the book is a valuable one and the minor irritations are more than outweighed by the author's insight into the basic problems of mathematical statistics.

TOM PITCHER, University of Southern California

Mathematical Theory of Connecting Networks and Telephone Traffic. By V. E. Beneš. Academic Press, New York, 1965. xiv+319 pp. $12.00.

This work would best be titled "A Selection of Reprints on Telephone Networks," since except for Chapter 3, it is an almost word for word reprint of papers published in the *Bell Systems Technical Journal* [Chap. 1, vol. 41, p. 1201; Chap. 2, vol. 41, p. 1249; Chap. 4, vol. 32, p. 406; Chap. 5, vol. 36, p. 939; Chap. 6, vol. 40, p. 117; Chap. 7, vol. 42, p. 567; Chap. 8, vol. 42, p. 2795]. This is unfortunate in several ways, especially since it is never explicitly mentioned and since Chap. 4 is essentially the writing of C. Clos. Of more concern to the reader are the facts that: (1) it is a bit hard to understand the beginning without having read the middle (notations are not clearly introduced until well into the book, etc.), (2) it is divided into somewhat isolated portions, (3) it is very specialized and omits alternate approaches (as graph theory synthesis) and applications (as to air traffic routes). On the other hand the ideas treated are fascinating and represent extremely useful applications of somewhat advanced mathematics. For example a typical portion of the book, Chap. 3, is concerned with representation of telephone connecting networks through permutation groups; if appropriate group factorizations were on hand this would lead to economical synthesis of such networks.

The work represents what appears as a conscientious effort to combine engineering concepts with a rigorous mathematical treatment. As such it is fairly successful, but the reader does need considerable background in probability and group theory with some knowledge of topology being useful.

In summary, the work will most likely become a useful reference to workers in the telephone industry. Still, the lack of editorial effort is regrettable; with relatively little editing it could become a classic reference on very practical uses of semiabstract mathematics.

R. W. NEWCOMB, Stanford University

Linear Equations and Matrices. By J. B. Johnston, G. B. Price and F. van Vleck. Addison-Wesley, Massachusetts, 1966. 308 pp. $8.50.

This is an elementary text with an emphasis on matrix algebra and its applications. Basic concepts are clearly defined and are accompanied by examples which give added meaning to the explanations. Considerable attention is given to computational procedures. Students interested in a good understanding of the mathematical development of the subject must pay close attention to the problems in the book.

The topics treated are: systems of linear equations, algorithms, flow charts, and matrices and their applications.

B. VISWANATHAN, University of New Brunswick

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 15 (junior level)—18 (second graduate year).

Algebra

Lectures on Real and Complex Vector Spaces. By Frank S. Cater. Saunders, Philadelphia, 1966. x+167 pp. $5.00. Twenty-eight lectures, most followed by problems, divided into five subsets: fundamental concepts, matrices, determinants, operators, operators on vector spaces, finite dimensional vector spaces, finite-dimensional unitary spaces. T (15–17), S, P.

Linear Transformations and Matrices. By F. A. Ficken. Prentice-Hall, Englewood Cliffs, N. J. 1967. xii+398 pp. $10.95. Preliminaries (sets, logic, real numbers, classical algebra and geometry of vectors in "naive physical space," algebraic structures), linear spaces, transformations, functionals, duality, matrices, linear systems, matric equivalence, bilinear and guadratic functionals and forms, determinants, similar operators, unitary space, 760 problems, and bibliography. The publishers, a few years behind in their homework, announce on the dust jacket that the author has served as Associate Editor of this MONTHLY. T (13–15), S. P.

Algebraic Structures. By Serge Lang. Addison-Wesley, Reading, Mass., 1967. vii+169 pp. $6.95. Another in what is becoming a *lang* series of short books, this volume complements the author's *Linear Algebra* and could be used independently or for half (preferably the second half) of a year course. Topics include the integers, groups, rings, polynomials, vector spaces and modules, field theory, the real and complex numbers, sets (through cardinals, well-ordering, proof of Zorn's lemma). "The desire to avoid encyclopedic proportions, and specialized emphasis, and to keep the book short, explains the omission of some theorems which some teachers will miss and may want to include in the course." T (14–15).

The Abstract Theory of Groups. By the late O. U. Schmidt. Translated from the Russian by Fred Holling and J. B. Roberts. Edited by J. B. Roberts. Freeman, San Francisco, 1966. vi+174 pp. $5.00. This translation of Schmidt's classic work of 1916 is based on the 2nd edition of 1933 which differed from the first by "only insignificant editorial changes." The editor offers it for its "selection of topics . . . treatment . . . numerous historical references" and describes it as a "beautiful book." In his preface (1933) Schmidt observed that only one comprehensive treatise (Burnside) existed when he wrote in 1916 and that between the two editions there were only two "noteworthy books of general nature" (Speiser and Miller-Blickfeldt-Dickson). One notes that Schmidt signs his preface of 1933 on the Icebreaker Chelyuskin, and Roberts signs his of 1966 in Dar es Salaam, Tanzania. S. P. L.

Introduction to Linear Programming, with Applications. By William R. Smythe, Jr. and Lynwood A. Johnson. Prentice-Hall, Englewood Cliffs, N. J., 1966. vii+221 pp. $9.00. Topics from linear algebra, algebraic solutions of linear programming problems (simplex method), networks and flows, applications. T (14–15), S.

Analysis

Transcendental Functions Satisfying Nonhomogeneous Linear Differential Equations. By A. W. Babister. Macmillan, New York, 1967. xi+414 pp. $14.95. Methods of finding particular integrals and "a comprehensive treatment" of their properties, including results published for the first time in book form. Presupposes an elementary course in differential equations. There are problem sets. T, S, P.

Differential and Difference Equations. By Louis Brand. Wiley, New York, 1966. xvi+698 pp. $11.95. Not just another book on its subjects, but a parallel treatment of difference and differential equations bringing out similarities in both theory and methods of procedure. The inspiration of Maxime Bôcher's course at Harvard on linear differential equations of the second order is acknowledged by the author. Topics include the Laplace transform, solution in series, Mikusinski's operational calculus, existence and uniqueness theorems, and numerical methods. T (15–17), S, P, L.

Problems in Differential Equations. By J. L. Brenner. Freeman, San Francisco, 1966. ix+172 pp. $2.50. (paper). To the first edition have been added two appendices, one on the formalism of the Laplace transform and one giving a "logically complete discussion of the existence and uniqueness of the solution of a linear differential equation with constant coefficients . . . without using the ideas of Wronskian and linear dependence." T (14), S.

The Theory of Cluster Sets. By E. F. Collingwood and A. J. Lohwater, Cambridge University Press, New York, 1966. xi+211 pp. $8.50. An introductory treatise on cluster sets, a notion first introduced by Painlevé in 1895. (The cluster set of a function at a boundary point of its domain is the set of limit points of the function there.) Bibliography. P, L.

Introduction to Analysis and Abstract Algebra. By John E. Hafstrom. Saunders, Philadelphia, 1967. xii+344 pp. $6.50. Logic, the ordered field of real numbers and its subsets, completeness, sets, relations, mappings, operations, groups, rings, integral domains, fields, construction of the real numbers, sequences, point sets, limits, continuity, integration, polynomials and continuous functions—in short "what every young mathematician ought to know" by the end of his junior year at least. T (14–15), S.

Linear Differential Equations and Function Spaces. By José Luis Massera and Juan Jorge Schäff. Academic, New York, 1966. xx+404 pp. $16.00. A treatise "presenting in systematic and detailed form the recent studies of the authors concerning linear ordinary differential equations in the real domain" based "on an idea going back to the work of O. Perron (1930) . . . " Assumes no special knowledge of differential equations but a working acquaintance with Banach-space theory, both "soft" and "hard" (not including operatory theory). P.

Multiple Integrals in the Calculus of Variations. By Charles B. Morrey, Jr. Springer-Verlag, New York, 1966. xii+506 pp. $19.50. A comprehensive treatise based on the author's research over more than three decades. P, L.

Infinite Series. By Earl D. Rainville. Macmillan, New York, 1967. xii+265 pp. $7.95. The usual topics in an elementary introduction plus uniform convergence, orthogonal sets of functions, Fourier series, asymptotic series, infinite products and much material not included in the sketchy treatment offered in the usual calculus course. T (14–15), S.

Value Distribution Theory. By Leo Sario and Kiyoshi Noshiro. Van Nostrand, Princeton, N. J., 1966. xi+256 pp. $7.50. A "modern value distribution theory for complex analytic mappings between abstract Riemann surfaces" inspired by classical value distribution theory, which was initiated by Picard's celebrated theorem, "one of the most beautiful results in classical analysis," and was further developed by Borel, Hadamard, Valiron, Julia, Collingwood, Littlewood, and others. There are historical remarks and an extensive bibliography. P, L.

Introductory Complex Analysis. By Richard A. Silverman. Prentice Hall, Englewood Cliffs, N. J., 1967. xi+372 pp. $8.75. The author describes this as a scaled-down version of the three volume translation *Theory of Functions of a Complex Variable* by A. I. Markushevich of Moscow State University, published by Prentice-Hall under his editorship 1965–1967. The distillation is intended for the "usual one-year graduate course" given in the United States. T (17).

Applications

Space Mathematics, Parts I, II, III. Volumes 5, 6, 7 of Lectures in Applied Mathematics. American Mathematical Society, 1966. I: 315 pp. $12.80. II: 258 pp. $12.80. III: 315 pp. $12.80. Twenty-seven papers based mostly on lectures at the 1963 Summer Seminar in Applied Mathematics and covering "current state of research . . . pressing unsolved problems . . . mathematical techniques which are currently useful in attacking" problems on the behavior of non-propulsive space vehicles. P, L.

Fourier Transforms and the Theory of Distributions. By J. Arsac. Translated by Allen Nussbaum and Gretchen C. Heim. Prentice Hall, Englewood Cliffs, N.J., 1966. xv+318 pp. $12.75. ". . . I have tried to merge the viewpoints of the mathematicians and the physicists and run the middle course between extreme rigor and none at all . . ." Topics include transforms of summable functions and square-summable functions, modified distributions and their transforms, multidimensional space, diffraction at infinity, complex impedances, applications in mathematics physics, linear filters, numerical methods. S, P.

Mathematical Methods in the Physical Sciences. By Mary L. Boas. Wiley, New York, 1966. xix+778 pp. $11.95. ". . . coverage that includes nearly every area of mathematical physics needed in the junior, senior, and first year of graduate work . . ." Series, complex numbers, determinants and matrices, partial derivatives and multiple integrals, vector analysis, Fourier series, ordinary and partial differential equations, calculus of variations, special functions, transformations and tensor analysis, complex functions, series solutions, integral transforms, probability. T (14). Reviewed in *Science* 24 March 1967.

A Deductive Theory of Space and Time. By Saul A. Basri. North Holland, Amsterdam, 1966. xi+163 pp. $7.00. "The theory starts with the primitive concept of the class of living humans and ends with the derivation of the equations of motion of a particle in an arbitrary gravitational field." Making use of the "fruitful and exciting developments in mathematics during the last half century," particularly symbolic logic, that "have not been utilized in physics to the extent they could," the author "develops the concepts of particles, events, clocks, length measuring instruments, and all other concepts of space-time geometry in terms of only eight primitive concepts." P.

Mathematical Systems Theory. A new serial published quarterly by Springer Verlag, New York. Executive Editors: D. Bushaw, A. J. Lohwater, M. D. Mesarovic, and G. P. Szegö. ". . . will feature articles in such fields as topological dynamics, theory of dynamical polysystems, general systems theory, formal systems theory, mathematical theory of automata and algorithms, mathematical linguistics, algebraic systems theory, ergodic theory, several aspects of qualitative theory of differential equations and the like. In addition, papers in functional analysis, topology, algebra and logic directed related to the theory of mathematical systems will be published." Did they overlook anything?

Stochastic Processes. By M. Girault. Vol. III of Econometrics and Operations Research. Springer-Verlag, New York, 1966. ix+122 pp. $7.00. "The practical worker must . . . have at his disposal a set of original and varied stochastic models . . ." Poisson pro-

cesses, numerical processes with independent random increments, Markov processes, Laplace processes and second order processes, processes on continuous-time. S, P.

Mathematical Theory of Elementary Particles. Proceedings of the Conference held at Endicott House, in Dedham, Mass. Sept. 12–15, 1965. Edited by Roe Goodman and Irving Segal. M.I.T. Press, Cambridge, Massachusetts, 1966. vii+188 pp. $6.00. Thirteen papers representing "a quite broad spectrum of mathematical approaches" including papers by "quantum field theorists of both axiomatic and constructivist persuasions (some of whom have legs in both outposts): and group-theoretical single-particleers." The editors express the hope that the volume "has sufficient breadth, topicality, and elementary clarity to be useful for orientation purposes, as well as of interest to experts . . ." P.

An Introduction to Dynamic Programming. The Theory of Multistage Decision Processes. By O. L. R. Jacobs. Chapman & Hall, London, 1967. x+123 pp. Obtainable in U.S.A. only from Barnes and Noble, New York. $5.50. A simple introduction, requiring only elementary calculus and based on examples, to the theories initiated by the work of R. E. Bellman in the 1950's. S, P.

Econometric Techniques and Problems. By C. E. Leser. Hafner, New York, 1966. viii+119 pp. $4.45 (Paper). ". . . based on a course of lectures which were held . . . at the Economic Research Institute, Dublin, in the winter of 1963–64. Participants . . . were mainly university students of economics in their honours year and economics graduates in the public service, in business, and in research institutes . . . will be of interest to a wider public . . . who wish to get a good grasp of what econometrics is and how it is used." S, P.

Formulas and Theorems for the Special Functions of Mathematical Physics. By Wilhelm Magnus, Fritz Oberhettinger, and Raj Pal Soni. Third enlarged edition. Springer-Verlag, New York, 1966. vii+508 pp. $16.50. A new and enlarged English edition of the book that originally appeared in German in 1946. The gamma, hypergeometric, Bessel, Legendre, Kummer's, Whittaker, parabolic, parabolic cylinder, theta, elliptic and incomplete gamma functions, orthogonal polynomials, elliptic integrals, integral transforms, coordinate transformations. L.

The Mathematical Apparatus for Quantum-Theories, Based on the Theory of Boolean Lattices. By Otton Martin Nikodym. Springer-Verlag, New York, 1966. x+952 pp. $36.00. "The purpose of this book is to give the theoretical physicist a geometrical, visual and precise mathematical apparatus . . . based on the theory of Boolean lattices, whose elements are closed subspaces in the separable and complete Hilbert-Hermite-space." This is a systematic synthesis of the author's work beginning with his paper of 1947 (see *Zentralblatt*, vol. 37 (1951) 278). The price of 3.7¢ per page for quality printing and binding of material with a high density of formulas and special symbolism contrasts favorable with the 7¢ a page charged for some recent books produced by cheaper methods. P, L.

The Method of Summary Representation for Numerical Solution of Problems of Mathematical Physics. By G. N. Polozhii. Translated from the Russian by G. J. Tee. Translation edited by K. L. Stewart. Pergamon, New York, 1965. xx+283 pp. $10.00. "The book will be useful to scientific workers and engineers who are interested in the numerical solution of problems of mathematical physics and technology, particularly in those cases where the accuracy of an approximate solution is a matter of great importance." P.

Great Ideas in Information Theory, Language and Cybernetics. By Jagjit Singh. Dover, New York, 1966. ix+338 pp. $2.00 (paper). A popularization of interest to mathe-

maticians by the author of *Great Ideas of Modern Mathematics* and of *Great Ideas and Theories of Modern Cosmology*, which were honored by the UNESCO Kalinga Prize for outstanding achievement in the dissemination of science. S, P, L.

Boundary Value Problems of Mathematical Physics. Vol. I. By Ivar Stakgold. Macmillan, New York, 1967. viii+340 pp. $12.95. This first volume deals with the mathematics required for the methods of eigenvalue expansion and Green's function and with applications to ordinary differential and integral equations. The second will deal with partial differential equations. Prerequisites are advanced calculus and elementary complex variables. T (17, science and engineering), P.

Applied Methods of the Theory of Random Functions. By A. A. Sveshnikov. Translated by J. Berry. Translation edited by L. Haller. Pergamon, New York, 1966. ix+321 pp. $10.00. ". . . methods are presented of random functions analysis with their applications in various branches of technology: in the theory of ships, in the theory of automatic regulation and control, in radio engineering and so on . . . the mathematical rigour of the treatment has been pursued only to the extent to which it is necessary for understanding the text and for obtaining a mastery of the methods given . . . intended for engineers and scientists . . ." General properties, spectral theory, optimal dynamical systems, experimental methods for determining characteristics, method of envelopes. P.

Mathematics of Automatic Control. By Toshie Takahashi. Translated by Scripta Technica, Inc. Translation edited by George M. Kranc. Holt, Rinehart and Winston, New York, 1966. xx+454 pp. $14.50. Transient response, frequency response, functions of complex variables, Laplace transformation, transfer functions, stability. ". . . for beginners intending to understand the essential interconnection between the mathematical theory of complex functions and dynamical theory of linear control systems. This book has grown out of my lectures on applied mathematics for students of mechanical engineering . . ." T (15–16), S, P.

Spacetime Physics. By Edwin F. Taylor and John Archibald Wheeler. Freeman, San Francisco, 1966. 208 pp. $4.75. Though "developed for the first month of a freshman physics course," it is of interest because it provides "an introduction to relativity and brings closer the time when the student of physics will be as much at home with the geometry of spacetime as the student in an earlier century was with Euclidean geometry." P, L.

Topology

Topology Seminar Wisconsin. Edited by R. H. Bing and R. J. Bean. Princeton Univ. Press, Princeton, N.J. 1966. 246 pp. $5.00. "This volume contains the proceedings of the topology seminar held under the direction of R. H. Bing at the University of Wisconsin in the summer of 1965. Most of the articles were presented as talks to the topologists assembled, then expanded to make expository articles with the details of proofs frequently omitted . . ." P, L.

Elements of Mathematics, General Topology, Parts 1 and 2. ByNicolas Bourbaki. Addison-Wesley, Reading, Mass., 1966. I: 438 pp. $18.50. II: 368 pp. $18.50. The first two of eight planned volumes in which the publishers, with the gracious permission of M. Bourbaki, plan to issue an English revised version of the thirty-one paperbound booklets in which *Les Eléments* has appeared piecemeal and profoundly affected contemporary mathematical thinking. The identical price for volumes of significantly different size illustrates once more the arbitrary pricing policies of publishers when

offering books that they estimate (correctly in this case!) must be bought by libraries. (Addison-Wesley informs us that the price was set by the French publisher and is approximately equal to the total cost of the original edition when bought in separate pamphlets.) P, L.

First Concepts of Topology. The Geometry of Mappings of Segments, Curves, Circles, and Disks. Number 18 in the New Mathematical Library. By W. G. Chinn and N. E. Steenrod. Random House and Singer, New York, 1966. viii+160 pp. $1.95 (paper). Although written for high school students and laymen, this series is of such high quality that it belongs in any mathematician's library. This volume deals with existence theorems in one and two dimension in order to "show how topology arose, develop a few of its elements, and present some of its simpler applications." P, L.

Topological Groups. By L. S. Pontryagin. Translated from the Russian by Arlen Brown. Gordon & Breach, New York, 1966. xv+543 pp. $32.50 (ref) $17.50 (pro). ". . . does not demand . . . a broad mathematical knowledge . . . (only) significant mathematical maturity . . . a knowledge of the most elementary mathematical material, such as analytic geometry, the theory of matrices, the theory of ordinary differential equations, etc." For a cheaply produced photo-offset book the price of 6¢ a page is outrageous!

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Dr. S. M. Ulam, Professor of computing science and mathematics at the University of Colorado, has received an Alfred Jurzykowski Foundation Polish Millennium Award for science.

Arizona State University: Mr. C. W. Arnquist, University of California, has been appointed Assistant Professor; Associate Professors J. B. Kelly and N. W. Savage have been promoted to Professors.

California State Polytechnic College: Assistant Professor T. J. Cullen, California State College, has been appointed Assistant Professor; Mr. Albert Konigsberg has been promoted to Assistant Professor; Assistant Professor E. J. Vought has been promoted to Associate Professor; Associate Professors K. B. Kriege and H. F. Simmons have been promoted to Professors.

City College of New York: Associate Professors Solomon Hurwitz and Abraham Schwartz have been promoted to Professors.

Dartmouth College: Associate Professor R. Z. Norman has been promoted to Professor; Assistant Professor W. E. Slesnick has been promoted to Associate Professor; Assistant Professor G. W. Day has been appointed Assistant Professor at the University of Wyoming; Dr. J. E. Graver has been appointed Assistant Professor at Syracuse University.

University of Dayton: Mr. J. W. Friel, University of Nebraska, has been appointed Assistant Professor; Dr. T. E. Gantner, Purdue University, has been appointed Assistant Professor; Dr. R. C. Steinlage, Ohio State University, has been appointed Assistant Professor.

Dominican College: Sister Agnes Rose Kokke has been promoted to Assistant Professor and Chairman of the Mathematics Department; Assistant Professor Sister Mary David Kutchey has been appointed Chairman of the Mathematics Department at St. Catherine's High School.

Eastern Kentucky University: Associate Professor B. R. Lane, George Peabody College, has been appointed Professor and Chairman of the Mathematics Department; Associate Professor F. G. Scorsone has been promoted to Professor.

Louisiana State University: Dr. W. L. Bynum, University of North Carolina, has been appointed Assistant Professor; Dr. R. L. Pendleton, Massachusetts Institute of Technology, has been appointed Assistant Professor; Assistant Professor J. R. Retherford has been promoted to Associate Professor.

New York University: Associate Professor Melvin Hausner has been promoted to Professor; Dr. V. G. Vitale, New York University, has been appointed Assistant Professor.

Ohio University: Dr. I. A. Barnett, University of Cincinnati, has been appointed Visiting Professor; Mr. Hari Shankar has been promoted to Assistant Professor.

University of Oregon: Dr. L. C. Grove, John Wesley Young Research Institute, has been appointed Assistant Professor; Assistant Professors G. W. Struble and C. R. Wright have been promoted to Associate Professors; Associate Professor F. W. Anderson has been promoted to Professor.

St. John Fisher College: Assistant Professor D. L. Muench, U. S. Naval Academy, has been appointed Associate Professor; Miss Helen B. Boulas has been promoted to Assistant Professor and Acting Chairman of the Mathematics Department.

San Diego State College: Dr. E. J. Howard, Idaho State University, has been appointed Assistant Professor; Dr. R. P. Kopp, University of Michigan, has been appointed Assistant Professor; Assistant Professors H. A. Gindler and Betty Kvarda have been promoted to Associate Professors; Associate Professor Margaret F. Willerding has been promoted to Professor.

Southern Illinois University: Assistant Professor R. E. Hughs, Carleton College, has been appointed Associate Professor; Mr. L. S. Holden has been promoted to Assistant Professor; Assistant Professor R. B. Rutledge has been promoted to Associate Professor.

State University of New York at Binghamton: Assistant Professor E. P. Rozycki, Canisius College, has been appointed Assistant Professor; Dr. N. J. Sterling, Syracuse University, has been appointed Assistant Professor; Professor W. R. Transue, Kenyon College, has been appointed Professor.

Professor Emeritus V. G. Grove, Michigan State University, died on January 23, 1967. He was a member of the Association for forty-seven years.

Dr. Jack Indritz, University of Minnesota, died on November 30, 1966. He was a member of the Association for fourteen years.

Professor Emeritus Eugenie M. Morenus, Sweet Briar College, died on October 15, 1966. She was a Charter Member of the Association.

Mr. G. A. Seubert, Supervising Principal, Unity Township School District, died on December 3, 1966. He was a member of the Association for forty-four years.

Professor C. A. Shook, Lehigh University, died on December 28, 1966. He was a member of the Association for fifty years.

Professor A. K. Snyder, Valley Forge Military Jr. College, died on December 29, 1966. He was a member of the Association for seventeen years.

PUBLICATION OF *EDUCATION IN APPLIED MATHEMATICS*

The Society for Industrial and Applied Mathematics has available reprints of the proceedings of a Conference sponsored by SIAM under a grant from the National Science Foundation at Aspen, Colorado, May 24–27, 1966. The proceedings were published as part of the April 1967 issue of the *SIAM Review*. Topics include: Objectives of Applied Mathematics Education; The Mathematics Curriculum; The Science Curriculum; Training in Applied Mathematics Research; The Research Program in Applied Mathematics; and, Post Doctoral Experience. Only a limited number of reprints are still available at the special price of $2.40 each. Orders should be addressed to: SIAM, 33 S. 17th Street, Philadelphia, Pa. 19103. Payment should accompany order.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The forty-seventh meeting of the Southern California Section of the MAA was held at San Diego State College, San Diego, California, on March 11, 1967. The registered attendance was 128, including 107 members of the Association. Professor V. C. Harris, Chairman of the Section, presided at the morning and afternoon sessions.

At the business meeting Dr. R. E. Horton, Chairman of the Nominating Committee, reported the election of the following officers, who are to serve beginning July 1, 1967: Chairman, Professor F. A. Valentine, University of California, Los Angeles; Vice-Chairman, Professor C. W. Seekins, Occidental College; Secretary-Treasurer, Professor D. H. Potts, San Fernando Valley State College. The Chairman and Vice-Chairman will serve for one year; the Secretary will serve for three years. The following members of the Program Committee for the 1968 meeting of the Section were also elected: Professor Richard Dean, Cal Tech, Chairman; Professors Courtney Coleman, William Lambert, L. J. Paige, and Sidney Spital.

Professor R. C. James reported as Governor for the Section on items and actions of the Board of Governors for the January, 1967 meeting at Houston, Texas. Dr. J. M. Huffman of the Office of Schools for San Diego County, reported on the National High School Mathematics Contest, as conducted in the Section by the Committee on the High School Contest. There was increased participation on the part of schools and their students in southern California, especially in the San Diego area. Many schools are benefitting from the program.

It was announced that the next meeting of the Section is scheduled to be held on March 9, 1968, at Loyola University of Los Angeles.

The following program was presented:

1. *Distributions in n-dimensions whose support is a point*, by Leon Nower, San Diego State College, introduced by the Secretary.

The concepts of distribution and support, and associated fixed-point theorems, were discussed for two-dimensional examples, and for higher dimensions.

2. *Digital bracelets*, by Dean C. W. Trigg, retired, Los Angeles City College.

A bracelet is one period of a simply periodic series considered as a closed sequence with terms equally spaced around a circle. Application to exhaustion of the 100-pair set of ordered digits by $u_{n+2}=u_n+u_{n+1}$ produces six digital bracelets. Operation of one bracelet on another by addition (reduced modulo ten) causes rotation or conversion. The recurrence $u_{n+2}=au_n+bu_{n+1}$ produces plain bracelets when $a=1, 3, 7, 9$. Bracelet digits have 1-digit charms attached for even a, and odd b. Charms are bifurcated for a, b even. Bracelet digits are multicharmed when $a=5$, $b\neq 5$, with quinfurcation of charms when $a=b=5$. In systems with prime number bases, all bracelets are plain.

3. *Remarks on Banach's contraction mapping principle*, by J. B. Diaz, University of California, Riverside, introduced by the Secretary.

The talk was based on the paper: "Remarks on a generalization of Banach's principle of contraction mappings," by Sherwood C. Chu and J. B. Diaz, Journal of Mathematical Analysis and Applications, Vol. 11, 1965, pp. 440–446, where it is shown that a generalization of Banach's principle of contraction mappings is a special case of an elementary fact. The contraction mapping principle appears in the book: *Elements of the Theory of Functions and Functional Analysis*, Volume I, *Metric and Normed Spaces*, by S. Banach.

4. *Course work for the general college student*, a panel discussion chaired by C. J. A. Halberg, Jr., University of California, Riverside, concerning recommendations of the W. L. Duren panel of CUPM.

The discussion centered around a three-page report which had been submitted to members of the Section. In general the Committee concurs with the Duren Panel, but it suggests that problems peculiar to the junior college and to students qualified for advanced placement, are in need of further study, with regard to the area of mathematics in general education.

5. *Correlation properties of sequences*, by L. R. Welch, University of Southern California.

This talk iilustrated applications of finite fields and quadratic forms over finite fields, to communication theory. Sequences defined by linear recurrences with constant coefficients over $GF(2)$ are of interest to communications engineers, and the analysis of such sequences is a natural extension of the customary analysis over the field of real numbers. Properties of interest are periodicity, autocorrelation, and cross-correlation for sequences. The first two properties are easily predicted, using extension fields of $GF(2)$. Although the last property is generally more difficult, for a special class of cross-correlations, it can be reduced to considering quadratic forms over $GF(2^n)$.

6. *Dihedral groups arising in Fourier transform theory*, by R. W. Preisendorfer, Research Mathematician, Scripps Institute of Oceanography, University of California, San Diego.

The operations of taking a Fourier transform T, complex conjugator C, and reversal in the real line R, with $T\cdot T=R$, form a *dihedral group* of order eight. The elements of the group are in the set, $\{I, T, R, R\cdot T, C, T\cdot C, R\cdot C, \text{and } C\cdot T\}$. There are two related groups of four elements: a cyclic group generated by T, and a *four group*, $\{I, R, T\cdot C, C\cdot T\}$. The dihedral group has been used to generate many variations of convolution, Parseval, and power spectrum theorems.

R. B. HERRERA, *Secretary-Treasurer*

SUMMER DUES PAYMENTS

Since dues in the Association are payable on a calendar year basis, the Buffalo office is faced each year with a great overload of work in December and January since so many members pay their annual dues at that time.

It would help to spread the work of the office and to reduce the cost of operation if most members would pay dues in advance during the preceding summer. We suggest, therefore, that members plan to pay their 1968 dues of $8 sometime this summer. Payments should be sent to: Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, N. Y. 14214.

CALENDAR OF FUTURE MEETINGS

Forty-eighth Summer Meeting, University of Toronto, Toronto, Ontario, Canada, August 28–30, 1967.

Fifty-first Annual Meeting, San Francisco, California, January 25–27, 1968.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS

INDIANA

IOWA, Wartburg College, Waverly, Spring 1968.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Mississippi State College for Women, Columbus, February 16–17, 1968.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA

MISSOURI

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 26–27, 1968.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO

OKLAHOMA-ARKANSAS, Federal Aviation Agency, Oklahoma City, March 29–30, 1968.

PACIFIC NORTHWEST

PHILADELPHIA, University of Delaware, Newark, Delaware, November 18, 1967.

ROCKY MOUNTAIN

SOUTHEASTERN, East Carolina College, Greenville, North Carolina, March 29–30, 1968.

SOUTHERN CALIFORNIA, Loyola University of Los Angeles, Los Angeles, March 9, 1968.

SOUTHWESTERN, New Mexico State University, University Park, Spring 1968.

TEXAS, Texas A&M University, College Station, April 19–20, 1968.

UPPER NEW YORK

WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, New York, New York, December 26–31, 1967.

AMERICAN MATHEMATICAL SOCIETY, Toronto, Ontario, Canada, August 29–Sept. 1, 1967.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY Sheraton-Park, Washington, D. C., August 29–31, 1967.

ASSOCIATION FOR SYMBOLIC LOGIC, Sheraton-Boston Hotel, December 27–28, 1967.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 23–25, 1967.

INSTITUTE OF MATHEMATICAL STATISTICS

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, McGill University, Montreal, August 21–24, 1967.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Sheraton-Chicago Hotel, November 1–3, 1967.

PI MU EPSILON, Toronto, Ontario, Canada, August 29–30, 1967.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, University of California at Santa Barbara, November 29–December 2, 1967. (Symposium on combinatorial mathematics.)

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THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 74 NUMBER 7

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AUGUST-SEPTEMBER 1967

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED in 1894 BY BENJAMIN F. FINKEL)

R. A. ROSENBAUM, *Editor*

EDITORIAL CORRESPONDENCE AND MANUSCRIPTS OF PAPERS should be addressed to the Editor, R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

General Office: SUNY at Buffalo, Buffalo, N. Y. 14214. *Executive Director:* H. M. GEHMAN

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $6.00 ($8.00 in 1968). For nonmembers the subscription price is $10.00 ($15.00 in 1968).

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July,
August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

J. H. McKAY, Oakland University and University of California, Berkeley

The following results of the twenty-seventh William Lowell Putnam Mathematical Competition held on November 19, 1966, have been determined in accordance with the regulations governing the Competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Douglas E. Burke, Irwin Gaines, and Neal I. Koblitz; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were Theodore C. Chang, Gerald S. Gras, and Michael R. Rolle; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of University of Chicago, Chicago, Illinois. The members of the team were James Coulthard, Louis L. Crane, and Dennis A. Hejhal; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of University of Michigan, Ann Arbor, Michigan. The members of the team were Ronald J. Evans, Robert L. Scott and Paul E. Weiss; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of Princeton University, Princeton, New Jersey. The members of the team were James K. Baker, Daniel I. A. Cohen and Robert J. Weber; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are Marshall W. Buck, Harvard University; Theodore C. Chang, Massachusetts Institute of Technology; Robert E. Maas, University of Santa Clara; Richard C. Schroeppel, Massachusetts Institute of Technology; and Robert S. Winternitz, Massachusetts Institute of Technology. To each of these a prize of seventy-five dollars is awarded.

The ten persons ranking second highest in the examination, named in alphabetical order, are Gerald S. Gras, Massachusetts Institute of Technology; Ronald A. Hunsinger, Humboldt State College; Neal I. Koblitz, Harvard University; William L. Lepowsky, Harvard University; Theodore S. Martner, University of Colorado; Roger E. Poore, Carleton College; William F. Prange, University of Nebraska; Robert L. Scott, University of Michigan; John C. Stewart, Virginia Polytechnic Institute; and William G. Unruh, University of Manitoba.

The following teams, named in alphabetical order, won honorable mention: California Institute of Technology, the members of the team being Glenn E. Engebretsen, Kim D. Gibson and Stacy G. Langton; Clarkson College of Technology, the members of the team being John G. Alvord, David Boss, and Philip C. Washburn; Dartmouth College, the members of the team being J. Lawrence Carter, Sidney W. Marshall, and David L. Salsbury; Stanford University, the members of the team being A. Craig Franklin, Alan R. Siegel, and Andrew J. Vince; and Swarthmore College, the members of the team being Mary E. Kramer, John C. Mather and Stephen B. Maurer.

Honorable mention is given to the following twenty individuals, named in alphabetical order: Roger K. Alexander, University of Kansas; Clifton L. Anderson, Yale University; Douglas E. Burke, Harvard University; J. Lawrence Carter, Dartmouth College; James E. Coulthard, University of Chicago; Paul R. Dippolito, Kenyon College; Glenn E. Engebretsen, California Institute of Technology; Robert M. Ephraim, Columbia University; Gerald B. Folland, Harvard University; Robert S. Fowler, Swarthmore College; Irwin Gaines, Harvard University; Jacob B. Gross, Yeshiva University; Michael E. Grost, Michigan State University; Christopher A. Gurwood, New York University; John R. Hall, Jr., Brown University; Douglas A. Lind, University of Virginia; John M. Malcolm, Harvey Mudd College; Stephen B. Maurer, Swarthmore College; William C. Mitchell, California Institute of Technology; and Pieter B. Visscher, Harvard University.

One thousand five hundred twenty-six contestants from two hundred fifty-one colleges and universities participated in the examination on November 19, 1966.

A listing of the top five hundred contestants may be obtained from the Director. The list, which includes addresses and expected dates of graduation, may be helpful to departments of mathematics in selecting graduate students.

The Questions Committee, consisting of A. M. Garsia (chairman), R. E. Greenwood, and N. D. Kazarinoff, prepared the problems (listed below) for the competition.

PROBLEMS PART A

A-1. Let $f(n)$ be the sum of the first n terms of the sequence 0, 1, 1, 2, 2, 3, 3, 4, $\cdots$, where the nth term is given by

$$a_n = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Show that if x and y are positive integers and $x>y$ then $xy=f(x+y)-f(x-y)$.

A-2. Let a, b, c be the lengths of the sides of a triangle, let $p=(a+b+c)/2$, and r be the radius of the inscribed cricle. Show that

$$\frac{1}{(p-a)^2}+\frac{1}{(p-b)^2}+\frac{1}{(p-c)^2}\geqq\frac{1}{r^2}.$$

A-3. Let $0<x_1<1$ and $x_{n+1}=x_n(1-x_n)$, $n=1, 2, 3, \cdots$. Show that

$$\lim_{n\to\infty} nx_n = 1.$$

A-4. Prove that after deleting the perfect squares from the list of positive integers the number we find in the nth position is equal to $n+\{\sqrt{n}\}$, where $\{\sqrt{n}\}$ denotes the integer closest to $\sqrt{n}$.

A-5. Let C denote the family of continuous functions on the real axis. Let T be a mapping of C into C which has the following properties:

1. *T is linear*, i.e. $T(c_1\psi_1+c_2\psi_2)=c_1T\psi_1+c_2T\psi_2$, for c_1 and c_2 real and ψ_1 and ψ_2 in C.

2. *T is local*, i.e. if $\psi_1 \equiv \psi_2$ in some interval I then also $T\psi_1 \equiv T\psi_2$ holds in I.
Show that T must necessarily be of the form $T\psi(x) = f(x)\psi(x)$ where $f(x)$ is a suitable continuous function.

A-6. Justify the statement that

$$3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}}.$$

PART B

B-1. Let a convex polygon P be contained in a square of side one. Show that the sum of the squares of the sides of P is less than or equal to 4.

B-2. Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

B-3. Show that if the series

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

is convergent, where $p_1, p_2, p_3, \cdots, p_n, \cdots$ are positive real numbers, then the series

$$\sum_{n=1}^{\infty} \frac{n^2}{(p_1 + p_2 + \cdots + p_n)^2} p_n$$

is also convergent.

B-4. Let $0 < a_1 < a_2 < \cdots < a_{mn+1}$ be $mn+1$ integers. Prove that you can select either $m+1$ of them no one of which divides any other, or $n+1$ of them each dividing the following one.

B-5. Given $n(\geqq 3)$ distinct points in the plane, no three of which are on the same straight line, prove that there exists a simple closed polygon with these points as vertices.

B-6. Show that all solutions of the differential equation $y'' + e^x y = 0$ remain bounded as $x \to \infty$.

SOLUTIONS. PART A

The number in parentheses, immediately following the problem number, is the number of participants who received a score of 8, 9 or 10 on the problem. The maximum possible score on a problem is 10.

A-1 (840) It is easily verified by induction that

$$f(n) = \begin{cases} n^2/4 & \text{when } n \text{ is even,} \\ (n^2 - 1)/4 & \text{when } n \text{ is odd.} \end{cases}$$

Therefore, since $x+y$ and $x-y$ always have the same parity, in any case we must have

$$f(x+y) - f(x-y) = \frac{(x+y)^2 - (x-y)^2}{4} = xy.$$

A-2 (34) The area of the given triangle can be calculated in two ways, $A = pr$ and $A = \sqrt{p(p-a)(p-b)(p-c)}$. Squaring and equating we get $p^2r^2 = p(p-a)(p-b)(p-c)$. Setting $x = 1/(p-a)$, $y = 1/(p-b)$, $z = 1/(p-c)$ we can write this equation in the form

$$\frac{1}{r^2} = pxyz = xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

Thus we need only show that $yz+xz+xy \leqq x^2+y^2+z^2$. However this follows from the trivial inequalities $y^2+z^2 \geqq 2yz$, $x^2+z^2 \geqq 2xz$, $x^2+y^2 \geqq 2xy$.

A-3 (6) Multiplying the defining relation by $(n+1)$ we get

$$(1) \qquad (n+1)x_{n+1} = nx_n + x_n - (n+1)(x_n)^2 = nx_n + x_n[1-(n+1)x_n].$$

To prove that nx_n is increasing, we need to show that $1-(n+1)x_n \geqq 0$. From the graph of $x(1-x)$ we note that $x_2 \leqq \frac{1}{4}$ and that $x_n \leqq a \leqq \frac{1}{2}$ implies $x_{n+1} \leqq a(1-a)$. So by induction,

$$(n+1)x_n \leqq (n+1)\frac{1}{n}\left(1-\frac{1}{n}\right) = 1-\frac{1}{n^2} \leqq 1.$$

Furthermore, $nx_n < (n+1)x_n \leqq 1$ and so nx_n is bounded above by 1. Thus nx_n converges to a limit α with $0 < nx_n < \alpha \leqq 1$. Now summing (1) from 2 to n we obtain

$$(2) \qquad \begin{aligned} 1 &\geqq (n+1)x_{n+1} \\ &= 2x_2 + x_2(1-3x_2) + x_3(1-4x_3) + \cdots + x_n[1-(n+1)x_n]. \end{aligned}$$

If $\alpha \neq 1$ then $[1-(n+1)x_n] \geqq (1-\alpha)/2$ for all large n and thus (2) would show that $\sum x_n$ is convergent. However $nx_n \geqq x_1$ and so $\sum x_n \geqq x_1 \sum(1/n)$.

A-4 (160) To prove the formula by induction, it suffices to show that the difference $\Delta = n+\{\sqrt{n}\}-(n-1+\{\sqrt{n-1}\}) = 1$ or 2, with the value 2 occurring if and only if the number $n+\{\sqrt{n-1}\}$ is a perfect square. For convenience, let $\{\sqrt{n-1}\} = q$. Then of course $q-\frac{1}{2} < \sqrt{n-1} < q+\frac{1}{2}$ or better $q^2-q+\frac{1}{4} < n-1 < q^2+q+\frac{1}{4}$. This gives $q^2+5/4 < n+\{\sqrt{n-1}\} < (q+1)^2+\frac{1}{4}$. Therefore the number $n+\{\sqrt{n-1}\}$ is a perfect square if and only if $n=(q+1)^2-q$. However, then and only then $\sqrt{n} > q+\frac{1}{2} > \sqrt{n-1}$. In other words then and only then $\{\sqrt{n}\}-\{\sqrt{n-1}\} = 1$, because this difference is never greater than 1.

A-5 (17) We shall show that $T\psi(x) = f(x)\psi(x)$ for all $\psi \in C$, where f is the image under T of the function 1 which sends x into 1. For each x_0 and ψ, define ψ' by

$$\psi'(x) = \begin{cases} \psi(x) & \text{if } x \leqq x_0, \\ \psi(x_0) & \text{if } x > x_0. \end{cases}$$

Since T is local we must have $T\psi'(x) = T\psi(x)$ for all $x \leqq x_0$. On the other hand, for $x > x_0$, $T\psi'(x) = \psi(x_0)T1(x) = \psi(x_0)\cdot f(x)$. By continuity of $T\psi'$, $T\psi(x_0) = \psi(x_0)\cdot f(x_0)$.

Comment: The condition (1) is needed only in the case where $c_2 = 0$. Also the interval I in condition (2) should be required to have positive measures since the problem is trivial if I can be taken as a point.

A-6 (1) We understand the statement to mean that

$$3 = \lim_{n\to\infty} \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots \sqrt{1 + (n-1)\sqrt{1+n}}}}}.$$

We see that

$$3 = \sqrt{1 + 2\cdot 4} = \sqrt{1 + 2\sqrt{16}} = \sqrt{1 + 2\sqrt{1 + 3\sqrt{25}}}$$
$$= \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{36}}}}.$$

This leads us to conjecture the relation

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots + \sqrt{1 + n\sqrt{(n+2)^2}}}}} \quad \text{for all } n \geqq 1.$$

Proceeding by induction we verify that $(n+2)^2 = n^2+4n+4 = 1+(n+1)(n+3)$ $= 1+(n+1)\sqrt{(n+3)^2}$. This given, we must have

$$3 \geqq \sqrt{1 + 2\sqrt{1 + 3\sqrt{\cdots \sqrt{1 + (n-1)\sqrt{(1+n)}}}}}.$$

To set an inequality in the other direction, observe that for any $\alpha > 1$

$$\sqrt{1 + n\alpha} \leqq \sqrt{\alpha}\sqrt{1+n}.$$

A repetition of this inequality gives then

$$3 \leqq (n+2)^a \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots \sqrt{1 + (n-1)\sqrt{1+n}}}}}, \quad \text{where} \quad a = 2^{1-n}.$$

SOLUTIONS. PART B

B-1 (278) Let $P_1, P_2, \cdots, P_n$ be the vertices of P. Let $P_1', P_2', \cdots, P_n'$ be the projections of $P_1, P_2, \cdots, P_n$ upon one of the sides of the squares, and let $P_1'', P_2'', \cdots, P_n''$ be the projections of $P_1, P_2, \cdots, P_n$ upon a side that is orthogonal to the previous one. Since P is convex, the first side will be covered at most twice by the segments $\overline{P_1'P_2'}, \cdots, \overline{P_{n-1}'P_n'}, \overline{P_n'P_1'}$. We thus deduce the inequality $\overline{P_1'P_2'}^2 + \cdots + \overline{P_n'P_1'}^2 \leqq 2$. Similarly $\overline{P_1''P_2''}^2 + \cdots + \overline{P_n''P_1''}^2 \leqq 2$. Adding these two inequalities and using the Pythagorean theorem the assertion follows.

B-2 (343) Any common factor of two of such numbers would have to be divisible by 2, 3, 5 or 7. So it is sufficient to prove that among any ten consecutive integers there is at least one that is not divisible by 2, 3, 5 or 7. We get such an integer by elimination as follows. Strike out those divisible by 3. There may be either 3 or 4 of them. Among these there is either at least one or two respectively that are divisible also by 2. Thus if we strike off also those that are divisible by 2 we will have eliminated at most seven of the integers. Note that by so doing we have stricken off at least one number divisible by five. Thus we are left with three integers only two of which can be divisible by 5 or 7.

B-3 (0) Set $q_n = p_1 + p_2 + \cdots + p_n (q_0 = 0)$. We are led to estimate $S_N = \sum_{n=1}^{N} (n/q_n)^2 (q_n - q_{n-1})$ in terms of $T = \sum_{n=1}^{\infty} 1/p_n$. Note that

$$S_N \leqq \frac{1}{p_1} + \sum_{n=2}^{N} \frac{n^2}{q_n q_{n-1}} (q_n - q_{n-1}) = \frac{1}{p_1} + \sum_{n=2}^{N} \frac{n^2}{q_{n-1}} - \sum_{n=2}^{N} \frac{n^2}{q_n}$$

$$= \frac{1}{p_1} + \sum_{n=1}^{N-1} \frac{(n+1)^2}{q_n} - \sum_{n=2}^{N} \frac{n^2}{q_n} \leqq \frac{5}{p_1} + 2\sum_{n=2}^{N} \frac{n}{q_n} + \sum_{n=2}^{N} \frac{1}{q_n}.$$

By Schwarz's inequality,

$$\left(\sum_{n=2}^{N} \frac{n}{q_n}\right)^2 \leqq \sum_{n=2}^{N} \frac{n^2}{q_n^2} p_n \sum_{n=1}^{\infty} \frac{1}{p_n}$$

and thus

$$S_N \leqq \frac{5}{p_1} + 2\sqrt{S_N T} + T.$$

This quadratic inequality implies that $\sqrt{S_N} \leqq \sqrt{T} + \sqrt{2T + 5/p_1}$.

B-4 (15) Let, for each $1 \leqq i \leqq mn+1$, n_i denote the length of the longest chain, starting with a_i and each dividing the following one, that we can select out of $a_i, a_{i+1}, \cdots, a_{mn+1}$. If no n_i is greater than n then there are at least $m+1$ n_i's that are the same. However, the integers a_i corresponding to these n_i's cannot divide each other, because $a_i \mid a_j$ implies that $n_i \geqq n_j + 1$.

B-5 (144) Let these points be denoted by $P_1, P_2, \cdots, P_n$. To every permutation $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ of $(1, 2, 3, \cdots, n)$ we associate a closed polygon, namely $P_{\sigma_1}P_{\sigma_2} \cdots P_{\sigma_n}P_{\sigma_1}$. This way we obtain $(n-1)!$ distinct closed polygons some of which may have selfintersections. We claim that anyone of these polygons whose length is the shortest possible is simple. By the hypothesis that no three P_i's are on the same line, a selfintersection occurs if and only if two segments say $\overline{P_{\sigma_1}P_{\sigma_2}}$ and $\overline{P_{\sigma_m}P_{\sigma_{m+1}}}$ cross each other. However, then the closed polygon $P_{\sigma_2} \cdots P_{\sigma_{m-1}}P_{\sigma_m}P_{\sigma_1}P_{\sigma_n}P_{\sigma_{n-1}} \cdots P_{\sigma_{m+1}}P_{\sigma_2}$ would have shorter length. Thus there can't be a cross if the length of $P_{\sigma_1} \cdots P_{\sigma_n}P_{\sigma_1}$ is shortest possible.

Alternate solution: Take two points P_1 and P_2 such that all the other points $P_3, P_4, \cdots, P_n$ are on the same side of the line connecting P_1 and P_2. Each point P_i, $i>2$, determines an angle θ_i between P_1P_2 and P_1P_i, with $0<\theta_i<\pi$. By hypothesis, $\theta_i \neq \theta_j$ if $i \neq j$. Let $(i_3, i_4, \cdots, i_n)$ be the permutation of $(3, 4, \cdots, n)$ such that $\theta_{i_3} < \theta_{i_4} < \cdots < \theta_{i_n}$. Then $P_1P_2P_{i_3}P_{i_4} \cdots P_{i_n}P_1$ is a closed simple polygon.

B-6 (0) Multiplying the equation by $e^{-x}y'$ and integrating from 0 to T we get, after rearranging terms,

$$y^2(T) + 2\int_0^T e^{-x}y'y''\,dx = y^2(0).$$

Note then that for some $0 \leqq \zeta \leqq T$ we must have

$$2\int_0^T e^{-x}y'y''\,dx = 2\int_0^\zeta y'y''\,dx = \{y'(\zeta)\}^2 - \{y'(0)\}^2.$$

We then obtain $y^2(T)+\{y'(\zeta)\}^2=y^2(0)+\{y'(0)\}^2$.

Acknowledgements

The Director would like to acknowledge the assistance of the Questions Committee in preparing the above solutions and acknowledge the services of the following persons, who were graders for the competition: D. C. Benson, R. L. Bishop, D. Donohue, R. M. Dudley, D. J. Ewy, S. I. Goldberg, A. W. Hales, G. A. Heuer, V. E. Hoggatt, Jr., J. B. Kelly, J. A. Larrivee, A. E. Livingston, E. A. Maier, D. G. Mead, M. H. Protter, B. J. Stone, J. J. Stone, J. W. Woll, Jr., P. B. Yale, and A. Yaqub.

METRIC POSTULATES FOR SPACE GEOMETRY

ROLAND BROSSARD, Université de Montréal

1. Introduction. Metric postulates for plane geometry have been given by Saunders MacLane in [4]. We shall show that it is also possible to define high school space geometry in a very similar way, leaving open the possibility of extension to higher dimensional spaces. More precisely, this is an attempt to find a simple and intuitive set of facts relating distances and angles which will suffice to characterize space geometry. The axioms are formulated in such a way as to exhibit a certain symmetry between the distance (measure of segments) and the measure of angles. They consist of four existence or incidence axioms, five axioms on distance, five axioms on angles, the continuity axiom, and the similarity axiom. We shall prove that the axioms in [5] are theorems or axioms in the axiomatic structure given here, and consequently the proof given in [5] for categoricity establishes the categoricity of the system of axioms given here. The development outlined below is oriented toward this proof of categoricity, and would have indeed no interest for high school teaching.

2. Primitive notions. *Points* are abstract undefined objects. Primitive terms are: point, distance, line, ray or half-line, half-bundle of rays, and angular measure. The set of all points will be denoted by the letter S and some subsets of S are called *lines*. The plane as well as the three-dimensional space shall not be taken as primitive terms but will be constructed.

3. Axioms on points, lines, and distance. The axioms on the points and on the lines are:

E_1. *There exist at least two points in S.*
E_2. *A line contains at least two points.*
E_3. *Through two distinct points there is one and only one line.*
E_4. *There exist points not all on the same line.*

A set of points is said to be *collinear* if this set is a subset of a line. Two sets are *collinear* if the union of these sets is collinear.

The axioms on distance are:

D_1. *If A and B are points, then $d(AB)$ is a nonnegative real number.*

D_2. *For points A and B, $d(AB)=0$ if and only if $A=B$.*

D_3. *If A and B are points, then $d(AB)=d(BA)$.*

For distinct points A,B,C, B is *between* A and C if and only if $d(AB)+d(BC)=d(AC)$. If B is between A and C we write $[A, B, C]$. If $A\neq B$, the class of points P between A and B is the *interval* AB. If $A\neq O$, the class of points $P\neq O$ such that $d(AP)=|d(OA)-d(OP)|$ is the *ray* OA (or *half-line* OA). O is the *end-point* of the ray OA. We have furthermore the two following axioms concerning the distance.

D_4. *Three distinct points belong to a line if and only if one of them is between the other two.*

D_5. *On each ray OA and for each positive real number β there exists a point B such that $d(OB)=\beta$.*

We have then in [4] the proof of the following theorem.

THEOREM 1. (a) *If $[O, A, B]$ then interval OA is contained in the interval OB,*

(b) *on each ray OA and to each positive real number β there exists exactly one point B such that $d(OB)=\beta$,*

(c) *if $A\neq O$ and $B\in$ ray OA, then ray $OA=$ ray OB,*

(d) *two rays with the same end-point are either equal or disjoint,*

(e) *if O is a point of a line l, then l contains exactly two disjoint rays with end-point O and each point $P\neq O$ of l belongs to exactly one of these rays,*

(f) *on a line l, for every distinct points O and A there exists a one-to-one mapping x of l onto the set of real numbers such that $x(O)=0$, $x(A)>0$, and for all points P, Q of l $d(PQ)=|x(P)-x(Q)|$,*

(g) *a point Q lies between P and R if and only if either $x(P)<x(Q)<x(R)$ or $x(R)<x(Q)<x(P)$.*

If O is a point of a line l, we know that there exist exactly two distinct rays with end-point O; any one is *opposite* to the other. If r is a ray, the opposite ray is denoted $\bar{r}$. Parts (f) and (g) of Theorem 1 imply that the betweenness relation has the usual properties. We shall group under Theorem 2 the properties of the betweenness relation that will be used later.

THEOREM 2. (a) $[A, B, C]$ *and* $[B, C, D]$ *imply* $[A, B, D]$,

(b) $[A, B, C]$ *and* $[B, D, C]$ *imply* $[A, B, D]$,

(c) $[A, B, C]$ *implies* $[C, B, A]$,

(d) $[A, B, C]$ *implies* $\sim[B, C, A]$ (*not* $[B, C, A]$),

(e) $[A, X, B]$ *and* $[A, Y, B]$ *imply* $[A, X, Y]$ *or* $[A, Y, X]$,

(f) *if $A\neq B$, then there exists a point C such that* $[A, B, C]$.

4. The axioms on angles, half-bundles, and the continuity axiom. An *angle* is an ordered couple of rays with the same end-point. The rays are the *sides* of the angle and the common end-point is the *vertex* of the angle. An angle is *flat* in case the sides are distinct and belong to a line. We shall introduce the measure $\angle rs$ of an angle rs with the axioms A_1, A_2, A_3. The measure of an angle is intuitively conceived as the number obtained with the use of a protractor. If $A \neq O$ and $B \neq O$, we write $\angle AOB$ for $\angle(\text{ray } OA)(\text{ray } OB)$.

A_1. *If rs is an angle, then $\angle rs$ is a real number such that $0 \leqq \angle rs \leqq 180$.*

A_2. *For an angle rs, $\angle rs = 0$ if and only if $r = s$.*

A_3. *If rs is an angle, then $\angle rs = \angle sr$.*

A *half-bundle* ab, where a and b are rays with the same end-point not belonging to a line, is the class of rays x such that $x \neq a$ and $\angle bx = |\angle ab - \angle ax|$. The common end-point of the rays is the *vertex* of the half-bundle. For the half-bundle ab we use the notation HB_{ab}. A ray b is *between* the rays a and c if and only if a, b, c have a common end-point, no two of them belong to a line, and $\angle ab + \angle bc = \angle ac$; if b is between a and c we write $[a, b, c]$. With those notations we have that $x \in HB_{ab}$ if and only if $[a, b, x]$ or $[a, x, b]$ or $x = b$.

The axioms on the half-bundles are:

A_4. *Three distinct rays a, b, c belong to a half-bundle if and only if one of them is between the other two.*

A_5. *If β is a real number between 0 and 180, then there exists on each half-bundle HB_{oa} one ray b such that $\angle ob = \beta$.*

We shall now add the so-called continuity axiom.

CONTINUITY AXIOM. *If $[a, b, c]$ and if A, C are points on rays a and c respectively, then there exists a point B on ray b such that $[A, B, C]$. Conversely, if a and c are noncollinear rays with the same end-point O, $A \in a$, $C \in c$, and if B is such that $[A, B, C]$ then the ray $OB = b$ is between the rays a and c (i.e. $[a, b, c]$).*

We have used the word "betweenness" for the ternary relation $[a, b, c]$; we shall now justify this terminology and prove that this relation defined on the elements of a half-bundle has the properties of the formal betweenness relation as defined for instance in [3]. Those properties coincide also with those of the betweenness relation for the points of a line as mentioned in section 3.

THEOREM 3. *The betweenness relation for the elements of a half-bundle has the characteristic properties of the formal betweenness relation, that is to say,*

(a) *$[a, b, c]$ implies that a, b, c are distinct and belong to a half-bundle,*

(b) *if a, b, c are distinct elements of a half-bundle, then $[a, b, c]$ or $[b, c, a]$ or $[c, a, b]$,*

(c) *$[a, b, c]$ implies $\sim[b, c, a]$,*

(d) *if $[a, b, c]$ then $[c, b, a]$,*

(e) $[a, b, c]$ *and* $[b, d, c]$ *imply* $[a, b, d]$,
(f) *if* $[a, b, c]$ *and* $[b, c, d]$ *then* $[a, b, d]$.

Proof. (a) and (b) are immediate consequences of axiom A_4, and the definition of the betweenness relation for the rays. To prove (c) and (d) we consider the rays a, b, c such that $[a, b, c]$. The continuity axiom implies the existence of points A, B, C on rays a, b, c respectively such that $[A, B, C]$. If $[a, b, c]$ and $[b, c, a]$, then $[A, B, C]$ and there exists a point $C' \in c$ such that $[B, C', A]$, but as A, B, C, C' are collinear points, $C = C'$ and we have a contradiction; consequently $[a, b, c]$ implies $\sim[b, c, a]$. Moreover $[A, B, C]$ implies $[C, B, A]$ and the continuity axiom implies $[c, b', a]$, but as B is on b and on b', then $b = b'$ and consequently $[a, b, c]$ implies $[c, b, a]$.

Let $[a, b, c]$ and $[b, d, c]$. $[a, b, c]$ and the continuity axiom imply the existence of points A, B, C on rays a, b, c respectively such that $[A, B, C]$, furthermore $[b, d, c]$ and the continuity axiom imply the existence of a point D on d such that $[B, D, C]$. But $[A, B, C]$ and $[B, D, C]$ imply $[A, B, D]$ and the continuity axiom finally gives $[a, b, d]$. This proves part (e).

We shall now give an indirect proof of the last part of the theorem. Let $[a, b, c]$, $[b, c, d]$, and $\sim[a, b, d]$. a, b, d being distinct elements of a half-bundle, part (b) implies that $\sim[a, b, d]$ is equivalent to $[b, d, a]$ or $[d, a, b]$.

If we have $[a, b, c]$, $[b, c, d]$, and $[b, d, a]$ we observe that parts (d) and (e) applied to $[a, b, c]$ and $[b, d, a]$ give $[c, b, d]$ which contradict $[b, c, d]$.

If we have $[a, b, c]$, $[b, c, d]$, and $[d, a, b]$, we observe first that $[b, c, d]$ implies the existence of points B, C, D on rays b, c, d respectively such that $[B, C, D]$. Moreover $[d, a, b]$ implies the existence of a point $A \in a$ such that $[D, A, B]$. But $[B, A, D]$ and $[B, C, D]$ imply $[B, C, A]$ or $[B, A, C]$ and consequently we have $[b, c, a]$ or $[b, a, c]$ which in either case contradict $[a, b, c]$. This completes the proof of Theorem 3.

We shall use from now on the properties of the betweenness relation for the elements of a half-bundle without further reference. In particular we shall use the following ones:

$[a, b, c]$ and $[b, d, c]$ imply $[a, d, c]$;
$[a, b, c]$ and $[a, b, d]$ imply $[a, d, c]$ or $[a, c, d]$, and $[b, c, d]$ or $[b, d, c]$;
$[a, b, c]$ and $[a, d, c]$ imply $[a, b, d]$ or $[a, d, b]$, and $[b, d, c]$ or $[d, b, c]$.

6. Properties of the half-bundles. If HB_{oa} is a half-bundle we shall call *opposite half-bundle* the class of rays x such that the opposite ray $\bar{x}$ is in HB_{oa}. The opposite half-bundle of HB_{oa} is denoted $\overline{HB}_{oa}$. We shall prove in Lemma 4 below that the opposite half-bundle of a half-bundle is itself a half-bundle.

LEMMA 1. *If b is an element of HB_{oa}, then $HB_{ob} = HB_{oa}$.*

Proof. If $b = a$ the lemma is obvious. If $b \neq a$, then $b \in HB_{oa}$ if and only if $[o, a, b]$ or $[o, b, a]$. But as "$[o, a, b]$ or $[o, b, a]$" is a symmetric sentence in a and b, $b \in HB_{oa}$ if and only if $a \in HB_{ob}$, and consequently it is sufficient to prove that $HB_{ob} \subset HB_{oa}$. Let $x \in HB_{ob}$ that is to say let x such that $[o, b, x]$ or $[o, x, b]$ or $x = b$.

Case I: $[o, a, b]$. $[o, a, b]$ and $[o, b, x]$ imply $[o, a, x]$ and $x \in HB_{oa}$. If we have $[o, a, b]$ and $[o, x, b]$ then $[o, a, x]$ or $[o, x, a]$ and again $x \in HB_{oa}$.

Case II: $[o, b, a]$. If $[o, b, a]$ and $[o, b, x]$, then $[o, a, x]$ or $[o, x, a]$and $x \in HB_{oa}$. Furthermore $[o, b, a]$ and $[a, x, b]$ imply $[o, x, a]$ and again $x \in HB_{oa}$.

Finally if $x = b$, then in both cases $x \in HB_{oa}$, and the proof is complete.

LEMMA 2. *On each half-bundle HB_{oa} and to each real number β between* 0 *and* 180 *there is only one ray b such that $\angle ob = \beta$.*

Proof. If b and b' are rays of HB_{oa} such that $\angle ob = \angle ob' = \beta$ then Lemma 1 implies that $HB_{oa} = HB_{ob}$ and consequently we have $[o, b', b]$ or $[o, b, b']$ or $b = b'$. But as $[o, b', b]$ or $[o, b, b']$ imply that $\angle ob' + \angle b'b = \angle ob$ or $\angle ob + \angle bb' = \angle ob'$ respectively, $\angle bb' = 0$, and $b = b'$.

LEMMA 3. *If a, b, c are distinct rays with the same end-point O, then*
(a) $[a, b, c]$ *is equivalent to* $[b, c, \bar{a}]$,
(b) $[a, b, \bar{c}]$ *if and only if* $[a, c, \bar{b}]$,
(c) $[a, \bar{b}, c]$ *is equivalent to* $[a, \bar{c}, b]$,
(d) $[a, \bar{b}, \bar{c}]$ *if and only if* $[\bar{a}, b, c]$.

Proof. $[a, b, c]$ and the continuity axiom imply the existence of points A, B, C on rays a, b, c respectively such that $[A, B, C]$. Let A' be on ray $\bar{a}$ (Fig. 1).

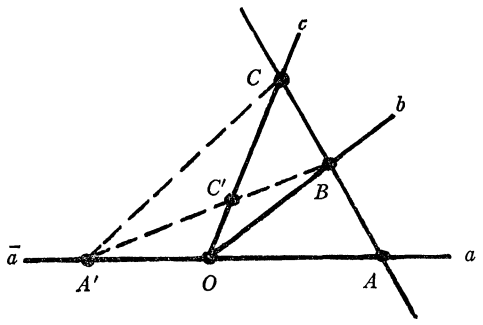


FIG. 1

Then we have $[A', O, A]$ (where O denotes the common end-point of the rays a, b, c) and the continuity axiom applied twice to the rays $A'A$, $A'B$, $A'C$ with end-point A' implies the existence of a point $C' \in c$ such that $[O, C', C]$. Furthermore, $C \notin$ line AA', $[A', O, A]$ imply $[CA', CO, CA]$, and consequently we have $[A', C', B]$. But $[A', C', B]$ implies $[b, c, \bar{a}]$ and the proof that $[a, b, c]$ implies $[b, c, \bar{a}]$ is complete. Five successive applications of this implication to $[b, c, \bar{a}]$ give $[a, b, c]$ and part (a) is proved.

Parts (b), (c), (d) result from part (a) and Theorem 3 part (d).

LEMMA 4. (a) $HB_{\bar{o}a} = HB_{oa}$ *and* $\bar{o} \notin HB_{oa}$,
(b) $HB_{o\bar{a}} = \overline{HB}_{oa}$,
(c) *the opposite half-bundle of a half-bundle is a half-bundle.*

Proof. (a) $x \in HB_{oa}$ if and only if $[o, x, a]$ or $[o, a, x]$ or $x = a$ and $x \in HB_{\bar{o}a}$ if and only if $[\bar{o}, x, a]$ or $[\bar{o}, a, x]$ or $x = a$, but as Lemma 3 implies that $[\bar{o}, x, a]$ and $[o, a, x]$ are equivalent as well as $[\bar{o}, a, x]$ and $[o, x, a]$ are equivalent $HB_{\bar{o}a} = HB_{oa}$. As $\bar{o} \notin HB_{\bar{o}a}$, then $\bar{o} \notin HB_{oa}$.

(b) $x \in HB_{o\bar{a}}$ if and only if $[o, x, \bar{a}]$ or $[o, \bar{a}, x]$ or $x = \bar{a}$, and $x \in \overline{HB}_{oa}$ if and only if $[o, \bar{x}, a]$ or $[o, a, \bar{x}]$ or $\bar{x} = a$. But Lemma 3 implies that $[o, \bar{x}, a]$ and $[o, \bar{a}, x]$ are equivalent; Lemma 3 implies also that $[o, a, \bar{x}]$ and $[o, x, \bar{a}]$ are equivalent. As we know that $\bar{x} = a$ if and only if $x = \bar{a}$, the proof of part (b) is complete.

(c) Part (c) is an immediate consequence of part (b).

We shall call o and $\bar{o}$ the *end-rays* of the half-bundle HB_{oa}.

THEOREM 4. *If x and o are two noncollinear rays with the same end-point, then* $\angle ox + \angle x\bar{o} = 180$.

Proof. Let y be an element of the half-bundle HB_{ox} distinct from x. We have either $[o, x\ y]$ or $[o, y, x]$ and we shall consider first the case $[o, x, y]$. $[o, x, y]$ implies $[x, y, \bar{o}]$ (Lemma 3); thus $\angle ox + \angle xy = \angle oy$ and $\angle xy + \angle y\bar{o} = \angle x\bar{o}$. Consequently $\angle ox + \angle x\bar{o} = \angle oy + \angle y\bar{o}$ and as $[o, y, x]$ leads to the same result we have that in the half-bundle HB_{ox} the sum of the measures of "supplementary" angles with sides o, $\bar{o}$ is a constant $\alpha = \angle ox + \angle x\bar{o}$. We shall now prove that $\alpha = 180$.

Let $\angle ox + \angle x\bar{o} = \alpha < 180$ and let z be an element of HB_{ox} such that $\angle oz = \alpha$ (axiom A_5). Then $\angle oz + \angle z\bar{o} = \alpha$, $\angle z\bar{o} = 0$ and $z \notin HB_{ox}$ which is a contradiction.

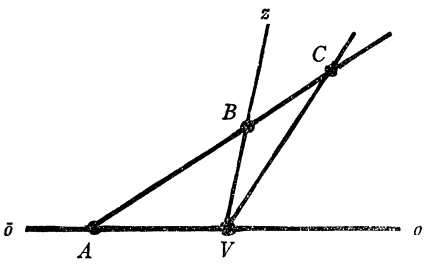


FIG. 2

Let now $\angle ox + \angle x\bar{o} = \alpha = 180 + \beta$ where $\beta > 0$, and let z be an element of HB_{ox} such that $\angle oz = \beta$. Then $\angle oz + \angle z\bar{o} = 180 + \beta$ so that $\angle z\bar{o} = 180$. If A and B are points of rays $\bar{o}$ and z respectively then there exists a point C such that $[A, B, C]$; consequently $[VA, VB, VC]$, where V is the vertex of the half-bundle HB_{ox}, and $\angle \bar{o}z + \angle z$ (ray VC) $= \angle \bar{o}$ (ray VC). But as $\angle \bar{o}z = 180$, $\angle \bar{o}$ (ray VC) > 180 which is a contradiction. This completes the proof of Theorem 4.

7. Bundles of rays. A *bundle* ab, denoted B_{ab}, is the class of rays $HB_{ab} \cup \overline{HB}_{ab} \cup \{a\} \cup \{\bar{a}\}$. We have immediately that $x \in B_{ab}$ if and only if

$$[a, b, x] \text{ or } [a, x, b] \text{ or } [a, b, \bar{x}] \text{ or } [a, \bar{x}, b] \text{ or } x = a \text{ or } \bar{x} = a \text{ or } x = b \text{ or } \bar{x} = b. \tag{1}$$

We shall use the notation $P(a, \bar{a}, b, \bar{b}, x, \bar{x})$ for this last sentence. In the following theorem we shall prove that a bundle is uniquely defined by two noncollinear rays. To simplify the proof we shall first establish four symmetries of the sentence $P(a, \bar{a}, b, \bar{b}, x, \bar{x})$.

LEMMA 5. *The sentence* $P(a, \bar{a}, b, \bar{b}, x, \bar{x})$ *is*: (a) *symmetric in the letters* $a, \bar{a}$, (b) *symmetric in the letters* $b, \bar{b}$, (c) *symmetric in the letters* $x, \bar{x}$, *and* (d) *symmetric in the letters* a, b.

Proof. It is sufficient to prove the symmetries only for the part "$[a, b, x]$ or $[a, x, b]$ or $[a, b, \bar{x}]$ or $[a, \bar{x}, b]$." The symmetry in $x, \bar{x}$ is obvious and, since $[\bar{a}, b, x]$, $[\bar{a}, x, b]$, $[\bar{a}, b, \bar{x}]$, $[\bar{a}, \bar{x}, b]$ are respectively equivalent to $[a, x, b]$, $[a, b, x]$, $[a, \bar{x}, b]$, and $[a, b, \bar{x}]$, we have the symmetry in the letters $a, \bar{a}$.

Furthermore, as $[a, \bar{b}, x]$, $[a, x, \bar{b}]$, $[a, \bar{b}, \bar{x}]$, $[a, \bar{x}, \bar{b}]$ are respectively equivalent to $[a, \bar{x}, b]$, $[a, b, \bar{x}]$, $[a, x, b]$, and $[a, b, x]$, we have the symmetry in the letters $b, \bar{b}$. Finally, as $[b, a, x]$, $[b, x, a]$, $[b, a, \bar{x}]$, $[b, \bar{x}, a]$ are respectively equivalent to $[a, b, \bar{x}]$, $[a, x, b]$, $[a, b, x]$, and $[a, \bar{x}, b]$, we have the symmetry in the letters a, b. All the above equivalences are readily obtained with Lemma 3 and Theorem 3 part (d).

THEOREM 5. *If* x *and* y *are noncollinear elements of a bundle* B_{ab}, *then* $B_{ab} = B_{xy}$.

Proof. With the given hypothesis on the rays a, b, x, y, we have to prove that

$$P(a, \bar{a}, b, \bar{b}, x, \bar{x}) \,\&\, P(a, \bar{a}, b, \bar{b}, y, \bar{y}) \,\&\, P(a, \bar{a}, b, \bar{b}, z, \bar{z}) \Rightarrow P(x, \bar{x}, y, \bar{y}, z, \bar{z})$$

and

$$P(a, \bar{a}, b, \bar{b}, x, \bar{x}) \,\&\, P(a, \bar{a}, b, \bar{b}, y, \bar{y}) \,\&\, P(x, \bar{x}, y, \bar{y}, z, \bar{z}) \Rightarrow P(a, \bar{a}, b, \bar{b}, z, \bar{z}).$$

Using the symmetry of $P(a, \bar{a}, b, \bar{b}, x, \bar{x})$ in the letters $x, \bar{x}$, in the letters $b, \bar{b}$ and in the letters a, b, we can easily show that $P(a, \bar{a}, b, \bar{b}, x, \bar{x})$ reduces to

$$x = a \quad \text{or} \quad [a, b, x]. \tag{2}$$

Using the symmetry in the letters $a, \bar{a}$, and the symmetry in the letters $y, \bar{y}$, we can also show that $P(a, \bar{a}, b, \bar{b}, y, \bar{y})$ reduces to

$$y = a \quad \text{or} \quad y = b \quad \text{or} \quad [a, y, b]. \tag{3}$$

The symmetry in the letters $z, \bar{z}$, implies that the sentences $P(a, \bar{a}, b, \bar{b}, z, \bar{z})$ and $P(x, \bar{x}, y, \bar{y}, z, \bar{z})$ reduce respectively to

$$z = a \quad \text{or} \quad z = b \quad \text{or} \quad [a, b, z] \quad \text{or} \quad [a, z, b], \tag{4}$$

and

(5) $$z = x \quad \text{or} \quad z = y \quad \text{or} \quad [x, y, z] \quad \text{or} \quad [x, z, y].$$

We have then to prove that (2), (3), (4) imply $P(x, \bar{x}, y, \bar{y}, z, \bar{z})$ and conversely that (2), (3), (5) imply $P(a, \bar{a}, b, \bar{b}, z, \bar{z})$. We shall now prove the first of these implications, that is to say, $B_{ab} \subset B_{xy}$.

If z is collinear with a or b then we have the following possible cases ($x = y = a$ should be excluded):

$$x = a \quad \text{and} \quad y = b \quad \text{and,} \quad z = a \quad \text{or} \quad z = b,$$
$$x = a \quad \text{and} \quad [a, y, b] \quad \text{and,} \quad z = a \quad \text{or} \quad z = b,$$
$$[a, b, x] \quad \text{and} \quad y = a \quad \text{and,} \quad z = a \quad \text{or} \quad z = b.$$
$$[a, b, x] \quad \text{and} \quad y = b \quad \text{and,} \quad z = a \quad \text{or} \quad z = b,$$
$$[a, b, x] \quad \text{and} \quad [a, y, b] \quad \text{and,} \quad z = a \quad \text{or} \quad z = b.$$

In the four first cases, we can easily see that $z \in B_{xy}$. In the last case we have $[x, y, z]$ or $[x, z, y]$ and consequently $z \in B_{xy}$.

If z is not collinear with a or b then we have the five following cases ($x = y = a$ should also be excluded):

(6) $$x = a \quad \text{and} \quad y = b \quad \text{and,} \quad [a, b, z] \quad \text{or} \quad [a, z, b],$$

(7) $$x = a \quad \text{and} \quad [a, y, b] \quad \text{and,} \quad [a, b, z] \quad \text{or} \quad [a, z, b],$$

(8) $$[a, b, x] \quad \text{and} \quad y = a \quad \text{and,} \quad [a, b, z] \quad \text{or} \quad [a, z, b]$$

(9) $$[a, b, x] \quad \text{and} \quad y = b \quad \text{and,} \quad [a, b, z] \quad \text{or} \quad [a, z, b],$$

(10) $$[a, b, x] \quad \text{and} \quad [a, y, b] \quad \text{and,} \quad [a, b, z] \quad \text{or} \quad [a, z, b].$$

If $x = a$, $y = b$, and $[a, b, z]$ or $[a, z, b]$, then $[x, y, z]$ or $[x, z, y]$ and $z \in B_{xy}$. If $x = a$, $[a, y, b]$, and $[a, b, z]$ or $[a, z, b]$, then $[x, y, z]$ or $[x, z, y]$ and in each case $z \in B_{xy}$. If $y = a$, $[a, b, x]$, and $[a, b, z]$ or $[a, z, b]$, then we have $[x, y, \bar{z}]$ or $[x, z, y]$ and $z \in B_{xy}$. In case (9), $y = b$, $[a, b, x]$, and $[a, b, z]$ or $[a, z, b]$ imply $[x, y, \bar{z}]$ or $[x, z, y]$ or $[x, y, z]$ and consequently $z \in B_{xy}$.

Finally in case (10): $[a, b, x]$ and $[a, y, b]$ imply $[y, b, x]$; moreover $[a, y, b]$ and $[a, b, z]$ imply $[y, b, z]$; $[y, b, z]$ and $[y, b, x]$ imply $[x, y, \bar{z}]$ or $[x, z, y]$ and $z \in B_{xy}$. Furthermore $[a, b, x]$ and $[a, z, b]$ imply $[z, b, x]$. Then $[y, b, x]$ and $[z, b, x]$ imply $[x, y, z]$ or $[x, z, y]$ and again $z \in B_{xy}$. This completes the proof of $B_{ab} \subset B_{xy}$.

In a very similar way we prove that $B_{xy} \subset B_{ab}$.

8. Triangles, parallel lines, and the concept of plane. A *triangle* is an unordered set of three distinct points; the points are the *vertices* of the triangle. The three closed intervals or segments defined by the vertices of a triangle are the *sides* of the triangle. We shall denote by AB (or BA) the distance between the points A and B of the side AB. The three angles defined by the sides of a triangle are the *angles* of the triangle. In the context of triangles, for instance in a triangle ABC, the measure of an angle, say angle ABC with vertex B, will be

denoted $\angle B$ instead of $\angle ABC$. Two triangles are *similar* if the vertices can be labelled A, B, C and A', B', C' in such a way that

$$AB/A'B' = BC/B'C' = CA/C'A',$$

and $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$. A triangle is *proper* if the vertices are noncollinear.

SIMILARITY AXIOM. *If two triangles ABC and $A'B'C'$ are such that $AB/A'B' = BC/B'C'$ and $\angle B = \angle B'$, then they are similar.*

THEOREM 6. *The sum of the measures of the angles of a proper triangle is equal to* 180.

The proof of this theorem can be found in [2]. The following four lemmas correspond to Lemmas 1, 2, 4, and 5 in [2] but the proofs are somewhat different. From now on it will be convenient to use also for the bundles the notation B_0 where the right subscript denotes the vertex of the bundle.

LEMMA 6. *Let l be an element of a bundle B_0 and let α, β be two numbers between* 0 *and* 180. *If m is a ray of B_0 such that $\angle lm = \alpha$, then there exists a ray n of B_0, more precisely of $\overline{HB}_{lm}$, with the following properties*: (a) $\angle ln = \beta$, (b) *for all points $M \in m$ and for all points $N \in n$ the interval MN has a point P in common with the unique line containing the ray l.*

Proof. Let l, m be two rays of B_0 such that $\angle lm = \alpha$. Let n be an element of $\overline{HB}_{lm}$ such that $\angle ln = \beta$. But $n \in \overline{HB}_{lm}$ if and only if $[l, m, \bar{n}]$ or $[l, \bar{n}, m]$ or $\bar{n} = m$ that is to say if and only if $[m, l, n]$ or $[n, \bar{l}, m]$ or $\bar{n} = m$. If $M \in m$ and if $N \in n$, then the continuity axiom implies the existence of a point P such that $P \in l$ or $P \in \bar{l}$ if $[m, l, n]$ or $[n, \bar{l}, m]$; if $\bar{n} = m$ then P is the vertex of the bundle.

Two distinct lines having a point in common determine four angles for which the sides are noncollinear; if those angles have all the same measure then the lines are *perpendicular*. It is easy to show that this common measure is 90.

LEMMA 7. *If l is a line and if P is a point not on the line l, then there exists one and only one line containing P and perpendicular to l.*

Proof. Let A be a point of l, and let h be one of the rays defined by A on l. Furthermore let HB_{ha} (where a denotes the ray AP) be the half-bundle defined by h and a (Fig. 3). If b is a ray of $\overline{HB}_{ha}$ such that $\angle bh = \angle ha$, and if P' is the point of ray b such that $AP' = AP$ then Lemma 6 implies that the interval PP' has a point I in common with the line l. If $I = A$ the points P, A, and P' belong to a line and this line is perpendicular to l. If $I \neq A$, the similarity axiom together with Theorem 4 imply that the triangles PAI and $P'AI$ are similar and $\angle AIP = \angle AIP' = 90$ and PI is a line perpendicular to l. If there exists another perpendicular PI' from P to l, then the triangle PII' is a proper triangle with two angles with measure 90 which is a contradiction.

The set consisting of the points belonging to the rays of a bundle B_o and the point O (vertex of B_o), denoted $\{B_o\}$, is called a *plane*. Any ray a of B_o deter-

mines exactly two half-bundles, one of them containing a ray b of B_o and the other one containing the ray $\bar{b}$. The set of points belonging to the rays of a half-bundle HB_{ab}, denoted $\{HB_{ab}\}$, is called a *half-plane.*

THEOREM 7. *If M and N are two distinct points of a plane $\{B_o\}$ then the line MN belongs to the plane $\{B_o\}$.*

Proof. The theorem is obvious if M, N, and the vertex O of B_o are collinear. If m, n are the noncollinear rays of B_o such that $M \in m$, $N \in n$, then X is a point of the line MN distinct from M and distinct from N if and only if $[X, M, N]$ or $[M, X, N]$ or $[M, N, X]$. Consequently the continuity axiom implies $[x, m, n]$ or $[m, x, n]$ or $[m, n, x]$ where x denotes the ray OX, and then, as $[x, m, n]$ is equivalent to $[m\ n, \bar{x}]$, x is an element of either HB_{mn} or $\overline{HB}_{mn}$. Finally Theorem 5 implies that $x \in B_o$ and $X \in \{B_o\}$.

LEMMA 8. *If two bundles B_o and $B_{o'}$ with vertices O and O' are such that $O \neq O'$, if there exists a point A such that A is not collinear with the points O, O', and if A, O, O' belong to $\{B_o\}$ and to $\{B_{o'}\}$, then the planes $\{B_o\}$ and $\{B_{o'}\}$ coincide.*

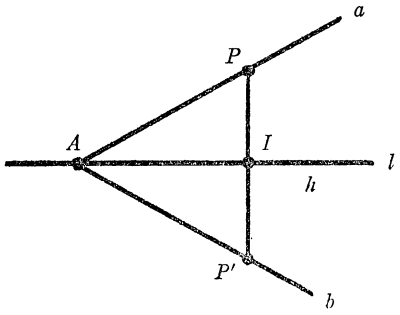


FIG. 3

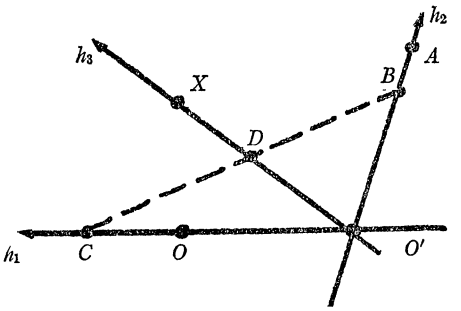


FIG. 4

Proof. Let h_1 and h_2 denote the rays $O'O$ and $O'A$ respectively (Fig. 4). If X is a point of the plane $\{B_{o'}\}$ we shall denote h_3 the ray $O'X$. With those notations, X belongs to the plane $\{B_{o'}\}$ if and only if $[h_1, h_2, h_3]$ or $[h_1, h_3, h_2]$ or $[h_1, h_2, \bar{h}_3]$ or $[h_1, \bar{h}_3, h_2]$ or $h_3 = h_1$ or $\bar{h}_3 = h_1$ or $h_3 = h_2$ or $\bar{h}_3 = h_2$. If h_3 or $\bar{h}_3$ coincides with h_1 then $X \in \{B_o\}$. If h_3 or $\bar{h}_3$ coincides with h_2 then as A and O' are in $\{B_o\}$, Theorem 7 implies that X is an element of $\{B_o\}$. In the cases $[h_1, h_3, h_2]$ or $[h_1, \bar{h}_3, h_2]$, if $B \in h_2$ and if $C \in h_1$ then the continuity axiom implies the existence of a point D on h_3 or on $\bar{h}_3$ such that $[B, D, C]$. In the cases $[h_1, h_2, h_3]$ or $[h_1, h_2, \bar{h}_3]$ then we have $[h_2, h_3, \bar{h}_1]$ or $[h_2, \bar{h}_3, \bar{h}_1]$ and if $B \in h_2$, $C \in \bar{h}_1$, then the continuity axiom implies the existence of a point D on h_3 or on $\bar{h}_3$ such that $[B, D, C]$. A and O' being in the plane $\{B_o\}$, B also belongs to the plane $\{B_o\}$. In the same way B and C being in $\{B_o\}$, D is in $\{B_o\}$. Finally D and O' being in $\{B_o\}$, X belongs to $\{B_o\}$ and $\{B_{o'}\} \subset \{B_o\}$. In the same way $\{B_o\} \subset \{B_{o'}\}$ and the proof is complete.

If two lines of a plane coincide or have no point in common, then they are *parallel.*

LEMMA 9. *If B_o is a bundle, if l is a line defined by a ray h of B_o, and if P is a point of the plane $\{B_o\}$ not on l, then there exists one and only one line in the plane $\{B_o\}$ containing P and parallel to l.*

Proof. Lemma 7 implies the existence of a unique perpendicular PD from P to l. P and D being distinct elements of $\{B_o\}$, the perpendicular PD is in the plane $\{B_o\}$. If $D \neq O$ then there exists in the plane $\{B_{PD,PO}\}$ a unique perpendicular m from P to the line PD. But Lemma 8 implies that $\{B_o\} = \{B_{PD,PO}\}$ and $m \subset \{B_o\}$. m is parallel to l for, if T is on m and on l, then we would have from T two distinct perpendiculars to the line PD.

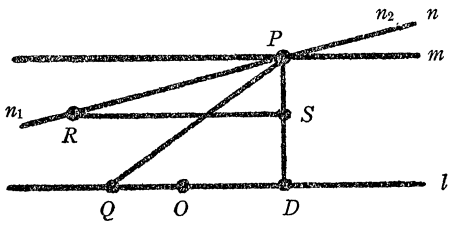


FIG. 5

If n is another parallel from P to l in the plane $\{B_o\}$, then $n \subset \{B_{PD,PO}\}$ and we can distinguish on n two opposite rays n_1, n_2 with end-point P in $B_{PD,PO}$; one of those rays, say n_1, is in $HB_{PD,PO}$ and the other one is in $\overline{HB}_{PD,PO}$. We have then either $\angle(\text{ray } PD)\ (n_1) < 90$ or $\angle(\text{ray } PD)\ (n_1) > 90$.

Case 1: $\angle(\text{ray } PD)\ (n_1) < 90$. Let R be a point of n_1 and let RS be the unique perpendicular from R to the line PD. Then S is a point of the ray PD (otherwise the sum of the measures of the angles of the triangle PSR would be greater than 180). If Q is a point of the ray DO such that $DQ = (PD \cdot RS)/PS$, then the similarity axiom implies that the triangles PSR and PDQ are similar and consequently that $\angle DPR = \angle DPQ$. But $Q \in$ ray DO if and only if $Q = O$ or $[D, O, Q]$ or $[D, Q, O]$, and as $[D, O, Q]$ or $[D, Q, O]$ imply $[PD, PO, PQ]$ or $[PD, PQ, PO]$, in each case ray PQ belongs to $HB_{PD,PO}$. Consequently Lemma 2 implies that ray PR and ray PQ coincide, and we have from Q two perpendiculars to the line PD which is a contradiction.

Case 2: $\angle(\text{ray } PD)\ (n_1) > 90$. In that case $\angle(\text{ray } PD)\ (n_2) < 90$ and using the preceding argument in the half-bundle $\overline{HB}_{PD,PO}$ (containing the ray n_2) with the ray DO we obtain also a contradiction and consequently m is the unique parallel from P to l in $\{B_o\}$.

If D coincides with O then the same argument applies by choosing a point $O' \neq O$ on the line l.

THEOREM 8. *Two planes coincide if and only if they have three noncollinear points in common.*

The proof of this theorem can be found in [2]. The results of lemmas 7 and 9 can be formulated in a more general form as follow.

THEOREM 9. *In a plane, from a given point there exists one and only one perpendicular to a given line, and from a given point there exists one and only one parallel to that line.*

The following theorem is an immediate consequence of the continuity axiom.

THEOREM 10. *If three distinct points A, B, and C do not lie on the same line, and if D and E are two points such that C is between B and D, and E is between A and C, then there exists between A and B a point F such that D, E, F lie on the same line.*

9. 3-dimensional euclidean space. The following axiom is now added to the structure.

S. *There exists a point not on a given plane.*

The 3-dimensional euclidean space is introduced as defined by O. Veblen in [5]. A set of four noncoplanar points is called a *tetrahedron* whose *faces* are the interior of the triangles defined by the elements of the tetrahedron (the interior of a proper triangle ABC is defined to be the class of points P between X and Y, where X and Y belong to different intervals defined by the points A, B, and C). A *3-space* $ABCD$ is the set of all points collinear with any two points of the faces of the tetrahedron $ABCD$.

Except for the Axiom X, the remaining eleven axioms given in [5] are either a property of real numbers, a consequence of a definition given here, or one element of the following list of axioms and theorems: E_1, E_3, E_4, D_4, S, Theorem 2, Theorem 9, and Theorem 10. The Axiom X says that all points belong to the same 3-space. We have then in [5] the proof of the following property concerning categoricity.

THEOREM 11. *If M_1 and M_2 are two models of a given 3-space, then they are isomorphic.*

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RODRIGUES FORMULA SOLUTIONS TO SOME SECOND ORDER DIFFERENTIAL EQUATIONS

J. M. HORNER, General Motors Research Laboratories, Warren, Michigan
(Now at the University of Alabama in Huntsville)

In the following paragraphs we continue our investigation (see [3], [4]) of the second order differential equation (DE)

$$p_2(x)y'' + p_1(x)y' + a_0y = q(x), \tag{1}$$

where

$$\begin{cases} p_2(x) = a_2x^2 + b_2x + c_2 \not\equiv 0 \\ p_1(x) = a_1x + b_1 \\ |a_2| + |a_1| \neq 0 \end{cases} \tag{2}$$

and a_2, b_2, c_2, a_1, b_1, and a_0 are constants.

Among the special cases of (1) are the Legendre equation, the hypergeometric equation, the confluent hypergeometric equation, the Gegenbauer equation, the Jacobi equation, the Hermite equation, the two Tchebicheff equations, the equations of the Laguerre and Sonine polynomials, and, by well-known transformations, the Bessel equation and numerous other classical equations (see [5] and [7]).

For convenience, let

$$g(j, x) = p_2^j \exp\left[-\int (p_1/p_2)dx\right]. \tag{3}$$

Unless otherwise stated, x is a real variable and j in (3) is an integer. $g(0, x)$ is of course a constant multiple of the Wronskian of any fundamental system for (1) [8, pp. 117–119]. The integration constant in (3) is arbitrary so that the indefinite integral may be replaced by a definite integral.

In [2], [3], and [4] we defined the *associated equation* of the DE (1) to be the quadratic equation

$$A(t) = a_2t^2 + (a_2 - a_1)t + a_0 = 0 \tag{4}$$

and frequently encountered the first order DE

$$p_2v' + (p_1 - up_2')v = h(x). \tag{5}$$

In order to avoid repetitious hypotheses, j, k, n, and p will always be used to represent nonnegative integers.

One of the principal results of [2] and [4] is that when $A(n)=0$, $n>0$, and $q(x)\equiv 0$, the DE (1) has the $n+1$ solutions

$$y_1(x) = D^{n-1}[g(n, x)] \tag{6}$$

and

$$y_j(x) = D^{n-1}\left\{g(n,x)\int [x^{j-2}/g(n+1,x)]dx\right\}, \tag{7}$$

where $D \equiv d/dx$, $2 \leqq j \leqq n+1$, and the integration constants are arbitrary.

The primary purposes of the present paper are to extend these results by a transformation and to answer the question of if and when this set of solutions contains a fundamental system, i.e., a pair of nontrivial linearly independent solutions.

One might question the importance of studying (1) for the special case in which $A(t)=0$ has positive integral roots. Actually, this is an extremely important case. A few well-known equations which have integral associated roots are the differential equations for the Legendre, Sonine, Tchebicheff, Jacobi, Hermite, Gegenbauer and hypergeometric polynomials, and, by elementary transformations, the equations of the associated Legendre polynomials and spherical Bessel functions [7].

We will show that the following result is true:

THEOREM 1. *Let* $q(x) \equiv 0$, $n>0$, *and* $A(n)=0$ *so that* (1) *has the solutions* $\{y_j(x): 1 \leqq j \leqq n+1\}$ *of* (6) *and* (7). *Then*:

1. y_1 *and* y_2 *constitute a fundamental system for* (1) *if and only if* n *is the smallest positive integral root of* $A(t)=0$.

2. *If* n *and* k, $k<n$, *are positive integral roots of* $A(t)=0$, *then* y_i *and* y_{n-k+2} *form a fundamental system for* (1), *where* y_i *is a nontrivial choice of* y_1 *or* y_2. *This choice is always possible.*

The proof of this result will be given in several steps.

LEMMA 1. *If, in* (5), $h(x)$ *is a polynomial of degree* $\leqq j$, $v(x)$ *is a polynomial of exact degree* j *and* $A(u)=0$, *then also* $A(u-j-1)=0$.

Proof. Under the assumed conditions, differentiation of (5) $j+1$ times yields

$$(j+1)[p_1' - (u - \tfrac{1}{2}j)p_2'']v^{(j)} = 0.$$

Since $j+1 \neq 0$ and $v^{(j)} \neq 0$, then $a_1 = a_2(2u-j)$. Since we have assumed that $|a_1| + |a_2| \neq 0$, $a_2 \neq 0$. Also, $A(u) = a_2u^2 + (a_2-a_1)u + a_0 = 0$. Combination of these two results gives $a_0 = a_2u(u-j-1)$. Thus, a_1 and a_0 are expressed in terms of a_2, u and j and $A(t)$ becomes $A(t) = a_2(t-u)(t-u+j+1)$.

LEMMA 2. *If* (1) *has a polynomial solution of exact degree* j *and* $q(x) \equiv 0$, *or is a polynomial of degree* $\leqq j-1$, *then* $A(-j)=0$.

Proof. Under the assumed conditions, differentiation of (1) j times gives $[a_2j(j-1)+a_1j+a_0]y^{(j)}=0$. Since $y^{(j)} \neq 0$, then $A(-j)=0$.

We are now in a position to prove part 1 of Theorem 1.

LEMMA 3. *If* $q(x) \equiv 0$ *and* n *is the smallest positive integral root of* $A(t)=0$, *then* y_1 *and* y_2 *form a fundamental system for* (1).

Proof. If $n=1$, the conclusion is evident so suppose that $n\geqq 2$.

If $y_1(x)\equiv 0$, then $g(n, x)$ is a polynomial of exact degree $k\leqq n-2$ and satisfies (5) when $u=n$ and $h(x)\equiv 0$. But then, by Lemma 1, $A(n-k-1)=0$. Since $0\leqq k\leqq n-2$, $n>n-1\geqq n-k-1\geqq 1$ and $n-k-1$ is a positive integral root of $A(t)=0$ which is smaller than n. Thus, $y_1(x)\not\equiv 0$.

If $n\geqq 2$ and $y_2(x)\equiv 0$, then $g(n, x)\int[1/g(n+1, x)]dx$ is a polynomial of exact degree $k\leqq n-2$ and satisfies (5) when $u=n$ and $h(x)\equiv 1$. As before, this implies that $1\leqq n-k-1<n$ and $A(n-k-1)=0$ so $y_2(x)\not\equiv 0$.

Finally, suppose that $K_1y_1(x)+K_2y_2(x)\equiv 0$ for nonzero constants K_1 and K_2. Then

$$v(x) = g(n, x)\left\{K_1 + \int [K_2/g(n+1, x)]dx\right\}$$

is a polynomial of exact degree $k\leqq n-2$ which satisfies (5) when $u=n$ and $h(x)\equiv K_2$. Again this leads to a contradiction of the assumption that n is the smallest positive integral root of $A(t)=0$ and completes the proof.

So long as the integration constants remain arbitrary in (7), it is easy to find DE's for which y_1 and y_2 are the only nontrivial solutions in the set given by (6) and (7).

Example 1. For the DE $x^2y''+4xy'-4y=0$, the associated roots are $t=4$ and $t=-1$. Corresponding to the root $t=4$, the five solutions of (6) and (7) are $y_1(x)=D^3(x^4)=24x$, $y_2(x)=D^3[x^4\int x^{-6}dx]=6x^{-4}/5$ and, for $j=3$, 4, 5, $y_j(x)=D^3[x^4\int x^{j-8}dx]=0$ when the integration constants are taken to be zero.

LEMMA 4. *Let* $A(n)=A(k)=0$, $n>k\geqq 1$, *and let* $\{y_j\colon 1\leqq j\leqq n+1\}$ *and* $\{y_j^*\colon 1\leqq j\leqq k+1\}$ *be the solutions of* (6) *and* (7) *corresponding respectively to* n *and* k. *Then there exist constants* K_p, $1\leqq p\leqq n+1$, *such that*

$$y_p = K_py_1^*, \qquad 1\leqq p\leqq n-k+1 \tag{8}$$

and, for any p *such that* $n-k+2\leqq p\leqq n+1$,

$$y_p = K_py_1^* + \binom{p-2}{n-k}(n-k)!\,y_{p-n+k}^*. \tag{9}$$

Proof. For each y_j in (6) or (7), let v_j be the corresponding right member before application of the operator D^{n-1} and let v_j^* have the same relation to y_j^*.

If $1\leqq p\leqq n-k+1$, v_p satisfies (5) when $u=n$, and $h(x)\equiv 0$ if $p=1$ or $h(x)=x^{p-2}$ if $p\geqq 2$. If $p\geqq 2$, then $0\leqq p-2\leqq n-k-1$ so that in either case differentiation of (5) $n-k$ times yields

$$p_2v_p^{(n-k+1)} + (p_1 - kp_2')v_p^{(n-k)} + (n-k)[a_1 - (n+k+1)a_2]v_p^{(n-k-1)} = 0.$$

Now, $A(n)=A(k)=0$ so

$$a_2k^2 + (a_2 - a_1)k = -a_0 = a_2n^2 + (a_2 - a_1)n$$

and $(n-k)[a_1-(n+k+1)a_2]=0$. Hence, $D^{n-k}(v_p)$ satisfies (5), when $u=k$ and

$h(x) \equiv 0$, and there exist constants K_p such that $D^{n-k}(v_p) = K_p v_1^*$. Differentiation of this result $k-1$ times gives (8).

If $n-k+2 \leqq p \leqq n+1$, v_p satisfies (5) when $u=n$ and $h(x) = x^{p-2}$. Duplication of the above argument shows that $D^{n-k}(v_p)$ satisfies (5) when $u=k$ and

$$h(x) = \binom{p-2}{n-k}(n-k)!\, x^{p-n+k-2}.$$

So, for some constants K_p,

$$D^{n-k}(v_p) = K_p v_1^* + \binom{p-2}{n-k}(n-k)!\, v_{p-n+k}^*$$

and (9) follows by differentiating this result $k-1$ times.

COROLLARY. *If n is the larger of two positive integral roots n and k of $A(t)=0$ and $q(x) \equiv 0$, then no pair of the solutions $y_1, y_2, \cdots, y_{n-k+1}$, corresponding to the root $t=n$, constitute a fundamental system for* (1).

Proof. This result is immediate from (8) and, with Lemma 3, proves the first conclusion of Theorem 1.

It is possible, however, to select from among the set of solutions given by (6) and (7) a fundamental system for (1) when $q(x) \equiv 0$ and n is the larger of two positive integral roots of $A(t)=0$.

LEMMA 5. *Let $q(x) \equiv 0$, $A(n) = A(k) = 0$, $n > k > 0$ and $\{y_j\colon 1 \leqq j \leqq n+1\}$ be the solutions of* (6) *and* (7) *corresponding to the root n. Then y_{n-k+2} and one of y_1 and y_2 form a fundamental system for* (1).

Proof. With the notation of Theorem 2, it follows from Theorem 1 that y_1^* and y_2^* form a fundamental system for (1) when $q(x) \equiv 0$.

It is then evident from (8) and (9) that if there exists among the solutions $y_1, y_2, \cdots, y_{n-k+1}$ one which is nontrivial, then this solution and y_{n-k+2} constitute a fundamental system for (1) when $q(x) \equiv 0$. We will show that $y_1 \not\equiv 0$ when $k=n-1$ and $p_1 \not\equiv np_2'$, that $y_2 \not\equiv 0$ when $k=n-1$ and $p_1 \equiv np_2'$, and that y_1 and y_2 are not both trivial when $k \neq n-1$.

Suppose first that $k=n-1$. Then, as in the proof of Lemma 4, $a_1 = 2na_2$ and $p_1 - np_2' = K$ for some constant K. If $p_1 \equiv np_2'$, then $K=0$ and y_2 is equivalent to

$$D^{n-1}\left[\int p_2^{-1}dx\right] = D^{n-2}(p_2^{-1}) \not\equiv 0.$$

If $p_1 \not\equiv np_2'$, then $K \neq 0$ and y_1 is equivalent to

$$D^{n-1}\left[\exp\left(-\int Kp_2^{-1}dx\right)\right] \not\equiv 0.$$

Thus $y_1 \not\equiv 0$ when $k = n-1$ and $p_1 \not\equiv np_2'$, and $y_2 \not\equiv 0$ when $k = n-1$ and $p_1 = np_2'$.

Now suppose that $k \neq n-1$ and assume that $y_2 \equiv 0$. Then, from (8), $K_2 = 0$ since $y_1^* \not\equiv 0$. But in the proof of Lemma 4 it was shown that $D^{n-k}(v_2) = K_2 v_1^*$. Thus v_2 is a polynomial of exact degree $j \leqq n-k-1$ which satisfies (5) when $u = n$ and $h(x) \equiv 1$. But then $h(x)$ has degree $\leqq j$ and $A(n) = 0$, so Lemma 1 shows that also $A(n-j-1) = 0$. However $A(t)$ has only the two zeroes $t = n$ and $t = k$. So $j = n-k-1$ and v_2 has exact degree $n-k-1$. Similarly, if $y_1 \equiv 0$ then v_1 is a polynomial of exact degree $n-k-1$. So, if both y_1 and y_2 are trivial and $k \neq n-1$, then both v_1 and v_2 are polynomials of exact degree $n-k-1$. But, from (6) and (7) it is seen that $v_2 = Kv_1 \int (p_2 v_1)^{-1} dx$, where K is a nonzero constant. But this implies that $\int (p_2 v_1)^{-1} dx$ is constant, which is clearly false. Thus, y_1 and y_2 are not both trivial when $k \neq n-1$. Since y_2, for an appropriate choice of the integration constant, contains a multiple of y_1, a fundamental system can always be obtained from y_{n-k+2} and y_2 by a proper choice of the constant.

This gives the second conclusion of Theorem 1 which has now been completely proved.

We will now show by examples that without additional hypotheses, the various cases which arose in the proof of the previous lemma can produce the only fundamental system for (1) when $q(x) \equiv 0$.

Example 2. ($k \neq n-1$, $y_1 \equiv 0$),

$$x^2 y'' + 5xy' + 3y = 0.$$

For this Euler DE the associated roots are $t = 1$ and $t = 3$. When $x > 0$, the larger root gives the solutions

$$y_1 = D^2(x) = 0, \qquad y_2 = D^2\left(x \int x^{-3} dx\right) = -x^{-3},$$

$$y_3 = D^2\left(x \int x^{-2} dx\right) = 0 \quad \text{and} \quad y_4 = D^2\left(x \int x^{-1} dx\right) = x^{-1}.$$

The smaller root gives the solutions $y_1^* = x^{-3}$ and $y_2^* = x^{-3} \int x dx = x^{-1}/2$.

Example 3. ($k \neq n-1$, $y_2 \equiv 0$),

$$(x^2 + 1)y'' + 5xy' + 3y = 0.$$

For this DE the associated roots are $t = 1$ and $t = 3$. The larger root gives the solutions

$$y_1 = D^2[(x^2 + 1)^{1/2}],$$

$$y_2 = D^2\left[(x^2 + 1)^{1/2} \int (x^2 + 1)^{-3/2} dx\right]$$

$$= D^2\left\{(x^2+1)^{1/2}\int D[x(x^2+1)^{-1/2}]dx\right\} = 0,$$

$$y_3 = D^2\left[(x^2+1)^{1/2}\int x(x^2+1)^{-3/2}dx\right] = 0,$$

and

$$y_4 = D^2\left[(x^2+1)^{1/2}\int x^2(x^2+1)^{-3/2}dx\right].$$

The smaller root gives the two solutions $y_1^* = (x^2+1)^{-3/2}$ and

$$y_2^* = (x^2+1)^{-3/2}\int (x^2+1)^{-1/2}dx = (x^2+1)^{-3/2}\sinh^{-1} x.$$

This example illustrates an important property of the method described. In many cases solutions are obtained in closed form which would be extremely difficult to recognize from the Frobenius series and which are valid in a larger region than the series solution.

Example 4. ($k=n-1$, $p_1 \not\equiv np_2'$),

$$x(x-1)y'' + (4x-1)y' + 2y = 0.$$

For this DE the associated equation has the roots $t=1$ and $t=2$, and $p_1 - np_2' \equiv 1$. When $0<x<1$, we obtain, for the larger root $t=n=2$, $y_1 = D[x(1-x)^{-1}] = (1-x)^{-2}$,

$$y_2 = D\left[x(1-x)^{-1}\int (-x^{-2})dx = D\left\{x(1-x)^{-1}\int D[(1-x)x^{-1}]dx\right\} = 0,\right.$$

and

$$y_3 = D\left[x(1-x)^{-1}\int (-x^{-1})dx\right] = D[(-x\ln x)(1-x)^{-1}]$$

$$= (x-1-\ln x)(1-x)^{-2}$$

and for the smaller root $t=k=1$, $y_1^* = -(1-x)^{-2}$ and

$$y_2^* = (1-x)^{-2}\int (1-x)x^{-1}dx = -(x-\ln x)(1-x)^{-2}.$$

This example also illustrates an important property of this method. In many elementary DE's having logarithmic Frobenius solutions, which are very tedious to obtain, these solutions are obtained with relative ease by the above procedure.

Example 5. ($k=n-1$, $p_1 = np_2'$),

$$x(x+1)y'' + (4x+2)y' + 2y = 0.$$

For this DE the associated roots are $t=1$ and $t=2$, and $p_1 = 2p_2'$. For $x>0$, the larger root $t=n=2$ yields the solutions $y_1 = D(1) = 0$,

$$y_2 = D\left[\int x^{-1}(x+1)^{-1}dx\right] = x^{-1}(x+1)^{-1}$$

and

$$y_3 = D\left[\int (x+1)^{-1}dx\right] = (x+1)^{-1}$$

while the smaller root $t=k=1$ gives the solutions $y_1^*=x^{-1}(x+1)^{-1}$ and $y_2^* = (x+1)^{-1}$.

Mambriani [9, pp. 38–39] obtained the two solutions y_1 and y_2, given in (6) and (7), for the case that $n \geqq 2$. His method was considerably different however, and the exact relation between n and the coefficients in (1) was apparently not noticed. Using operator methods he "concludes that the general integral of

$$p_2y'' + p_1y' - n[a_2(n+1) - a_1]y = 0$$

is given by $y=K_1y_1+K_2y_2$ where K_1, K_2 are arbitrary constants" (notation of and translation by the present author). As we have seen, this conclusion can be false and, in fact, is always false unless n is the smallest positive integral root of $A(t)=0$.

In the remaining paragraphs we will consider a transformation of (1) into another DE of the same type which has the attractive feature of reversing the role of positive and nonpositive associated roots.

THEOREM 2. *Let $q_1(x)=a_2x+b_0$ and let $v(x)$ be a nontrivial solution to $p_2v' = q_1v$. If there exists a constant c_0 such that*

$$q_1^2 + q_1(p_1 - p_2') + q_1'p_2 = c_0p_2$$

then $y(x)$ is a solution to (1) *if and only if $w(x)=y(x)/v(x)$ is a solution to*

$$p_2w'' + (p_1 + 2q_1)w' + (a_0 + c_0)w = q(x)/v(x).$$

Proof. The proof consists of elementary computation, using the relation $p_2v'=q_1v$ to eliminate terms involving v' and v''.

COROLLARY. *$y(x)$ is a solution to* (1) *if and only if $w(x)=y(x)/g(1, x)$ is a solution to*

$$p_2w'' + (2p_2' - p_1)w' + (2a_2 - a_1 + a_0)w = q(x)/g(1, x) \tag{10}$$

whose associated equation is

$$A^*(t) = a_2t^2 + (a_1 - 3a_2)t + (2a_2 - a_1 + a_0) = 0$$

and $A(r)=0$ if and only if $A^*(1-r)=0$.

Proof. The first part of this result is a special case of Theorem 2, while the second part is quite easily verified. We note also that, when $q(x)\equiv 0$, (10) is the adjoint of (1).

This result, along with Theorem 1, leads to the following companion to Theorem 1.

THEOREM 3. *Suppose that* $q(x)\equiv 0$, $n\geqq 0$ *and* $A(-n)=0$.

1. *The* DE (1) *is satisfied by the* $n+2$ *functions*

$$y_1 = g(1, x)D^n\{[g(1-n, x)]^{-1}\}$$

and, for $2\leqq j\leqq n+2$, $y_j=g(1, x)D^n\{[g(1-n, x)]^{-1}\int x^{j-2}g(-n, x)dx\}$.

2. y_1 *and* y_2 *constitute a fundamental system for* (1) *if and only if* n *is the smallest nonnegative integer such that* $A(-n)=0$.

3. *If* n *and* k, $k<n$, *are nonnegative integers such that* $A(-n)=A(-k)=0$, *then* y_i *and* y_{n-k+2} *form a fundamental system for* (1), *where* y_i *is a nontrivial choice of* y_1 *or* y_2. *This choice is always possible.*

Proof. In Theorem 1, replace n by $n+1$, k by $k+1$ and let the function $g(j, x)$ which corresponds to (10) be denoted by $g^*(j, x)$. Also, let the coefficient of w' in (10) be denoted by p_1^*. Then Theorem 1 applied to (10) becomes: *Let* $q(x)/g(1, x)\equiv 0$ *and* $n+1$ *be a positive integer such that* $A^*(n+1)=0$. *Then* (10) *is satisfied by the* $n+2$ *functions* $w_1=D^n[g^*(n+1, x)]$ *and*

$$w_j = D^n\left\{g^*(n+1, x)\int x^{j-2}[g^*(n+2, x)]^{-1}dx\right\},$$

where $2\leqq j\leqq n+2$. w_1 *and* w_2 *form a fundamental system for* (10) *if and only if* $n+1$ *is the smallest positive integral root of* $A^*(t)=0$. *If* $k+1$ *and* $n+1$, $k+1<n+1$, *are positive integral roots of* $A^*(t)=0$, *then* w_i *and* w_{n-k+2} *form a fundamental system for* (10), *where* $i=0$ *or* $i=2$.

In this case $p_1^*=2p_2'$ and $g^*(j, x)$ is a constant multiple of $[g(2-j, x)]^{-1}$ (they satisfy the same first order DE). The desired result then follows from the previous corollary.

Numerous special cases of the solutions y_1 of Theorem 1 and y_1 of Theorem 3 are well known, but the only systematic method (known to the author) of obtaining solutions of this form is given by Bateman [1, pp. 110–115]. He observed that $y=(x-a)^v(x-b)^u$ satisfied the DE

$$(x-a)(x-b)y'' - [(v-1)(x-b) + (u-1)(x-a)]y' - (u+v)y = 0$$

and, by differentiating this equation n times, obtained

$$(x-a)(x-b)y^{(n+2)} - [(v-n-1)(x-b) + (u-n-1)(x-a)]y^{(n+1)} + [n^2 - n(v+u-1) - u - v]y^{(n)} = 0.$$

By choosing appropriate values of the parameters a, b, v, u, and n, Bateman obtained the solutions corresponding to y_1 of Theorem 1 for several classical equations although he concentrated his attention on the hypergeometric equation. By following this procedure with a transformation, Bateman obtained the solution y_1 of Theorem 3 for a certain subclass of the DE's of the form (1).

The corollary to Theorem 2, along with [4, Theorems 2 and 3], provides the following results:

THEOREM 4. *If* $n \geqq 0$ *and* $A(-n) = 0$, *then a particular solution to* (1) *is*

$$y(x) = g(1, x)D^n\left\{g(1-n, x)\int g(-n, x)D^{-n-1}[q(x)/g(1, x)]dx\right\},$$

where $D^{-k}(f(x)) = \int\int \cdots \int f(x)(dx)^k$.

Proof. This result is obtained from [4, Theorem 2], just as Theorem 3 is obtained from Theorem 1, by making use of the corollary to Theorem 2. As in [4, Theorem 2], it is often convenient to replace $D^{-n-1}[q(x)/g(1, x)]$ by the integral

$$I_{n+1}(x) = (1/n!)\int_a^x [(x-v)^n q(v)/g(1, v)]dv,$$

where the point a need only be a point for which the integral exists and satisfies $D^{n+1}(I_{n+1}(x)) = q(x)/g(1, x)$.

THEOREM 5. *If* $n > 0$, $A(n) = 0$ *and* $q(x)$ *is of class* C^{n-1}, *then every solution to* (1) *has the property that, for some constant* K.

$$D^n[y(x)/g(1, x)] = [g(n+1, x)]^{-1}\left\{K + \int g(n, x)D^{n-1}[q(x)/g(1, x)]dx\right\}.$$

Proof. [4, Theorem 3], and the corollary to Theorem 2 above.

In [5], these results are extended to the case of nonintegral associated roots, using both series and contour integral generalizations, and the entire collection of results is then extended to the general Pochhammer DE.

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CONULL AND OTHER MATRICES WHICH SUM A BOUNDED DIVERGENT SEQUENCE

ROBERT WHITLEY, University of Maryland

We generalize a theorem due to Wilansky [6, Theorem 3, page 243]. Our proof is straightforward and we obtain as a corollary the result that there is no matrix isomorphism of c onto c_0; a recent problem in this MONTHLY failed to elicit a simple proof of this [1]. We also point out the relation between our result and Berg's [2] characterization of those conservative matrices which sum a bounded divergent sequence.

The Banach space of bounded scalar sequences (a_n) with sup norm $\|(a_n)\| = \sup |a_n|$ is denoted by m, the closed subspace of sequences converging (to zero) by c (c_0), and the Banach space of sequences (a_n) with finite norm $\|(a_n)\| = \sum |a_n|$ by l.

A conservative matrix is an infinite matrix of scalars (b_{ij}), $i, j = 1, 2, \cdots$, which maps c into c by taking (a_i) into $(\sum_{j=1}^{\infty} b_{ij}a_j)$. The classical Silverman-Toeplitz conditions [7, page 4 and page 118] show that the conservative matrices (b_{ij}) are exactly those satisfying the conditions:

(1) $\sup_n \sum_{k=1}^{\infty} |b_{nk}|$ is finite and

(2) the limits $b_{0k} \equiv \lim_n b_{nk}$, $k = 1, 2 \cdots$ and $b_{00} \equiv \lim_n \sum_{k=1}^{\infty} b_{nk}$ exist.

Recall that each continuous linear functional x' on c corresponds to an element (c_i) in l where for every (a_i) in c we have $x'(a_i) = c_1 \lim a_i + \sum_{i=1}^{\infty} a_i c_{i+1}$ and $\|x'\| = \sum |c_i|$ [7, page 92]. It follows that a conservative matrix T has the form $Tx = (x_n'(x))$, for all x in c, where x_n' is the functional on c which corresponds as above to the sequence $(0, b_{n1}, b_{n2}, \cdots)$ in l. It is easy to see that T is a continuous linear operator mapping c into c with $\|T\| = \sup \|x_n'\| = \sup_n \sum_{k=1}^{\infty} |b_{nk}|$. For a conservative matrix (b_{ij}) we have defined the scalars b_{0n}, $n = 0, 1, \cdots$, in condition (2) of the Silverman-Toeplitz conditions and from condition (1) it follows that the sequence $(b_{00}, b_{01}, b_{02}, \cdots)$ is in l. Let x_0' be the continuous linear functional on c which corresponds to $(b_{00} - \sum_{j=1}^{\infty} b_{0j}, b_{01}, b_{02}, \cdots)$ and note that $\lim x_n'(x) = x_0'(x)$ for all x in c. Even though the first coordinate of each sequence $(0, b_{n1}, b_{n2}, \cdots)$ is zero, the first coordinate of x_0' need not be zero; if it is, then the matrix is *conull* while if not, then the matrix is *coregular* [7, page 93]. The distinction between conull and coregular matrices seems minor and yet the two types of matrices are really quite distinct. For example, the identity mapping c onto c is a one-to-one coregular map of c onto itself and in contrast Wilansky [6, Theorem 3, page 243] has shown that a one-to-one conull matrix cannot have a closed range.

1. THEOREM. *Let T be a conull matrix mapping c into c. If the null manifold of T is finite dimensional, then T cannot have a closed range.*

Proof. Let T be written in the form $Tx = (x_n'(x))$, $\lim x_n'(x) = x_0'(x)$, as we have done above. The conjugate (adjoint) operator T' maps l into l and we calculate $T'(a_i) = \sum_{i=1}^{\infty} a_i x_{i-1}'$, for (a_i) in l. Let f be the continuous linear functional

defined on l by $f(a_i)=a_1$. The functionals x_1', x_2', $\cdots$, are all in the null manifold $N(f)$ of f and, since T is conull, x_0' also lies in $N(f)$. Thus the range of T' lies in $N(f)$.

Now suppose that the null manifold $N(T)=T^{-1}(0)$ is finite dimensional and spanned by the vectors $x_1, x_2, \cdots, x_n$. Assume that the range of T is closed. Then [3, Theorem 2, page 487] the range of T' is equal to the set $N(T)^{\perp}=\{x' \text{ in } l: x'(y)=0 \text{ for all } y \text{ in } N(T)\}$ and thus $N(T)^{\perp}\subseteq N(f)$. Let J be the natural map of c into c'', so the linear functional Jy is defined by $Jy(x')=x'(y)$ for all functionals x' on c. From the fact the $N(T)$ is the span of $x_1, x_2, \cdots, x_n$ it follows that

$$N(T)^{\perp} = N(Jx_1) \cap N(Jx_2) \cap \cdots \cap N(Jx_n) \subseteq N(f).$$

Now [7, Theorem 3, page 39] there are scalars $b_1, b_2, \cdots, b_n$ with $f=\sum b_i Jx_i = J(\sum b_i x_i)$. But this is a contradiction since $f \neq Jy$ for any y in c. In fact, letting e_m be the vector in l whose nth coordinate is δ_{mn}, $f(e_{m+1})=0$ while $J(a_i)(e_{m+1})=a_m$ and so if we were to suppose that $f=J(a_i)$, we would find that $(a_i)=0$, obtaining the false conclusion that $f=0$. Hence the range of T cannot be closed.

2. COROLLARY. *There is no matrix isomorphism of c onto c_0.*

Proof. Let T be a matrix mapping c onto c_0. The map T has the form $Tx=(x_n'(x))$ and since the range of T is in c_0 we have $x_0'(x)=\lim x_n'(x)=0$ for all x in c. So not only is T conull, but $x_0'=0$.

We also see from the theorem that there is no matrix projection of c onto c_0, which is noted in [6, page 250].

A conull matrix may have closed range if its null manifold is infinite dimensional. In fact [6, page 250] the matrix

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & \cdot & \cdot \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

is a conull map of c onto itself.

A matrix T is said to *sum a sequence* x, say in m, if Tx is in c. It is interesting that a conull matrix always sums a divergent sequence in m [2, see the last paragraph on page 650] and this property is related to the property of Theorem 1. Indeed, Theorem 1 is a consequence of Theorem 3 below and the fact that a conull matrix sums a bounded divergent sequence—a circuitous but informative proof.

The following theorem is given by Berg [2, Theorems 1 and 2, page 651] in the case where T is one-to-one, which is the hard part.

3. THEOREM. *A matrix T mapping c into c sums no bounded divergent sequence if and only if it has a finite dimensional null manifold and a closed range.*

Proof. Suppose that the range of T is not closed. Let

$$M_p = \{(a_i) \text{ in } c: a_1 = a_2 = \cdots = a_p = 0 \text{ and } \lim a_i = 0\}.$$

Then M_p is a closed subspace of c and $c = M_p \oplus F$ where F is a finite dimensional subspace of c. Since $Tc = TM_p + TF$ is the translation of TM_p by a finite dimensional subspace, it is closed if TM_p is closed [7, Corollary 5, page 192]. Thus the range of T restricted to M_p, $T|_{M_p}$, is not closed. If T is not one-to-one on M_p, then we can find a z in M_p of norm one with $Tz = 0$. If T is one-to-one on M_p, then because the range of $T|_{M_p}$ is not closed there is no constant $b > 0$ with $\|Tx\| \geqq b\|x\|$ for all x in M_p. Thus given $\epsilon > 0$ we can find a z of norm two in M_p with $\|Tz\| < \epsilon$. The element $z = (z_i)$ is in c_0, $z = \lim(z_1, z_2, \cdots, z_n, 0, 0, \cdots)$, so for some integer m $\|(z_1, z_2, \cdots, z_m, 0, 0, \cdots)\| \geqq 1$ and $\|T(z_1, z_2, \cdots, z_m, 0, 0, \cdots)\| < \epsilon$. It follows from the lemma of [5] that T sums a bounded divergent sequence.

Suppose that the null manifold of T is infinite dimensional. Then $N(T) \cap M_p$ is not $\{0\}$ for any p and proceeding as above we see that T sums a bounded divergent sequence.

Finally suppose that T sums a bounded divergent sequence. For $x = (x_i)$ in c, let $s(x) = \{i: x_i \neq 0\}$. By the lemma in [5] (and as in the proof given there) we can select a sequence $\{x^n\}$ of elements of c with $s(x^n) \cap s(x^m) = \varnothing$ for $n \neq m$, $\|x^n\| = 1$, and $\|Tx^n\| \leqq 2^{-n}$. Then T, when restricted to the subspace $sp(x^n)$ spanned by the vectors $\{x^n\}$, is the limit in the norm of operators with finite dimensional range and so is compact on that subspace [3, Lemma 3, page 486] and hence is also compact on the closure $\overline{sp}(x^n)$ of that subspace.

Assume that T has a finite dimensional null manifold and a closed range. Write $c = N(T) \oplus M$, M a closed subspace. Then T is one-to-one on M and has a closed range and so, by the open mapping theorem, T has a bounded inverse when restricted to M. Because $N(T)$ is finite dimensional, $M \cap \overline{sp}(x^n)$ is a closed infinite dimensional subspace upon which T is compact and has a bounded inverse. This is a contradiction. (If $T|_M^{-1}$ is continuous, then the product $T|_M^{-1}T$ is compact on $M \cap \overline{sp}(x^n)$ [3, Theorem 4, page 486] and so the unit sphere of $M \cap \overline{sp}(x^n)$ is compact which tells us that the space is finite dimensional [3, Theorem 5, page 245].) Thus T cannot have a finite dimensional null manifold and a closed range if it sums a bounded divergent sequence.

Suppose that T sums no bounded divergent sequence. The operator T may not be a Fredholm operator, because the range of T may not have finite codimension, but it does belong to a class of operators to which the perturbation theory for Fredholm operators has been extended. We recommend [4] for an exposition of this theory. Using these results we can conclude, for example, that if T sums no bounded divergent sequence and B is a matrix mapping c into c with $\|B\| < \gamma(T)$, where $\gamma(T)$ is Kato's perturbation constant (the constant $\gamma(T)$ is called the minimum modulus of T and for one-to-one T is given by $\gamma(T) = 1/\|T^{-1}\|$), then $T + B$ also sums no bounded divergent sequence.

These theorems have also been obtained by J. P. Crawford in a June 1966, Lehigh University Ph.D. thesis.

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ELEMENTARY PATH COUNTS

DAVID BLACKWELL AND J. L. HODGES, JR., University of California, Berkeley

1. Introduction. We present new elementary proofs for two known path counts. The material seems suitable for advanced high school classes.

A *path of length* s is a sequence $x=(x_1, \cdots, x_s)$ with each x_i equal to ± 1. We think of x_i as the amount a gambler wins from the ith toss of a coin, so that his net gain from the first i tosses is

$$\sigma_i = x_1 + \cdots + x_i \qquad (\sigma_0 = 0).$$

The polygonal line $\pi = Q_0Q_1 \cdots Q_s$, where $Q_i=(i, \sigma_i)$ will also be called a path of length s, and the line segment $Q_{i-1}Q_i$ will be called the ith *step* of the path. The ith step is *up* or *down* according as $x_i=1$ or $x_i=-1$. The ith step will be called *positive* if both Q_{i-1} and Q_i are on or above the horizontal axis, i.e. if $\sigma_{i-1}\geqq 0$ and $\sigma_i\geqq 0$. The number of positive steps in a path will be called its *lead* and denoted by λ, and a path with lead s, i.e. with every step positive, will be called a *positive path.* The gambler's net gain σ_s at the end of the path will be denoted by γ.

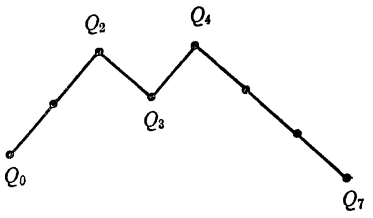


FIG. 1

For instance the path $(1, 1, -1, 1, -1, -1, -1)$ shown in Figure 1 has $s=7$, three steps up, $\lambda=6$, and $\gamma=-1$.

Here are two interesting facts:

A. *The number of positive paths of length s equals the number of paths of length s with* $\gamma = 0$ *if s is even, and equals the number of paths of length s with* $\gamma = 1$ *if s is odd.*

B. *For s even, among the paths of length s with* $\gamma = 0$, *there are equally many with each of the possible leads* $0, 2, \cdots, s$.

Both A and B are known; see Chapter III of Feller's book [2] for example. A is classical and B was discovered by Chung and Feller in 1949. We shall present new and elementary proofs for A and B.

2. Proof of A. For any path of length s, denote by μ the maximum of the numbers $\sigma_0, \sigma_1, \cdots, \sigma_s$: μ is the maximum height of the path, i.e. the maximum net gain the gambler ever experiences.

THEOREM. *Let h be any nonnegative integer. Associate with each path x of length s with* $\mu \geqq h$ *a corresponding path* $\phi(x)$ *of length s, defined as follows. Find the first i for which* $\sigma_i(x) = h$ *and write* $x = yz$, *where y is the initial segment of x with length i. Then* $\phi(x) = zw$, *where w is the path with* $s - i$ *steps obtained from y by reflecting y through a vertical line: if* $y = (a_1, \cdots, a_{s-i})$, $z = (-a_{s-i}, \cdots, -a_1)$. *Then for any integers g, m with* $m \geqq \max(g, h)$, ϕ *is a one-one mapping from the set of all paths of length s with* $\mu = m$, $\gamma = g$ *onto the set of all paths of length s with* $\mu = m - h$, $\gamma = g - 2h$, *so that, denoting by* $N(m, g)$ *the number of paths of length s with* $\mu = m$, $\gamma = g$, *we have*

$$N(m, g) = N(m - h, g - 2h) \quad \text{for} \quad h \geqq 0, m \geqq \max(g, h). \tag{1}$$

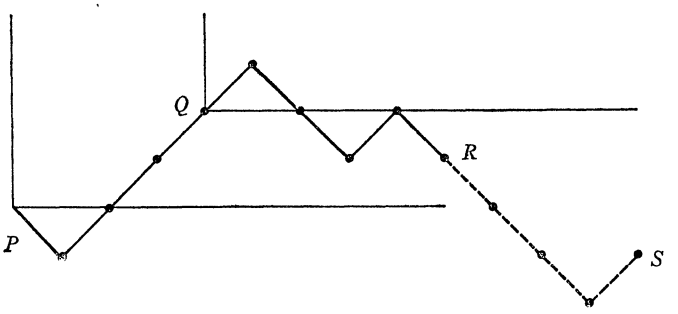


FIG. 2

Figure 2 shows the construction ϕ with $h = 2$, applied to a path with $s = 9$. The polygonal lines PQ, QR, RS are the paths y, z, w respectively. The function ϕ maps the path PQR into the path QRS.

Proof of Theorem. Clearly ϕ decreases μ by h and γ by $2h$. Moreover, any path v of length s with $\gamma \leqq \mu - h$ is the ϕ-image of a unique x. To find x, find the shortest final segment w of v with $\gamma(w) = -h$. Say $v = zw$. Then $x = yz$, where y is obtained from w by reflection in a vertical line.

To obtain A from (1), for s even, take any $m \geqq 0$, set $h=m$ and $g=0$ in (1), obtaining

$$N(m, 0) = N(0, -2m). \tag{2}$$

Now sum (2) over $m \geqq 0$ obtaining, for s even,

$$\#\{\gamma = 0\} = \#\{\mu = 0\}. \tag{3}$$

But the paths with $\mu=0$ are just the paths with no positive steps, which are clearly equinumerous, by reflection in a horizontal line, with the positive paths.

To obtain A from (1), for s odd, take any $m \geqq 1$, set $h=m$ and $g=1$ in (1), obtaining

$$N(m, 1) = N(0, 1 - 2m). \tag{4}$$

Sum (4) over $m \geqq 1$ obtaining, for s odd, $\#\{\gamma=1\} = \#\{\mu=0\}$. As with even s, $\#\{\mu=0\}$ equals the number of positive paths.

3. Proof of B. Take any path π of even length $s=2k$ and gain 0. Then π has exactly k steps up. Denote any one of these steps up by p, write $\pi=apb$, and say that $\pi_1=bpa$ is obtained from π by *switching at* p. The path π_1 will also have gain 0. We assert that π_1 and π have different leads. To see this, note that $0=\gamma(\pi)=\gamma(a)+1+\gamma(b)$, so that exactly one of the two numbers $\gamma(a)$, $\gamma(b)$ is nonnegative. Suppose that $\gamma(a) \geqq 0$, $\gamma(b)<0$. We check that $\lambda(\pi)>\lambda(\pi_1)$ by comparing the positive steps in each of the three parts a, p, b. In π, a starts at height 0 while in π_1 it starts at height $\gamma(b)+1 \leqq 0$, so any step in a which is positive in π_1 is also positive in π. Similarly b starts at height 0 in π_1 and at height $\gamma(a)+1>0$ in π, so every b step which is positive in π_1 is also positive in π. Finally p is itself a positive step in π, starting at $\gamma(a) \geqq 0$, but not in π_1, starting at $\gamma(b)<0$. So $\lambda(\pi)>\lambda(\pi_1)$.

Next note that if π_1, π_2 are obtained from π by switching at different p's, either is obtained directly from the other by switching:

$$\pi = apbpc$$
$$\pi_1 = bpcpa$$
$$\pi_2 = cpapb.$$

So π_1, π_2 have different leads.

Thus if we call two paths *equivalent* if they are identical or if one is obtained from the other by switching, each equivalence class contains $k+1$ paths, no two of which have the same lead. Since the only possible leads are the $k+1$ numbers $0, 2, \cdots, 2k$, there is exactly one path in each class with each lead.

The special case of lead $2k$ (or 0) is obtained in essentially this way by Dvoretzky and Motzkin [1].

The first authors research sponsored (partially) by the Air Force Office of Scientific Research, Office of Aerospace Research, USAF, under AFOSR Grant No. 1312-67.

The second authors research prepared with the partial support of the Office of Naval Research under Contract No. Nonr-222(43).

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MATHEMATICAL NOTES

A NOTE ON MATRIX QUADRATIC RESIDUES

AZRIEL ROSENFELD, University of Maryland

1. Introduction. Let $A=(a_{ij})$, $B=(b_{ij})$ be n by n matrices of integers; we shall call A and B congruent modulo k (written $A\equiv B \pmod{k}$) if $a_{ij}\equiv b_{ij} \pmod{k}$ for all $1\leqq i, j\leqq n$. Evidently, there are just k^{n^2} n-by-n integral matrices which are incongruent mod k.

The study of matrix congruences, as just defined, is a natural generalization of the study of congruences in residue rings of the integers, but few of the methods used in classical congruence theory have direct generalizations to matrix congruences.

This note deals with the problem of enumerating the 2-by-2 integral matrices which are congruent to squares of 2-by-2 integral matrices modulo an odd prime p; we call such matrices "squares" in what follows.

2. Scalar Squares.

PROPOSITION 1. *Every scalar matrix is a square. Furthermore, the number of incongruent matrix square roots of*

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

is

(A) p^2, *if* $k\equiv 0$,
(B) p^2+p+2, *if* $k\not\equiv 0$ *is a quadratic residue of* p,
(C) p^2-p, *if* k *is a quadratic nonresidue of* p.

Proof. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \equiv \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix};$$

then $a^2+bc\equiv d^2+bc\equiv k$ and $b(a+d)\equiv c(a+d)\equiv 0$, so that $b\equiv c\equiv 0$ or $a+d\equiv 0$. In the former case, $a^2\equiv d^2\equiv k$, so that k must be a quadratic residue of p and a, d (independently) $\equiv \pm k$. In the latter case $a^2\equiv d^2$ and b, c can be any integers

satisfying $bc \equiv k - a^2$. The enumeration of the square roots in the three cases of the proposition now proceeds as follows:

(A) $k \equiv 0$: Here $b \equiv c \equiv 0$ requires $a^2 \equiv d^2 \equiv 0$ or $a \equiv d \equiv 0$; it thus suffices to consider the case $a + d \equiv 0$, which here implies $bc \equiv -a^2$. If $a \equiv 0$, this is satisfied by $2p - 1$ incongruent pairs (b, c), namely those for which b or $c \equiv 0$ (this includes the pair $b \equiv c \equiv 0$). If $a \not\equiv 0$, it is satisfied by just $p - 1$ such pairs, since b can be anything $\not\equiv 0$ and c is then determined as $-a^2/b$. Since there are $p - 1$ a's $\not\equiv 0$, the total number of incongruent square roots of the zero matrix is thus $2p - 1 + (p-1)^2 = p^2$.

(B) $k \not\equiv 0$, a quadratic residue: The case $b \equiv c \equiv 0$ yields just four square roots. If $a + d \equiv 0$, there are two subcases: If $a^2 \equiv k$, b or c (but not both, since this case has already been counted) must $\equiv 0$; thus, for each of the two incongruent a's such that $a^2 \equiv k$, there are $2(p-1)$ incongruent pairs (b, c) which yield a square root. If $a^2 \not\equiv k$ (there being $p - 2$ such a's), there are as in case (A) just $p - 1$ incongruent pairs (b, c) such that $bc \equiv k - a^2 \not\equiv 0$. The total number of incongruent square roots for this type of k is thus $4 + 2 \cdot 2(p-1) + (p-2) \cdot (p-1) = p^2 + p + 2$.

(C) k a quadratic nonresidue: Here $b \equiv c \equiv 0$ yields no square roots, while $a + d \equiv 0$ yields $p - 1$ incongruent pairs (b, c) for each of the p incongruent a's, since $k - a^2$ cannot be $\equiv 0$. The number of incongruent square roots in this case is thus just $p(p-1) = p^2 - p$.

3. Nonscalar squares. In contrast to Proposition 1, by which a scalar matrix has at least $p^2 - p \geqq 6$ square roots, we have for nonscalar matrices

PROPOSITION 2. *A nonscalar square has at most four incongruent square roots.*

Proof. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \equiv \begin{pmatrix} e & f \\ g & h \end{pmatrix}^2 \equiv M,$$

so that

(1) $a^2 + bc \equiv e^2 + fg$, (3) $b(a + d) \equiv f(e + h)$,

(2) $d^2 + bc \equiv h^2 + fg$, (4) $c(a + d) \equiv g(e + h)$.

If M is nonscalar, $a + d$ and $e + h$ cannot $\equiv 0$. Suppose that $b \not\equiv 0$; then $f \not\equiv 0$, and we can solve (3) for $e + h$, (1) and (2) for $e^2 - h^2$, thus obtaining $e - h$, whence e and h, in terms of a, b, d, f as

$$e + h = b(a + d)/f, \qquad e - h = f(a - d)/b, \qquad g = fc/b.$$

Substituting for g and e in (1) (or g and h in (2)) yields a quadratic congruence in $(f^{-1}b)^2$ which has the roots 1 and

$$\frac{(a - d)^2 + 4bc}{(a + d)^2},$$

where it will be noted that $D=(a-d)^2+4bc$ is the discriminant of the characteristic polynomial of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If D is a quadratic residue, we thus have $f\equiv \pm b$, $\pm b(a+d)/\sqrt{D}$; if D is a nonresidue, only $f\equiv \pm b$, and in either case there are at most four possible matrices

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

If $b\equiv 0$ but $c\not\equiv 0$, an analogous argument leads to the same conclusion. But if $b\equiv c\equiv 0$, since $e+h\not\equiv 0$ we have $f\equiv g\equiv 0$, $a^2\equiv e^2$, $d^2\equiv h^2$, so that $e\equiv \pm a$, $h\equiv \pm d$ (where the signs are independent), and again M has at most four incongruent square roots.

4. Diagonalizable squares.

PROPOSITION 3. *A diagonalizable matrix which is not a scalar matrix is a square if and only if its eigenvalues are quadratic residues. Furthermore, such a square diagonalizable matrix has exactly four incongruent square roots unless one of its eigenvalues is congruent to zero, in which case it has exactly two incongruent square roots.*

Proof. If M is diagonalizable but not scalar, there exists a nonsingular matrix P such that

$$PMP^{-1} \equiv \begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix},$$

where $h\not\equiv k$ are the eigenvalues of M. If h and k are quadratic residues, say $h\equiv r^2$, $k\equiv s^2$, then

$$\begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix} \equiv \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}^2, \quad \text{and} \quad M \equiv P^{-1}\begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix} P \equiv \left[P^{-1}\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} P\right]^2.$$

Conversely, suppose that M is a square, say $M\equiv N^2$; then

$$\begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix} \equiv PMP^{-1} \equiv (PNP^{-1})^2.$$

Now

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \equiv \begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix}$$

implies $b(a+d)\equiv c(a+d)\equiv 0$ and $a^2+bc\equiv h$, $d^2+bc\equiv k$. Since $h\not\equiv k$, $a+d$ cannot

$\equiv 0$; hence $b \equiv c \equiv 0$ and $h \equiv a^2$, $k \equiv d^2$. Furthermore,

$$\begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix}$$

has just the incongruent square roots

$$\begin{pmatrix} \pm a & 0 \\ 0 & \pm d \end{pmatrix},$$

where the signs are independent; there are four of these if neither h nor $k \equiv 0$, and two if h or $k \equiv 0$. Since the square roots of M are in one-to-one correspondence with those of

$$\begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix},$$

the proof of the proposition is complete.

5. Classification of squares. Since $(-M)^2 = M^2$, it is clear that every square except the zero matrix has an even number of (in particular, at least two) incongruent square roots. This observation is sharpened by

PROPOSITION 4. *A matrix has more than four incongruent square roots if and only if it is scalar; exactly four incongruent square roots, if and only if it has incongruent nonzero eigenvalues which are quadratic residues; and no or two incongruent square roots, otherwise.*

Proof. The first part is just Propositions 1–2. If M has eigenvalues which are incongruent to each other and to zero, it must be diagonalizable and hence has exactly four incongruent square roots by Proposition 3. Conversely, by the proof of Proposition 2, if

$$M \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2$$

is nonscalar and $b \equiv c \equiv 0$, M itself is diagonal, and its square roots

$$\begin{pmatrix} \pm a & 0 \\ 0 & \pm d \end{pmatrix}$$

are all incongruent if and only if $a, d \not\equiv 0$, while $a \not\equiv d$ since M is nonscalar. But if $b \not\equiv 0$ (and similarly if $c \not\equiv 0$), M can have four square roots only if D, the discriminant of the characteristic polynomial of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

is a nonzero quadratic residue, which is equivalent to the existence of two incongruent eigenvalues for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since M is nonscalar, it too then has two incongruent eigenvalues, the squares of those of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and so is diagonalizable; by Proposition 3, its four square roots are thus incongruent only if neither of its eigenvalues is zero. Finally, if M has less than four incongruent square roots, it has either none or tow by the remark just preceding this Proposition.

6. Enumeration of squares. Propositions 1–4 may be applied to an enumeration of the 2 by 2 matrices which are congruent to squares mod p as follows:

By Proposition 1, the p incongruent scalar matrices have a total of

$$p^2 + \frac{p-1}{2}(p^2+p+2) + \frac{p-1}{2}(p^2-p) = p^3 + p - 1$$

incongruent square roots. By Proposition 3, the diagonalizable matrices which have incongruent nonzero quadratic residue eigenvalues each have exactly four square roots. To enumerate these explicitly, we use

LEMMA 1. *There are* $p(p+1)(p-1)^2$ *incongruent nonsingular matrices.*

Proof.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is nonsingular if and only if $ad \not\equiv bc$. If $b \equiv 0$, this is equivalent to $ad \not\equiv 0$; this is satisfied by any c and by any $a \not\equiv 0$, $d \not\equiv 0$—a total of $p(p-1)^2$ incongruent possibilities. If $b \not\equiv 0$, this is equivalent to $c \not\equiv adb^{-1}$, which is satisfied by $p-1$ incongruent c's for any a, any d, and any $b \not\equiv 0$—a total of $p^2(p-1)^2$ incongruent possibilities. The number of incongruent nonsingular matrices is thus $p(p+1)(p-1)^2$.

LEMMA 2. *A nonscalar diagonal matrix is similar to* $p(p+1)$ *incongruent matrices.*

Proof. Let

$$D = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

be diagonal, let P and Q be nonsingular, and suppose that $PDP^{-1} \equiv QDQ^{-1}$; then $Q^{-1}P$ commutes with D. Now

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \equiv \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

requires $ub \equiv vb$, $uc \equiv vc$. Since D is nonscalar, $u \not\equiv v$; hence this implies $b \equiv c \equiv 0$, so that $Q^{-1}P$ must be diagonal,

$$\text{say} \equiv \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

where $ad \not\equiv 0$ insures nonsingularity. It follows that for each nonsingular Q, all the P's of the form

$$Q\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

where $ad \not\equiv 0$, have the property that $QDQ^{-1} \equiv PDP^{-1}$. The number of incongruent matrices which are similar to D is thus equal to the number of nonsingular Q's (found in Lemma 1) divided by the number of such P's, which is clearly the same as the number of such

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\text{'s}$$

or $(p-1)^2$; this immediately proves the Lemma.

Now the number of incongruent diagonal matrices which have incongruent nonzero quadratic residues as diagonal terms is clearly

$$\frac{p-1}{2} \cdot \frac{p-3}{2};$$

however, since

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

is similar to

$$\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix},$$

only half of these should be counted. It follows by Propositions 3 and 4 and Lemma 2 that the number of incongruent matrices which have exactly four incongruent square roots is

$$\frac{p(p+1)(p-1)(p-3)}{8}.$$

The number of incongruent square roots of such matrices is thus

$$\frac{p(p+1)(p-1)(p-3)}{2} \quad \text{or} \quad \frac{p^4-3p^3-p^2+3p}{2}.$$

By Proposition 4, finally, if a matrix is not the square root of a scalar matrix or of a matrix which has incongruent nonzero quadratic residue eigenvalues, its square has exactly two incongruent square roots. Now the number of such matrices is just

$$p^4-(p^3+p-1)-\frac{p^4-3p^3-p^2+3p}{2}=\frac{p^4+p^3+p^2-5p+2}{2};$$

hence the number of incongruent squares not yet counted is just half of this. It follows that the total number of incongruent squares is

$$p+\frac{p^4-3p^3-p^2+3p}{8}+\frac{p^4+p^3+p^2-5p+2}{4}=\frac{3p^4-p^3+p^2+p+4}{8}.$$

It is of interest to enumerate separately the singular and nonsingular squares. To this end, note that the scalar matrices are all nonsingular with the exception of the zero matrix, and that the matrices which have exactly four incongruent square roots are all nonsingular. Since a matrix is singular if and only if its square is singular, it follows by Lemma 1 and Proposition 1 that the number of incongruent singular matrices whose squares have exactly two square roots is $p^4-p(p+1)(p-1)^2-p^2$ or p^3-p. The number of incongruent singular squares is thus $\{(p^3-p)/2\}+1$. Combining this with the total number of incongruent squares found above, the results of this section may be summed up in

PROPOSITION 5. *There are*

$$\frac{p^3-p+2}{2}=\frac{p(p^2-1)}{2}+1$$

incongruent singular squares, and

$$\frac{3p^4-5p^3+p^2+5p-4}{8}=\frac{p^2-1}{8}(3p^2-5p+4)$$

incongruent nonsingular squares.

The author wishes to express his appreciation to the referee for calling his attention to "Scalar Polynomial Equations for Matrices over a Finite Field" by John H. Hodges [Duke Math. J., 25 (1958) 291–296], which treats the more general problem of enumerating the roots of an arbitrary polynomial matrix equation over a finite field, with emphasis on the case of matrix roots of unity.

ON THE NUMBER OF PAIRS OF GENERALIZED INTEGERS WITH LEAST COMMON MULTIPLE NOT EXCEEDING x

E. M. HORADAM, The University of New England, Australia

Generalized Integers are defined as follows: Suppose there is given a finite or infinite sequence $\{p\}$ of real numbers which we will call generalized primes such that $1 < p_1 < p_2 < \cdots$. Form the set $\{l\}$ of all p-products, i.e., products $p_1^{v_1}p_2^{v_2}\cdots$, where $v_1, v_2, \cdots$, are integers $\geqq 0$ of which all but a finite number are 0. Call these numbers generalized integers and suppose that the semigroup consisting of all these products is free. Then assume $\{l\}$ may be arranged as an increasing sequence: $1 = l_1 < l_2 < l_3 < \cdots, < l_n < \cdots$. Let $[x]$ denote the number of generalized integers $\leqq x$ and assume that

$$[x] = x + R(x) \quad \text{where} \quad R(x) = O(x^\alpha) \quad \text{and} \quad 0 < \alpha < 1. \tag{1}$$

This assumption, incidentally, ensures that the number of generalized primes is infinite. We say $d \mid l_n$ if $d \in \{l\}$ and there exists $\delta \in \{l\}$ such that $d\delta = l_n$.

Denote by (l_r, l_s) the greatest common divisor of l_r and l_s. Then the number d is a unitary divisor of l_n if $(d, \delta) = 1$. Let $[l_r, l_s]$ denote the least common multiple of l_r and l_s; $\tau^*(l_n)$ denote the number of unitary divisors of l_n; $t(l_n)$ denote the number of ordered pairs of generalized integers l_r and l_s with $[l_r, l_s] = l_n$ and $\theta(x)$ the number of ordered pairs of generalized integers with least common multiple not exceeding x.

The object of this note is to find an estimate for $\theta(x)$ using the result given in [1]; namely

$$T^*(x) = \sum_{l_n \leqq x} \tau^*(l_n) = \frac{x}{\zeta(2)}\left\{\log x - \frac{2\zeta'(2)}{\zeta(2)} + 2\gamma_1 - 1\right\} + O(x^{(1+\alpha)/2}), \tag{2}$$

where

$$\zeta(s) = \sum_{l_n=1}^{\infty} l_n^{-s} \quad (s > 1),$$

and $\zeta'(s)$ is its derivative.

Again, as described in [2], Abel's theorem is used together with assumption (1) to give the following results, (γ_β, γ_1 and δ_1 are constants),

$$\sum_{l_n \leqq x} \frac{1}{l_n^\beta} = \frac{x^{1-\beta}}{1-\beta} + \gamma_\beta + O(x^{\alpha-\beta}), \qquad \beta \neq 1, \tag{3}$$

$$\sum_{l_n \leqq x} \frac{1}{l_n} = \log x + \gamma_1 + O(x^{\alpha-1}), \tag{4}$$

$$\sum_{l_n \leqq x} \frac{\log l_n}{l_n} = \frac{1}{2}\log^2 x + \delta_1 + O(x^{\alpha-1}\log x). \tag{5}$$

THEOREM.

$$\theta(x) = \frac{x \log x}{\zeta(2)}\left\{\frac{\log x}{2} - 2\frac{\zeta'(2)}{\zeta(2)} + 3\gamma_1 - 1\right\} + O(x).$$

Proof. The proof given by Cesàro in [3] that $t(n) = \sum_{d|n} \tau^*(d)$ is immediately applicable to generalized integers giving

$$t(l_n) = \sum_{d|l_n} \tau^*(d). \tag{6}$$

Then $\theta(x) = \sum_{l_n \leqq x} t(l_n)$ (by definition), $= \sum_{l_n \leqq x} \sum_{d|l_n} \tau^*(d)$ (from (6)), $= \sum_{d\delta \leqq x} \tau^*(d) = \sum_{\delta \leqq x} \sum_{d \leqq x/\delta} \tau^*(d) = \sum_{\delta \leqq x} T^*(x/\delta)$ and, after substitution from (2)

$$\begin{aligned}
\theta(x) &= \sum_{\delta \leqq x}\left[\frac{x}{\zeta(2)}\cdot\frac{1}{\delta}\left\{\log x - \log\delta - \frac{2\zeta'(2)}{\zeta(2)} + 2\gamma_1 - 1\right\} + O\left(\frac{x}{\delta}\right)^{(1+\alpha)/2}\right] \\
&= \frac{x}{\zeta(2)}\left(\log x - 2\frac{\zeta'(2)}{\zeta(2)} + 2\gamma_1 - 1\right)\sum_{\delta \leqq x}\frac{1}{\delta} \\
&\quad - \frac{x}{\zeta(2)}\sum_{\delta \leqq x}\frac{\log\delta}{\delta} + O\left\{x^{(1+\alpha)/2}\sum_{\delta \leqq x}\left(\frac{1}{\delta}\right)^{(1+\alpha)/2}\right\} \\
&= \frac{x}{\zeta(2)}\left(\log x - 2\frac{\zeta'(2)}{\zeta(2)} + 2\gamma_1 - 1\right)(\log x + \gamma_1 + O(x^{\alpha-1})) \\
&\quad - \frac{x}{\zeta(2)}\left(\tfrac{1}{2}\log^2 x + \delta_1 + O(x^{\alpha-1}\log x)\right) + O(x^{(1+\alpha)/2}\cdot x^{1-(1+\alpha)/2})
\end{aligned}$$

(from (3), (4) and (5))

$$= \frac{x \log x}{\zeta(2)}\left(\frac{\log x}{2} - 2\frac{\zeta'(2)}{\zeta(2)} + 3\gamma_1 - 1\right) + O(x).$$

This is the required result.

A more exact estimation of $\theta(x)$ for natural numbers using nonelementary methods is given by H. Jager in [4].

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A RING OF SEQUENCES GENERATED BY RATIONAL FUNCTIONS

D. A. KLARNER, McMaster University, Canada

1. A Ring of Sequences. Let D represent the set of all sequences of complex numbers which satisfy linear homogeneous difference equations with constant coefficients: that is, a sequence of complex numbers $\{a_n\}$ is in D if, and only if, there exist constants $\alpha_1, \alpha_2, \cdots, \alpha_k$ such that $\alpha_1 \neq 0$ and

$$\alpha_1 a_n + \alpha_2 a_{n+1} + \cdots + \alpha_k a_{n+k-1} = a_{n+k}, \qquad n = 0, 1, \cdots. \tag{1}$$

Another necessary and sufficient condition that $\{a_n\}$ belongs to D is that there exist polynomials $p(x)$ and $q(x)$ such that

$$p(x)/q(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{2}$$

with the degree of q greater than the degree of p; we will refer to p/q as the generator of $\{a_n\}$.

The following theorem, though apparently known by some mathematicians, seems to appear nowhere in the literature.

THEOREM 1. *The sequences of D form a commutative ring under term by term addition and multiplication.*

Before giving a proof of this theorem, we note that the sequences of D do not form an integral domain under these operations since the sequences $\{1, 0, 1, 0, \cdots\}$ and $\{0, 1, 0, 1, \cdots\}$ generated by $1/(1-x^2)$ and $x/(1-x^2)$, respectively, have the product $\{0, 0, 0, 0, \cdots\}$ which is the zero of the ring.

Proof of the Theorem. The additive identity of the ring is $\{0\}$ and the multiplicative identity is $\{1\}$, generated by $1/(1-x)$. Since the generator of $\{a_n+b_n\}$ is found by adding the generators of $\{a_n\}$ and $\{b_n\}$, it is clear that the sequences of D form an additive Abelian group. In fact, the only ring property which is not immediately obvious is that the term by term product of any two sequences in D is also in D. To prove this we recall an observation due to Hadamard [5]: If $A(x) = \sum a_n x^n$, $B(x) = \sum b_n x^n$, and $C(x) = \sum a_n b_n x^n$, then

$$C(x) = \frac{1}{2\pi i} \int_\gamma A\left(\frac{x}{s}\right) B(s) \frac{ds}{s}, \tag{3}$$

where γ is a contour in the s plane which includes the singularities of $A(x/s)/s$ and excludes the singularities of $B(s)$.

Suppose $p(x)$ is a polynomial of degree less than n and let $A(x) = p(x)/(1-\theta_1 x)^{e_1} \cdots (1-\theta_k x)^{e_k}$ with the θ_j distinct and $e_1+e_2+\cdots+e_j=n$; then

$$\frac{1}{2\pi i} \int_\gamma A\left(\frac{x}{s}\right) B(s) \frac{ds}{s}$$

$$(4) \qquad = \frac{1}{2\pi i}\int_{\gamma} \frac{s^{n-1}p(x/s)B(s)ds}{(s-\theta_1 x)^{e_1}\cdots(s-\theta_k x)^{e_k}}$$

$$= \sum_{j=1}^{k} \frac{1}{e_j!}\frac{\partial^{e_j-1}}{\partial s^{e_j-1}}\left\{\frac{s^{n-1}p(x/s)B(s)}{(s-\theta_1 x)^{e_1}\cdots(s-\theta_{j-1}x)^{e_{j-1}}(s-\theta_{j+1}x)^{e_{j+1}}\cdots}\right\}.$$

Now if B is a rational function, then so is the last member of (4); in fact, each of the summands in the last member of (4) generate sequences in D. Hence $\{a_n \cdot b_n\}$ is in D if $\{a_n\}$ and $\{b_n\}$ are in D.

Rational functions p/q, with the coefficients of p and q real numbers or integers and the degree of p less than the degree of q, generate sequences of real or rational numbers respectively; these subsets of D also form subrings of the ring D. The subset K of D which contains only sequences of integers is also a subring of D. A characterization of the generating functions of the sequences in K was given by Fatou.

THEOREM 2. *If $\{a_n\}$ is a sequence of integers generated by an irreducible rational function p/q, then p and q are polynomials with integer coefficients and $q(0)=1$.*

A proof of this theorem can be found in Salem's book [4].

One of the implications of the fact that K is a ring under term by term addition and multiplication is that $\{a_n^k\} \in K$ if $\{a_n\} \in K$. If p_k/q_k is the generating function of $\{a_n^k\}$ we can use (4) to find a recurrence formula for $\{p_k/q_k\}$; in fact, going back to (3),

$$(5) \qquad \frac{p_{k+1}(x)}{q_{k+1}(x)} = \frac{1}{2\pi i}\int_{\gamma} \frac{p_1(x/s)}{q_1(x/s)}\frac{p_k(s)}{q_k(s)}\frac{ds}{s},$$

where γ is defined as in (3); the integration is trivial if we know the order of the zeros in the s plane of $q_1(x/s)$.

Riordan [3] found generating functions for the kth powers of the Fibonacci numbers ($F_0=0$, $F_1=1$, $F_{n+2}=F_{n+1}+F_n$), and Carlitz [1] generalized these results to arbitrary second order sequences. Gould [2] has also discussed the generating functions of powers of certain second order sequences. We should note that the methods employed by these authors could be generalized to treat the sequences under discussion here; our introduction of the contour integral may be justified, however, from the point of view that proofs are shortened.

2. Some Algebraic Properties of K. We now turn our attention to the problem of characterizing the associates of the multiplicative unit in K. To do this we prove

THEOREM 3. *Let $\{a_n\} \in D$ and suppose $\{a_n\}$ has only a finite number of distinct range elements, then $\{a_n\}$ is periodic and is generated by the not necessarily irreducible rational function $p(x)/(1-x^p)$, where p is the length of some period of $\{a_n\}$.*

Proof. Suppose $\{a_n\} \in D$ with

$$\alpha_1 a_n + \alpha_2 a_{n+1} + \cdots + \alpha_j a_{n+j-1} = a_{n+j}, \qquad n = 0, 1, \cdots, \tag{6}$$

and that the range of $\{a_n\}$ contains exactly k elements.

The maximum number of distinct ordered sets $A_n = (a_n, a_{n+1}, \cdots, a_{n+j-1})$ is less than or equal to k^j since a_{n+i} can be selected in k ways for $i = 0, \cdots, j-1$. Thus, some pair of the ordered sets $A_0, A_j, \cdots, A_{jk^j}$ must be identical, say $A_{rj} = A_{sj}$ with r minimal. Using (6) we see that

$$(a_{rj}, \cdots, a_{sj-1}) = (a_{sj}, \cdots, a_{2sj-rj-1});$$

consequently, $\{a_n\}$ is periodic from a_{rj} onward with period of length $sj-rj$. This gives

$$\begin{aligned} \sum_{n=0}^{\infty} a_n X^n &= \sum_{n=0}^{rj-1} a_n X^n + \sum_{n=0}^{\infty} (a_{rj}X^{rj} + \cdots + a_{sj-1}X^{sj-1})X^{jn(s-r)} \\ &= \frac{a_{rj}X^{rj} + \cdots + a_{sj-1}X^{sj-1} + (1 - X^{js-jr}) \sum_{n=0}^{rj-1} a_n X^n}{1 - X^{js-jr}}. \end{aligned} \tag{7}$$

Collecting the coefficients of x^n in the numerator of the rational function in (7) we have

$$\begin{aligned} &(a_{sj-1} - a_{rj-1})X^{sj-1} + (a_{sj-2} - a_{rj-2})X^{sj-2} + \cdots \\ &\qquad + (a_{sj-j} - a_{rj-j})X^{sj-j} + \cdots; \end{aligned} \tag{8}$$

but not all of the quantities $a_{sj-i} - a_{rj-i} = 0$ for $i = 1, 2, \cdots, j$ since this would contradict the minimal property of r. Hence the degree of the polynomial in (8) is $js-i$ where $i < j$. Since $\{a_n\} \in D$ we must have $js-jr > js-i$, hence $r = 0$. Thus,

$$\sum_{n=0}^{\infty} a_n x^n = \frac{p(x)}{1 - x^{js}}, \tag{9}$$

where the degree of $p(x)$ is less than js, which implies $\{a_n\}$ is periodic. This completes the proof.

COROLLARY. *A sequence $\{a_n\} \in K$ is a divisor of $\{1, 1, \cdots\}$ if, and only if, $\{a_n\}$ is a periodic sequence of positive and negative units.*

Proof. This is an immediate consequence of Theorem 3 and the definition of K.

Ordinarily, in a commutative ring R with a unit, we say an element $r \in R$ is irreducible if, and only if, every factorization $r = r_1 \cdot r_2$ implies r_1 or r_2 is a unit of R. Adopting this convention for the ring K, it turns out that the only irreducible elements in K are the units of K. To prove this suppose the contrary; then if $\{a_n\}$ is not a unit and is irreducible there are at least two elements a_i, a_j of $\{a_n\}$ such that $|a_i|, |a_j| \neq 1$. It is easy to check that the sequences

$$A = \{a_0, \cdots, a_i, \cdots, a_{j-1}, 1, 1, \cdots, 1, a_{2j}, \cdots, a_{2j-1}, 1, 1, \cdots, 1, a_{4j}, \cdots\} \tag{10}$$

(11) $B=\{1, \cdots, 1, \cdots, 1, a_j, a_{j+1}, \cdots, a_{2j-1}, 1, 1, \cdots, 1, a_{2j}, \cdots, a_{4j-1}, 1, \cdots\}$

are elements of K; since $a_i \in A$, $a_j \in B$ and $A \times B = \{a_n\}$ we have a contradiction.

Another interesting example of factorization in K is the following: Let $\{F_n\}$ denote the Fibonacci sequence and recall that F_a divides F_{ab} for all natural numbers b. The sequence $\{a_n\}$ defined by

$$a_n = \begin{cases} F_n/F_j, & \text{if } j \text{ divides } n \\ F_n, & \text{otherwise} \end{cases} \tag{12}$$

is generated by a rational function since the periodic sequence $\{1, 1, \cdots, 1/F_j, 1, 1, \cdots, 1/F_j, \cdots\}$ and $\{F_n\}$ are both generated by rational functions; furthermore $\{a_n\}$ is a sequence of integers and is therefore in K. Also, the periodic sequence $F=\{1, 1, \cdots, F_j, 1, 1, \cdots, F_j, \cdots\}$ is in K; clearly $\{a_n\} \times F = \{F_n\}$. In this and the previous example, certain periodic subsequences have made it possible to factor sequences of K in a highly artificial way. For example, it would be more interesting to say an increasing sequence in K is irreducible if it has no factors which are increasing sequences in K. It seems difficult to ascertain whether or not the Fibonacci or Lucas sequences are irreducible in this sense.

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SUMS OF SQUARES OF POLYNOMIALS WITH COEFFICIENTS IN A FINITE FIELD

WILLIAM LEAHEY, University of Illinois

Let F be a finite field of odd characteristic p and denote by $F[X]$ the ring of polynomials with coefficients in F. Suppose $f(X) \in F[X]$. When do there exist $g(X)$, $h(X) \in F[X]$ such that $f(X) = (g(X))^2 + (h(X))^2$? This question and the questions of the representation of $f(X)$ as a sum of an arbitrary number of squares and the number of such representations have been studied in great detail in a series of papers by L. Carlitz and E. Cohen (see bibliography). These authors obtain necessary and sufficient conditions for representability and the

number of possible representations. In particular, in [5, section 5] Cohen deals explicitly with the problem of expressing a polynomial as a sum of two squares.

The object of this paper is to give a simple and somewhat different criterion for deciding when a polynomial can be written as a sum of two squares. In fact the theorem to be proved is very much analogous to the well-known theorem of Fermat [7, Chapter V] on the representability of a positive integer as a sum of two squares. The irreducible polynomials of even degree play the role of the primes congruent to 1 modulo 4 and the irreducible polynomials of odd degree play the role of the primes congruent to 3 modulo 4. The following lemma will be used.

LEMMA. *Let F be a finite field and suppose $a \in F$. Then there exist $b, c \in F$ such that $a = b^2 + c^2$.*

Proof. Let F have order p^n. If $p = 2$ then the map $x \to x^2$ is onto. Thus for any $a \in F$ there exists an $x \in F$ such that $a = x^2 = x^2 + 0^2$.

For p odd the image of $x \to x^2$ contains $t = (p^n - 1)/2 + 1$ elements. Let these elements be $x_1^2, x_2^2, \cdots, x_t^2$. Now for $a \in F$ the elements $a - x_1^2, a - x_2^2, \cdots, a - x_t^2$ are all distinct. Since $2t = p^n + 1 > p^n$ it follows that $\{a - x_1^2, \cdots, a - x_t^2\} \cap \{x_1^2, \cdots, x_t^2\}$ is not empty. Therefore there exist j and k such that $a - x_j^2 = x_k^2$, i.e. $a = x_j^2 + x_k^2$. This completes the proof of the lemma.

Before stating the theorem, we note that if $i = \sqrt{-1} \in F$, then any $f(X) \in F[X]$ can always be written as a sum of two squares. Thus one can write

$$f(X) = [(f(X) + 1)/2]^2 + (i)^2 \cdot [(f(X) - 1)/2]^2.$$

If $p \equiv 1 \pmod 4$ then F contains i. On the other hand if $p \equiv 3 \pmod 4$ then F contains i if and only if the order of F is p^n with n even. Thus the fields F of interest are those of order p^n with $p \equiv 3 \pmod 4$ and n odd.

THEOREM. *Let F be a finite field of order p^n where p is a prime congruent to 3 modulo 4 and n is odd. Let $f(X) \in F[X]$ and suppose that*

$$f(X) = a(f_1(X))^{e_1} \cdots (f_s(X))^{e_s}$$

with $a \in F$, and $f_j(X) \in F[X]$ is the factorization of $f(X)$ into an element of F and monic irreducible polynomials in $F[X]$. Then $f(X)$ can be written as a sum of two squares if and only if e_j is even for those polynomials $f_j(X)$ which are of odd degree.

Proof. Suppose that $f(X) = (g(X))^2 + (h(X))^2$. Let $f_j(X)$ be an irreducible factor of $f(X)$ of odd degree k and suppose that e_j is odd. If neither $g(X)$ nor $h(X)$ is divisible by $f_j(X)$ then $(g(X))^2 \equiv -(h(X))^2 \pmod{f_j(X)}$. Furthermore, since $f_j(X) \mid h(X)$, there exists a polynomial $h_1(X)$ such that $h_1(X) \cdot h(X) \equiv 1 \pmod{f_j(X)}$. Then $(g(X) \cdot h_1(X))^2 \equiv -1 \pmod{f_j(X)}$.

If F' is the extension of F obtained by adjoining a root of $f_j(X)$ to F then this last equation says that F' contains i. But this is a contradiction since F' contains p^{nk} elements with $p \equiv 3 \pmod 4$ and both n and k odd.

If $g(X)$ or $h(X)$ is divisible by $f_j(X)$ then, for some positive integer k, $f(X)=(f_j(X)^{2i}[(g'(X))^2+(h'(X))^2]$, where neither $g'(X)$ nor $h'(X)$ is divisible by $f_j(X)$. (Recall that an odd power of $f_j(X)$ divides $f(X)$, i.e. e_j was assumed to be odd.) Then, after dividing out by $(f_j(X))^{2i}$, one can repeat the above argument.

Thus, if $f(X)$ is a sum of two squares it is necessary that the irreducible factors of $f(X)$ of odd degree occur to even powers.

By virtue of the lemma and the identity

$$(x^2 + y^2)(z^2 + w^2) = (xz + yw)^2 + (xw - yz)^2,$$

in order to show that the conditions of the theorem are sufficient it suffices to show that any monic irreducible polynomial $f(X)\in F[X]$ of even degree can be written as a sum of two squares.

First let $f(X)$ be any monic irreducible polynomial in $F[X]$ and let F_1 be the extension of F obtained by adjoining i to F. Suppose $f(X)=p_1(X)p_2(X)\cdots p_s(X)$ is the factorization of f in $F_1[X]$ into monic irreducible polynomials. For any $g(X)\in F_1[X]$ denote by $\bar{g}(X)$ the conjugate polynomial obtained by replacing each coefficient $\alpha+\beta i$, $\alpha, \beta\in F$ of g by $\alpha-\beta i$. Then $f=\bar{f}=\bar{p}_1\bar{p}_2\cdots\bar{p}_s$. Multiplying these two factorizations of f together yields $f^2=(p_1\bar{p}_1)\cdots(p_s\bar{p}_s)$. Since f is irreducible and each $p_j\bar{p}_j\in F[X]$ there are only two possibilities: (i) $s=1$ and $f=p_1$ or (ii) $s=2$ and $f=p_1\bar{p}_1$. Thus any monic irreducible polynomial in $F[X]$ either remains irreducible in $F_1[X]$ or splits into a product of two irreducible polynomials which are conjugate over F.

Now suppose that $f(X)\in F[X]$ is a monic irreducible polynomial of even degree k. Let F' be the field obtained by adjoining a root of f to F. The order of F' is p^{nk}. Since nk is even F' contains i and therefore the equation $(g(X))^2\equiv -1 \pmod{f(X)}$ is solvable for $f(X)\in F[X]$. It follows that $f(X)$ could not remain irreducible in $F_1[X]$; for otherwise one would have either

$$f(X) \mid (g(X) + i) \quad \text{or} \quad f(X) \mid (g(X) - i)$$

in $F_1[X]$. But this is not possible. For example, if $g(X)+i=f(X)\cdot h(X)$ with $h(X)\in F_1[X]$, writing

$$h(X) = h_1(X) + ih_2(X) \quad \text{with} \quad h_j(X) \in F[X] \quad \text{for } j = 1, 2$$

leads to the impossible equation $f(X)\cdot h_2(X)=1$. Thus an irreducible polynomial $f(X)\in F[X]$ of even degree must split into a product of two conjugate polynomials in $F_1[X]$, i.e.

$$\begin{aligned} f(X) &= (g(X) + ih(X))\cdot(g(X) - ih(X)) \\ &= (g(X))^2 + (h(X))^2, \qquad g(X), h(X) \in F[X]. \end{aligned}$$

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THE NUMBER OF k-SIMPLICES IN THE mth BARYCENTRIC SUBDIVISION OF AN r-SIMPLEX

D. G. HOEL, University of North Carolina, Chapel Hill

In this note some techniques from combinatorial probability are employed to determine the number $f(r, k, m)$ of k-simplices in the mth barycentric subdivision $\sigma^m A_r$ of the complex A_r consisting of an r-simplex together with its faces.

The problem reduces to studying the abstract complex $S=(A, \mathcal{P}(A))$ of which A_r is the geometric realization. Here $A=\{a_0, \cdots, a_r\}$ denotes the set of 0-simplices of A_r and $\mathcal{P}(A)$, the power set of A, is the set of (abstract) simplices of S. Following L. Bers ([1], p. 53), the barycentric subdivision σS of S is defined as follows: $\sigma S=(\mathcal{P}(A), L)$ where $L\subseteq\mathcal{P}(\mathcal{P}(A))$ and $\{S_{i_0}, \cdots, S_{i_j}\}\in L$ if and only if there exists a permutation of i, say k, such that $S_{k_0}\subseteq S_{k_1}\subseteq \cdots \subseteq S_{k_j}$. Thus $f(r, k, 1)$ is equal to the number of distinct collections $\{S_0, \cdots, S_k\}$ such that $S_0\subset S_1\subset \cdots \subset S_k$ where $S_i\in\mathcal{P}(A)$. Letting $E_0=S_0$, $E_i=S_i-S_{i-1}$ for $i=1, \cdots, k$ and $E_{k+1}=A-S_k$, we see that $f(r, k, 1)$ is equal to the number of distinct collections $\{E_0, \cdots, E_{k+1}\}$ where the sets $E_0, \cdots, E_{k+1}$ are pairwise disjoint and $E_0, \cdots, E_k$ are nonempty. Hence $f(r, k, 1)$ is equal to the number of ways of arranging $k+1$ balls (elements of A) in $k+2$ cells $(E_0, \cdots, E_{k+1})$ such that the first $k+1$ cells are not empty. Thus we have

$$f(r, k, 1) = N(r+1, k+2) + N(r+1, k+1), \tag{1}$$

where $N(r, k)$ denotes the number of ways of distributing r balls in k cells such that no cell is empty. Now it can be shown (see [2], p. 58) that

$$N(r, k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^r.$$

Thus, by (1),

$$\begin{aligned} f(r, k, 1) &= \sum_{i=0}^{k+1} (-1)^i \binom{k+2}{i} (k+2-i)^{r+1} + \sum_{i=0}^{k} (-1)^i \binom{k+1}{i} (k+1-i)^{r+1} \\ &= (k+2)^{r+1} + \sum_{i=1}^{k+1} (-1)^i \binom{k+2}{i} (k+2-i)^{r+1} \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{k+1} (-1)^i \binom{k+1}{i-1} (k+2-i)^{r+1} \\
&= (k+2)^{r+1} + \sum_{i=1}^{k+1} (-1)^i \binom{k+1}{i} (k+2-i)^{r+1} \\
&= \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (k+2-i)^{r+1}.
\end{aligned} \tag{2}
$$

Let b_{jk} denote the number of k-simplices of σA_j whose interior intersects the interior of A_j. Then we have

$$f(r, k, m) = \sum_{i=0}^{r} f(r, i, m-1) b_{ik}$$

and, by letting

$$F_m = \begin{bmatrix} f(0,0,m) & \cdots & f(0,r,m) \\ \vdots & & \vdots \\ f(r,0,m) & \cdots & f(r,r,m) \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{00} & \cdots & b_{0r} \\ \vdots & & \vdots \\ b_{r0} & \cdots & b_{rr} \end{bmatrix},$$

we see that $F_m = F_{m-1}B$. Therefore, $F_m = F_0(F_0^{-1}F_1)^m$, where

$$f(i, j, 0) = \binom{i+1}{j+1}.$$

One notices that the general term of F_0^{-1} is $(-1)^{i+j}\binom{i}{j}$. Thus it is a matter of straightforward calculations similar to those in (2), to obtain

$$F_m = F_0 N^m,$$

where the general term of N is $N(i, j)$, which was defined above.

The author gratefully acknowledges the advice of Dr. I. M. Chakravarti and of Dr. M. E. Watkins, who posed this problem.

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AN ALMOST-MEASURE OF SYMMETRY

R. L. TENNISON, University of Texas at Arlington

An *affine invariant measure of symmetry* [1, p. 234] is a real-valued function f defined on the family K^n of all convex bodies in E^n satisfying:
(1) $0 \leqq f(X) \leqq 1$ for all $X \in K^n$;
(2) $f(X) = 1$ if, and only if, X has a center of symmetry;

(3) $f(X)=f[t(X)]$ for every $X\in K^n$ and every nonsingular affine transformation t of E^n onto itself;
(4) $f(X)$ is a continuous function of X.

There are many examples of measures of symmetry which appear in the literature. (See [1], pages 245–51.) Among these is the Kovner-Besicovitch measure of symmetry [1, p. 254] which is of particular interest, and which is defined as follows.

For a convex body $X\in K^n$ and a point $p\in X$, let $X(p)=X\cap(2p-X)$, and let $F(X, p)$ be the ratio of the volumes of $X(P)$ and X. The Kovner-Besicovitch measure is then the maximal element of the set $\{F(X, p):p\in X\}$.

It is the purpose of this paper to introduce a function which has all the properties listed above except possibly that of continuity. For this reason, it is called an *almost-measure of symmetry*. The family K^n will be modified so that the singletons (i.e. sets consisting of a single point) will be treated as members of K^n.

The almost-measure of symmetry to be defined is different from most measures in that its origin is algebraic rather than geometric in nature. This almost-measure depends more on the symmetry of the summands of a given set than on the set itself.

For X and Y in K^n, $X+Y$ denotes the vector sum of X and Y. X and Y are called summands of $X+Y$. The *homothetic image* λX of X by the scalar λ is the set, $\{\lambda x:x\in X\}$. A set X is 0-symmetric if, and only if, $X=(-1)X$ and is symmetric if, and only if, there is a vector $x\in E$ such that $X-x$ is 0-symmetric. The vector x is called the *center of symmetry* of X.

Let $S(X)=\{Y: Y$ is a 0-symmetric summand of X and $Y\in K^n\}$. Clearly $S(X)\neq\varnothing$. Let $V(X)$ denote the n-dimensional Lebesgue measure of X and define $V^*(X)$ as sup $\{V(Y):Y\in S(X)\}$. The almost-measure of symmetry, f, is defined by

$$f(X) = V^*(X)/V(X),\ X\neq\{x\} \quad \text{and} \quad f(\{x\}) = 0.$$

An immediate consequence of this definition is that $0\leqq f(X)\leqq 1$ for each $X\in K^n$.

Suppose that X has a center of symmetry x. Then $X=(X-x)+x$ with $X-x\in S(X)$ and $V^*(X)=V(X-x)=V(X)$ which implies that $f(X)=1$. Note that not both summands are necessarily in K^n, unless modified.

Conversely, suppose that $X\in K^n$ with $f(X)=1$. It will be shown by a sequence of lemmas that X is symmetric. The proof uses the topology of K^n induced by the Hausdorff metric,

$$\Delta(X, Y) = \inf\{\lambda:X\subset Y+\lambda B \text{ and } Y\subset X+\lambda B\},$$

where B denotes the n-dimensional unit ball.

LEMMA 1. *For each $X\in K^n$, $S(X)$ is convex. That is, if Y_1 and $Y_2\in S(X)$ and $0\leqq\lambda\leqq 1$, then $\lambda Y_1+(1-\lambda)Y_2\in S(X)$.*

Proof. There exist sets Z_1 and Z_2 such that $X=Y_1+Z_1=Y_2+Z_2$. Therefore

$$X = \lambda X + (1-\lambda)X = \lambda(Y_1 + Z_1) + (1-\lambda)(Y_2 + Z_2)$$
$$= (\lambda Y_1 + (1-\lambda)Y_2) + (\lambda Z_1 + (1-\lambda)Z_2)$$

which proves the lemma.

LEMMA 2. *For each* $X \in K^n$, $S(X)$ *is a closed set.*

Proof. Any convergent sequence $\{X_n\}$ of 0-symmetric sets has a 0-symmetric limit. Now, if Y_0 is a limit point of $S(X)$, then there is a sequence $\{Y_n\}$ with $Y_n \in S(X)$ for each n which converges to Y_0, and corresponding to each Y_n there is a summand Z_n. Therefore $X = Y_n + Z_n = Y_0 + Z_0$ implies that $Y_0 \in S(X)$.

LEMMA 3. *For each* $X \in K^n$ *there exists a* $Y \in S(X)$ *such that* $V^*(X) = V(Y)$.

Proof. If $S(X)$ is finite then the theorem is true since $V^*(X)$ is simply the maximum of a finite set of real numbers. Otherwise a sequence $\{Y_n\}$ may be selected from $S(X)$ with the properties that $V(Y_i) < V(Y_j)$ for $i<j$ and $\lim V(Y_n) = V^*(X)$. By the Blaschke Selection Theorem [2, p. 64] there is a subsequence $\{Y_{n'}\}$ of $\{Y_n\}$ which has a limit in $S(X)$, say Y_0. Now $Y_{n'} \to Y_0$ and the continuity of the volume V imply that $V(Y_n) \to V(Y_0) = V^*(X)$, completing the proof.

LEMMA 4. *If* Y *and* Z *belong to* $S(X)$ *with* $V(Y) = V(Z) = V^*(X)$ *then* $Y = Z$.

Proof. The set $\frac{1}{2}Y + \frac{1}{2}Z \in S(X)$ by Lemma 1. Thus

$$V(\tfrac{1}{2}Y + \tfrac{1}{2}Z) \leqq V^*(X). \tag{1}$$

The Brunn-Minkowski inequality [2, p. 97] yields

$$[V(\tfrac{1}{2}Y + \tfrac{1}{2}Z)]^{(1/n)} \geqq \tfrac{1}{2}[V(Y)]^{(1/n)} + \tfrac{1}{2}[V(Z)]^{(1/n)} \tag{2}$$

with equality holding if, and only if, Y is a positive homothetic image of Z. But $V(Y) = V(Z) = V^*(X)$ and inequality (2) imply that

$$V(\tfrac{1}{2}Y + \tfrac{1}{2}Z) \geqq V^*(X). \tag{3}$$

By (1) and (3) Y and Z are positive homothetic images. But if Y and Z are both 0-symmetric and have the same volume then $Y = Z$.

THEOREM 1. *If* $f(X) = 1$ *then* X *is a nonzero symmetric set.*

Proof. $X \neq \{0\}$ since $f(0) = 0$. Now as $f(X) = 1$, $V^*(X) = V(X)$. Let Y be the unique element of $S(X)$ such that $V(Y) = V^*(X)$ and let Z be the corresponding summand. The Brunn-Minkowski inequality gives

$$[V(X)]^{(1/n)} \geqq [V(Y)]^{(1/n)} + [V(Z)]^{(1/n)}.$$

But $V(X) = V(Y)$ implies that equality must hold and therefore X must be a translate of a homothetic image of Y. Hence X is symmetric.

THEOREM 2. *If t is a nonsingular affine transformation from E^n onto itself then* $f(X)=f[t(X)]$.

Proof. If t is a simple translation then the theorem is true by the invariance of volume under a translation. Suppose then that t is a nonsingular linear transformation, and let $|\det t|$ denote the absolute value of the determinant of a matrix representation for t. Then $V[t(X)]=|\det t|\ [V(X)]$ and for each $Y\in S(X)$, $t(Y)\in S[t(X)]$ which imply that

$$f[t(X)] = V^*[t(x)]/V[t(x)] = |\det t|\ V^*(X)/|\det t|\ V(X) = f(X).$$

Hence f is affine invariant.

One shortcoming of this measure was pointed out by the referee, namely, it is sensitive only for a subclass of convex bodies closely related to symmetric bodies. Also, as has been indicated, the question of continuity relative to the Hausdorff metric was not settled.

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A SERIES REPRESENTATION FOR EULER'S CONSTANT

A. W. ADDISON, Ohio State University

If we let γ denote Euler's constant, then

$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \sum_{m=2^{n-1}}^{2^n-1} \frac{n}{(2m)(2m+1)(2m+2)}.$$

Write $f(x)=-(x-[x])+\frac{1}{2}$, so that [1] if Re $s>0$, then

$$s\int_1^{\infty} \frac{f(x)}{x^{s+1}}\,dx = -\left(\frac{s}{s-1} - \zeta(s)\right) + \frac{1}{2}, \tag{1}$$

which for $s=1$ is equal to $\gamma-\frac{1}{2}$. Define

$$g(x) = f(x) - \frac{f(2x)}{2}, \tag{2}$$

so that

$$g(x) = \begin{cases} \frac{1}{4} & \text{if } 0 \leqq x-[x] < \frac{1}{2} \\ -\frac{1}{4} & \text{if } \frac{1}{2} \leqq x-[x] < 1. \end{cases} \tag{3}$$

In view of (2) we have $f(x)=\sum_{v=0}^{\infty} g(2^v x)/2^v$, so that for Re $s>0$

$$s\int_1^{\infty} \frac{f(x)}{x^{s+1}}\,dx = \sum_{v=0}^{\infty} \frac{1}{2^v}\, s\int_1^{\infty} \frac{g(2^v x)}{x^{s+1}}\,dx. \tag{4}$$

Write $y=2^{\nu}x$ in each integral within the summation sign of (4) so that (4) reduces to

$$\sum_{\nu=0}^{\infty} 2^{\nu(s-1)}s\int_{2^{\nu}}^{\infty}\frac{g(y)}{y^{s+1}}\,dy=\sum_{\nu=0}^{\infty}c_{\nu}(s)\cdot s\int_{2^{\nu}}^{2^{\nu+1}}\frac{g(y)}{y^{s+1}}\,dy, \tag{5}$$

where $c_{\nu}(s)=\sum_{k=0}^{\nu}2^{k(s-1)}$. Put $s=1$ in (5) and note that $c_{\nu}(1)=1+\nu$ so that

$$\int_{1}^{\infty}\frac{f(x)}{x^2}\,dx=\sum_{\nu=0}^{\infty}(1+\nu)\int_{2^{\nu}}^{2^{\nu+1}}\frac{g(y)}{y^2}\,dy,$$

But from (3) it follows that

$$\int_{m}^{m+1}\frac{g(y)}{y^2}\,dy=\frac{1}{4}\int_{m}^{m+\frac{1}{2}}\frac{dy}{y^2}-\frac{1}{4}\int_{m+\frac{1}{2}}^{m+1}\frac{dy}{y^2}=\frac{1}{(2m)(2m+1)(2m+2)},$$

from which the result follows easily.

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A PROPERTY OF SYLOW SUBGROUPS OF FINITE GROUPS

H. S. Leonard, Jr., Carnegie Institute of Technology

Our purpose is to derive the following consequence of a theorem of Philip Hall [2, Theorem 14.4.1].

Theorem. *Let G be a finite group, p an odd prime, P a p-Sylow subgroup of G, N the normalizer of P, and H a subgroup of G containing N. Suppose G is generated by its elements of orders relatively prime to p. Suppose that P has a normal subgroup C of nilpotence class less than $p-1$. Suppose that for all $x\in G-H$, $C\not\subseteq x^{-1}Hx$. Then $P\subseteq H'$, the commutator subgroup of H.*

Remarks. Here H is its own normalizer in G, since H contains N. Hence the condition that $C\not\subseteq x^{-1}Hx$ means that H is the only conjugate of H containing C. The case of the theorem when C is abelian, and hence of class one, was formulated independently by J. G. Thompson in a letter to the author. The proof of this case is the essence of the proof of [1, Theorem 25.4], and, as pointed out to me by Professor Thompson, is contained in the proof of [1, Lemma 17.2]. The proof of the present theorem uses the same argument. For the concepts involved we refer to [2, pp. 207–211].

Proof. Suppose the hypotheses are satisfied and that $P\not\subseteq H'$. Then the group $u_p(H)$ generated by elements of H of orders relatively prime to p is a proper

GENERAL EXISTENCE CRITERIA FOR THE INVERSE OF AN OPERATOR

A. M. OSTROWSKI, University of Basel, Switzerland

It is well known that if S and T are two linear bounded operators on a Banach space B into B, and if S^{-1} exists on B, then the inequality

$$\|S - T\| < \frac{1}{\|S^{-1}\|} \tag{1}$$

implies the existence of T^{-1}.

In [1] it was noted that (1) can be replaced in the above statement by

$$\|(S - T)(I - T)\| < \frac{1}{\|S^{-1}\|}. \tag{2}$$

(See Theorem 3, pp. 273–274 of [1], where the condition is written in a different form obtained from the above replacing S by $I-S$ and T by $I-T$.)

In the following I prove the more general

THEOREM. *If S and T are two linear bounded operators on the Banach space B with their values in B, if S^{-1} exists on B and if for a polynomial $P(x)$ with $P(0) = 1$ we have*

$$\|S^{-1}(S - T)P(T)\| < 1, \tag{3}$$

then the left-sided inverse of T, T^{-1}, exists on B^*, the image of B by T, and is bounded on B^*.

Our condition contains for $P(x) \equiv 1$ the condition (1) and for $P(x) \equiv 1 - x$ the condition (2), since

$$\|S^{-1}(S - T)\| \leqq \|S^{-1}\| \, \|(S - T)\|,$$
$$\|S^{-1}(S - T)(I - T)\| \leqq \|S^{-1}\| \, \|(S - T)(I - T)\|.$$

Proof. By the assumption about $P(x)$ there exists a polynomial $Q(x)$ such that

$$P(x) - 1 = xQ(x). \tag{4}$$

Putting

$$U = S^{-1}(S - T)P(T), \tag{5}$$

we see from (3) that $(I-U)$ has an inverse on B, $(I-U)^{-1} = \sum_{\nu=0}^{\infty} U^{\nu}$.

(Cf. the comments on the use of the Liouville-Neumann series in this connection in [2], p. 4.) On the other hand it follows from (5) that, identically,

$$I - U = S^{-1}[T - (S - T)(P(T) - I)] = S^{-1}[I - (S - T)Q(T)]T$$

and, therefore, on B^* the left-sided inverse of T is

$$T^{-1} = (I - U)^{-1}S^{-1}[I - (S - T)Q(T)]. \tag{6}$$

From (6) we have further

$$\|T^{-1}\| \leqq \frac{1}{1-\|U\|}\|S^{-1}\|\,\|I-(S-T)Q(T)\|, \tag{7}$$

and we see that T^{-1} is bounded on B^*.

In applying the above theorem it is important to emphasize particularly the cases in which B^* is identical to B. This is certainly true if B is a finite dimensional space of dimension n and the operators S and T can be considered as $n\times n$ matrices. In this case, since the mapping of B on B^* by T is a one-to-one transformation, B^* must be identical to B and our theorem reads:

If S is a nonsingular matrix and (3) *is satisfied, then T is nonsingular, too.*

More generally this argument holds if the Fredholm alternative is valid for the mapping by T, for instance, if T or a power of T is compact. (Cf. the discussion of the conditions for the Fredholm alternative in [3].)

Obviously, by inverting the order of the factors in (3) we obtain the completely symmetric condition for the right-sided inverse.

Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin, under Contract No.: DA-11-022-ORD-2059.

The author is grateful to Mr. Victor Pereyra of the MRC staff for helpful discussions.

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PRIMARY ABELIAN GROUPS WITH A HEIGHT RESTRICTING CONDITION

H. E. HEATHERLY, Texas A & M University

The purpose of this paper is to consider primary abelian groups with a certain restriction on the height of the elements. This restriction, though not ruling out elements of infinite height, is sufficient to insure that any such group is the direct sum of cyclic groups or is divisible. Throughout the paper the word group will mean a p-primary abelian group.

DEFINITION. *A group G is said to be of contained height if and only if $h(x+y)\leqq h(x)+h(y)$ for each $x, y\in G$ such that $x+y\neq 0$. (Here $h(u)$ denotes the height of u in G.)*

LEMMA 1a. *If G is of contained height then every element of zero height has order p.*

Proof. If $h(y)=0$ and $py\neq 0$, we have

$1 \leqq h(py) = h[(p-1)y+y] \leqq h((p-1)y)+h(y) = 2h(y) = 0$, a contradiction.

LEMMA 1b. *If G is of contained height, every element of finite height has zero height.*

Proof. If $0 < h(x) = a < \infty$, we have $x = p^a y$, where $h(y) = 0$; by Lemma 1a, y has order p, so that $x = p^a y = 0$, a contradiction.

Thus in a group of contained height every element of finite height has zero height and order p.

Recall that a group G is reduced if and only if it has no divisible subgroups; a subgroup H of G is pure if and only if whenever $x \in H$ is divisible in G, then it is divisible in H.

LEMMA 2. *If G is reduced and of contained height, every nonzero element has zero height and order p.*

Proof. The set I of elements of G which have infinite height (in G) is a subgroup. If I is not pure, some nonzero $x \in I$ must have finite height a in I. But since x has infinite height in G, there exists $g \in G$, $g \notin I$, such that $p^{a+1}g = x$. Since $g \notin I$, it has finite height and we have

$$\infty = h(x) = h(p^{a+1}g) \leqq h((p^{a+1}-1)g)+h(g) = 2h(g) < \infty,$$

contradiction; hence I is pure. But then every element of I has infinite height in I, making I divisible; since G is reduced, this means that $I = (0)$. Thus every nonzero element of G has finite height and hence by Lemmas 1b and 1a the desired conclusion follows.

THEOREM 1. *If G is of contained height, it is either divisible or reduced.*

Proof. Write $G = D \oplus H$, where D is the maximal divisible subgroup of G and H is reduced. If H has no elements of finite height in G, then G has no elements of finite height. Either G has all elements of infinite height and hence is divisible or H has an element of finite height. We turn our attention to the case where G is not divisible. Suppose that D and H are each nontrivial. For any $z \in D$, $h(z) = \infty$; also there exists $x \in H$ such that $h(x) < \infty$, which by Lemma 1a and Lemma 1b implies that $h(x) = 0$ and x has order p. Then

$$\infty = h(z) = h((p-1)x+(x+z)) \leqq h((p-1)x)+h(x+z) = 0+0 = 0.$$

The supposition that both D and H are nontrivial is false and since G is not divisible we see that G is reduced.

It is well known that any group of bounded order is a direct sum of cyclic groups [2, Th. 6, pp. 17–18]. Instead of making use of this result the following special case will be proven in an elementary manner.

LEMMA 3. *If every element of G has order p, then G is a direct sum of cyclic groups.*

Proof. We say a subset $\{x_1, \cdots, x_n\}$ of G is linearly independent if and only if $a_1x_1+\cdots+a_nx_n=0$ implies $a_ix_i=0$, $i=1, \cdots, n$; an infinite subset A of G is linearly independent if and only if every finite subset of A is linearly independent. Let F be the set of all linearly independent subsets of G. Partially order F by set inclusion. Using Zorn's lemma we establish the existence of a maximal element M of F. Let $C(M)=\{T:T$ is a cyclic group generated by $x\in M\}$. Let $G'=\Sigma T$, where $T\in C(M)$. From the independence of M it follows that the sum is direct. Now suppose that $x\in G$, $x\notin G'$. Then $qx\notin G'$, where $0<q<p$. Consider $M\cup(x)$, where (x) is the subgroup generated by x. M is a proper subset of $M\cup(x)$. If $n_1x_1+\cdots+n_kx_k+nx=0$, where $0<n_i<p$, $0<n<p$, and $x_i\in M$, then $n_ix_i=nx=0$, for otherwise we would have $nx\in M$. Thus $M\cup(x)$ is linearly independent, contrary to the maximality of M. We see now that $G=G'=\Sigma T$, where the T are cyclic groups and the sum is direct.

COROLLARY. *If G is of contained height then it is either divisible or a direct sum of cyclic groups.*

Proof. By Theorem 1, if G is not divisible it is reduced; by Lemma 2, every nonzero element of G has zero height; by Lemma 1a, every element of G has order p, so that the desired conclusion follows from Lemma 3.

If the property of contained height is redefined so that $h(x+y)\leqq\theta_1h(x)+\theta_2h(y)$ where $0<\theta_1<\infty$ and $0<\theta_2<\infty$, then the above results still hold and the proofs remain essentially the same.

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A k-CRITICAL GRAPH OF GIVEN DENSITY

L. C. HOUSE, University of Connecticut

In this note a *graph* is a pair (G, E) where G is a finite set and E is a set of unordered pairs of distinct elements of G. The elements of G are called *vertices* and those of E *edges*. We will refer to the graph simply by G if the edge set is clear by the context. Graphs which are isomorphic by an obvious definition will not be distinguished. The notation xEy (or yEx) will be used to abbreviate $\{x, y\}\in E$. Also $E(x)=\{y\in G:xEy.\}$ A *subgraph of* (G, E) is a graph (H, F) with $H\subseteq G$ and $F\subseteq E$.

A subset A of G is called *independent* if and only if xEy is false for every x, $y\in A$. A *k-coloring of G* is a partition $\mathfrak{C}$ of G into k cells $A_1, A_2, \cdots, A_k$ where each A_i is independent. The indices 1, 2, $\cdots$ k are the "colors" and vertices x

and y "have the same color (in $\mathfrak{A}$)" means that they belong to the same cell. The *chromatic number of* G is the smallest positive integer $k(G)$ such that there exists a $k(G)$-coloring of G.

A graph is called a *complete graph on n vertices* and designated $\{n\}$ if $|\{n\}| = n$ and xEy for all distinct $x, y \in \{n\}$. The *density of* G is the largest positive integer $d(G)$ such that G has $\{d(G)\}$ as a subgraph. For any graph it is clear that $k(G) \geqq d(G)$.

A graph (G, E) is called a *k-critical graph* if and only if $k(G) = k$ and whenever (H, F) is a subgraph of (G, E) such that either H is a proper subset of G or F is a proper subset of E then $k(H) < k$. Clearly $\{k\}$ is a k-critical graph since whenever any edge $\{x, y\}$ is removed there exists a $(k-1)$-coloring of G in which x and y have the same color.

Zykov [1] has shown that for any positive integers d and k with $1 < d \leqq k$ there exists a graph G with $d(G) = d$ and $k(G) = k$. The purpose of this note is to show how we may construct a k-critical graph of density d. The construction also requires fewer vertices than Zykov's—(11 for a 4-critical graph of density two as compared to 140 in the Zykov construction, for example).

If $\mathfrak{A} = \{A_1, A_2, \cdots, A_k\}$ is a k-coloring of G and $x \in A_i$ then x is called *complete* (mod $\mathfrak{A}$) if and only if $E(x) \cap A_j \neq \varnothing$ for all $A_j \in \mathfrak{A}$ with $j \neq i$.

LEMMA 1. *Let $k(G) = k$ and $\mathfrak{A}$ be a k-coloring of G. Then for each $i = 1, 2, \cdots, k$ there exists a vertex $x \in A_i$ that is complete* (mod $\mathfrak{A}$).

Proof. Suppose that for some A_i no such vertex exists. Then any vertex in A_i can be replaced (recolored) in another cell of $\mathfrak{A}$ to produce a new coloring $\mathfrak{A}'$ in which A_i still has no complete vertex. Repeating this process eventually leads to a $k-1$ coloring of G.

LEMMA 2. *If (G, E) is k-critical and xEy then there exists a k-coloring of G in which no other vertex in G has the same color as x and no other vertex in $E(x)$ has the same color as y.*

Proof. If we remove $\{x, y\}$ from E we may color the resulting graph in $k-1$ colors. In such a coloring x and y will necessarily be the same color since otherwise the same coloring would suffice for the original graph G. We may thus recolor x using a kth color and reinsert $\{x, y\}$ to obtain the desired coloring.

THEOREM. *Given any positive integers d and k with $1 < d \leqq k$ there exists a k-critical graph of density d.*

Proof. We first note that for $d = k$ the complete graph on k vertices will suffice. The theorem is thus proved by showing that the construction is possible for the integers d and $k+1$ if it is possible for d and k. Thus, assume that (G, E) is a k-critical graph of density d. We also assume that G has r vertices.

We construct the graph (G', E') as follows. Let $G = \{g_1, g_2, \cdots, g_r\}$ and select r more vertices $X = \{x_1, x_2, \cdots x_r\}$ disjoint from G, and a final vertex z not in either G or X. Let $G' = G \cup X \cup \{z\}$. E' is defined by the rules $x_i E' g_j$ and

$g_iE'g_j$ whenever g_iEg_j and also $zE'x_i$ for each $x_i \in X$. Note that X is independent in G'.

Clearly $k(G') \geqq k$. To show that $k(G') = k+1$ we first note that a $k+1$ coloring exists by using any k-coloring of G, then coloring x_i the same as g_i and using a $(k+1)$st color on z. Thus we need only show that G' cannot be colored in k colors. But this is clear since any k-coloring of G' determines a k-coloring of G and, by Lemma 1, for each color there exists a complete vertex of that color considering only the vertices in G. If g_i is complete then x_i must have the same color. Thus all k colors are used for the vertices of X and there is no color remaining for z.

To show that G' is k-critical we consider three cases: (i) the removal of an edge connecting two vertices of G, (ii) the removal of any edge having z as one end point, and (iii) the removal of any edge connecting a vertex in G with a vertex in X. In each of these cases the remaining graph can be colored in k colors. In case (i) we can color the vertices of G in $k-1$ colors and the vertices of X in the same $k-1$ colors (by coloring x_i the same as g_i) and then a kth color for z. In case (ii) if $\{z, x_i\}$ is removed then, using only part of Lemma 2, we can color the vertices of G in k colors in such a way that no other vertex in G has the same color as g_i. The vertices of X may then be colored correspondingly so that no other vertex of X has the same color as x_i. Then z may be colored the same as x_i. In case (iii) suppose $\{x_i, g_j\}$ is removed. By Lemma 2 we may color the vertices of G with k colors in such a way that no other vertex in G has the same color as g_i and no other vertex of $E'(g_i) \cap G$ has the same color as g_j. Then the vertices of X may be colored the same as the corresponding colors of G except for x_i which is colored the same as g_j. Thus only $k-1$ colors are used on the vertices of X and z may be colored by the remaining color.

To show that $d(G') = d(G)$ where $d(G) > 1$ we note that clearly $d(G') \geqq d(G)$. If $d(G') > d(G)$ then $d(G') \geqq 3$ and there must exist a complete subgraph $\{m\}$ of G' with $m \geqq 3$, and $m > n$ for any complete subgraph $\{n\}$ of G. Clearly z is not a vertex of $\{m\}$ and no two vertices of X belong to $\{m\}$. Since the vertices of $\{m\}$ cannot all belong to G there must be one vertex x_i of X in $\{m\}$ and $m-1$ vertices from G. But then if x_i were replaced by g_i the resulting graph would be a complete $\{m\}$ in the original graph G. This contradiction completes the proof.

We note in particular that when we commence with a pentagon and repeat the above construction we can construct a critical k-chromatic graph without triangles having $(2^{k-3} \cdot 6) - 1$ vertices. An interesting question posed by Ore [2] is what is the minimum number of vertices in such a graph.

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ON THE NUMERICAL RADIUS OF A LINEAR OPERATOR

J. P. WILLIAMS and T. CRIMMINS, Institute of Science and Technology, University of Michigan

If T is a bounded linear operator on a complex Hilbert space $\mathfrak{H}$ the *numerical range* of T is defined [1] as

$$W(T) = \{(Tf, f) \mid f \in \mathfrak{H}, \|f\| = 1\},$$

and the *numerical radius* $|W(T)|$ is given by

$$|W(T)| = \sup_{\lambda \in W(T)} |\lambda|.$$

It is well known that $\frac{1}{2}\|T\| \leqq |W(T)| \leqq \|T\|$. It is also well known that these inequalities are the best possible. Equality on the right holds for any normal operator, the simplest example being an operator on a 1-dimensional Hilbert space. The simplest operator for which $|W(T)| = \frac{1}{2}\|T\|$ is the "2-dimensional shift" given by the matrix

$$S_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to show that if T is any operator on a Hilbert space such that $|W(T)| = \|T\| = 1$, and which attains its numerical radius, then there exists a 1-dimensional reducing subspace for T. We prove the analogous result:

THEOREM. *If* $|W(T)| = \frac{1}{2}$ *and* $\|T\| = 1$ *and* T *attains its norm, then there exists a 2-dimensional reducing subspace on which* T *is the 2-dimensional shift.*

Proof. By assumption, there is $f_1 \in \mathfrak{H}$ such that $\|Tf_1\| = \|f_1\| = 1$. Let $f_2 = Tf_1$. Then

$$(T^*f_2, f_1) = (f_2, Tf_1) = \|f_2\|^2 = 1.$$

Since $\|T^*\| \leqq 1$, this implies $T^*f_2 = f_1$.

For any operator A, $([\mathrm{Re}\, A]f, f) = \mathrm{Re}\,(Af, f)$ and therefore, since $\mathrm{Re}\, A$ is self-adjoint,

$$\|\mathrm{Re}\, A\| = |W(\mathrm{Re}\,[A])| \leqq |W(A)|.$$

Thus,

$$\|\mathrm{Re}[e^{i\theta}T]\| \leqq |W(e^{i\theta}T)| = |W(T)| = \tfrac{1}{2}.$$

In particular,

$$\|e^{i\theta}Tf_2 + e^{-i\theta}T^*f_2\|^2 \leqq 1,$$
$$\|Tf_2\|^2 + 2\,\mathrm{Re}[e^{2i\theta}(Tf_2, T^*f_2)] + \|T^*f_2\|^2 \leqq 1,$$
$$\|Tf_2\|^2 \leqq -2\,\mathrm{Re}[e^{2i\theta}(Tf_2, T^*f_2)]$$

for all θ. Hence $Tf_2 = 0$. Similarly, the inequality

$$\|e^{i\theta}Tf_1 + e^{-i\theta}T^*f_1\|^2 \leqq 1$$

implies that $T^*f_1 = 0$. Furthermore,

$$(f_1, f_2) = (f_1, Tf_1) = (T^*f_1, f_1) = 0.$$

Let $\mathfrak{M}$ be the subspace spanned by f_1 and f_2. Then $\mathfrak{M}$ reduces T and the matrix of $T|M$ relative to the basis $\{f_1, f_2\}$ is S_2.

As a corollary, note that the theorem yields a characterization of the 2-dimensional shift, namely, it is the only operator which is irreducible, has numerical radius $\frac{1}{2}$, norm 1 and which attains its bound.

In conclusion we point out that it is easy to construct counterexamples to show that the hypothesis that T attains its bound cannot be omitted.

Research supported in part by the National Science Foundation under grant GP-524.

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A CONTINUOUS PERIODIC FUNCTION HAS EVERY CHORD TWICE

J. B. Diaz and F. T. Metcalf, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland

A continuous periodic function has horizontal chords of all lengths; that is, if the real valued continuous function f, defined for all real x, has period $p>0$, and a is any real number, then there is at least *one* x such that $f(x+a)-f(x)=0$. The proof given in [1, p. 78] consists of two parts. It is shown there that: (1) the function $f(x+a)-f(x)$ must "change sign" (unless f also has the period a, when there is no more to be said) and (2) if $f(x+a)-f(x)$ "changes sign once in a period it must change sign at least twice," which means, actually, that Boas proves the numerically stronger result (3) f "has *two* horizontal chords of any given length, with their left-hand endpoints at different points of $[0, p)$." While the proof of (2) is elementary, based only on the intermediate value property of continuous functions, the proof of (1) involves the extraneous notions of Riemann integration and change of variable in an integral, plus the fact that if a definite integral vanishes, the continuous integrand must be identically zero or it must "change sign."

It is the purpose of this note to present an alternative direct proof of (3), based upon the precise meaning of the phrase "changes sign," which has the same elementary character as the proof of (2) just mentioned, and resembles the proof of the "universal chord theorem" [1, p. 79].

Given a, either the function $f(x+a)-f(x)$ "changes sign" or it does not "change sign." If the function $f(x+a)-f(x)$ "changes sign" (i.e., it is >0 for one value of x and <0 for another value of x) then (3) follows from the intermediate value property of continuous functions, as in the argument for (2) mentioned above. On the other hand, if $f(x+a)-f(x)$ does not "change sign," it may be supposed, without loss, that $f(x+a)-f(x)\geqq 0$ for *all* x. Let n_i and m_i,

where $m_i \geqq 1$ and $i=1, 2, 3, \cdots$, be integers such that $\lim_{i\to\infty}(m_i a - n_i p) = 0$. The existence of these integers follows, in a known way, from the elements of Diophantine approximation, by means of the "Dirichlet box principle." (Let i be a positive integer, and consider the $i+1$ differences

$$k\frac{a}{p} - \left[k\frac{a}{p}\right], \quad \text{for } k = 0, 1, 2, \cdots, i;$$

where, as usual, $[y]$ denotes the integer such that $[y] \leqq y < [y]+1$. These $i+1$ differences all belong to the interval $[0, 1)$. Therefore, at least two of them must belong to one of the i subintervals (each of length $1/i$): $[0, 1/i)$, $[1/i, 2/i)$, $\cdots$, $[(i-1)/i, 1)$; that is,

$$\left| k_1\frac{a}{p} - \left[k_1\frac{a}{p}\right] - \left(k_2\frac{a}{p} - \left[k_2\frac{a}{p}\right]\right)\right| < \frac{1}{i},$$

where, without loss, $k_1 > k_2$, say. Then one can choose $m_i = k_1 - k_2 \geqq 1$ and

$$n_i = \left[k_1\frac{a}{p}\right] - \left[k_2\frac{a}{p}\right],$$

to obtain the inequality $|m_i a - n_i p| < p/i$.) Then, for any x, one has

$$0 \leqq f(x+a) - f(x) \leqq \sum_{k=1}^{m_i} [f(x+ka) - f(x+(k-1)a)]$$
$$= f(x+m_i a) - f(x) = f(x+m_i a - n_i p) - f(x).$$

Therefore, by the continuity of f at x, it follows (upon letting i tend to ∞) that $f(x+a) - f(x) \equiv 0$, so that (3) again holds.

It follows immediately from (3) *alone* (with $\Delta^{n-1}f$ taking the place of f) that the nth order difference $\Delta^n f(x) = \Delta \cdot \Delta^{n-1} f(x)$, where $\Delta f(x) = f(x+a) - f(x)$ and $n = 2, 3, \cdots$, vanishes at least *twice* in $0 \leqq x < p$. (See [1, p. 79 and p. 174] for another argument, using integration, when $n=2$.)

Suppose now, instead, that f is merely a real-valued function, defined for all real x, having period $p>0$; and that a is any real number. The argument given above shows that either: (a) the function $f(x+a) - f(x)$ changes sign (since now f need not be continuous, it cannot be concluded that the difference $f(x+a) - f(x)$ must change sign more than once); or (b) the function $f(x+a) - f(x)$ vanishes at every x where f is continuous. Therefore, if one supposes, in addition, that: (a) the difference function $f(x+a) - f(x)$ is a Darboux function (i.e., this function always assumes every real value which lies between any two of its values); (b) f is continuous for at least one x; then one can conclude that f has at least *one* chord of length a.

This research was supported by the Air Force Office of Scientific Research under Grant AFOSR 400-64 with the University of Maryland, and by the U. S. Naval Ordnance Laboratory, White Oak, Maryland. (Present address: J.B.D., Rensselaer Polytechnic Inst. F.T.M., University of California, Riverside.)

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UNIQUENESS IN THE DIVISION ALGORITHM

M. A. JODEIT, JR., Rice University

The division algorithm for polynomials with coefficients in a field gives unique quotient and remainder. This note shows that such uniqueness characterizes polynomial rings among Euclidean domains.

An integral domain E which admits an integral-valued function g satisfying the following two conditions is called a *Euclidean domain* ([1], Section 18):

(1) $g(ab) \geqq g(a) \geqq 0$ for all $a \neq 0 \neq b$;

(2) For each $a \neq 0$ and b in E there exist q, r such that

$$b = qa + r, \qquad r = 0 \quad \text{or} \quad g(r) < g(a).$$

Quotient and remainder are *unique* if only one pair q, r satisfies (2) for given a, b. We recall these results from [1]: Each Euclidean domain has an identity 1; $g(a) < g(b)$ if a is a proper divisor of b; $g(a) = g(1)$ iff a is a unit.

THEOREM. *Let E be a Euclidean domain in which $1 \neq 0$. If quotient and remainder in E are unique the units (with zero) form a field F; if $E \neq F$, $E \cong F[x]$.*

The main point is contained in

LEMMA 1. *Quotient and remainder are unique iff* $g(a + b) \leqq \max(g(a), g(b))$.

Proof. If $a \neq 0 \neq b$ are such that $g(a + b) > \max(g(a), g(b))$, then

$$\begin{aligned} a &= 0(a + b) + a, & g(a) &< g(a + b), \\ &= 1(a + b) - b, & g(b) &< g(a + b). \end{aligned}$$

This contradicts uniqueness.

If the inequality holds and

$$\begin{aligned} c &= qa + r, & r &= 0 \quad \text{or} \quad g(r) < g(a), \\ &= q'a + r', & r' &= 0 \quad \text{or} \quad g(r') < g(a), \end{aligned}$$

with $r \neq r'$ and $q \neq q'$, then

$$g(a) \leqq g((q - q')a) = g(r' - r) < g(a).$$

Thus $r = r'$ or $q = q'$. Since either of these implies the other, uniqueness holds.

In what follows we may assume that $g(1) = 0$; if not let $g'(a) = g(a) - g(1)$; g' has all the other properties of g.

To see that the set F of units, with zero added, is a field, it is enough to show that a nonzero sum of units is a unit, and this follows from

$$0 \leqq g(u_1 + u_2) \leqq \max(g(u_1), g(u_2)) = g(1) = 0.$$

Finally, if $E \neq F$ we must find an isomorphism $E \to F[x]$. This comes from

LEMMA 2. *If $a \neq 0$ is such that $g(a)$ is a positive minimum, for each $0 \neq b \in E$ there exist unique $q_0, \cdots, q_k$ in F such that*

$$b = q_k a^k + \cdots + q_1 a + q_0, \qquad q_k \neq 0.$$

Then the mapping $b \to \sum q_j x^j$ is an isomorphism.

Proof. The sequence $g(a^k)$, $k = 0, 1, \cdots$ is strictly increasing since a^k is a proper divisor of a^{k+1}.

If $b \neq 0$, $g(a^k) \leqq g(b) < g(a^{k+1})$ for some $k \geqq 0$. Then $b = q_k a^k + r$, $r = 0$ or $g(r) < g(a^k)$. $q_k \neq 0$; if q_k were not a unit we would have

$$q_k = la + m, \qquad m \in F \quad \text{and} \quad l \neq 0.$$

By Lemma 1, $g(b) \geqq g(b - ma^k - r) = g(la^{k+1}) \geqq g(a^{k+1})$, contrary to assumption.

If $r \neq 0$, we apply the argument again. By induction

$$b = \sum_{j=0}^{k} q_j a^j, \qquad q_j \in F. \tag{3}$$

Finally, if $0 = \sum_{j=m}^{n} s_j a^j$, $s_j \in F$, $s_m \neq 0$, we have

$$-s_m = a\left(\sum_{j=m+1}^{n} s_j a^{j-m-1}\right)$$

which implies $g(1) \geqq g(a)$, and contradicts the choice of a. Thus all the s_j are zero. It follows that the representation (3) is unique.

The author wishes to thank the referee for his comments.

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A NOTE ON A CONVOLUTION-TYPE COMBINATORIAL IDENTITY

MICHAEL SKALSKY, Southern Illinois University

Using the results developed in a series of papers, H. W. Gould [2] proved the following identity

$$\sum_{k=0}^{n} (-1)^k A_k(c, tb) A_{n-k}(a + bk - k, (1 - t)b) = A_n(a - c, (1 - t)b), \tag{1}$$

where

$$A_k(a, b) = \frac{a}{a + bk}\binom{a + bk}{k}.$$

The identity (1) is valid for any numbers a, b, c, t, and for $a=c$ reduces to an orthogonality relation

$$\sum_{k=0}^{n} (-1)^k A_k(a, tb) A_{n-k}(a + bk - k, (1 - t)b) = \binom{0}{n}.$$

The purpose of this note is to derive (1) directly by use of Lagrange's formula for the inversion of power series. We state without proof Lagrange's formula [1]:

If $w=f(z)=z/\phi(z)=\sum_{n=1}^{\infty} a_n z^n$ is an analytic function, regular in a neighborhood of the point $z=0$, and satisfying the condition $a_1=f'(0)\neq 0$, then the equation $f(z)=w$ has a unique solution, regular in a neighborhood of the point $w=0$ of the form

$$z = g(w) = \sum_{n=1}^{\infty} \frac{w^n}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}(\phi(z))^n\right]_{z=0}.$$

More generally, if $F(z)$ is an analytic function regular in a neighborhood of the point $z=0$, then there is a neighborhood of the point $w=0$ in which

$$F(z) = F(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}(F'(z)\{\phi(z)\}^n)\right]_{z=0}. \tag{2}$$

With (2) given, the proof of (1) becomes simple. Choosing $\phi(z)=(1+z)^{tb}$, and $F(z)=(1+z)^c$, where b, c, t are any complex numbers, and applying (2), we obtain

$$\begin{aligned}(1 + z)^c &= 1 + \sum_{k=1}^{\infty} \frac{w^k}{k!}\left[\frac{d^{k-1}}{dz^{k-1}}\{c(1 + z)^{c-1}(1 + z)^{tbk}\}\right]_{z=0} \\ &= 1 + \sum_{k=1}^{\infty} c\frac{w^k}{k!}(c + tbk - 1)_{k-1} = \sum_{k=0}^{\infty} w^k \binom{c + tbk}{k}\frac{c}{c + tbk} \\ &= \sum_{k=0}^{\infty} w^k A_k(c, tb),\end{aligned} \tag{3}$$

where $w=z/(1+z)^{tb}$.

Replacing in (3) $1+z$ by $(1+z)^{-1}$ we get

$$(1 + z)^{-c} = \sum_{k=0}^{\infty} (-1)^k A_k(c, tb) z^k (1 + z)^{tbk-k}. \tag{4}$$

Multiplying both sides of (4) by $(1+z)^a$ we obtain

$$\begin{aligned}(1 + z)^{a-c} &= \sum_{k=0}^{\infty} (-1)^k A_k(c, tb) z^k (1 + z)^{a+tbk-k} \\ &= \sum_{k=0}^{\infty} (-1)^k A_k(c, tb)[z/(1 + z)^{b-tb}]^k (1 + z)^{a+bk-k}.\end{aligned} \tag{5}$$

Using (2) once more, with $F(z)=(1+z)^{a+bk-k}$ and $\phi(z)=(1+z)^{b-tb}$, we get from (5)

$$(1+z)^{a-c}=\sum_{k=0}^{\infty}(-1)^k A_k(c,tb)\left[z/(1+z)^{b-tb}\right]^k$$

$$\cdot\sum_{j=0}^{\infty}A_j(a+bk-k,\,b-tb)\left[z/(1+z)^{b-tb}\right]^j \tag{6}$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^k A_k(c,tb)A_j(a+bk-k,\,b-tb)v^{k+j},$$

where $v=z/(1+z)^{b-tb}$. Letting $k+j=n$, we find from (6)

$$(1+z)^{a-c}=\sum_{n=0}^{\infty}v^n\sum_{k=0}^{n}(-1)^k A_k(c,tb)A_{n-k}(a+bk-k,\,b-tb). \tag{7}$$

On the other hand, the application of (2) with $F(z)=(1+z)^{a-c}$ and $\phi(z)=(1+z)^{b-tb}$ yields

$$(1+z)^{a-c}=\sum_{n=0}^{\infty}v^n A_n(a-c,\,b-tb). \tag{8}$$

Equating the coefficients of v^n in (7) and (8), we obtain

$$\sum_{k=0}^{n}(-1)^k A_k(c,tb)A_{n-k}(a+bk-k,\,b-tb)=A_n(a-c,\,b-tb).$$

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ON A THEOREM OF H. PÉTARD

CHRISTIAN ROSELIUS, Tulane University

In a classical paper [4], H. Pétard proved that it is possible to capture a lion in the Sahara desert. He further showed [4, no. 8, footnote] that it is in fact possible to catch every lion with at most one exception. Using completely new techniques, not available to Pétard at the time, we are able to sharpen this result, and to show that *every* lion may be captured.

Let $\mathfrak{L}$ denote the category whose objects are lions, with "ancestor" as the only nontrivial morphism. Let $\mathfrak{l}$ be the category of caged lions. The subcategory $\mathfrak{l}$ is clearly complete, is nonempty (by inspection), and has both a generator and cogenerator [3, vii, 15–16]. Let $F\colon \mathfrak{l}\to\mathfrak{L}$ be the forgetful functor, which forgets the cage. By the Adjoint Functor Theorem [1, 80–91] the functor F has a coadjoint $C\colon \mathfrak{L}\to\mathfrak{l}$, which reflects each lion into a cage.

We remark that this method is obviously superior to the Good method [2], which only guarantees the capture of one lion, and which requires an application of the Weierkäfig Preparation Theorem.

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CLASSROOM NOTES

EDITED BY GEORGE RANEY, University of Connecticut

Material for this department should be sent to George Raney, Department of Mathematics, University of Connecticut, Storrs, CT 06268.

ON (WHAT SHOULD BE) A WELL-KNOWN THEOREM IN GEOMETRY

DANIEL PEDOE, University of Minnesota

The following theorem is not as well known as it should be:

(a) *If ABC is a given triangle, V a point in the plane of ABC which does not lie on a side of the triangle, and $A'B'C'$ is a triangle which has $B'C'$ parallel to VA, $C'A'$ parallel to VB and $A'B'$ parallel to VC, then lines through A' parallel to BC, through B' parallel to CA and through C' parallel to AB will be concurrent at a point V'.*

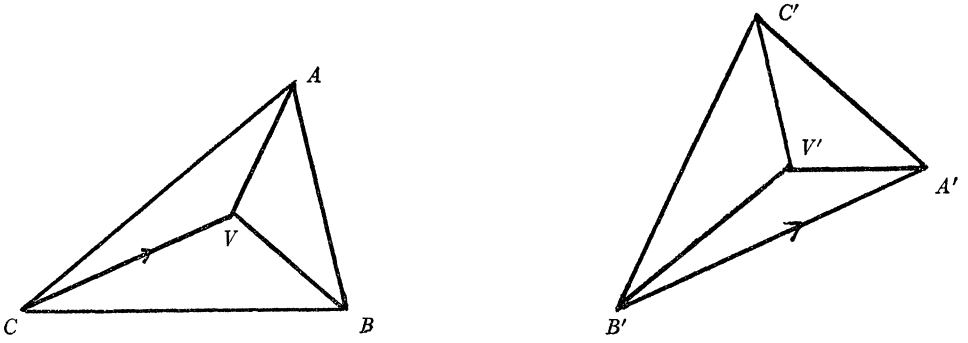


FIG. 1

The theorem is, however, more familiar to geometers in the form:

(b) *If ABC and $A'B'C'$ are two triangles which are such that lines through A*

parallel to $B'C'$, through B parallel to $C'A'$ and through C parallel to $A'B'$ meet in a point V, then the lines through A', B' and C' parallel to BC, CA and AB respectively will also meet in a point V'.

It is in this second form that this theorem appears as an exercise in the older books on coordinate geometry. There is a tendency for this theorem to be rediscovered from time to time. It has a respectable history, and is at least a hundred years old. It should be better known, perhaps.

In the form (a) it can be found in books on statics. The quadrangles $ABCV$, $A'B'C'V'$ are said to form *reciprocal figures*, and the interesting thing about the two figures is that in any one figure three lines forming the sides of a triangle are parallel to three concurrent lines in the other.

The connection with graphical statics is evident, to anyone who has studied force-diagrams. To quote Clerk Maxwell (Philosophical Magazine, (1864), p. 258): "If forces represented in magnitude by the lines of the first figure be made to act between the extremities of the corresponding lines of the reciprocal figure, then the points of the reciprocal figure will all be in equilibrium under the action of these forces."

This statement refers to general reciprocal figures. Not every figure of straight lines and points in the plane admits a reciprocal figure. Maxwell obtained his general reciprocal figures by the orthogonal projection of polyhedra reciprocal with respect to a paraboloid of revolution. After the projection, one figure has to be rotated through a right angle. The great Italian geometer Cremona, who published a book called *Reciprocal figures in graphical statics* (Oxford, 1890), preferred to use polyhedra reciprocal with respect to a null system, on the grounds that after a suitable projection the reciprocal figures were produced without one having to be rotated through a right angle!

Clerk Maxwell's early papers were on geometry, and a knowledge of elementary geometry does not seem to have hurt him! It is interesting to see how he proves Theorem (a). He considers circles with centers at the points A, B, C and V and notes that the chord of intersection of two circles is perpendicular to the line joining the centers and the chords of intersection of three circles, taken in pairs, all pass through a point. Hence the common chords of the four circles are six lines which pass in threes through four points A', B', C' and V', where $B'C' \perp VA$, $C'A' \perp VB$, $A'B' \perp VC$, $V'A' \perp BC$, $V'B' \perp CA$ and $V'C' \perp AB$. All we have to do now is to rotate the figure $A'B'C'V'$ through a right angle, and the figure reciprocal to $ABCV$ is in existence!

This involvement with right angles suggests another theorem, which also appears in books on coordinate geometry:

(c) *If the perpendiculars from the vertices A, B and C of a triangle ABC onto the sides $B'C'$, $C'A'$ and $A'B'$ of a triangle $A'B'C'$ are concurrent, then so are the perpendiculars from the vertices A', B' and C' of $A'B'C'$ onto the sides BC, CA and AB of triangle ABC.*

This theorem can be found in Steiner (Gesammelte Werke, 1 (1881) p. 157, G. Reimer, Berlin), but it is immediately derived from Theorem (b). Rotate the triangle $A'B'C'$ through a right angle. If it be called $A^*B^*C^*$ in its new position, we are told that lines through A, B and C parallel to B^*C^*, C^*A^* and A^*B^* respectively meet in a point V^*. Hence lines through A^*, B^* and C^* respectively parallel to BC, CA and AB will be concurrent in a point V. Rotate $A^*B^*C^*$ back to $A'B'C'$, and let V' be the new position of V. Then lines through A', B' and C' perpendicular to BC, CA and AB respectively intersect at V'.

To make the theorem apparently harder, we can rotate the triangle $A'B'C'$ through an angle α. The formulation of the theorem can be left to the student. Rotation brings relief, in any case.

There are a number of different proofs of (a), not as ingenious as Maxwell's proof. That version of Ceva's theorem which introduces the ratio of the sines of the angles which VA, VB and VC make with the sides of triangle ABC gives an immediate proof on rearrangement, since the Ceva condition for concurrency is also a sufficient condition. But a more fundamental proof can be given if we use projective geometry.

The six sides of the quadrangle $ABCV$ intersect any line in the plane in three pairs of points in involution, and this is also true if the line be the line at infinity. Let the intersections with the line at infinity of BC and AV be x and x', of CA and BV be y and y' and of AB and CV be z and z'. Then since $B'C'$ is parallel to VA, $C'A'$ to VB and $A'B'$ to VC we know that $B'C'$ passes through x', that $C'A'$ passes through y' and that $A'B'$ passes through z'. The fact that $A'x$, $B'y$ and $C'z$ are concurrent is immediate. For if $A'x$ meets $B'y$ in V', then $C'V'$ meets the line at infinity in the mate of z' in the involution which has (x, x') and (y, y') as pairs, and this involution is the one we are considering already. Hence $C'V'$ passes through z, and so $A'x$, $B'y$ and $C'z$ are concurrent.

ON THE INTERSECTION OF A NESTED SEQUENCE OF COMPACT SETS

B. L. McAllister, South Dakota School of Mines and Technology

1. One of the requirements of a T_1 topological space is that the intersection of any two open sets should be open. The purpose of this note is to present an example to show that the following theorem depends upon this requirement.

Theorem. *In a T_1 topological space, the elements of any nested sequence of nonempty closed sets, at least one of which is countably compact, have a nonempty intersection.*

2. To avoid possible ambiguity, terms are defined here in the sense in which they will be used. (If $\mathfrak{I}$ is any collection of sets, X will denote the underlying space, viz. $\cup\mathfrak{I}$.) We say that X (or, properly, $\mathfrak{I}$) is T_1 if for each two members p and q of X, some member of $\mathfrak{I}$ contains p but not q. A member p of X is a *limit point* of a subset S of X provided that every member of $\mathfrak{I}$ that contains p

meets $S-p$. A set is *closed* if it contains all its limit points, and *countably compact* provided that every infinite subset of it has a limit point. A sequence p_n *converges* to p if every member of $\mathfrak{I}$ that contains p contains p_n for all but a finite set of values of n.

3. Let S be the set of all polynomials with coefficients equal to 0 or 1 (including the 0-polynomial). For each k, R_k^1 is the set of all members of S such that the term x^k has coefficient 1, and R_k^0 is the set of all members of S such that the coefficient of x^k is 0. (If the degree of a polynomial is less than k, the coefficient of x^k is taken as 0.) Thus $R_k^0 \cap R_k^1 = \varnothing$. Our "pseudo topology" $\mathfrak{I}$ is the collection of all subsets of S that are unions of R_k^j's. It is at once evident that $\mathfrak{I}$ is T_1, since unequal polynomials differ in at least one coefficient.

4. Now, for $i=1, 2, \cdots$, let $M_i = \bigcap_{k=0}^{i} R_k^1$. Clearly $M_i \supset M_{i+1}$, since $\bigcap_{k=0}^{i} R_k^1 \supset \bigcap_{k=0}^{i+1} R_k^1$. Thus the sequence of M_i's is nested. Each M_i is closed, since if p is a member of $S-M_i$, then for some $k \leqq i$, $p \in R_k^0$. But $R_k^0 \in \mathfrak{I}$, and does not meet M_i (since if it did it would have to meet R_k^1). Then p is not a limit point of M_i. That is, no point outside M_i is a limit point of M_i.

On the other hand, each M_i is countably compact. For let Z be an infinite subset of M_i. Since S is countable, so is Z, say

$$Z = \{p_1, p_2, \cdots\} \quad \text{where} \quad p_k = \sum_{j=0}^{m_k} a_j^k x^j.$$

Let μ be the largest of the degrees of p_1, p_2, and p_3. We determine a polynomial $q = \sum_{j=0}^{\mu} b_j x^j$ by the following rule: For all

$$j \leqq \mu \qquad \text{if } a_j^1 = a_j^2 \quad \text{or} \quad a_j^1 = a_j^3, \quad b_j = a_j^1.$$

Otherwise, $b_j = a_j^2$ (hence also $b_j = a_j^3$). Then every member of $\mathfrak{I}$ that contains q contains either an R_k^0 or an R_k^1 that contains q and each such R_k^0 or R_k^1 contains two of the polynomials p_1, p_2, p_3, at least one of which differs from q.

Finally note that $\bigcap_{i=0}^{\infty} M_i = \varnothing$. For $0 \notin \bigcap_{i=0}^{\infty} M_i$, and for each $p \in S$, if $p \neq 0$ then p has some degree, say n. Thus

$$p \notin R_{n+1}^1, \qquad p \notin M_{n+1}, \quad \text{and} \quad p \notin \bigcap_{i=0}^{\infty} M_i.$$

REMARK. The space described above is countable and satisfies all of the requirements of a Hausdorff space except the requirement that the intersection of any two open sets be open. The author obtained this example several years ago when he was a student in a class with Professor C. E. Burgess.

The author thanks the referee for suggesting several improvements.

THE BOUNDARY VALUES OF A HOLOMORPHIC FUNCTION

P. R. CHERNOFF, Harvard University

"A function, continuous on the closed unit disc, holomorphic inside, and vanishing on an open subset of the boundary, vanishes identically. A simple proof of this well-known proposition is obtained by considering its Cauchy integral representation," [1].

Whatever the authors of [1] had in mind, it is difficult to imagine a proof simpler than the following:

Because $f(e^{i\theta})$ vanishes on an open interval, there are finitely many θ's, say $\theta_1, \cdots, \theta_n$, such that the product

$$f(ze^{i\theta_1}) \cdot f(ze^{i\theta_2}) \cdots f(ze^{i\theta_n})$$

vanishes identically on the boundary of the disc. By the maximum principle it must vanish on the entire disc. Hence one of its factors must vanish identically.

Of course, a much stronger result is known: the boundary zeros of a holomorphic function cannot have positive Lebesgue measure unless the function vanishes identically [2], but the available proofs of this fact are not nearly so elementary.

The author is an NSF graduate fellow.

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CONDITION FOR A LOOP TO BE A GROUP

DONALD WATSON, RAAF Academy, Point Cook, Australia

Zassenhaus' rectangle rule (1 and 2) provides a method for deciding whether a loop is a group. I give below a generalization which improves the rule in two ways: the multiplication table of the loop need not be in any special form; the identification is made simply from the *pattern* of the table, no multiplications being necessary.

The improved rule is this. In the multiplication table of the loop choose any four places forming the vertices of a rectangle. Suppose that the entries in these places are

$$\begin{matrix} q & r \\ p & s \end{matrix}$$

If the loop is a group then all other rectangles having p, q, r, as entries at successive vertices, with p and q sharing a column, will have s as the entry at the fourth vertex. The converse is also true.

In order to prove the rule we first express it in algebraic form. The relevant extract from the table is:

	c	d
a	q	r
b	p	s

The rule then states that a necessary and sufficient condition for a loop to be a group is that for any a, b, c, d, A, B, C, D in the loop the relations

$$(1) \qquad bc = BC, \qquad ac = AC, \qquad ad = AD,$$

should imply that $bd = BD$.

The condition is necessary since if the loop is a group, multiplication is associative and therefore

$$\begin{aligned} bd &= bc(c^{-1}a^{-1})ad = bc(ac)^{-1}ad \\ &= BC(AC)^{-1}AD = BC(C^{-1}A^{-1})AD = BD. \end{aligned}$$

To prove the sufficiency of the condition we must show that the condition implies that the loop is associative, that is,

$$x(yz) = (xy)z,$$

where x, y, z are any members of the loop. To do this we choose

$$a, b, c, d, A, B, C, D \qquad \text{as } i, x, yi, yz, iy, xy, i, z,$$

where i is the identity element of the loop. These substitutions will satisfy the relations (1) and it follows that

$$x(yz) = (xy)z.$$

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1. H. Zassenhaus, The Theory of Groups, Chelsea, New York, 1949.
2. F. D. Parker, When is a loop a group?, this MONTHLY, 72 (1965) 765–766.

Editor's Note. Professor Parker, author of the note [2] referred to in the bibliography of the preceding paper, has pointed out that the rule he stated in his paper provides a necessary, but not a sufficient, condition for a loop to be a group. This point was not made clear in Professor Parker's paper.

GRAPH EXTENSION PRESERVING CHROMATIC NUMBER

T. K. SHENG, University of Malaya, Kuala Lumpur

Let G be any finite graph without isolated vertices. In this note we give a method for extending G to a larger finite regular graph H, with the same chromatic number as G. The graph H is interesting insofar as it is entirely composed of copies of G, and the method of forming H can be likened to the way in which

chemical crystals grow. Our result was conjectured by H. H. Teh, who gave (unpublished lecture notes) an infinite extension of G.

We first illustrate the method with an example. Suppose G is given by Fig. 1, and colored with 2 colors α, β. Then we take 4 copies of G as shown in Fig. 2, and combine those vertices contained in dotted lines to obtain H in Fig. 3.

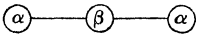


FIG. 1. Graph G.

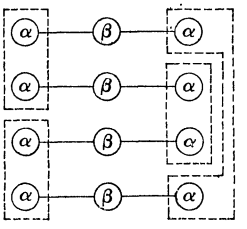


FIG. 2

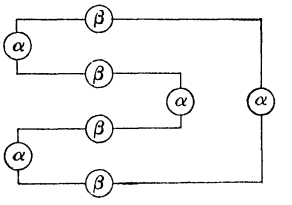


FIG. 3. Graph H.

Moreover, H is colored with α and β. It can in fact be seen that H has the properties listed in our general theorem:

THEOREM. *Let G be any finite graph with vertices $p_1, p_2, \cdots, p_n$ of nonzero local degrees $\rho_1, \rho_2, \cdots, \rho_n$. Then G can be extended to a finite graph H with the following properties*:

(a) *H is entirely composed of copies of G and any two of the copies have at most one vertex in common in H.*

(b) *H is connected (disconnected) if G is connected (disconnected).*

(c) *The chromatic number of H is the chromatic number of G.*

(d) *H is regular of degree N = least common multiple $(\rho_1, \rho_2, \cdots, \rho_n)$.*

(e) *H contains at least s subgraphs which are copies of G, where*

$$s = N^n/\rho_1\rho_2 \cdots \rho_n. \tag{1}$$

(f) *The number of vertices in H is $v = N^{n-1}(\rho_1+\rho_2+\cdots+\rho_n)/(\rho_1\rho_2\cdots\rho_n)$.*

Proof. We take s identical copies of G, where s is defined by (1), and index them as $G(x_1, x_2, \cdots, x_n)$, where for

$$i = 1, 2, \cdots, n, \qquad \text{we have } x_i = 1, 2, \cdots, N/\rho_i. \tag{2}$$

Then we combine vertices out of these copies of G to form H as follows. For given fixed i and fixed $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$ satisfying (2), we combine as a single vertex of H, the ith vertices p_i of N/ρ_i of the copies of G, namely the copies

$$G(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n); \qquad x_i = 1, 2, \cdots, N/\rho_i.$$

The resulting vertex of H has local degree $\rho_i(N/\rho_i) = N$. We do this combining process for all choices of i and $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$ satisfying (2).

Consider now the resulting graph H. By construction H is entirely composed

of copies of G. Let G_1, G_2 be any two copies, say

$$G_1 = G(x_1', x_2', \cdots, x_n') \quad \text{and} \quad G_2 = G(x_1'', x_2'', \cdots, x_n''). \tag{3}$$

Suppose G_1 and G_2 have their ith vertices p_i combined in H. Then $x_j' = x_j''$ for $1 \leqq j \leqq n$; $j \neq i$. Therefore if G_1 and G_2 are distinct they have at most one vertex in common, and so (a) holds.

Next we prove (b). If G is a disconnected graph, then since in forming vertices of H, we combined only corresponding vertices p_i, p_i of the copies of G, we must have H disconnected. Suppose on the other hand that G is connected, and let G_1, G_2 be the two copies (3) of G. Then in the sequence of copies

$$G(x_1', x_2', \cdots, x_n'), \cdots, G(x_1'', \cdots, x_{i-1}'', x_i', \cdots, x_n') \cdots,$$
$$G(x_1'', \cdots, x_i'', x_{i+1}', \cdots, x_n'), \cdots, G(x_1'', x_2'', \cdots, x_n''),$$

any two consecutive copies have one point in common. Hence H is connected.

The result (c) follows from the fact that only corresponding vertices of the copies of G are combined. We have already mentioned that each vertex of H has local degree N, and we start the construction with s copies of G, so (d) and (e) hold.

Finally to prove (f), we note that s copies of G contain

$$\tfrac{1}{2}(\rho_1 + \rho_2 + \cdots + \rho_n)s = \tfrac{1}{2}N^n(\rho_1 + \rho_2 + \cdots + \rho_n)/(\rho_1\rho_2 \cdots \rho_n)$$

edges and H contains $\frac{1}{2}Nv$ edges. Hence it follows that

$$\tfrac{1}{2}Nv = \tfrac{1}{2}N^n(\rho_1 + \rho_2 + \cdots + \rho_n)/(\rho_1\rho_2 \cdots \rho_n).$$

This proves (f).

The author would like to thank Prof. E. S. Barnes, Prof. D. E. Daykin and Prof. H. H. Teh for their encouragement and assistance in writing this paper.

ON A WELL-KNOWN IMPROPER INTEGRAL

HIROSHI HARUKI, Osaka University (Now at University of Waterloo, Canada)

We consider the well-known improper integral (a real)

$$f(a) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-x^2) \cos axdx, \tag{1}$$

which is an extension of the probability integral $(1/\sqrt{\pi})\int_{-\infty}^{+\infty} \exp(-x^2)dx (=1)$.

The usual methods of evaluating this integral ($f(a) = \exp(-a^2/4)$) are the following three:

(A) The method of using integration by parts and deducing the differential equation $f'(a) = -\frac{1}{2}af(a)$ with the initial condition $f(0) = 1$. (See [2] p. 433.)

(B) The method of using power series and termwise integration. (See [2] p. 460.)

(C) The method of using analytic function theory. (See [3] p. 87.)

Now we shall prove that $f(a)=\exp(-a^2/4)$ from the standpoint of functional equation theory by using the following lemma (see [1] pp. 74, 96):

LEMMA. *If a real-valued function $f(x)$ is continuous in $-\infty<x<+\infty$, and satisfies the functional equation $f(s)f(t)=f(\sqrt{s^2+t^2})$ for real values of s and t, then*

$$f(x) = 0 \quad \text{or} \quad f(x) = \exp(\alpha x^2),$$

where α is an arbitrary real constant.

Proof of $f(a)=\exp(-a^2/4)$: By (1) we have

$$f(s)f(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp(-x^2-y^2)\cos sx\cos ty\,dxdy, \tag{2}$$

where s, t are arbitrary real numbers. By (2) we have

$$\begin{aligned} f(s)f(t) = \frac{1}{2\pi}\Bigg(&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp(-x^2-y^2)\cos(sx+ty)\,dxdy \\ &+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp(-x^2-y^2)\cos(sx-ty)\,dxdy\Bigg) = \frac{1}{2\pi}(I_1+I_2). \end{aligned} \tag{3}$$

Putting in I_1 (when $s^2+t^2>0$)

$$x = \frac{s}{\sqrt{s^2+t^2}}u - \frac{t}{\sqrt{s^2+t^2}}v, \qquad y = \frac{t}{\sqrt{s^2+t^2}}u + \frac{s}{\sqrt{s^2+t^2}}v,$$

we have (Jacobian $=1$, $x^2+y^2=u^2+v^2$)

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp(-u^2-v^2)\cos(\sqrt{s^2+t^2}\,u)\,dudv \\ &= \int_{-\infty}^{+\infty}\exp(-v^2)dv\int_{-\infty}^{+\infty}\exp(-u^2)\cos(\sqrt{s^2+t^2}\,u)\,du \\ &= \sqrt{\pi}\sqrt{\pi}f(\sqrt{s^2+t^2}) = \pi f(\sqrt{s^2+t^2}). \end{aligned} \tag{4}$$

By the above result we have

$$I_2 = \pi f(\sqrt{s^2+(-t)^2}) = \pi f(\sqrt{s^2+t^2}). \tag{5}$$

By (3), (4), (5) we have

$$f(s)f(t) = f(\sqrt{s^2+t^2}). \tag{6}$$

By the continuity of $f(a)$, (6) and $f(0)=1$, the above lemma gives

$$f(a) = \exp(\alpha a^2), \tag{7}$$

where α is a real constant. By (1) we have

$$f''(0) = -(1/\sqrt{\pi})\int_{-\infty}^{+\infty} x^2 \exp(-x^2)dx = -\tfrac{1}{2}. \tag{8}$$

By (7), (8) we have $\alpha = -\frac{1}{4}$. Hence, by (7) we have $f(a) = \exp(-a^2/4)$.

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ON THE MATRIX EQUATION $AB=I$

PAUL HILL, University of Houston

In a course in linear algebra, and elsewhere, it is a great computational convenience to know that if A and B are n-square matrices such that $AB=I$, where I is the identity matrix of order n, then necessarily $BA=I$ and B is the inverse of A. A standard proof of this result is the identification of a matrix with its corresponding linear transformation and the proof that a linear transformation of a finite dimensional vector space is onto if and only if it is one-to-one. There is, however, an elementary direct proof of an even stronger result which is almost an immediate consequence of an interesting theorem about matrices that is not so well known, although its proof is quite simple.

Suppose that R is a ring and that n is a positive integer. Let R_n denote the ring of all n-square matrices over R. We emphasize that R need not be commutative.

THEOREM. *If R satisfies the ascending chain condition for right ideals, then so does R_n.*

Proof. Denote by $f_{i,j}$ the function from R_n to R that sends a matrix onto its (i, j)-component. Observe that if S is a nonempty subset of R_n which is closed with respect to addition, subtraction, and scalar multiplication on the right, then $f_{i,j}(S)$ is a right ideal of R.

Assume that $\{E_k\}$ is a sequence of right ideals of R_n such that E_k is properly contained in E_{k+1} for each positive integer k. For each positive integer t not exceeding n^2 let $E_{k,t}$ denote the subset of E_k consisting of those matrices in E_k that have zero in each of the first t components—for definiteness, count the components by rows. Now suppose that t is less than n^2 and that we have already shown that $E_{k,t}$ is properly contained in $E_{k+1,t}$ for all but a finite number of k. We wish to show that the latter result must also hold for $t+1$. Since $f_{i,j}(E_{k,t})$ is a right ideal of R and since R satisfies the ascending chain condition, we know that $f_{i,j}(E_{k,t}) = f_{i,j}(E_{k+1,t})$ for all but a finite number of k. If this equality holds, however, and if $E_{k,t}$ is proper in $E_{k+1,t}$, then $E_{k,t+1}$ is proper in $E_{k+1,t+1}$; for if A is in $E_{k+1,t}$ but not in $E_{k,t}$, then there exists B in $E_{k,t}$ having the same $(t+1)$-

component as A and $B-A$ is in $E_{k+1,t+1}$ but not in $E_{k,t+1}$. We are led to the conclusion that the set consisting of only the zero matrix in R_n is a proper subset of itself. Thus our assumption must be denied, and R_n satisfies the ascending chain condition for right ideals.

The next theorem is a weak version of Theorem 4 in [1].

THEOREM. *If R is a ring with unit which satisfies the ascending chain condition for right ideals, then the equation $ab=1$ in R implies that $ba=1$.*

Proof. Define a mapping π from R into itself by: $x\to ax$. Since $\pi(bx)=x$ for each x in R, π is onto. Let $K_n=\{x\in R\mid \pi^n(x)=0\}$ denote the kernel of π^n for each positive integer n. Since K_n is a right ideal of R, there must be a positive integer n such that $K_n=K_{n+1}$. For any such n, we see that $\pi(\pi^n(x))=0$ implies that $\pi^n(x)=0$. Since π^n is onto and since π preserves addition, π must be one-to-one. Since $\pi(1)=\pi(ba)$, $ba=1$ and the theorem is proved.

COROLLARY. *If R is a field or division ring and if A and B are n-square matrices over R such that $AB=I$, then $BA=I$.*

Proof. R has only the two trivial right ideals.

Both of the above theorems can be proved, in much the same way, for the descending chain condition and also for left ideals, so we actually have

THEOREM. *If R is a ring with unit which satisfies the ascending (descending) chain condition for right (left) ideals and if A and B are n-square matrices over R such that $AB=I$, then $BA=I$.*

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TITCHMARSH'S THEOREM FOR ANALYTIC FUNCTIONS

M. R. VIRGA, University of Michigan (Undergraduate student)

The convolution of two functions f and g written $f*g$ is a function defined on $[0,\ \infty)$ by $f*g(t)=\int_0^t f(u)g(t-u)du$. An important property of convolutions, which was first proved (in [1]) by E. Titchmarsh, is: If f and g are continuous on $[0,\ \infty)$ and if $f*g=0$, then $f=0$ or $g=0$.

Jan Mikusinski makes use of this property of convolution products to construct convolution quotients, the theory of which provides a rigorous basis for Heaviside's operational calculus. He presents a different proof of Titchmarsh's theorem in [2].

An elementary proof can be given for the following special case of Titchmarsh's theorem:

THEOREM. *If f and g are analytic functions on an interval I which includes the origin and if $f*g=0$ on I, then either $f=0$ or $g=0$ on I.*

The proof of this theorem will make use of the following properties of convolutions. First, by making the change of variable $v=t-u$, we obtain

$$f * g = \int_0^t f(u)g(t-u)du = \int_0^t f(t-v)g(v)dv = g * f. \tag{1}$$

Next, using Leibnitz's rule for derivatives of integrals depending on a parameter, we obtain

$$\begin{aligned}(f * g)' &= \frac{d}{dt}\left(\int_0^t f(u)g(t-u)du\right) \\ &= f(t)g(t-t) + \int_0^t f(u)\frac{\partial}{\partial t}g(t-u)du \\ &= f(t)g(0) + f * g'.\end{aligned} \tag{2}$$

Since $f * g = g * f$, we also have

$$(f * g)' = f(0)g(t) + f' * g. \tag{2'}$$

Finally if $h(t) = f * g(t)$, then

$$h(0) = \int_0^0 f(u)g(-u)du = 0. \tag{3}$$

LEMMA 1. *If $f * g = 0$ and if $f(0) \neq 0$, then $g(0) = 0$, $f * g' = 0$, and consequently $g^{(n)}(0) = 0$ for $n = 0, 1, 2, \cdots$.*

Proof. From (2) and (2') we have

$$(f * g)' = f(0)g(t) + f' * g \tag{4}$$

$$(f * g)' = g(0)f(t) + f * g'. \tag{5}$$

Both of these expressions are identically equal to zero since $(f * g)' = 0$. Thus, by (4), $g(t) = (-1/f(0))f' * g$. Hence, by (3), $g(0) = 0$, and consequently $f * g' = 0$ from (5). By successively substituting $g^{(k)}$ for $g^{(k-1)}$ in this argument, we obtain $g^{(k)}(0) = 0$ and $f * g^{(k+1)} = 0$, for values of $k = 1, 2, \cdots$. Thus $g^{(n)}(0) = 0$ for $n = 0, 1, 2, \cdots$.

LEMMA 2. *If $f * g = 0$ and $f(0) = 0$, then $f' * g = 0$ and either $f'(0) = 0$ or $g^{(n)}(0) = 0$ for $n = 0, 1 \cdots$.*

Proof. From (2') we have $(f * g)' = f(0)g(t) + f' * g = f' * g$, since $f(0) = 0$. Since $(f * g)' = 0$, it follows that $f' * g = 0$. If $f'(0) \neq 0$ we can then substitute f' for f in Lemma 1 and obtain the result that $g^{(n)}(0) = 0$ for $n = 0, 1, \cdots$.

Proof of Theorem. We shall show that under our hypotheses $f^{(n)}(0) = 0$ or $g^{(n)}(0) = 0$ for all n. This will complete the proof. If $f * g = 0$ and $f(0) \neq 0$, then it follows from Lemma 1 that $g^{(n)}(0) = 0$ for all n. If on the other hand $f(0) = 0$, and $g^{(n)}(0) \neq 0$ for some n, then $f'(0) = 0$ and by successively substituting $f^{(k)}$

for $f^{(k-1)}$ in Lemma 2, we obtain the result that $f^{(k+1)}(0)=0$ and that $f^{(k+1)} * g=0$, for values of $k=1, 2, \cdots$. Thus in this case $f^{(n)}(0)=0$ for all n.

Acknowledgement: I wish to acknowledge my sense of gratitude to Professor J. A. Smoller for his guidance during the preparation of this paper.

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MATRICES WHICH TAKE A GIVEN VECTOR INTO A GIVEN VECTOR

M. MACHOVER, St. John's University

A slight variation of the usual problem for a system of linear equations is to solve for the coefficient matrix when the solution vector is known. More abstractly, if V and W are n- and m-dimensional vector spaces over the same field, and $x \in V$ and $y \in W$, find all linear transformations L such that $Lx=y$. In this note we answer the question completely and in fact give a procedure for writing down the solution from inspection.

Since we are dealing with the finite-dimensional case we may assume x and y are given column vectors and L is an $m \times n$ matrix. When $x=0$ a solution clearly exists if and only if $y=0$, and is in fact any $m \times n$ matrix. The case for $x \neq 0$ is governed by the following:

THEOREM. *Let $x \neq 0$ and y be column vectors of n and m components, respectively (with elements from the same field). Suppose the p-th component of x is not zero. The set of all matrices L such that $Lx=y$ is nonempty, and is given by*

$$L = L^0 + \sum_{r=1}^{n}{}' \sum_{q=1}^{m} \alpha_{qr} L^{qr} \tag{1}$$

(the prime denotes that r does not take on the value p), where the α_{qr} are arbitrary constants (scalars of the field), and

$$\begin{aligned} L^0_{ij} &= (y_i \delta_{jp})/x_p, && i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, n, \\ & && (\delta_{ij} \text{ is the "Kronecker delta"}) \\ L^{qr}_{ij} &= x_r \delta_{iq} \delta_{jp} - x_p \delta_{iq} \delta_{jr}, && i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, n. \end{aligned} \tag{2}$$

Proof. The equations $Lx=y$ or

$$\begin{aligned} L_{11}x_1 + L_{12}x_2 + \cdots + L_{1n}x_n &= y_1 \\ L_{21}x_1 + L_{22}x_2 + \cdots + L_{2n}x_n &= y_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ L_{m1}x_1 + L_{m2}x_2 + \cdots + L_{mn}x_n &= y_m \end{aligned} \tag{3}$$

may be rewritten as

$$(4)\qquad \begin{bmatrix} x_1 & x_2 & \cdots & x_n & 0 & 0 & \cdots & 0 & 00 & \cdots & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & x_1 & x_2 & \cdots & x_n & 00 & \cdots & 0 & \cdots & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdots & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 00 & \cdots & 0 & x_1x_2 & \cdots x_n \end{bmatrix} \begin{bmatrix} L_{11} \\ L_{12} \\ \cdot \\ \cdot \\ L_{1n} \\ L_{21} \\ L_{22} \\ \cdot \\ \cdot \\ L_{2n} \\ \cdot \\ \cdot \\ L_{mn} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_m \end{bmatrix}$$

Denoting this $m \times mn$ coefficient matrix by X, we see that its rank is m since $x_p \neq 0$. Since the rank of the augmented matrix is also m there is a solution L to this system. Also the associated homogeneous system has $nm-m=(n-1)m$ linearly independent solutions. Thus the general solution is the sum of a particular solution plus an arbitrary linear combination of $(n-1)m$ linearly independent solutions of the homogeneous system. One may readily check that L^0 and the L^{qr} as defined by (2) are these required solutions. (They may be obtained by applying Gaussian elimination to (4).)

We observe that L^0 and the L^{qr} may be immediately written down from an inspection of (3). The pth column of L^0 is y/x_p, all the other entries are zero. The qth row of L^{qr} contains x_r in the pth column and $-x_p$ in the rth column. All its other rows are zero. Thus the set of all matrices A satisfying

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

is immediately seen to be (we will work with $p_1=1$)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix} + \alpha_{12} \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha_{22} \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$
$$+ \alpha_{13} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} + \alpha_{23} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

Although the question is easily answered in the finite-dimensional case it offers new difficulties when we proceed to infinite dimensions, especially when we restrict the class of permissible transformations. A typical question might be to find all kernels of the integral equation $\int_a^b k(s,t)f(t)dt = g(s)$ once f and g are specified.

PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

S. WOLFENSTEIN, Université de Paris

The so-called Fundamental Theorem of Algebra states that every nonconstant polynomial with complex coefficients has at least one complex zero; or alternatively, that if p is any nonconstant polynomial function of a complex variable, then the range of p contains every complex number.

The purpose of the present note is to sketch a proof simpler than appears currently available and in any case employing no mathematical tools more sophisticated than are already required for elementary calculus. The proof is based upon the convergence of an iterative procedure, a notion which should be familiar to beginning students (Newton's method, Picard's method). Prerequisites are as follows:

(A) A nonzero polynomial has only a finite number of zeros.

(B) If f: $R^m \to R^n$ is a continuous function such that $|f(x)|$ goes to infinity with $|x|$ (in particular if f is a nonconstant polynomial function of a complex variable), then the range of f is closed. (This follows at once from Weierstrass's Theorem that every bounded sequence has a convergent subsequence.)

(C) For $n>1$, the complement of a finite subset of R^n is connected. In addition only the most elementary properties of complex numbers are needed.

LEMMA. *Let p be a nonconstant polynomial function of a complex variable, S the range of p and $T=\{p(z): p'(z)=0\}$. Then $S-T$ is open.*

Proof. We first show that if $|c|$ is sufficiently small the equation

$$w + \sum_{m=2}^{n} b_m w^m = c \tag{1}$$

has a root that can be approximated by iteration. Let $w_0=0$,

$$w_{k+1} = c - \sum_{m=2}^{n} b_m w_k^m,$$

clearly if the sequence w_k converges at all it converges to a root of (1). Suppose now that $|c|$ is small; specifically, let α be any positive number less than 1, $\rho=\min(\alpha/[\sum_{m=2}^{n} m|b_m|], 1)$ and $|c|<\rho-\sum_{m=2}^{n}|b_m|\rho^m$ (this is a positive number). A straightforward inductive calculation shows that, for all k, $|w_k|\leqq\rho$. Further, for $k>0$,

$$\begin{aligned} |w_{k+1}-w_k| &= |w_k-w_{k-1}|\left|\sum_{m=2}^{n} b_m \sum_{s=1}^{m} w_k^{m-s} w_{k-1}^{s-1}\right| \\ &\leqq |w_k-w_{k-1}| \sum_{m=2}^{n} m|b_m|\rho^{m-1}, \quad \text{by the foregoing,} \\ &\leqq \alpha|w_k-w_{k-1}|, \end{aligned}$$

so that, for c as stated, w_k does converge to a root of (1).

Now let p be any nonconstant polynomial, z_0 any complex number such that $p'(z_0)\neq 0$. With the substitutions $z=z_0+w$, $a=p(z_0)+p'(z_0)c$, the result just obtained shows that the equation $p(z)=a$ has a root for all values of a close enough to $p(z_0)$. In other words, if z_0 is not a zero of p', $p(z_0)$ is an interior point of S. This proves the Lemma.

THEOREM. *With the notation of the lemma, S is the set of all complex numbers.*

Proof. The complement of S is open (B), and $S-T$ is open by the lemma. But as T is finite (A) its complement cannot consist of two disjoint nonempty open sets (C). Hence the complement of S must be empty.

AN EASY PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

CHARLES FEFFERMAN, University of Maryland

THEOREM. *Let $P(z)=a_0+a_1z+\cdots+a_nz^n$ be a complex polynomial. Then P has a zero.*

Proof. We shall show first that $|P(z)|$ attains a minimum as z varies over the entire complex plane, and next that if $|P(z_0)|$ is the minimum of $|P(z)|$, then $P(z_0)=0$.

Since $|P(z)|=|z|^n|a_n+a_{n-1}/z+\cdots+a_0/z^n|$ $(z\neq 0)$, we can find an $M>0$ so large that

$$|P(z)| \geqq |a_0| \qquad (|z|>M). \tag{1}$$

Now, the continuous function $|P(z)|$ attains a minimum as z varies over the compact disc $\{|z|\,|z|\leqq M\}$. Suppose, then, that

$$|P(z_0)| \leqq |P(z)| \qquad (|z|\leqq M). \tag{2}$$

In particular, $|P(z_0)|\leqq P(0)=|a_0|$ so that, by (1), $|P(z_0)|\leqq|P(z)|$ $(|z|>M)$. Comparing with (2), we have

$$|P(z_0)| \leqq |P(z)| \qquad \text{(all complex } z\text{)}. \tag{3}$$

Since $P(z)=P((z-z_0)+z_0)$, we can write $P(z)$ as a sum of powers of $z-z_0$, so that for some complex polynomial Q,

$$P(z) = Q(z-z_0). \tag{4}$$

By (3) and (4),

$$|Q(0)| \leqq |Q(z)| \qquad \text{(all complex } z\text{)}. \tag{5}$$

We shall show that $Q(0)=0$. This will establish the theorem since, by (4), $P(z_0)=Q(0)$.

Let j be the smallest nonzero exponent for which z^j has a nonzero coefficient in Q. Then we can write $Q(z)=c_0+c_jz^j+\cdots+c_nz^n$ $(c_j\neq 0)$. Factoring

z^{j+1} from the higher terms of this expression, we have

$$Q(z) = c_0 + c_j z^j + z^{j+1}R(z), \tag{6}$$

where $c_j \neq 0$ and R is a complex polynomial.

If we set $-c_0/c_j = re^{i\theta}$, then the constant $z_1 = r^{1/j}e^{i\theta/j}$ satisfies

$$c_j z_1^j = -c_0. \tag{7}$$

Let $\epsilon > 0$ be arbitrary. Then, by (6),

$$Q(\epsilon z_1) = c_0 + c_j \epsilon^j z_1^j + \epsilon^{j+1} z_1^{j+1} R(\epsilon z_1). \tag{8}$$

Since polynomials are bounded on finite discs, we can find an $N > 0$ so large that, for $0 < \epsilon < 1$, $|R(\epsilon z_1)| \leqq N$. Then, by (7) and (8) we have, for $0 < \epsilon < 1$,

$$\begin{aligned} |Q(\epsilon z_1)| &\leqq |c_0 + c_j \epsilon^j z_1^j| + \epsilon^{j+1} |z_1|^{j+1} |R(\epsilon z_1)| \\ &\leqq |c_0 + \epsilon^j (c_j z_1^j)| + \epsilon^{j+1}(|z_1|^{j+1} N) \\ &= |c_0 + \epsilon^j (-c_0)| + \epsilon^{j+1}(|z_1|^{j+1} N) \\ &= (1 - \epsilon^j)|c_0| + \epsilon^{j+1}(|z_1|^{j+1} N) \\ &= |c_0| - \epsilon^j |c_0| + \epsilon^{j+1}(|z_1|^{j+1} N). \end{aligned} \tag{9}$$

If $|c_0| \neq 0$, then we can take ϵ so small that $\epsilon^{j+1}(|z_1|^{j+1} N) < \epsilon^j |c_0|$. In that case, by (9)

$$|Q(\epsilon z_1)| \leqq |c_0| - \epsilon^j |c_0| + \epsilon^{j+1}(|z_1|^{j+1} N) < |c_0| = |Q(0)|,$$

contradicting (5). So $|c_0| = 0$, and therefore $Q(0) = c_0 = 0$.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

Material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D.C. 20005.

SEARCHING FOR MATHEMATICAL TALENT IN WISCONSIN, III.

D. W. CROWE, University of Wisconsin

The Wisconsin High School Mathematical Talent Search operated for its third year in 1965–66, with continuing financial support from the National Science Foundation. Reports for preceding years are given in references [1], [2].

The following five problem sets were used. In parenthesis after each problem is the number of students who submitted essentially correct solutions.

Problem Set 1—Sept. 20, 1965

1. Prove that $\sqrt[3]{15}$ is not a rational number. (16)

2. (a) A *cute* girl only eats cake cut in pieces the shape of *acute* triangles. Show how to cut a rectangular cake (of unequal sides) to serve cute girls, with no pieces left over. (The pieces do not all have to be the same size, but must all be acute triangles.)

(b) Show how to do the same for a square cake. (49)

The symbol "y^{z^w}" means "y raised to the power z^w." The symbol "$x^{y^{z^w}}$" means "x raised to the power y^{z^w}."

3. Prove that $2^{2^{2^2}}-2^{2^2}$ divides $n^{2^{2^2}}-n^{2^2}$ for all integers n. (42)

4. (a) Three wrestlers find themselves in the (30 foot square) ring at the same time. Each, being deathly afraid of the other two, wants to keep as far from them as possible. Prove: They can stand so that no two of them are closer than $60\sqrt{2-\sqrt{3}}$ feet from each other.

(b) Prove that the number $60\sqrt{2-\sqrt{3}}$ cannot be replaced by a larger number in the last statement. (19)

Food for thought: What if there are more wrestlers?

5. A student "solved" the quadratic equation $x^2+x-2=0$ as follows:

$$\begin{aligned} & x^2 + x - 2 = 0, \\ \therefore\ & -x^2 - x + 6 = 4, \\ \therefore\ & (x+3)(2-x) = 4, \\ \therefore\ & \text{Either } x+3 = 4, \quad \text{hence } x = 1 \\ & \text{or } 2 - x = 4, \quad \text{hence } x = -2. \end{aligned}$$

These are, in fact, the roots of the original equation.

Can you find a quadratic equation with integral roots which cannot be "solved" this way?

More exactly: Can you find a quadratic equation, $x^2+bx+c=0$, having integral roots, which cannot be rewritten as

$$(x-d)(-e-x) = f, \qquad (f \neq 0),$$

where $d+f$ and $-(e+f)$ are the roots of the original equation? (20)

Problem Set 2—October 29, 1965

1. It is known that no root of $x^3=15$ is rational (Problem Set 1, problem #1).

(a) Prove that if $a+b\sqrt{c}$ (a, b, c rational numbers, $\sqrt{c}$ not rational) satisfies the equation $x^3=15$ then $a-b\sqrt{c}$ satisfies the same equation.

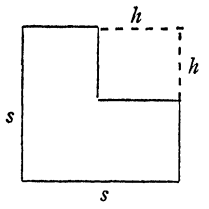
(b) Deduce that $\sqrt[3]{15}$ is not of the form $a+b\sqrt{c}$ if a, b, c are rational numbers. (14)

2. A cake is in the shape of an obtuse triangle. Show that it can be cut to serve cute girls, with no pieces left over. (That is, show how to dissect an obtuse triangle into a finite number of acute triangles.) (20)

3. Find all solutions, in nonnegative integers m, n, of the equation

$$3 \cdot 2^m + 1 = n^2. \qquad (35)$$

4. A square house of side h has been built at the corner of a square lot of side $s>h$, (see figure).



What is the largest square garage which can be built on the remainder of the lot? Prove your result. (1)

5. (a) Let a, b, c, d be integers for which $ad \neq bc$. Show that it is always possible to write the fraction $1/[(ax+b)(cx+d)]$ in the form

$$\frac{r}{ax+b}+\frac{s}{cx+d},$$

where r and s are rational numbers.

(b) Find the sum

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{2998\cdot 3001}. \qquad (7)$$

Problem Set 3—December 6, 1965

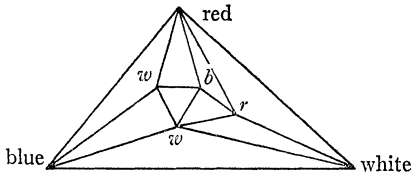
1. How many right triangles are there whose legs have whole number lengths and whose hypotenuse has the length $\sqrt{2}\cdot 2^n$ (where n is a given positive integer)? (22)

2. Three parallel lines (not necessarily equally spaced) are drawn in the plane. Show how to construct (with ruler and compass) an equilateral triangle with one vertex on each line. (46)

3. Use each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once to form two five digit numbers having the largest possible product. (Proof?) (63)

4. What is the least number of soldiers which can march in exactly 64 different rectangular formations? (For example, six soldiers can march in exactly 4 different formations: 6×1, 3×2, 2×3, 1×6.) (44)

5. At the three corners of a large triangular field are red, white, and blue posts, respectively. The field is subdivided into smaller triangular fields, with fenceposts at each corner. (However, no fencepost is added along an already existing fence.) Each of these fenceposts is then painted one or another of the three colors red, white and blue. Prove that no matter how this is done there is always at least one small field having no two corner posts the same color. (15)



Problem Set 4—February 1, 1966

1. Six cute girls, each wearing a new hat, met at a beauty salon. After having their hair done, each girl borrowed another girl's hat and each left the salon with a hat different from the one she arrived in. That hat borrowed by Ann belonged to the borrower of Betty's hat. The owner of the hat borrowed by Cathy borrowed the hat belonging to the borrower of Debby's hat. If the borrower of Ellen's hat was not the owner of that borrowed by Flora, who borrowed Ann's hat? Whose hat did Ann borrow? How do you know? (57)

2. A square cake has frosting on its top and all four sides. Show how to cut it to serve nine people so that each of them gets exactly the same amount of cake and exactly the same amount of frosting. (The people you serve don't have to be cute girls!) (52)

3. Prove or disprove: There are two numbers x, y such that

$$x + y = 1, \quad x^2 + y^2 = 2, \quad \text{and} \quad x^3 + y^3 = 3.$$

Let v be the number of vertices of a polyhedron, e the number of edges and f the number of faces. Then Euler's theorem is: *If a polyhedron is convex, then* $v-e+f=2$. (54)

4. (a) By counting the edges of each face of a convex polyhedron, and the edges at each vertex, show that $3f \leqq 2e$ and $3v \leqq 2e$.

(b) Prove or disprove: It is possible to cut a potato into a (convex) polyhedron having exactly seven edges. (22)

5. Prove: If $1+4^d+4^{2d}+\cdots+4^{(n-1)d}$ is a perfect square then n must be odd. (6)

Problem Set 5—March 7, 1966

1. A boy has k girl friends. For each two of these girls he makes a decision as to which is cuter. (However, the fact that he says Helen is cuter than Penelope and Penelope cuter than Marilyn does not mean that he says Helen is cuter than Marilyn.)

(a) Is it then always possible to list these k girls in some order, say $g_1, g_2, \cdots, g_k$ so that g_1 is cuter than g_2, g_2 is cuter than g_3, $\cdots$, g_{k-1} is cuter than g_k? (Proof?)

(b) Suppose, in a given situation, it is possible to arrange the girls as in (a), and suppose also that g_1 is cuter than g_k. Is this arrangement necessarily the only one satisfying the original conditions? (Proof?) (11)

2. If x and y are fixed real numbers such that $x+y=1$ and $x^4+y^4=c$, find x^2+y^2 and x^3+y^3 (in terms of c). (10)

The *Fibonacci numbers* are 1,2,3,5,8,13, $\cdots$, where each one (after the second) is obtained by adding together the previous two numbers in the sequence.

3. Prove that every positive integer can be written as the sum of *distinct* Fibonacci numbers. (11)

4. 100 points are distributed in space so that no three are collinear. Prove that it is possible to draw 2500 segments joining them, so that no triangle (having 3 of the 100 points as vertices) is formed. (18)

5. Find all positive integers n which have the property that the product of their distinct (positive) divisors is n^2. (For example, 6 is such a number, since $1\cdot2\cdot3\cdot6=6^2$.) (19)

References

1. M. N. Bleicher, Searching for mathematical talent in Wisconsin, this MONTHLY, 72 (1965) 412–416.

2. J. R. Smart, Searching for mathematical talent in Wisconsin, II, this MONTHLY, 73 (1966) 401–406.

RESEARCH WORKERS IN MATHEMATICS EDUCATION

OTTO BASSLER, George Peabody College for Teachers

One of the critical needs in mathematics education is the need for trained researchers to investigate problems in how students learn mathematics, what the mathematics curriculum should include at various levels of instruction, and what methodological techniques are best suited for presentation of the specific content to be taught. This need was underscored by the Conference on Needed Research in Mathematical Education in their Summary Report. This conference concludes that there is a particular need for people who know both good mathematics and psychology.

The Institute on School Learning and Individual Differences at George Peabody College for Teachers has designed and instituted a doctoral program to train prospective research workers in several academic areas. The academic areas that are represented in the program at the present time are elementary education, English and mathematics. Administration of the program is a joint endeavor involving the academic departments and the department of psychology.

Mathematics students accepted in this program will pursue graduate study in mathematics and psychology. The graduate mathematics courses are designed to provide a firm foundation in algebra, geometry and analysis. A further emphasis in the mathematics curriculum will be placed upon the teaching of mathematics and research in mathematics education. The coursework in psychology emphasizes learning theory and research skills such as experimental design, statistical inference and problems in testing and measurement. The major goal of the coursework is to provide a firm foundation in mathematics and to prepare the student to become a researcher in mathematics education.

During the students' entire tenure in this program, they will actively participate on faculty-student research teams which investigate learning problems in mathematics. In this way the students will gain practical experience in planning, conducting, and evaluating meaningful research projects in their area of interest in conjunction with their coursework.

Graduates of this program will be prepared to teach mathematics at the college level and to conduct research on the learning of mathematics.

ON THE TRAINING OF TEACHERS OF ADVANCED PLACEMENT COURSES

G. N. GARRISON, Wesleyan University

The continuing growth of the need for high school teachers who are prepared to teach courses in calculus stimulates this note. The program which is described below has been essentially in its present form since 1962–63.

The program is administered by the Wesleyan University Graduate Summer School for Teachers as a small (12 participants) National Science Foundation

Academic Year Institute for experienced secondary school teachers. Participants are selected on the basis of (1) an indication of their opportunity to teach advanced placement courses in the near future; (2) their ability to handle the program; and (3) their professional need for the level of instruction proposed.

The most unusual feature of the program, and the one about which we have had the most comment, is a special kind of involvement of the Institute participants in the freshman calculus course. Each participant attends all the class meetings of this course, does the assigned homework, attends the problem sessions, and takes the tests and examinations (for a grade). In addition, depending upon his instructor's estimate of his competence to do so, each participant grades about seven homework papers from each assignment for the class he attends. This is not intended as work done for the Department of Mathematics, but rather, as experience for him. Actually, other graders are available in the Department to keep the load from being excessive. Finally, again dependent upon his ability, the participant is expected to help conduct one two-hour "help session" a week to which freshmen who are having difficulties with the course can come for assistance. The rationale here, as in the grading of homework papers, is to give the high school teacher experience without actually placing him in charge of a class.

A second feature of the program, intended as support for the one just described, is a seminar in analysis for the participants. In this seminar, the topics of first-year calculus are treated with more rigor and greater precision than is usual in a freshman course.

Each high school teacher also participates in a special seminar in topics other than analysis. His program is completed with an elective chosen from the mathematics department offerings.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P.O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before January 31, 1968. Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2002 [1967, 720] **Correction.** *Proposed by F. Leuenberger, Feldmeilen, Switzerland.*

Construct a triangle ABC, having given the two sides AB and AC, and knowing that the line joining the circumcenter and the incenter is perpendicular to the interior angle bisector of angle A.

E 2006. *Proposed by A. S. Howard, Eastern Kentucky State College*

Given two fixed points A, B in the Euclidean plane, let C be free to move on a circle in the plane with A as center. Find the locus of P the point of intersection of BC with the internal bisector of angle A of the triangle ABC.

E 2007. *Proposed by W. A. Donnell, North Texas State University*

Show that a cross-cancellative (i.e., $ab=bc$ implies $a=c$) semigroup S is a cancellative semigroup.

E 2008. *Proposed by Murali Rao, University of British Columbia*

Let $\{a_n\}$ be a sequence of positive numbers such that $\sum a_n$ converges. Find a necessary and sufficient condition for the existence of a sequence of positive numbers $\{b_n\}$ such that $\sum a_n/b_n$ and $\sum b_n$ both converge.

E 2009. *Proposed by R. A. Avelsgaard and R. E. Stockton, Bemidji State College, Minnesota*

It is well known, as an example of Dirichlet's Test, that the series $\sum_{n=2}^{\infty} \sin n/\log n$ converges. Does it converge absolutely?

E 2010. *Proposed by D. R. Rao, Osmania University, India*

Show that $[(m-1)!/m]$ is even for all $m>4$. Brackets indicate the greatest integer function.

E 2011. *Proposed by R. C. Entringer, University of New Mexico*

Solve the functional equation $f(x)=f(x/(1-x))$, $x\neq 1$, subject to the condition that f is continuous at $x=0$.

E 2012. *Proposed by R. S. Kulkarni, Harvard University*

Let $f(x_1, \cdots, x_r)$, $g(y_1, \cdots, y_s)$ be two homogeneous polynomials with integral coefficients, and suppose the degree of f is relatively prime to the degree of g. Show that there exist infinitely many integral solutions of the equation $f=g$.

E 2013. *Proposed by Walter Noll, Carnegie Institute of Technology*

m, n being positive integers, prove the identity

$$S_{n,m} = \sum_{r=1}^{n} (-1)^{n-r} \sum (x_{i_1} + \cdots + x_{i_r})^m = \begin{cases} n!x_1x_2 \cdots x_n & \text{if } m = n \\ 0 & \text{if } m < n, \end{cases}$$

where the second summation is taken over all possible selections $i_1, \cdots, i_r$ of r of the first n natural numbers.

E 2014. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New York City*

Show that $2^{2^n}-1$ has at least n distinct prime divisors.

SOLUTIONS OF ELEMENTARY PROBLEMS

Inequalities for Two Sequences

E 1860 [1966, 199]. *Proposed by Edward Thorpe and Fred Richman, New Mexico State University*

Consider the sequences n^n, $n^{(n^n)}$, etc., and $n!$, $n!!$, $n!!!$, etc., for a fixed integer $n \geqq 3$. Do the terms of one sequence eventually become larger, and stay larger, than the corresponding terms of the other? Suppose, instead, that we compare the corresponding terms of n^n, $n^{(n^n)}$, etc., and $n!!$, $n!!!$, etc., and ask the same question.

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa. Let $a_1 = n^n$, $a_{m+1} = n^{a_m}$ for $m > 0$; $b_1 = n!$, $b_{m+1} = b_m!$ for $m > 0$. We will prove

$$a_2 > 2b_2^2 (> b_2), \qquad n \geqq 2. \tag{1}$$

For $n = 2, 3, 4$, (1) is easily checked. For $n \geqq 5$,

$$\ln a_2 = n^n \ln n \text{ and } \ln 2b_2^2 = \ln 2 + 2 \ln n!!.$$

Now

$$n!! = \prod_{i=2}^{n!} i < \prod_{i=2}^{n} (i!)^{i!-(i-1)!} < \prod_{i=2}^{n} (i!)^{n!},$$

whence $\ln n!! < n! \sum_2^n \ln i!$. Also $\ln i! = \ln i + \cdots + \ln 2 < (i-1) \ln n$, so that

$$2 \ln n!! < 2n! \ln n \sum_{i=2}^{n} (i - 1) = (n - 1)n \cdot n! \ln n.$$

Thus

$$\begin{aligned} \ln 2b_2^2 &< \ln 2 + (n - 1)n \cdot n! \ln n < (n^2 - n + 1)n! \ln n \\ &< n^2 n! \ln n < n^n \ln n = \ln a_2, \end{aligned}$$

since, for $n \geqq 5$, $n^2 n! < n^n$. This implies (1).

For m, $n \geqq 2$, $b_m \geqq 2$ and for any $x \geqq 2$ it is easily seen that $x^x \geqq 2x!$. Thus for m, $n \geqq 2$, $b_m^{b_m} \geqq 2b_m! = 2b_{m+1}$. Also for m, $n \geqq 2$, $n^{b_m} > b_m$. Now suppose that $a_m > 2b_m^2$ for some m, $n \geqq 2$. (We have shown this for $m = 2$ and $n \geqq 2$.) Then

$$\begin{aligned} a_{m+1} = n^{a_m} &> n^{(2b_m^2)} = (n^{b_m})^{2b_m} > b_m^{2b_m} = (b_m^{b_m})^2 \\ &\geqq (2b_{m+1})^2 > 2b_{m+1}^2 (> b_{m+1}). \end{aligned}$$

By induction on m, $a_m > b_m$ for m, $n > 1$.

For the second question, we proceed as follows. Let $n \geqq 3$. From

$$n!! > n!(n! - 1) \cdots (n! - n) > n^{n+1} > n(n^{n-1} + 1) - 1$$

it follows that $n!! - n^n + 1 > n$, whence

$$b_3 = n!!! > n!!(n!! - 1) \cdots (n!! - n^n + 1) > n^{(n^n)} = a_2;$$

also $b_2 = n!! > n^n + n - 1 > n^n = a_1$.

Now suppose that $b_m > a_{m-1}$ for some $m \geqq 2$ for any $n \geqq 3$. (We have just shown this for $m = 2$, 3 and $n \geqq 3$.) Then $b_{m+2} = b_m!! > a_{m-1}!!$ by hypothesis; and, by problem E 1816 [1965, 902], $a_{m-1}!! > [(a_{m-1}-1)!]^{a_{m-1}!}$. Now for $m \geqq 2$ and $n \geqq 3$,

$$(a_{m-1} - 1)! \geqq (a_1 - 1)! = (n^n - 1)! > n^n - 1 > n$$

and also

$$\begin{aligned} a_{m-1}! &> a_{m-1}(a_{m-1} - 1) \cdots (a_{m-1} - n + 2) \cdots n > n^{a_{m-1}-n+1} \cdot n^{n-1} \\ &= n^{a_{m-1}} = a_m \end{aligned}$$

since $a_{m-1} - n + 2 \geqq a_1 - n + 2 = n^n - n + 2 > n^2$. Thus $b_{m+2} > a_{m-1}!! > n^{a_m} = a_{m+1}$. By induction on odd and even m separately, $b_{m+1} > a_m$ for all $m \geqq 1$ and $n \geqq 3$. (The inequality is reversed for $n = 2$ and replaced by an equality with 1 for $n = 1$.)

Also solved by the proposers.

Number of Representations

E 1862 [1966, 199]. *Proposed by Michael Fried, University of Tennessee*

Let $A = \{a_i\}_{i=1}^{\infty}$ be an ordered set of distinct positive integers. Let

$$P(x) = \prod_{i=1}^{\infty} (1 + x^{a_i}) = \sum_{i=0}^{\infty} N_A(i) x^i.$$

It is well known that $N_A(n)$ is the number of ways n can be expressed as a sum of distinct elements from A.

$N_A(n)$ is said to be unbounded if $\lim_n \sup N_A(n) = \infty$.

$N_A(n)$ is said to be uniformly unbounded if $\lim_n N_A(n) = \infty$.

(a) Find necessary and sufficient conditions on the set A so that $N_A(n) > 0$ for all n.

(b) Find necessary and sufficient conditions on the set A so that $N_A(n)$ is unbounded.

(c) Does there exist a set A for which $N_A(n)$ is unbounded but not uniformly unbounded?

Partial solution by E. S. Langford, U. S. Naval Post-graduate School, Monterey, Cal. (a) It is necessary and sufficient that $a_1 = 1$ and $a_{n+1} \leqq 1 + s_n$, where $s_n = a_1 + \cdots + a_n$. (Thus, as is well known, the 'most efficient' set is found by taking $a_n = 2^{n-1}$.) Call an integer n *attainable* if $N_A(n) > 0$.

Necessity. If for some n, $a_{n+1} > 1 + s_n$, then certainly $1 + s_n$ is not attainable.

Sufficiency. (By induction.) By definition, $s_1 = 1$ is attainable using only a_1, so assume that every integer $\leqq s_n$ is attainable using only $a_1, \cdots, a_n$. Suppose x is such that $s_n + 1 \leqq x \leqq s_{n+1}$. Since $a_{n+1} \leqq s_n + 1$, it follows that $0 \leqq x - a_{n+1} \leqq s_{n+1} - a_{n+1} = s_n$. Either $x = a_{n+1}$ or by the inductive assumption $x - a_{n+1}$ is attainable using only $a_1, \cdots, a_n$; in either case, x is attainable using only $a_1, \cdots, a_n, a_{n+1}$.

(c) Let A be the set of even positive integers. Then no odd number is attainable, so that lim inf $N_A(n)$ is 0. But evidently lim sup $N_A(n)$ is infinite.

Also solved by the proposer.

Part (b) was misstated—the proposer intended to include the restriction that $N_A(n)>0$ for all n. With this restriction, $N_A(n)$ will be unbounded iff s_n-a_{n+1} is unbounded.

Prime Power Divisors

E 1868 [1966, 309]. *Proposed by K. E. Eldridge, Ohio University*

Let p be any prime. Let t and s be integers such that $t>0$ and $0<s<p^t$. Define integer i as the exponent of the highest power of p dividing s, and integer a as the exponent of the highest power of p dividing the binomial coefficient $\binom{p^t}{s}$. Prove that $a=t-i$.

Solution by R. A. Patenaude, University of Heidelberg, Germany. Let $p^x \| y$ imply $p^x | y$ and $p^{x+1} \nmid y$. We first note that $p^x \| p^t - y$ if and only if $p^x \| y$ for $0<y<p^t$. Then

$$p^a \| \binom{p^t}{s} = \frac{(p^t)!}{s!(p^t-s)!} = \frac{p^t}{s} \prod_{j=1}^{s-1} \frac{p^t-j}{j} .$$

We are given $p^i \| s$, whence $p^a \| p^t/p^i$, and therefore $a=t-i$.

Also solved by J. C. Abad, P. N. Bajaj, M. T. L. Bizley (England), Leonard Carlitz, H. M. Edgar, M. G. Greening (Australia), F. T. Howard, Geoffrey Kandall, V. H. Keiser, Jr., Kenneth Kramer, Arthur Lieberman, J. E. MacDonald, Jr., Dan Marcus and Herrold Fischer, D. C. B. Marsh, Lieselotte Miller, Norman Miller, Wanda J. Mourant, Stanton Philipp, Stanley Rabinowitz, Perry Scheinok, R. J. Snelling, Al Somayajulu, Sidney Spital, I. Vrečko (Yugoslavia), Charles Wexler, and J. C. Williams.

Plane Subdivided by Lines

E 1869 [1966, 309]. *Proposed by J. A. Burslem, St. Louis University*

(a) Let L be a finite set of pairwise intersecting lines in the plane, and let P be the set of points determined by L. Define the multiplicity of a point $p \in P$ as the number of lines of L passing through p. For each integer $k \geq 2$, given i_k, the number of points of multiplicity k, determine the number N of regions into which L divides the plane.

(b) Do the analogous problem resulting when "great circle" and "sphere" are substituted for "line" and "plane" respectively.

Solution by Harley Flanders, Purdue University. (a) Changing slightly to more customary notation, let V be the number of vertices, E the number of edges, F the desired number ($=N$) of faces and L the number of lines which were originally placed to create this configuration. Let $V_k(=i_k)$ be the number of vertices on exactly k lines and let L_k be the number of lines on exactly k vertices. We are given only the numbers $V_k(k=2, 3, \cdots)$ and the hypothesis that no lines are parallel.

Since $V=V_2+V_3+V_4+\cdots$, the number V is determined.

Next we consider the incident triples (A, l, m) where A is a vertex and l and m are distinct lines. Counting two ways leads to

$$(*) \qquad \binom{2}{2}V_2+\binom{3}{2}V_3+\binom{4}{2}V_4+\cdots=\binom{L}{2},$$

whence L is determined.

We now count incident pairs (A, l) which gives us

$$2V_2+3V_3+4V_4+\cdots=L_1+2L_2+3L_3+\cdots.$$

But $L=L_1+L_2+L_3+\cdots$, $E=2L_1+3L_2+4L_3+\cdots$, since a line with k vertices contains $k+1$ edges. Hence $2V_2+3V_3+4V_4+\cdots=E-L$, and so E is determined.

Finally we determine F by the Euler formula in the plane,

$$V-E+F=1.$$

This gives $F=1+L+V_2+2V_3+3V_4+\cdots$, where L is given by (*).

(b) On the sphere it is easier. We have simply

$$V=V_2+V_3+\cdots, \qquad E=2V_2+3V_3+\cdots,$$

the latter by counting (vertex, edge) pairs, and $V-E+F=2$. More explicitly,

$$F=2+V_2+2V_3+3V_4+\cdots.$$

Also solved by C. P. Celenza, Michael Goldberg, Norman Miller, W. O. J. Moser, and R. J. Snelling.

Decimal Expansions with Missing Digits

E 1871 [1966, 310]. *Proposed by R. A. Jacobson, South Dakota State University*

Let $A=\{x \mid x\in[0, 1]$, and the infinite decimal expansion of x consists of not more than 9 distinct digits$\}$. Is A countable or uncountable? Find the measure of A.

I. *Solution by L. A. Steen, Saint Olaf College.* Let S_i be the subset of A consisting of all $x\in[0, 1]$ whose infinite decimal expansion does not contain the digit i $(i=0, 1, \cdots, 9)$. Then each S_i is equivalent to the set of all $x\in[0, 1]$ when represented to the base 9. Thus S_i is uncountable, and (since $S_i\subset A$) A also is uncountable.

Let the elements of S_i' (the complement of S_i) be partitioned into classes according to the earliest occurrence of the digit i and let that occurrence be in the kth place. Then the measure of the set of numbers with $k=1$ is $1/10$; with $k=2$ is $(9/10)/10$; with $k=3$ is $(9/10)^2/10$, etc. The measure of S_i' is therefore $\sum_0^\infty (9/10)^i/10=1$. Therefore S_i has measure 0. Now $A=\bigcup_{i=1}^9 S_i$, whence A has measure zero.

II. *Solution by Erwin Just, Bronx Community College.* If N is the set of normal numbers (I. Niven, *Irrational Numbers*, Carus Monograph, 1956, 94–95) then it follows from the definition of normal number that $A \cap N = \varnothing$. Since almost all real numbers are contained in N (*ibid.*, 103–104) it follows that A has zero measure.

Also solved by E. P. Emerson, G. J. Foschini, Michael Goldberg, Robert Groover III, G. A. Heuer, Edward Hook, D. H. Hutcheson, Arthur Lieberman, Daniel Marcus, Lieselotte Miller, Norman Miller, George Purdy (England), Donald Quiring, Perry Scheinok, G. F. Schumm, Stephen Soldz, Lynn A. Steen, G. B. Weiss, K. L. Yocom, and the proposer.

The result follows also from Theorem 145 of Hardy and Wright, *Introduction to the Theory of Numbers*, p. 122: *The set of points whose decimals, in any scale, miss any digit is null.*

Decimal Expansions with k Repeated Digits

E 1872 [1966, 310]. *Proposed by R. A. Jacobson, South Dakota State University*

(a) Let $U = \{x \mid x \in [0, 1]$, and at least two consecutive digits in the decimal expansion of x are identical. Find measure U.

(b) Let $V = \{x \mid x \in [0, 1]$, and at least three consecutive digits in the decimal expansion of x are identical. Find measure V.

(c) Generalize to an arbitrary base b and to k consecutive digits identical.

I. *Solution by E. P. Emerson, University of Minnesota.* Each of the sets in question has measure 1. This follows at once from Theorem 146, Hardy and Wright, *An Introduction to the Theory of Numbers*, p. 123: *Almost all decimals, in any scale, contain all possible sequences of any number of digits.*

II. *Solution by Michael Goldberg, Washington, D. C.* The numbers of the problem are normal numbers, according to the definition in Ivan Niven, *Irrational Numbers*, p. 95. On p. 103 is the theorem: *Almost all real numbers are normal to every base.* Accordingly, each of the stated sets has measure 1.

Also solved by G. J. Foschini, Erwin Just, Daniel Marcus, Lieselotte Miller, and George Purdy (England).

Defining Properties of a Group

E 1873 [1966, 310]. *Proposed by G. A. Heuer and Bruce Erickson, Concordia College*

Let S be a semigroup with cancellation. Assume that for every $a \in S$, there is an integer $n_a > 1$ such that $a^{n_a} = a$. Is S necessarily a group?

I. *Solution by J. C. Williams, Lackland Air Force Base, Texas.* Let $a, b \in S$. Since $n_a > 1$, $a^{n_a}b = a \cdot a^{n_a-1}b = ab$ implies $a^{n_a-1}b = b$. Similarly, $ba^{n_a-1} = b$. Hence a^{n_a-1} is an identity. There cannot be two identities, e and e^*, in S, since $e = e \cdot e^* = e^*$. Therefore $a^{n_a-1} = e$ for every $a \in S$. Further, if $n_a > 2$, then $a^{-1} = a^{n_a-2}$; if $n_a = 2$, then $a^2 = a = e$, so $a^{-1} = a$. Therefore S has an identity and every element has an inverse, whence S is a group.

II. *Solution by S. Lajos, Budapest, Hungary.* It is known that a regular cancellative semigroup is a group (Clifford-Preston, *The Algebraic Theory of Semigroups*, 1961, vol. I, p. 33, Ex. 5). The assumption implies solvability of the equation $axa=a$ for each a in S. Here $x=a$ if $n_a=2$, and $x=a^{n_a-2}$ when $n_a>2$. Thus S is regular and cancellative, whence by the above result, S is a group.

Also solved by R. A. Bell, E. R. Berger, W. E. Bodden, J. L. Brown, Jr., John Burslem, J. P. Celenza, R. W. Feldman, Neil Grabois, L. C. Graue, Charles Green, M. G. Greening (Australia), Thomas Hand, G. A. Heuer, R. J. Hursey, Jr., Erwin Just, R. N. Leggett, Jr., Arthur Lieberman, C. C. Lindner, D. C. B. Marsh, Lieselotte Miller, Alexander Morgan, Wanda J. Mourant, D. E. Myers, F. D. Parker, Stanton Philipp, M. S. R. K. Sastry, D. A. Sibley, D. R. Stark, Raymond, Stockton, P. K. Subramanian & L. R. Pujara, L. R. Vermani (India), Charles Wells, J. E. Wilkins, Jr., J. C. Williams, E. L. Wilson, P. R. Winter, C. N. Winton, K. L. Yocom, and the proposers.

Brown notes that if S is a finite semigroup, then cancellation alone suffices to imply that S is a group. See N. Jacobson, *Lectures in Abstract Algebra*, Vol. I, p. 24, problem 4.

Conjugate Subgroups of Order n in a Group of Order n^2

E 1874 [1966, 310]. *Proposed by W. A. McWorter, University of British Columbia*

Let G be a group of order n^2 and let H be a subgroup of order n. Prove that for any $x\in G$ we have $H\cap x^{-1}Hx\neq I$.

Solution by William Scott, University of Utah. Deny the stated conclusion, and let $K=x^{-1}Hx$. K is a subgroup of G of order n. Then

$$o(HK) = o(H)o(K)/o(H\cap K) = n^2 = o(G).$$

Hence $HK=G$ and $x=hk$, $h\in H$, $k\in K$. Thus $K=k^{-1}h^{-1}Hhk=k^{-1}Hk$, whence $H=kKk^{-1}=K$, a contradiction (unless $n=1$).

Also solved by W. E. Bodden, W. G. Dotson, Jr., L. C. Graue, M. G. Greening (Australia), Thomas Hand, Edward Hook, J. E. Joseph & C. B. Baytop, Erwin Just, Geoffrey Kandall, V. H. Keiser, Jr., Arthur Lieberman, C. C. Lindner, J. J. Martinez, Stephen Milles & Al Somayajulu, Wanda J. Mourant, Stanton Philipp, Bob Prielipp, L. R. Pujara, Simeon Reich (Israel), C. H. Sampson, M.S.R.K. Sastry, Stephen Ullom, C. N. Winton, Peter Yff (Lebanon), and the proposer.

Yff notes that, according to G. A. Miller (Annals of Mathematics, vol. 14, p. 95) the index of $H\cap x^{-1}Hx$ under H must be less than the index of H under G. This problem is a special case.

Coefficients in a Uniformly Convergent Series

E 1875 [1966, 410]. *Proposed by Samuel Zaidman, University of Montreal*

Let $\phi\in C^\infty$ $[0, 1]$ satisfy the following conditions: (1) $\phi\not\equiv 0$, (2) $\phi^{(k)}(0)=0$, $k=0, 1, 2, \cdots$, (3) The series $\sum a_k\phi^{(k)}(x)$ is uniformly convergent on $[0, 1]$ where $\{a_k\}$ is a given sequence of real numbers. Prove that $k!a_k\to 0$ as $k\to\infty$.

Solution by Stanton Philipp, University of California at Riverside. Choose $\epsilon>0$ and let $x_0\in(0, 1]$ be such that $\phi(x_0)\neq 0$. By Taylor's theorem and (2) we see that for each natural number n there is a number $\xi_n\in[0, 1]$ such that

$$\phi(x_0) = \frac{\phi^{(n)}(\xi_n)x_0^n}{n!};$$

equivalently,

(a) $$\phi^{(n)}(\xi_n) = n!\phi(x_0)/x_0^n.$$

By (3), there is a natural number N such that if $n>N$, then

(b) $$|\, a_n\phi^{(n)}(\xi_n)\,| < \epsilon|\,\phi(x_0)\,|\,.$$

Substituting from (a) into (b), we have for every $n>N$,

$$\left|\frac{a_n n!\phi(x_0)}{x_0^n}\right| < \epsilon|\,\phi(x_0)\,| \Rightarrow |\,n!a_n\,| < \epsilon x_0^n \leqq \epsilon.$$

Also solved by A. S. Galbraith, C. H. Sampson, Al Somayajulu & Steve Milles, Sidney Spital, and the proposer.

Ratios of Number Theoretic Functions

E 1876 [1966, 410]. *Proposed by K. K. Norton, University of Illinois*

Let $\phi(n)$ be Euler's totient function, let $d(n)$ be the number of divisors of n, and let $\sigma(n)$ be the sum of those divisors. Show that

(1) $$\liminf_{n\to+\infty}\frac{\phi(n)}{nd(n)} = \liminf_{n\to+\infty}\frac{\sigma(n)}{nd(n)} = 0,$$

(2) $$\limsup_{n\to+\infty}\frac{\phi(n)}{nd(n)} = \limsup_{n\to+\infty}\frac{\sigma(n)}{nd(n)} = \frac{1}{2}.$$

Solution by R. B. Eggleton, Avondale College, Cooranbong, Australia. All the functions concerned are multiplicative, so it is sufficient to consider the case $n=p^k$, where p is prime and k any positive integer. Familiar formulas for ϕ, d and σ give

$$\frac{\phi(p^k)}{p^kd(p^k)} = \frac{p^{k-1}(p-1)}{p^k(k+1)} = \frac{p-1}{p(k+1)} \to \begin{cases} 1/(k+1) & \text{as } p\to\infty \text{ if } k \text{ is fixed;} \\ 0 & \text{as } k\to\infty \text{ if } p \text{ is fixed.} \end{cases}$$

$$\frac{\sigma(p^k)}{p^kd(p^k)} = \frac{p^{k+1}-1}{p^k(p-1)(k+1)} \to \begin{cases} 1/(k+1) & \text{as } p\to\infty \text{ if } k \text{ is fixed;} \\ 0 & \text{as } k\to\infty \text{ if } p \text{ is fixed.} \end{cases}$$

The appropriate results follow, by taking $k=1$ and $p\to\infty$ for the suprema, and p fixed with $k\to\infty$ for the infima.

Also solved by Anders Bager (Denmark), P. N. Bajaj, Gabriel Bastien, Addison Fischer, M. G. Greening (Australia), Emil Grosswald, R. E. Giudici, R. J. Hursey, Jr. & Charles Zeigenfus, Betty Kvarda, E. S. Langford, Arthur Lieberman, Dan Marcus, D. C. B. Marsh, Stanton Philipp, Simeon Reich (Israel), Sister Marion Beiter, R. Sivaramakrishnan (India), Al Somayajulu, Sidney Spital, and A. M. Vaidya (India).

Convex Pentagon Inscribed in a Semicircle

E 1877 [1966, 410]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey*

Let $ABCDE$ be a convex pentagon inscribed in a unit circle with AE as diameter, and let $AB=a$, $BC=b$, $CD=c$, $DE=d$. Then prove that

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4.$$

Solution by Allan Wachs, Student at Far Rockaway (N. Y.) High School. Draw AC and CE and put $\theta = \sphericalangle CEA$. Then $\sphericalangle CAE = 90° - \theta$, $\sphericalangle CBA = 180° - \theta$, $\sphericalangle CDE = 180° - (90° - \theta) = 90° + \theta$. By the law of cosines,

$$(1) \qquad AC^2 = a^2 + b^2 - 2ab \cos (180° - \theta) = a^2 + b^2 + 2ab \cos \theta,$$

$$(2) \qquad CE^2 = c^2 + d^2 - 2cd \cos (90° + \theta) = c^2 + d^2 + 2cd \sin \theta.$$

From the right triangle ACE, $AC^2 + CE^2 = AE^2 = 4$, also $AC = 2 \sin \theta > b$ and $CE = 2 \cos \theta > c$ (because of the obtuse angles). Substitution of these results into the sum of (1) and (2) gives at once

$$4 > a^2 + b^2 + c^2 + d^2 + abc + bcd.$$

Also solved by Leon Bankoff, W. J. Blundon, L. Carlitz, Mannis Charosh, M. A. Ettrick, Michael Goldberg, M. G. Greening (Australia), Ned Harrell, Donald Jeffords, Erwin Just, J. D. E. Konhauser, Dan Marcus, Lieselotte Miller, Norman Miller, C. B. A. Peck, Al Somoyajulu, J. L. Standig, C. S. Venkataraman (India), J. C. Williams, Dale Woods, and the proposer.

Concyclic Points within a Regular Polygon

E 1878 [1966, 410]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey*

Let $A_1A_2 \cdots A_n$ be a regular polygon inscribed in circle (O) of radius R. Denote the incenter of the triangle $A_{i-1}A_iA_{i+1}$ (indices mod n) by I_i, and that of the triangle formed by A_iA_{i+2}, $A_{i+2}A_{i+1}$, $A_{i+1}A_{i+3}$ by J_i. Then show that the points $I_1, \cdots, I_n$ and $J_1, \cdots, J_n$ all lie on the same circle of radius $R' = R \cos (3\pi/2n)/\cos (\pi/2n)$.

Solution by Frank Kerr, Napa, California. Let the intersection of $A_{i-1}A_{i+1}$ with OA_i be P_i and that of $A_{i+1}A_{i+2}$ with line OJ_i be Q_i. Then

$$\begin{aligned} OI_i &= OP_i + P_iI_i \\ &= R \cos (2\pi/n) + R \sin (2\pi/n) \tan (\pi/2n) \\ &= R[2 \cos (2\pi/n) \cos (\pi/2n) + 2 \sin (2\pi/n) \sin (\pi/2n)]/2 \cos (\pi/2n) \\ &= R \cos (3\pi/2n)/\cos (\pi/2n), \end{aligned}$$

and

$$\begin{aligned} OJ_i &= OQ_i - Q_iJ_i \\ &= R\cos(\pi/n) - R\sin(\pi/n)\tan(\pi/2n) \\ &= R[2\cos(\pi/n)\cos(\pi/2n) - 2\sin(\pi/n)\sin(\pi/2n)]/2\cos(\pi/2n) \\ &= R\cos(3\pi/2n)/\cos(\pi/2n). \end{aligned}$$

Hence $R' = OI_i = OJ_i = R\cos(3\pi/2n)/\cos(\pi/2n)$.

Also solved by J. P. Celenza, Michael Goldberg, M. G. Greening (Australia), J. D. E. Konhauser, E. S. Langford, Dan Marcus, D. C. B. Marsh, Lieselotte Miller, Norman Miller, Robert Patenaude (Germany), J. M. Quoniam (France), Simeon Reich (Israel), Stephen Spindler, Sidney Spital, C. S. Venkataraman (India), and the proposer.

A Billiard Table Problem

E 1879 [1966, 411]. *Proposed by Agatha Himmelfarb, Fordham University*

A billiard ball is cued from a corner of an $n\times m$-foot billiard table at an angle of 45° (n and m are integers). How many cushions will the ball strike before it again goes into a corner?

Solution by D. L. Silverman, Hughes Aircraft Company, El Segundo, Calif. Tesselate the first quadrant with $m\times n$ rectangles. By reflection, each boundary intersected by the line $x=y$ corresponds to a cushion struck by a ball cued from the origin at a 45° angle. The line $x=y$ first intersects a vertex at (L, L), where L is the lowest common multiple of m, n, and en route intersects $L/n-1$ vertical and $L/m-1$ horizontal boundaries. Accordingly, the ball strikes $L/n+L/m-2=(m+n)L/mn-2$ cushions. The result may also be written $(m+n)/(m, n)-2$, where $(,)$ indicates the greatest common divisor.

Also solved by J. C. Abad, Merrill Barneby, J. A. Byrd, J. P. Celenza, Yu Chang, R. B. Eggleton (Australia), Michael Goldberg, Michael Goodman, M. G. Greening (Australia), H. S. Hahn, G. A. Heuer, Donald Jeffords, Michael Goldberg, R. S. Kaluzniacki, R. L. Kammerer, M. S. Kaplan, Frank Kerr, M. S. Klamkin, M. S. Knebelman, Kenneth Kramer, E. S. Langford, Dan Marcus, D. C. B. Marsh, Lieselotte Miller, Norman Miller, Edward Moylan, F. D. Parker, Robert Patenaude (Germany), C. B. A. Peck, Stanton Philipp, J. M. Quoniam (France), Steven Russ, J. J. Segedy, K. N. Sigmon, T. R. Smith, Stephen Spindler, Lawrence Somer, Paul Sugarman, Stephen Tice, P. S. Viita, Allan Wachs, John Wessner, Oswald Wyler, J. W. Zerger, and the proposer.

Klamkin notes that the problem is not new and that a generalization appeared in the 1959 fall edition of the Pi Mu Epsilon Journal. See also problem E 1704 [1965, 669] for a related result.

Marsh notes a second interpretation of the problem; since the table has at most four cushions to be hit, the number of different cushions struck during the ball's motion will be

0 if $q=p=1$, $q=n/(m,n)$, $p=m/(m,n)$;
1 if q, $p=2$, 1 in either order;
2 if one of q, p is 1 and the other is >2;
3 if the smaller of q, p is 2;
4 if the smaller of q, p is >2.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before March 31, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose a self-addressed stamped postcard.

5510. *Proposed by M. L. Glasser, Battelle Memorial Institute, Columbus, Ohio*

Show that

$$\sum_{k=-\infty}^{\infty} (-1)^k(x + kz) \exp(-k^2z^2/x) \cdot {}_1F_1[\tfrac{1}{2}, \tfrac{3}{2}; (x + kz)^2/x] = 0.$$

Does this generalize to other hypergeometric functions?

5511. *Proposed by R. S. Kulkarni, Harvard University*

Characterize groups of order $2n$ which contain exactly n elements of order 2 (i.e. $a^2=1$, $a\neq 1$). (Note that the symmetry group of a polygon with an odd number of sides has this form.)

5512. *Proposed by M. F. Neuts, Purdue University*

At time O an urn contains N balls ($N \geqq O$). Every time unit later, k balls are added and immediately thereafter, (i.e., at times $n+O$), r balls are drawn out at random and discarded ($O<r<k$).

1. What is the probability that a particular ball which is in the urn at time $n+\frac{1}{2}$ will be thrown out later than time $n+\nu+O$? $\nu=1, 2, \cdots$.

2. What is the probability that a particular ball will eventually be thrown out?

3. What is the expected number of drawings required before the given ball is thrown out?

5513. *Proposed by L. F. Meyers, Ohio State University*

If a is a real number and S is a subset of the set R of all real numbers, then the translate T_aS of S by a is defined to be $\{x+a : x \in S\}$. For certain subsets S of R it is possible to find a real number a such that T_aS is the complement S' of S in R. For example, if $S=\bigcup_{k=-\infty}^{\infty}[2k, 2k+1)$, then the translate of S by each odd integer is S'. For each set S allowing a translation onto its complement, let

$$f(S) = \inf\{a : a > 0 \text{ and } T_aS = S'\}.$$

In the above example, $f(S)=1$. Is it possible to find a set S such that $f(S)=0$? In other words, is there a subset of the real line which admits arbitrarily small nontrivial translations of itself onto its complement?

5514. *Proposed by J. W. Wyman, Pasadena College, California*

Find all positive valued functions defined on the positive real numbers such that $ab \leqq \frac{1}{2}\{af(a)+bf^{-1}(b)\}$ for every $a>0$, $b>0$.

5515. *Proposed by J. W. Wyman, Pasadena College, California*

Let f: $M_1 \to M_2$, where M_1 and M_2 are metric spaces, and define f to be *absolutely continuous* if for each $\epsilon>0$, there exists $\delta>0$ such that if $\{a_1, \cdots, a_n\}$ is any finite set of points of M_1 such that $\sum_{i=1}^{n-1} d(a_i, a_{i+1})<\delta$, then $\sum_{i=1}^{n-1} d(f(a_i), f(a_{i+1}))<\epsilon$. If f is continuous, do there exist metrics equivalent to the given metrics in M_1 and M_2 respectively such that f is absolutely continuous with respect to those metrics?

5516. *Proposed by D. J. Simanaitis and L. R. Anderson, Western Reserve University*

Let E be a nonzero unitary K-module which is generated by one element. Suppose the annihilator of E in K is $\{0\}$. Prove or disprove: E admits a basis.

5517. *Proposed by W. A. McWorter, University of British Columbia*

Let $\bigcirc$ be a relation on a set X. A subset A of X is said to be a kernel of $\bigcirc$ if (1) $x \not\bigcirc y$ for every x, y in A, and (2) for every x not in A, there is a y in A such that $x \bigcirc y$.

Suppose $\bigcirc$ is such that for every nonempty subset B of X there is an x in B such that $x \not\bigcirc b$ for every b in B. Prove that $\bigcirc$ has a unique kernel.

5518. *Proposed by P. J. Schweitzer, Institute for Defense Analyses, Arlington, Va.*

Let $a_k=\sum_{i=1}^{n}(x_i)^k$, $k=1, 2, \cdots$. Show that a_k, $(k=n+1, n+2, \cdots)$ can be expressed as a polynomial in $a_1, a_2, \cdots, a_n$. Stated differently, show that the traces of the higher powers of an $n \times n$ matrix can be expressed as polynomials in the traces of the first n powers.

5519. *Proposed by W. O. Egerland, U.S.A. Nuclear Defense Laboratory, Edgewood Arsenal, Md.*

Let $\sum_{n=1}^{\infty} a_n z^n$ be a power series in $z=x+iy$, $|a_n| \leqq cn^p$, c and p positive constants. Determine

$$\lim_{n\to\infty} n^{\alpha} \iint_{|z|\leqq 1} \log(1+|a_n z^n|)\,dxdy$$

for every $\alpha<1$.

SOLUTIONS OF ADVANCED PROBLEMS

A Functional Equation of Bernoulli

5410 [1966, 782]. *Proposed by H. W. Guggenheimer, University of Minnesota*

Prove that $x^{k+1}y(x^{-k}) = y(x)$, $x>0$, has no analytic nontrivial $(y(x) \neq x)$ solution for $k \neq 1$. (This functional equation was proposed by John Bernoulli in 1696, but no valid discussion seems to have been given.)

Solution by Robert Breusch, Amherst College. Let k be real, $k \neq \pm 1$. Then there cannot even be a solution which is just continuous for $x>0$ (actually continuous only for $x=1$).

Let $f(x) = y(x)/x$. Then the problem asks for continuous solutions of

$$\text{(A)} \qquad f(x) = f(x^{-k}), \qquad f(x) \neq a \text{ (a constant)}, \qquad k^2 \neq 1.$$

By (A), $f(x) = f(x^{k^2}) = f(x^{k^4}) = \cdots = f(x^{k^{2n}}) = \cdots$, and $f(x) = f(x^{k^{-2}}) = f(x^{k^{-4}}) = \cdots = f(x^{k^{-2n}}) = \cdots$.

Let $f(1) = c$ and $k^2 < 1$. Then $\lim_{n\to\infty} x^{k^{2n}} = 1$ for every $x>0$, thus by continuity $f(x) = \lim_{n\to\infty} f(x^{k^{2n}}) = f(1) = c$ which contradicts (A).

If $f(1) = c$ and $k^2 > 1$, the same contradiction develops from the fact that $\lim_{n\to\infty} x^{k^{-2n}} = 1$ for every $x>0$.

Also solved by D. R. Anderson, D. Ž. Djoković (Yugoslavia), L. Dorembus & S. Rosset (Israel), D. A. Hejhal, M. D. Mavinkurve (India), R. H. C. Newton (Australia), Stephen Soldz, P. C. Tonne, and the proposer.

Tonne extended the problem and noted for a very general class of functions $u(x)$, $|u'(1)| \neq 1$, $u(1) = 1$, that the analytic solutions of $xu(x)y(1/u(x)) = y(x)$ have the form $y(x) = cx$.

Groups with Direct Factor Subgroups

5411 [1966, 782]. *Proposed by Azriel Rosenfeld, University of Maryland*

Let G be a group every subgroup of which is a direct factor of G. Prove that G is a direct product of cyclic groups of prime orders.

Solution by R. W. Gilmer, Jr., Florida State University. We first observe that G is abelian. For, if $g \in G$, $G = (g) \oplus H$ for some subgroup H of G; g commutes with each element of H, and hence commutes with each element of G.

Zorn's lemma implies the existence of a maximal independent subset $S = \{s_\alpha\}$ of G. If T is the subgroup of G generated by S, we show $T = G$. By hypothesis, T is a direct factor of G: $G = T \oplus H$. And for any nonidentity element h of H, $S \cup \{h\}$ is independent. The maximality of S then implies H is the trivial subgroup of G and $G = T$. Hence G is the direct product of the collection $\{(s_\alpha)\}$ of cyclic groups generated by elements of S.

To complete the proof we show that each (s_α) is a direct product of cyclic groups of prime order.

For this purpose, we first prove that if H is a subgroup of G and if J is a subgroup of H, then J is a direct factor of H. By assumption, J is a direct factor of G: $G=J\oplus T$, and the modular law then implies that $H=J\oplus(H\cap T)$. In particular, if $g\in G$, each subgroup of (g) is a direct factor of (g). From the known structure of cyclic groups, it follows that if (g) is nontrivial, then the order of (g) is finite and is a product of distinct prime factors. Hence (g) is a direct product of cyclic groups of prime order, which completes the proof.

Also solved by M. R. Alipour (Iran), Frank Cannonito, D. Ž. Djoković (Yugoslavia), S. J. Farber & Marvin Tretkoff, P. Hantom, G. A. Heuer, Geoffrey Kandall, H. D. Keesing, M. D. Mavinkurve (India), Ka Menehune, Al Somayajulu, Gomer Thomas, Z. Z. Uoiea, R. Wiegandt (Hungary), Kenneth Yanosko, and the proposer.

Kandall and Wiegandt refer to a theorem of A. Kertész: *Each subgroup of a group G is a direct factor of G if and only if G is abelian and the order of each element of G is a squarefree number.* See A. Kertész, *On groups every subgroup of which is a direct summand*, Publ. Math. Debrecen, 2 (1951) pp. 74–75.

The Magnitude of Least Common Multiples

5412 [1966, 783]. *Proposed by Paul Erdös, Budapest, Hungary*

Let $a_1<a_2<\cdots$ be an infinite sequence of integers. Put A_n for the least common multiple of $a_1, a_2, \cdots, a_n$. Prove that $A_n>n^{c_1 \log\log n}$, but that there is a sequence $a_1<a_2<\cdots$ so that $A_n<n^{c_2 \log\log n}$.

Solution by J. H. van Lint, Technological University, Eindhoven, Netherlands. (a) Let $d(A_n)$ be the number of divisors of A_n. Trivially $n\leqq d(A_n)$. Now for every $\epsilon>0$ and m sufficiently large

$$d(m) < \exp\left\{(1+\epsilon)\log 2\frac{\log m}{\log\log m}\right\}$$

(cf. Prachar, *Primzahlverteilung*, p. 24). It follows that $\log A_n>c_1(\log n)(\log\log n)$ with a positive constant c_1 so that $A_n>n^{c_1 \log\log n}$.

(b) Let $P_i=\Pi_{p\leqq 3^i} p$. Then $P_i>(P_{i-1})^2$ and therefore P_i has at least $\frac{1}{2}\cdot 2^{\pi(3^i)}$ divisors which are greater than P_{i-1}. Let the sequence $\{a_n\}$ consist of these numbers; i.e., $\{a_n\}$ is the sequence of numbers larger than P_{i-1} and dividing P_i for some i, which is the sequence 1, 2, 3, 6, 7, 10, 14, $\cdots$. If $P_{\nu-1}<a_n\leqq P_\nu$, then $A_n\leqq P_\nu$, i.e., $\log A_n=O(3^\nu)$.

Furthermore $n>\frac{1}{2}\sum_{i=1}^{\nu-1} 2^{\pi(3^i)}$, i.e.,

$$\frac{\log n}{\log 2} > \frac{3^{\nu-1}}{(\nu-1)\log 3} - 1$$

from which we see $3^\nu=O(\log n\cdot\log\log n)$. This means that for the sequence considered we have $\log A_n=O(\log n\cdot\log\log n)$ completing the proof.

Also solved by Dieter Wolke (Marburg, Germany) who uses the sequence $\{a_i\}$ defined as follows: $n \in \{a_i\}$ if and only if $\mu(n \neq 0$ and $p(n) \geqq (7/2) \log n$, where $p(n)$ is the largest prime divisor of n.

A Pair of Singular Double Integrals

5414 [1966, 783]. *Proposed by L. Carlitz, Duke University*

Show that

$$(1) \quad \int_0^1 \int_0^1 \{(1-x^2)(1-y^2)(1-x^2y^2)\}^{-1/2} dxdy = \frac{(\Gamma(\frac{1}{4}))^4}{16\pi}.$$

$$(2) \quad \int_0^1 \int_0^1 \{(1-x^2)(1-y^2)(1+x^2y^2)\}^{-1/2} dxdy = \frac{\pi^2}{4} \left\{\frac{\Gamma(9/8)}{\Gamma(5/4)\Gamma(7/8)}\right\}^2.$$

I. *Solution by R. E. Shafer, University of California at Livermore.* A more general pair of integrals are

$$\int_0^1 \int_0^1 x^{4a-1}y^{2a-1/2}\{(1-x^2)(1-y^2)(1 \mp x^2y^2)\}^{-1/2} dxdy$$

with Re $(a) > 0$. Expanding $(1 \mp x^2y^2)^{-1/2}$ by the binomial theorem and integrating termwise, we obtain

$$\frac{\pi}{4} \frac{\Gamma(2a)\Gamma(a+1/4)}{\Gamma(2a+1/2)\Gamma(a+3/4)} \cdot {}_3F_2[1/2, 2a, a+1/4; 2a+1/2, a+3/4; \pm 1]$$

which is reducible, using Clausen's theorem (Erdelyi, *Higher Transcendental Functions*, vol. 1, Section 4.3) into

$$\frac{\pi}{4} \frac{\Gamma(2a)\Gamma(a+1/4)}{\Gamma(2a+1/2)\Gamma(a+3/4)} \{{}_2F_1[1/4, a; a+3/4; \pm 1]\}^2$$

which may be readily evaluated by Gauss' identity. Therefore

$$\int_0^1 \int_0^1 x^{4a-1}y^{2a-1/2}\{(1-x^2)(1-y^2)(1-x^2y^2)\}^{-1/2} dxdy = \frac{\Gamma(a)\{\Gamma(1/4)\}^2 2^{-7/2}}{\Gamma(a+1/2)};$$

$$\int_0^1 \int_0^1 x^{4a-1}y^{2a-1/2}\{(1-x^2)(1-y^2)(1+x^2y^2)\}^{-1/2} dxdy = \frac{\pi}{32} \frac{\Gamma(a/2)\Gamma(a/2+1/4)}{\Gamma(a/2+1/2)\Gamma(a/2+3/4)}.$$

II. *Solution by M. L. Glasser, Battelle Memorial Institute.* The integrals are the case $\alpha=\beta=\frac{1}{2}$, $\gamma=1$ of the formula

$$\int_0^1\int_0^1 x^{2(\alpha-\gamma)+1}y^{2\beta-1}[(1-x^2)(1-y^2)]^{\gamma-\beta-1}(1\pm x^2y^2)^{-\alpha}dxdy$$

$$= \tfrac{1}{4}B(\beta,\gamma-\beta)\begin{cases}\dfrac{\Gamma(\gamma-\beta)\Gamma(\alpha-\gamma+1)}{\Gamma(\alpha-\beta+1)}\,{}_3F_2\left[\begin{matrix}\alpha-\gamma+1,\ \alpha,\ \beta\\ \alpha-\beta+1,\ \gamma\end{matrix}\,\middle|\,-1\right],\\ \dfrac{\Gamma(1+\alpha/2)\Gamma(\gamma)\Gamma(\alpha-\gamma+1)\Gamma(\gamma-\alpha/2-\beta)}{\Gamma(1+\alpha)\Gamma(1+\alpha/2-\beta)\Gamma(\gamma-\alpha/2)}, \text{ Re } (\gamma-\alpha/2-\beta)>0,\end{cases}$$

for Re $(\gamma-\beta)>0$, Re $(\alpha-\gamma+1)>0$.

This formula is obtained, after a simple change of variables, by combining equations 7.512 (1), (12) and 9.111 of Gradshteyn and Ryzhik, *Table of Integrals, Series and Products* (Academic Press, 1965). The generalized hypergeometric series arising in this case can be found, e.g., in Bailey, *Generalized Hypergeometric Series* (Cambridge, 1935).

Also solved by M. G. Beumer, D. A. Breault, J. H. Heinbockel, E. L. Koh, Chih-Bing Ling, J. W. Mellender, F. W. Steutel (Netherlands), the late E. L. Whitney, V. E. Wood, and the proposer.

Breault obtains the first formula above from problem 612, Mathematics Magazine, Sept.–Oct. 1966, pp. 251 ff.

Whitney avoids hypergeometric series by establishing the first integral as $\frac{1}{2}\pi\int_0^1 F(x^2)(1-x^2)^{-1}dx$ and the second as $\frac{1}{2}\pi\int_0^1 xF(x^4)(1-x^2)^{-1/2}dx$, where

$$F(r^2)=\frac{1}{\pi}\int_0^\pi \frac{d\theta}{\sqrt{1+r^2+2r\cos\theta}}=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1+r)^2-4r(\sin\theta)^2}}.$$

Primitive Elements for K/Q

5416 [1966, 783]. *Proposed by W. J. Leahey, University of Illinois*

Let K be a number field and θ a primitive element for K/Q, i.e. $K=Q(\theta)$, where Q is the field of rational numbers. Let $f(X)$ be the minimal polynomial of θ over Q and denote by $f'(X)$ the derivative of $f(X)$. Is $f'(\theta)$ necessarily a primitive element for K/Q?

Solution by L. Carlitz, Duke University. Consider $f(x)=x^6-2x^3+2$, which is evidently irreducible over Q (by the Eisenstein criterion). Then for $f(\theta)=0$,

$$\begin{aligned}\delta &= f'(\theta) = 6(\theta^5-\theta^2) = 6\theta^2(\theta^3-1),\\ \delta^2 &= 36\theta^4(\theta^3-1)^2 = 36\theta^4(\theta^6-2\theta^3+1) = -36\theta^4,\\ \delta^3 &= -216\theta^3(\theta^6-\theta^3) = -216\theta^3(\theta^3-2)\\ &= -216(\theta^6-2\theta^3) = 2\cdot 6^3,\end{aligned}$$

so that δ is not a primitive element of K.

A Multivariate Distribution

5417 [1966, 783]. *Proposed by M. V. Chari and John Slivka, State University of New York at Buffalo*

Show that $\int \cdots \int_S (y-z)dx_1 \cdots dx_r = m-n$, where $y = \min_i\{m,\ x_i+\frac{1}{2}\}$, $z = \max_i\{n,\ x_i-\frac{1}{2}\}$, $n<m$, and

$$S = \{(x_1, \cdots, x_r);\ n - \tfrac{1}{2} < x_i < m + \tfrac{1}{2} \text{ for all } i (i = 1, \cdots, r) \text{ and } \max_i\{x_i\} - \min_i\{x_i\} < 1\}.$$

Solution by the proposers. Consider the statistical joint density

$$f(w, x_1, \cdots, x_r) = \begin{cases} 1/(m-n), & n < w < m,\ w - \frac{1}{2} < x_i < w + \frac{1}{2} \text{ for all} \\ 0, & \text{elsewhere.} \end{cases}$$

$$= \begin{cases} 1/(m-n), & z < w < y \\ 0, & \text{elsewhere.} \end{cases}$$

Finding the marginal density of $x_1, \cdots, x_r$ we have

$$g(x_1, \cdots, x_r) = \begin{cases} \dfrac{y-z}{m-n}, & (x_1, \cdots, x_r) \in S \\ 0, & \text{elsewhere,} \end{cases}$$

from which the desired result follows immediately.

Rank in a Symmetric Semi-definite Matrix

5419 [1966, 784]. *Proposed by F. D. Faulkner, U. S. Naval Postgraduate School*

Let A_{nn} be a real, symmetric, positive semi-definite matrix. Let A_{pn} be the matrix obtained from A_{nn} by omitting the last $n-p$ rows, and A_{pp} be obtained by omitting the last $n-p$ columns of A_{pn}. Prove that rank A_{pp} = rank A_{pn}.

I. *Solution by R. F. Rinehart, Naval Postgraduate School.* Since A is positive semidefinite, there exists a matrix C such that $C^TC = A$. Let

$$P = \begin{pmatrix} I_{n-p} & 0 \\ 0 & 0 \end{pmatrix}$$

and let $R(X)$ signify rank of X. Then by well-known rank theorems, including the theorem that $R(X^TX) = R(X)$, for X real,

$$R(A_{pn}) = R(P^TA) \geqq R([P^TA]P) = R(A_{pp})$$
$$= R(P^TC^TCP) = R(P^TC^T) \geqq R(P^TC^TC) = R(A_{pn}).$$

Therefore $R(A_{pn}) = R(A_{pp})$.

II. *Solution by W. L. Morris, Oak Ridge National Laboratory.* Remark:

Denoting rank by R, it is well known that $R(CD)=R(D)$ if and only if $CDx=0$ implies $Dx=0$. (Mirsky: *An Introduction to Linear Algebra*, p. 165.)

Let A be a nonnegative hermitian matrix and S a matrix of p columns and n rows. We prove the (slightly) more general result: $R(S^HAS)=R(S^HA)$, where the superscript H denotes the conjugate transpose. If $S^HASx=0$ then $(Sx)^HA(Sx)=0$ which implies that either (1) $Sx=0$ or (2) $(Sx)^HA(Sx)/(Sx)^H(Sx)=0$. If (2) holds then Sx minimizes the Rayleigh quotient of A which means that Sx is a proper vector of A associated with its smallest proper value, viz., zero (Gantmacher, *The Theory of Matrices*, vol. 1, p. 319). In either case, $S^HASx=0$ implies $ASx=0$. Applying the remark gives $R(S^HAS)=R(AS)$, and since A is hermitian, $R(AS)=R(S^HA)$. Taking S to be the first p columns of the identity solves the problem.

Also solved by D. Ž. Djoković (Yugoslavia), A. S. Householder, M. D. Mavinkurve (India), C. C. A. Sastri, G. W. Soules, and the proposer.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Correspondence about Reviews will be welcome.

Reviewing Policy. We are now publishing telegraphic reviews of all textbooks above the sophomore level and of all other mathematics books, including many advanced treatises, thought to be of interest to the readers of this magazine. The lag between receipt of books and publication is about three months, of which more than two are accounted for by the mechanics of printing and distribution. This coverage, in spite of the use of small type, requires a high selectivity of titles for extended review. We solicit comments and suggestions.

The problem of reviewing textbooks in fairly standard courses is a most difficult one. We feel that the most important thing is a prompt informative notice, since professors make their choices only after examining the book itself. Accordingly, telegraphic reviews are intended to give sufficient information for a decision as to whether to order an examination copy. An extended review of a textbook by someone who has not used it is of doubtful value, especially if it is delayed by a year or more, and such reviews in the past have served often as nothing more than belated notices. On the other hand, a review based on actual use would be valuable. We have tried to arrange for such reviews by locating and soliciting users. However, we have had little success during the current year, largely because of the difficulty of finding a professor who had actually used a given book. Perhaps the best way to secure reviews of this kind is to ask for volunteers. In order to avoid unintentional duplications we ask that those willing to review give us advance notice of their intentions.

Perhaps even more valuable than reviews of individual books would be survey articles analyzing the existing textbooks in a given area. Such articles would be helpful to the professor faced with an overwhelming number of competing textbooks among which it is difficult to distinguish without extended study. This type of a survey article

would be a substantial job, but much of such work is done by those assigned to choose textbooks. We hope that volunteers will let us know of their willingness to perform this valuable service to the profession, and we stand ready to help secure the books.

Universal Algebra. By P. M. Cohn. Harper and Row, New York, 1965. xi+333 pp. $9.75 (Teleg. Review April 1967).

Given a set A together with a set Ω of functions from Cartesian powers of A to A. If Ω can be partitioned into disjoint subsets $\Omega(0), \Omega(1), \cdots$ (some of which may be empty) such that each element of $\Omega(n)$ is a function from A^n to A, then the pair $\langle A, \Omega\rangle$ constitutes a *universal algebra*. The elements of $\Omega(n)$ are the n-ary operations of $\langle A, \Omega\rangle$. The 0-ary operations are interpreted as distinguished elements. Thus a group can be presented as a universal algebra with one 0-ary operator the identity, one unary operation inversion, and one binary operation product. There is no restriction on the cardinality of Ω, so that it is possible to view a vector space over the real field as a universal algebra with, among others, an infinite set of unary operations: the scalar multiplications

$$\theta_r\colon x \to rx$$

for each real number r.

The history of Universal Algebra began with a work of A. N. Whitehead, a Treatise on Universal Algebra with Applications I, Cambridge 1898, from which the subject got its name. Whitehead in his treatise attempted to produce a unified treatment of Hamilton's quaternions, Grassmann's calculus of extensions, and Boole's algebra of logic, which were the main algebraic systems extant at the time. The term "applications" in the title of his Treatise refers to applications to geometry and mechanics.

After Whitehead, the subject lay dormant until the early 1930's when it was taken up by G. Birkhoff and O. Ore. Each of these authors published papers on universal algebras which employed algebraic concepts and the somewhat crude metamathematical apparatus available at the time.

The main result of Birkhoff (part of Theorem 3.1 p. 169 in the book under review) gives a flavour of the results of that time: If the axioms of a set of universal algebras can be presented in the form of equations without quantifications, then the set is closed under the formation of direct products, homomorphisms, and subalgebras.

The refinements in metamathematics and category theory which were made in the 1940's and early 50's paved the way for the further development of universal algebra. This began with A. Robinson's *On the metamathematics of algebra* in 1951; later it was continued by Tarski, Mostowski, and their students, until today there is a flourishing literature on the subject. Much of the present day work in universal algebra involves the theory of models developed by Robinson. His most famous result, developed within this theory (Corollary 6.8 p. 219) is the following: An elementary sentence (a formula without free variables) which holds in every field of characteristic zero also holds in every field of sufficiently high characteristic.

The book under review is, to the reviewer's knowledge, the first book since Whitehead's which is devoted entirely to the subject in full generality, and therefore it deserves some attention. It has already been reviewed in the Math. Reviews by J. R. Isbell (MR 31, no. 224) and in the J. London Math. Soc. by Graham Higman (41 (1966), 760).

Since Isbell's review covers the content quite adequately, I will not detail a list of the subject matter covered. Instead, I shall concentrate on the relevance to mathematics of the subject as a whole and of the book in particular. In his review, Higman remarks: "Universal Algebra is something everyone ought to know about, though nobody should specialize in it." To a large extent this is also the present reviewer's opinion. To develop a new argument or result in, say, group theory and then to generalize it to a wider class

of universal algebras, puts a result of proven utility at the disposal of workers studying algebras of this wider class, and hence is a useful contribution to mathematics. On the other hand, to obtain a result about universal algebras as such, without reference to an application, may result in yet another hollow addition to the already overblown mathematical literature.

The book under review is concerned with results of the former kind, i.e. standard results and arguments given in their broadest generality and results proved in a general setting which have obvious application to well-known problems. The reader with a year of graduate work in algebra will find the book a compendium of familiar arguments and proofs arranged in such a way that it is easy for him to fit them into algebra (and mathematics) as a whole. In general, I would say that the book could be used as a basis for a first or perhaps second year graduate course in algebra, for those interested in the relationship between abstract algebra and the foundations of mathematics.

A word of caution is necessary however: The style of the book is very condensed; in fact, in places it is genuinely heavy going. Some of the results are stated very cleverly, as for example Proposition 4.1 on p. 21 where you are tempted to provide a counter-example before you realize that the infimum of the empty set should be the maximal element of the lattice. In other cases results are stated a little haphazardly, as for example Theorem 4.9 on p. 25 where in (i) the author means to say that the chain $a=a_0 \geqq a_1 \geqq a_2 \geqq \cdots$ is ultimately constant. The only genuine error which the reviewer found was Theorem 5.7 on p. 84 for which a counter example exists. This situation requires that Corollary 6.5 on p. 88 be proved directly, which is quite easily done. Other minor errors are mentioned in the Isbell review.

Despite some minor imperfections which can easily be rectified in future editions, this book is an important addition to mathematical literature and should be placed in every college and university mathematics library.

BARRON BRAINERD, University of Toronto

Harmonic Analysis. 1965 MAA Cooperative Summer Seminar. By Lynn H. Loomis, notes by Ethan Bolker. Mathematical Association of America, Buffalo, 1966. 350 pp. (paper) $3.00. (Teleg. Review January 1967)

As the author states in his preface, "The principal theme of these notes is the development of the heavy machinery required to manipulate the Fourier transform." The first half of the book presents basic theorems of linear functional analysis and harmonic analysis on a locally compact abelian group. In the second half attention is restricted to harmonic analysis on euclidean n-space where weak methods are applied, via the Fourier transform, to prove some theorems about linear partial differential equations.

Because of insufficient complementary results and exercises (only eight) which would deepen understanding of the principal theorems, these notes are not recommended as a primary text. The content, enhanced by an informal style, makes the book valuable for general reading. It provides insight into applications of harmonic analysis for the modern analyst and a different approach to linear differential equations for the classical analyst. Supposedly, the only prerequisites are first year graduate level topology, measure theory and algebra. However, since the text is loaded with "big" theorems, some previous knowledge of functional analysis or partial differential equations would make its reading more enjoyable and worthwhile.

RICHARD ILTIS, University of Toronto

Introduction to Computing. By T. E. Hull. Prentice-Hall, Englewood Cliffs, N. J., 1966. xi+212 pp. $7.75. (Teleg. Review March 1967)

In discussing the development of computers, one usually talks in terms of generations.

First generation computers contained tubes, second generation computers utilized solid state devices, and the current question is whether the third generation has been reached. Can introductory textbooks on computing be classified in the same way? The first generation would be the programming manuals initially supplied by the organization that manufactured the computer; these were usually incomprehensible to the novice. The second generation would be the programming texts designed for the beginner and usually published commercially. The central characteristic of a third generation text, in this classification, is the focus on the computer as a processor of symbols rather than solely as a computer of numerical values. In this sense this book belongs to the third generation.

This book is intended as an introductory text and, according to the author, requires nothing beyond high-school mathematics, except for one example and a few exercises which can be omitted. From a pedagogical viewpoint the book has several desirable characteristics: briefness, numerous exercises, a widely available language (FORTRAN IV) and material on nonnumerical applications. My personal preference is to delay machine language and assembly programming (covered in Chapter 2 and 3) as long as possible in an introductory course. The book appears to be one of the best candidates now available for freshman or sophomore courses.

D. TEICHROEW, Case Institute of Technology

The Logic Design of Transistor Digital Computers. By G. A. Maley and J. Earle. Prentice-Hall, Englewood Cliffs, N. J., 1963. x+322 pp. $12.95.

This book presents an introductory treatment of the logic design techniques used to derive NAND/NOR circuits currently in use in modern digital computers and digital devices. Extensive use is made of Karnaugh map methods for both logic design and minimization. They are applied principally to NAND/NOR combinational logic circuits and asynchronous sequential circuits. The mathematical and theoretical foundations of the subject are developed only in minimal fashion.

Use for a year as a supplemental text in the second semester of a one-year graduate course in the logical design of digital computers at the University of Southern California indicated that the text was too elementary and nonmathematical at this level.

I. S. REED AND C. R. WALLI, University of Southern California

A Survey of Geometry, Vol. II. By Howard Eves, Allyn and Bacon, Boston, 1965. xiii+496 pp. $9.95. (Teleg. Review Feb. 1967)

This volume contains Chapters 9–16 ("the analytical and the more abstract portions of the subject") of a two volume survey of geometry. Volume I was reviewed in this MONTHLY, 72 (1965) 691. All of the excellent features of Volume I (clear exposition, imaginative and extensive problem lists, and extensive use of historical information) are carried over to Volume II.

The topics developed at more than a superficial level include the geometry of complex numbers (Chapter 12), the classical differential geometry of curves and surfaces (Chapter 14), and the combinatorial topology of 2-manifolds (Chapter 15). Shorter treatments of more than just a descriptive nature include solvability and insolvability with Euclidean tools (Section 9.3), geometrical applications of the intermediate value theorem (Section 13.3), and pathological curves and surfaces (Section 13.4). These should not be omitted from the "basic course" as outlined by the author.

The major pedagogical difficulty of this text, but one of its great virtues as a reference, is the large number of definitions which appear in rapid succession for the student to learn. For example, under "curve(s)" there are over 100 entries in the index (including lituus, orthoptic, and orthotomic).

In the preface to Volume I and again in its review the question is raised as to whether

the author will do for geometry what Birkhoff and MacLane did for modern algebra in 1941. This reviewer's opinion is that he has not and that in order to do this a survey of geometry, especially one that claims to present the analytical and more abstract portions of geometry, must exploit linear algebra to present a more extensive and rigorous treatment of n-dimensional affine and projective spaces.

This text is excellent for a terminal course in geometry for prospective teachers. Although it is not appropriate as the basic text for a class of good mathematics majors preparing for graduate study, it is valuable as a source of supplementary reading assignments for such a class.

P. B. YALE, Pomona College

About Vectors. By Banesh Hoffmann. Prentice-Hall, Englewood Cliffs, N. J., 1966. ix+134 pp. (paper) $3.75.

The enormous proliferation of college mathematics textbooks, most of which are uninspired, unoriginal and dull, makes it especially pleasing to pick up an unpretentious paperback like "About Vectors" and find that it does have ideas, genuine originality and a flavor that will give delight to students and instructors alike. Instead of presenting a neat linear path through the subject of vectors, the author raises all sorts of questions which he hopes will "disturb and annoy." The provocative questions and roadblocks strewn in the path of the reader exhibit a wealth of experience in teaching the subject and thoughtful reflection upon the difficulties involved—and there seem to be very serious difficulties. For example, early in the game one is faced with the question: What is a vector? Is it any quantity with length and direction? or a directed segment? or an equivalence class? or an undefined algebraic entity? The spirit of raising such questions continues, giving the reader insights into the subject, *per se*, and to the peculiar ways in which it is applied (in physics). Of particular interest is the chapter on Vector Products and Quotients of Vectors. The embarrassing character of the cross-product is not kept secret; that it is a pseudovector rather than a vector paves the way for an introduction of quarternions and a subsequent chapter on tensors. There is also a very nice discussion relating cross-products to oriented areas and boundaries. This discussion will give students valuable assistance in understanding the integral theorems of Stokes and Gauss. Unfortunately, the chapter on tensors is too bare to provide the same sort of service as its predecessors.

No vector calculus appears. This is a mixed blessing, for it makes the book accessible to a large audience but precludes its use as the text for a standard vector analysis course. The principal use for "About Vectors" will probably be as an auxiliary text for any course (including physics courses) in which the subject of vectors plays a central role. However, its use in any manner is sure to give ultimate pleasure in spite of—or because of—the author's design to "disturb and annoy."

SEYMOUR SCHUSTER, University of Minnesota

Modern Algebra, vols. I and II. By Seth Warner. Prentice-Hall, Englewood Cliffs, N. J., 1965. viii+797 pp. vol. I, $12.95; vol. II, $11.95.

These volumes are intended primarily as a text, at the junior-senior level, in a two-semester introductory course in abstract algebra. By omitting appropriate portions, they can be used for students who have been introduced to linear algebra earlier in the undergraduate curriculum. The amount of space devoted to linear algebra is large; the text contains nearly all topics covered in such a text as Hoffman and Kunze's *Linear Algebra*.

There are six chapters in the first volume, five in the second. The chapter headings give a good idea of the diversity of topics. They are: I Algebraic Structures; II New

Structures from Old; III The Natural Numbers; IV Rings and Fields; V Vector Spaces; VI Polynomials; VII The Real and Complex Number Fields; VIII Algebraic Extensions of Fields; IX Linear Operators; X Inner Product Spaces; XI The Axiom of Choice. As the author points out, there is too much material in the two volumes to be covered in one year, so the instructor has freedom in choosing topics to cover. Although the author states that there are sufficiently many topics in the first volume for a year course, most people teaching such a course would want to include topics from both volumes.

The writing is lucid and, as the author hopes, the student ought to be able to read it easily, thus leaving class time for problems. There are occasions, however, when the notation becomes cumbersome. Also, there are places where use of the author's notation would waste a student's time; for example, see the definition of automorphism group of E over k, p. 527. References to earlier portions of the text are not always made easy for the reader. For example, exercise 35.21, p. 392, refers to (N), whereas there is no hint in the exercise as to where (N) can be found; actually, it is on p. 380. The text contains a reasonable number of illustrative examples and many good exercises. The exercises come in all degrees of difficulty, with hints for the more difficult ones. Many exercises are attributed to this MONTHLY. There is no bibliography, which seems a mistake. The index is unusually full, containing references not only to the text but to the exercises.

It might have been better for the author to have had a chapter on group theory. The material on groups is scattered throughout the text. The concept of a group is introduced at the end of the first chapter, while such an elementary result as Lagrange's theorem appears, on p. 293, in the last section of the chapter on Rings and Fields.

In summary, the volumes have many good features, which will make them very useful in the presentation of a first course in abstract algebra. Of the large number of algebra books which have appeared recently these may be included among the better written ones.

ALICE T. SCHAFER, Wellesley College

Numbers and Ideals. By Abraham Robinson. Holden-Day, San Francisco, 1965. 106 pp $5.95.

As the tremendous bulk of twentieth century mathematics continues to grow, so does the danger that the student will miss much of its beauty by being exposed to the latest abstractions and generalizations before he has much knowledge of the specific mathematical objects or theories that are being abstracted or generalized. Thus a student may become thoroughly adept with categories, functors and commutative diagrams but still know little of the theory of groups, rings and modules. Similarly an extensive course in theory of groups or rings sometimes leaves a student with little knowledge of specific groups or rings.

Professor Robinson's book has been written primarily to combat this type of mathematical training. In an elementary and readable style he has written the story of unique factorization in the ring of integers, its failure in rings of algebraic integers of certain quadratic fields and of its restoration in the form of unique factorization for ideals. This is a beautiful chapter of mathematics, in which much of abstract ideal theory has its roots, but which is often neglected in modern courses in abstract algebra. The exposition is kept as simple as possible by treating only quadratic fields. The book should prove a valuable addition to the growing mathematical literature addressed to able undergraduates. Some exercises are provided to test the reader's understanding. There are several typographical errors that should be corrected in a second printing. (*Editorial Note:* Since this review was written, a second edition has been printed with corrections and added problems.)

D. C. MURDOCH, The University of British Columbia

Elements of General Topology. By Sze-Tsen Hu. Holden-Day Series in Mathematics, 1964. 214 pp. $8.95.

The author whose interests in mathematics are extensive and varied (a book on algebra and one on homotopy theory have been published recently with rumours of more to come) shows his usual pedagogical skill in this text also. He writes with a clear style and adheres to a high standard of mathematical rigor. Supplying basic information first, the author begins with a treatment of the standard elements of general topology, emphasizing basic concepts and properties. The proofs are presented in a concise manner and carefully chosen examples keyed to the main line of development are given.

While textbooks in general topology proliferate as never before, there is something new presented in this text. The last three chapters (pages 111 to 200) deal with more specific topics which have not previously appeared in books or have appeared only in advanced treatises. Chapter IV discusses cellular polytopes. Chapters V and VI deal with function spaces and fundamental groups. These chapters should be of interest to all mathematicians. A bibliography of 46 books and 17 research papers as well as a list of special symbols and abbreviations concludes the book. The text contains many exercises of varying degrees of difficulty. According to the author, it is designed as a text for an introductory first course in general topology for advanced undergraduates and beginning graduate students. In the reviewer's opinion, the text might offer some difficulties to the average beginning graduate student, but it would be ideal for the mathematically more mature one.

DAGMAR HENNEY, The George Washington University

Complex Variable Methods in Science and Technology. By John Cunningham. Van Nostrand, Princeton, N. J., 1965. viii+178 pp. $7.50.

This textbook is primarily a compendium of techniques involving complex methods, purportedly designed to furnish the prospective scientist and engineer with the necessary tools for successful pursuit of their fields. Such a book might have been welcome a generation or so ago; today, however, there are several excellent books which meet the present-day needs of the scientifically oriented student far more adequately and admirably, thereby making the present book not too desirable.

I am certain that the style of presentation will be unnerving not only to mathematicians but to scientists as well. Theorems are not always precisely stated; proofs are often slip-shod or are completely omitted, even though the necessary background is available. Important concepts such as bounded function, region, curve, simple curve, boundary points, multiple points are not defined. Some concepts such as line integral and analytic function are not well defined. Such expressions as "continuity" has something to do with "smoothness" and "the shape of this function" jar the nervous system. My experience has shown that most students taking courses in mathematics insist on precision and clarity of expression.

For a book that is addressed to engineers and scientists, it is woefully lacking in substantial applications. The phrase " . . . plays an important role in many two-dimensional problems in classical electrostatics, electric current flow, and non-viscous hydrodynamics" is of typical occurrence. However, since the terminology of the applied fields is not defined, the sprinkling of the text with this jargon hardly enhances its effectiveness as a text in applied mathematics.

Even as a "methods book," it has some serious shortcomings. For example, a consideration of well-known general theorems in the residue theory (well within the grasp of the student at this level) would obviate the necessity of establishing certain results in every problem that is encountered.

The chapter headings are: Real Variable Theory; Complex Numbers, Analyticity;

Conformal Transformations; Integration; Improper Integrals; Many-valued Functions; Beta, Gamma and Delta Functions; Differential Equations. At the end of each chapter is a set of exercises, mostly standard; some are culled from British university examination papers—even these are not exceptionally challenging.

A. E. DANESE, State University of New York at Buffalo

Trends in Elementary School Mathematics. By Lloyd Scott. Rand McNally, Chicago, 1966. ix+215 pp. $3.50 (paper).

Among the readers of the *Monthly* who did not look past the review title there probably is a large number of mathematicians who should read this book. In particular, those who teach mathematics (content) courses for elementary teachers could use the book's map of the contemporary situation and obtain a perspective of the changes taking place in elementary school mathematics curricula. Some of these changes are subtle, some are dramatic; some depend upon pressures from mathematicians, some show the influence of learning theorists. Each of the most important trends is carefully delineated, along with an indication of the conflicts that have arisen recently. Within the various elementary school mathematics projects there are examples of mathematical writing which reflect attitudes and approaches with which most mathematics professors and elementary school teachers are unfamiliar. Consequently, there is a great value in the author's comprehensive survey of major projects and their distinguishing characteristics. This is the basis for the recommendation to *Monthly* readers.

The central section, which discusses mathematical content, is not meant to be sophisticated and it isn't. Consequently, this section will meet with a mixed reception from mathematicians. Nevertheless, they should find the book useful because of the strong points already mentioned, as well as for the later chapters dealing with materials, teaching strategies, and evaluation, plus a set of carefully selected references.

Would that this book were written with an incisive polemic style! Then perhaps mathematicians would react to it, and encourage each other to read it. As it stands, it is a noncontroversial, carefully done survey that should be read by those who have broad responsibilities in teaching mathematics to either elementary school students or to their teachers.

LEONARD FELDMAN, San Jose State College

The Theory of Numbers. By Neal H. McCoy. Macmillan, New York, 1965. 150 pp. $4.95.

The author wrote this short book as a text for a one-semester course at the undergraduate level. There are five chapters. The first deals with divisors and prime numbers. The second and third involve the properties of congruences, polynomial congruences and primitive roots. The fourth deals essentially with the law of quadratic reciprocity and the fifth treats continued fractions. There are some remarks addressed to a reader familiar with some abstract algebra but these may be omitted without affecting the understanding of the subsequent material.

I find the book somewhat meager for a regular undergraduate course. I see no reason why, in considering $f(x) \equiv 0 \bmod p^k$, the author limited himself to the case where $f(x) \equiv 0 \bmod p$ and $f'(x) \equiv 0 \bmod p$ have no solution in common. Quadratic reciprocity is never utilized to get any significant results. There is no general discussion of number theoretic functions, and diophantine equations are limited essentially to the linear congruence. There are not enough challenging problems for the better students.

In my opinion this would be an excellent text in number theory in a good many summer institutes. Professor McCoy knows that writing out a proof in a fairly elementary textbook requires a different style from the one used in writing a paper for a mathematical journal. This is the strong point of the book—the unusual clarity of its writing.

SOLOMON HURWITZ, City College, City University of New York

Uniform Spaces. By J. R. Isbell. Mathematical Surveys No. 12. American Mathematical Society, Providence, Rhode Island, 1964. xi+175 pp. $7.50.

This is the first book since Weil's (1937) to be entirely devoted to the subject of uniform spaces. The theory has grown greatly, in large part thanks to the author, and the book will be warmly welcomed. It starts at the beginning and is self-contained, carefully organized and well motivated. Nevertheless it is not easy reading. The statements and proofs of theorems are laconic; the reader must often amplify the details, contending with unfamiliar ideas and terms. (Luckily, there is a good index.) He will be rewarded for his trouble by meeting many striking new results and problems, and gaining new insights into many familiar ones, here generalized from topological to uniform spaces with an increase of significance.

The book begins with a foreword on categories (in the sense of Eilenberg-MacLane rather than of Baire or Lusternik-Schnirelmann). Category theory as such is not used, but categorical notions supply motivation, organization and terminology. After two chapters on fundamentals there come chapters on spaces of functions, polyhedra, compactifications, dimension and "locally fine" spaces (those for which every uniformly locally uniform covering is uniform). There is an extensive bibliography. Each chapter includes problems, many of them important results in their own right, which comment on or complete the results in the text; and it ends with notes giving references to the literature. Some unsolved research problems are stated. The author adds comments from time to time, which the reviewer found always interesting, sometimes baffling, sometimes highly illuminating. (This review has been abridged with the author's permission.—K.O.M.)

A. H. STONE, University of Rochester

Fortran IV. Programming and Computing. By James T. Golden. Prentice-Hall, Englewood Cliffs, N. J., 1965. 270 pp. $4.50.

This book is written as a college-level introduction to computing and programming. The first concern of the author, who is scientific marketing manager of IBM, is with programming rather than specific applications.

As each feature of Fortran is described, the need for it is discussed and examples are given which depict usage. The treatment of Fortran is comprehensive. Two important aspects of Fortran, often ignored by programming manuals, are given detailed attention. Program planning and program testing or "debugging" are discussed in detail and with precision. The importance of these phases is often overlooked by the beginning programmer.

Considered as a whole, this book will be useful not only to the reader who may wish to use it for self-study but also to those who desire to learn more about recently developed features of Fortran.

DAGMAR HENNEY, George Washington University

Functions of a Complex Variable. Tutorial Text No. 2. By D. S. Mitrinovic. P. Noordhoff Ltd., Groningen, The Netherlands, 1965. 114 pp. $4.90.

This book consists of 286 problems in the theory of functions of a complex variable. In the author's words, the problems were "collected over many years with the aim of providing students and teachers with material, the search for which would otherwise occupy much valuable time." Complete solutions are given for some of the problems and answers or hints are given for others, but many problems are left unsolved. The existing solutions present methods that apply to subsequent problems so the situation is not as calamitous as it sounds, although students would probably appreciate more answers.

The topics treated are classical in nature, emphasis being placed on the geometry of

the complex numbers and the elementary analytic functions. Integration theory is omitted as well as topics such as local properties of analytic functions, mapping theorems, and topological properties of families of analytic functions. There is only one short section concerning residues (defined via Laurent series), but a future volume in the Tutorial Text series will be devoted entirely to the calculus of residues.

D. R. SHERBERT, University of Illinois

Matrix Mechanics. By H. S. Green. P. Noordhoff Ltd., Groningen, The Netherlands, 1965. 118 pages. $5.50.

Although the title of this book is "Matrix Mechanics," the emphasis is the application of an operator method to the solution of a number of quantum-mechanical problems, e.g., the simple harmonic oscillator, the particle in a box, and the hydrogen atom (relativistic and nonrelativistic). The only matrices displayed are those associated with the harmonic oscillator, with the angular momentum operators, the Pauli matrices, and the α_j and β matrices in Dirac's theory of the electron.

The book should be available to students who have had a first year course in quantum mechanics and an introductory course on matrix algebra. The author carefully lays the mathematical foundations as they are required in the text, which is supplemented by numerous well chosen exercises. There is a short section on quantum mechanical paradoxes. This and other features should make the book of interest to more experienced readers also.

ERNEST IKENBERRY, Auburn University

Topological Analysis. By Gordon T. Whyburn. Rev. ed. Princeton University Press, Princeton, N. J., 1964. 125 pp. $5.00.

Topological analysis deals with certain basic results of analysis, particularly complex function theory, that are fundamentally topological in nature and can be developed largely by topological methods. In the book under review, the author concentrates on those results from complex function theory which can be obtained with the use of the topological index and openness of mappings.

The first four chapters present preliminary material. Basic set-theoretic topology, in the setting of metric spaces, is given in Chapters 1 and 2. Chapter 3 treats plane topology, providing a topological background for later developments. These three chapters are concise and represent considerable condensation of Chapters 1, 2, and 6 of the author's *Analytic Topology* (American Mathematical Society Colloquium Publications, vol. 28). Chapter 4 is an elementary review of properties of the complex number system.

In Chapter 5, the topological (or circulation) index is introduced and studied. The index is closely related to the winding number of a mapping, and is defined using exponential representations of mappings from a metric space X to the complex plane. Properties of the topological index are applied in Chapter 6, in which a number of standard results of complex function theory are established. These include continuity of the derivative, existence of second and higher derivatives as consequences of differentiability, the maximum modulus theorem, Cauchy's inequality, and series representation. These results are established by topological methods. The author gives, in an appendix, a simplified treatment of the topological index sufficient to yield the results of Chapter 6. Various other results of complex function theory are established in Chapter 7, including Rouché's theorem, and theorems of Hurwitz and Vitali.

The last two chapters concern local and global analysis of open and light open mappings. The main results concern such mappings on 2-manifolds, and are established with the aid of an analysis of the action of such mappings on closed 2-cells. Among the major

results are the theorem of Stoïlow that any light open map from a 2-manifold without boundary to the complex plane is topologically equivalent to an analytic function, and a necessary and sufficient condition that a 2-manifold without boundary be a Riemann surface.

In this revised edition of *Topological Analysis*, the study of differentiability properties of mappings has been simplified. The book is well organized and clearly written. It will, no doubt, stimulate further developments in an interesting and important area of topology, one to which the author has made numerous significant contributions.

STEVE ARMENTROUT, University of Iowa

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase) 15 (junior level)—18 (second graduate year).

Algebra

The Modified Algorithm of Jacobi-Perron. By Leon Bernstein. Memoirs of the American Mathematical Society, Number 67. Amer. Math. Soc., Providence, R. I. 1966. 44 pp. $1.40 (paper). A generalization of the Euclidean algorithm that yields solutions of algebraic equations. P, L.

Proceedings of the Conference on Categorical Algebra, La Jolla, 1965. Edited by S. Eilenberg, D. K. Harrison, S. MacLane, H. Röhrl. Springer-Verlag, New York, 1966. 562 pp. $14.50. Twenty-two papers which the editors hope give "a representative, if incomplete, coverage of the present activities . . . trend of research . . . in this country." P.

Categories of Fractions and Homotopy Theory. By P. Gabriel and M. Zisman. Springer-Verlag, New York, 1967. x+168 pp. $9.50. Intended as a possible introduction to algebraic topology for "beginners," the algebraist, and "the topologist eager to assimilate the category language." This book appears to presuppose familiarity with homology theory, category theory, and other matters that give a special meaning to "beginners." P.

Sequences, Vol. I. By H. Halberstam and K. F. Roth. Oxford, New York, 1966. xx+290 pp. $10.10. The first of two volumes concerned with "a substantial branch of number theory of which no connected account appears to exist . . . , " the arithmetical properties of particular integer sequences (such as the nature of the distribution of the sequence of primes among congruence classes). Reviewed in *Science* 27 January 1967. P, L.

Finite and Infinite Primes for Rings and Fields. By D. K. Harrison. Memoirs of the American Mathematical Society, Number 68. Amer. Math. Soc., Providence, R. I. 1966. 62 pp. $1.50. If the unusually clear and informative introductory paragraphs are typical, this memoir should be good reading for anyone familiar with "the basic rudiments of ring theory." P, L.

An Introduction to Nonassociative Algebras. By Richard D. Schafer, Academic Press, New York, 1966. x+166 pp. $7.95. Assumes linear and elementary abstract algebra. Bibliography. S (17–18), P, L.

Analysis

Lectures on Quasiconformal Mappings. By Lars V. Ahlfors; manuscript prepared with the

assistance of Clifford J. Earle, Jr., Van Nostrand, New York, 1966. 146 pp. $2.75 (paper). Based on notes for lectures given in 1964, this is the first English treatment of the subject. S (18), P.

Distributions and the Boundary Values of Analytic Functions. By Edward J. Beltrami and M. R. Wohlers. Academic Press, New York, 1966. xiv+116 pp. $6.50. A research monograph assuming abstract analysis and addressed to "applied mathematicians, theoretical physicists, research engineers, advanced graduate students." P.

Perturbation Theory for Linear Operators. By Tosio Kato. Springer-Verlag, New York, 1966. xix+592 pp. $19.80. "Perturbation theory for linear operators is a collection of diversified results in the spectral theory of linear operators, unified more or less loosely by their common concern with the behavior of spectral properties when the operators undergo a small change." The author assumes only linear algebra and real and complex analysis. Functional analysis is developed as needed. Very brief history and a bibliography. T (17), P, L.

Mixed Boundary Value Problems in Potential Theory. By Ian N. Sneddon. North-Holland, Amsterdam, 1966. vii+283 pp. $12.75 (through Wiley in U. S. and Canada). A survey of recent work with brief historical remarks and a short bibliography. P.

Computer Science

Advances in Computers. Edited by Franz L. Alt and Morris Rubinoff. Academic Press, New York, 1966. xiii+303 pp. $14.00. Highly parallel information processing systems, programming language processors, the man-machine combination for computer-assisted copy editing, computer-aided type-setting, programming language for computational linguistics, computer driven displays and their use in man-machine interactions. P, L.

Computer Programming and Related Mathematics. By Richard V. Andree. Wiley, New York, 1967. 284 pp. $6.50. "This book is prepared in the belief that you can learn to program a modern computer and that in today's world every educated person must have at least a rudimentary knowledge of how computers work." Computer languages, GOTRAN, programming logic, FORTRAN, computer-related mathematics, 1620 machine language, and symbolic programming via elementary exposition, "over a thousand small computer problems and myriad noncomputer problems as well as a set of a more difficult type of 'term project' long problems." The book was tried in two preliminary editions. T (first undergraduate course), S, P, L.

Numerical Inversion of the Laplace Transform. Applications to Biology, Economics, Engineering, and Physics. By Richard Bellman, Robert E. Kalaba and Jo Ann Lockett. American Elsevier, New York, 1966. vii+249 pp. $9.50. Fourth in the series *Modern Analytic and Computational Methods in Science and Mathematics* based on the thesis "that the modern computer has become so powerful that it requires reformulation of many classical processes . . ." P, L.

Control Systems Functions and Programming Approaches. Volume A: Theory. Volume B: Applications. By Dimitris N. Chorafas. Academic Press, New York, 1966. xxvi+395 pp. $16.00. xx+276 pp. $11.00. The author, a Parisian management consultant in the computer field, studied industrial control systems in both the U. S. and Europe as a basis for this book whose main purpose is "to discuss and explain in a fundamental manner some of the control systems functions which became important in industrial practice during very recent years." "Fundamental" refers to the inclusion of "guidance logic" and "mathematical understructure." P, L.

Functional Analysis and Numerical Mathematics. By Lothar Collatz. Translated by Hansjorg Oser. Academic Press, New York, 1966. xx+473 pp. $18.50. Mainly concerned with those parts of functional analysis which relate to numerical applications, "does not claim to be a textbook of either functional analysis or numerical analysis" but rather an account of changes in the latter due to computers and abstract analysis. S (17–18).

An Introduction to Algorithmic Methods Using the MAD language. By Alan B. Marcovitz and Earl J. Schweppe. Macmillan, New York, 1966. ix+433 pp. $6.95. ". . . Using a computer does not eliminate the need for careful work in the solution of problems; in fact, it places a premium on such work and adds some new aspects. In addition to the careful logical and mathematical analysis of problems, one must now consider the following: the structure of the available input data and the desired results; the structure and efficiency of the computational algorithm which describes the process of getting from the input to the results; the linguistic means of expressing this algorithm to the computer; the capacity and weakness of computers; the problems involved in debugging and running the algorithm on a computer; and finally, the careful verification of the results to assure that they are meaningful . . . " We may ask Alfred E. Neuman to comment on the MAD language (Michigan Algorithm Decorder). T (13–14), S, P.

Solution of Equations and Systems of Equations. By A. M. Ostrowski. Academic Press, New York, 1966. xiv+338 pp. $11.95. Numerical analysis of solving equations. This edition is "thoroughly revised" and is nearly double the size of the first edition which was favorably reviewed in *Mathematical Reviews* when it appeared in 1960. S, P, L.

Mathematical Methods for Digital Computers, Vol. 2. Edited by Anthony Ralston and Herbert S. Wilf. Wiley, New York, 1967. x+287 pp. $11.95. A sequel to volume 1 containing 13 papers under the topics of programming languages (Fortran and Algol), quotient-difference algorithm, numerical linear algebra, numerical quadrature, numerical solution of equations, random number generation, and rational Chebyshev approximation. P, L.

Information. A Scientific American Book. Freeman, San Francisco, 1966. 218 pp. $5.00 (cloth), $2.50 (paper). Twelve essays on the present and future of "information machines" (computers), touching on such topics as logic and memory, system analysis, time sharing, transmission of data, information storage and retrieval, artificial intelligence, applications in science, technology, organizations and education. S.

Education

Mappings of the Plane. With applications to trigonometry and complex numbers. By H. E. Chrestenson. Freeman, San Francisco, 1966. vi+171 pp. $2.50. Written for and used for several years in summer and inservice institutes for high school teachers. TT, S (Linear Algebra), P.

Contemporary mathematics for elementary teachers. By Howard F. Fehr and Thomas J. Hill. Heath, Boston, 1966. xii+394 pp. $7.95. "All the material is presented from the contemporary mathematical viewpoint as exemplified in the writings of SMSG and CUPM." TT (Pre- and in-service).

Major Topics in Modern Mathematics. Set and Group Theory By D. E. Mansfield and M. Bruckheimer, with a foreword by W. H. Cockcroft. Harcourt, Brace & World, New York, 1965. 252 pp. $6.50. This is a "working text for teachers," designed for inservicing reading by high school teachers but possibly suitable also for teacher training. TT.

Introduction to Modern Mathematics for Elementary Teachers. By Dora McFarland and Eunice M. Lewis. Heath, Boston, 1966. x+406 pp. $7.50. "The material used in this textbook grew out of conferences held by the CUPM and the broad teaching experience of the authors." TT (pre- and in-service).

History

Bibliography of Schlicht Functions. By S. D. Bernardi. New York University, 1966. ix +157 pp. Free from the Courant Institute, 251 Mercer Street, New York, N. Y. A superb bibliography of 1694 titles arranged alphabetically by author and followed by a partial index by periodical, a list of expository papers, and a subject index of 68 topics. Let there be more bibliographies like this! With them we might hope for more reading and less writing, more awareness of previous work and less duplication, and perhaps even higher standards of scholarship in mathematics! L.

Kūshyār Ibn Labbān, "Principles of Hindu Reckoning." A translation with introduction and notes by Martin Levey and Marvin Petruck of the Kitāb fī usūl hisāb al-hind. Univ. of Wisconsin Press, Madison, 1965. 114 pp. $6.00. Facsimile text, translation, extended commentary (41 pp.) and glossary. Perhaps the oldest surviving Arabic text using Hindu numerals, it contains material supporting the claim that the Arabs were more than mere transmitters. P, L.

The Selected Papers of E. S. Pearson. Issued by the Biometrika Trustees to celebrate his 30 years as editor. Univ. of California Press, Berkeley, 1966. vi+327 pp. $6.75. Twenty-one papers selected from 112 listed in a bibliography. Joint papers with J. Neyman will be published separately. P, L.

Studien zur Mathematik des John Wallis, (1616–1703). Winkelteilungen, Kombinationslehre und zahlentheoretische Probleme. By Christoph J. Scriba. Steiner, Wiesbaden, 1966. 144 pp. DM. 32. A scholarly analysis, based on books and unpublished manuscripts of the mathematical and educational role of Wallis. P, L.

Essays in Mathematical Economics. In honor of Oskar Morgenstern. Edited by Martin Shubik. Princeton Univ. Press, 1967. xx+475 pp. $12.50. Following a too-brief essay on Morgenstern's contributions and a bibliography of his works, appear twenty-seven papers on game theory, programming, decision, economic theory, management science, international trade, and econometrics. P, L.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor J. C. Eaves, University of Kentucky, represented the Association at the inauguration of President K. K. Wilson of Asbury College on March 28, 1967.

Professor J. C. Lanz, Harrisburg Area Community College, represented the Association at the Centennial Convocation at Lebanon Valley College on April 8, 1967.

Professor R. C. Meacham, Florida Presbyterian College, represented the Association at the inauguration of President J. V. Elmendorf of New College on February 22, 1967.

Professor Henry Sharp, Emory University, represented the Association at the Centennial Convocation at Morehouse College on February 18, 1967.

Professor R. F. Shortt, Keuka College, represented the Association at the inauguration of President G. W. Glick of Keuka College on April 15, 1967.

Professor R. W. Veatch, University of Tulsa, represented the Association at the inauguration of President G. O. Roberts and the Dedication Services of Oral Roberts University on April 2, 1967.

Professor Dale Woods, Northeast Missouri State Teachers College, represented the Association at the inauguration of President J. C. Weaver of the University of Missouri on April 18, 1967.

Abilene Christian College: Assistant Professor S. E. McReynolds has been promoted to Associate Professor; Associate Professor J. C. Bradford has been promoted to Professor and Head of the Mathematics Department.

Colorado School of Mines: Dr. A. J. Boes, Purdue University, has been appointed Assistant Professor; Mr. D. L. Bohmont, University of Wisconsin, has been appointed Assistant Professor; Associate Professor D. C. B. Marsh has been promoted to Professor.

East Tennessee State University: Mr. T. H. Jablonski, Jr., Castle Heights Military Academy, has been appointed Assistant Professor; Mr. G. G. Killough, Jr., University of Tennessee, has been appointed Assistant Professor; Dr. J. C. Pleasant, East Carolina College, has been appointed Associate Professor.

Florida State University: Dr. W. A. LaBach, Northwestern University, has been appointed Assistant Professor; Associate Professor H. C. Griffith has been promoted to Professor.

Fort Hays Kansas State College: Mr. Robert Gumm has been promoted to Assistant Professor; Assistant Professor Ellen C. Veed has been promoted to Associate Professor.

Marquette University: Dr. P. R. Bender, University of Iowa, has been appointed Assistant Professor; Associate Professor E. W. Swokowski has been promoted to Professor.

University of Minnesota: Associate Professors Alfred Aeppli and D. A. Storvick have been promoted to Professors.

University of Missouri at Rolla: Associate Professors C. E. Antle and S. J. Pagano have been promoted to Professors; Professor J. R. Foote, University of New Mexico, has been appointed Professor; Dr. A. J. Penico, Stanford Research Institute, has been appointed Professor; Dr. A. G. Haddock, Arkansas College, has been appointed Associate Professor.

Texas Woman's University: Mr. R. V. McPherson, University of Texas, has been appointed Assistant Professor; Assistant Professor B. L. Fincher has been named Acting Chairman of the Department of Mathematics.

Assistant Professor W. P. Cooke, West Texas State University, has been promoted to Associate Professor.

Assistant Professor A. L. Deal, III, Virginia Military Institute, has been promoted to Associate Professor.

Mr. W. H. Drake, Upper Iowa College, has been promoted to Assistant Professor.

Mr. J. E. Garner, Lambuth College, has been appointed Assistant Professor at Greensboro College.

Associate Professor S. P. Hoffman, Jr., Trinity College, has been appointed Professor and Chairman of the Department of Mathematics at State University College at Cortland.

Associate Professor Lloyd Montzingo, Seattle Pacific College, has been promoted to Professor.

Mr. C. O. Worm, Adams State College, has been promoted to Assistant Professor.

Professor J. A. Beane, SUNY at Buffalo, died on March 5, 1967. He was a member of the Association for thirteen years.

Professor Margaret R. Davis, Southeastern Louisiana College, died on January 5, 1967. She was a member of the Association for twenty-four years.

Professor A. G. Hill, North Dakota State University, died in January 1967. He was a member of the Association for nineteen years.

Professor L. A. MacColl, Polytechnic Institute of Brooklyn, died on January 20, 1967. He was a member of the Association for thirty-four years.

Professor Emeritus W. M. Miller, Dickinson College, died on September 23, 1966. He was a member of the Association for thirty-nine years.

Professor J. P. Russell, Polytechnic Institute of Brooklyn, died on April 15, 1967. He was a member of the Association for seventeen years.

GEOMETRY IN SECONDARY SCHOOLS—AN NCTM PUBLICATION

This 64-page booklet is a report of the joint meeting of the Association and the NCTM which was held at Houston on January 28, 1967. It is available for 50 cents from the National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington, D. C. 20036.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for the three year term July 1, 1967 to June 30, 1970, by a mail vote of the Association in the Sections indicated:

Kansas	R. N. Bradt, University of Kansas
Missouri	Paul Burcham, University of Missouri at Columbia
New Jersey	Joshua Barlaz, Rutgers—The State University
Northeastern	Miss Grace Bates, Mount Holyoke College
Ohio	H. M. MacNeille, Case Western Reserve University
Pacific Northwest	Joseph Hashisaki, Western Washington State College
Southeastern	R. W. Ball, Auburn University
Southwestern	A. P. Hillman, University of New Mexico
Upper New York State	F. D. Parker, St. Lawrence University

The highest percentage of voters was 46% in the Kansas Section, followed by the Missouri Section with 43%.

RAOUL HAILPERN, *Associate Secretary*

MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on Saturday, March 18, at the University of Michigan, Ann Arbor, in

conjunction with the meeting of the Michigan Academy of Arts and Letters. Professor James Powell of Western Michigan University presided at the sessions. A total of 123 persons attended the meeting.

At the business meeting, the following officers were elected: Professor Beauregard Stubblefield, Oakland University, Chairman; Professor E. A. Nordhaus, Michigan State University, Vice-Chairman; Professor Lester Serier, Central Michigan University, Secretary-Treasurer.

Professor Lyle Mehlenbacher reported on the activity of the Board of Governors and announced that funds were available for section projects.

Professor W. E. Deskins gave a report on the tenth annual Michigan Prize Competition stating that 26,000 High School students took part in the contest.

Professor Phillip Jones pointed out that financial support for the High School Visiting Lecturers program is being discontinued by the National Science Foundation; it was decided that a committee be appointed to take steps to reinstate the program with Michigan Council support.

The following program was presented:

1. *Geometries of the Erlanger program*, by I. E. Vance, Michigan State University.

2. *The use of the telephone in the teaching of elementary college mathematics*, by Beauregard Stubblefield, Oakland University.

3. *Regular-faced polyhedra with holes*, by B. M. Stewart and N. W. Johnson, Michigan State University.

4. *The decimal expansion of log n*, by J. S. White, General Motors Research Laboratory, Warren.

5. *The domesticated integral test*, by George Piranian, University of Michigan.

6. *Functions of a matrix*, by J. S. Frame, Michigan State University.

7. Panel discussion.

L. H. SERIER, *Secretary-Treasurer*

APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The 1967 Spring Meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held at the University of Virginia, Charlottesville, Virginia, on Saturday, April 22, 1967. Chairman T. L. Reynolds presided. Total attendance was 127.

During the morning session, four papers were presented:

1. *Green's Matrix for a Linear Difference System*, by A. L. Deal, Virginia Military Institute, Lexington, Virginia.

Green's matrix is obtained for the difference analog of a linear differential system. The differential Green's matrix is shown to be the limit, in a certain sense, of the difference Green's matrix.

2. *On the Brown method of fictitious play for two-person zero sum games*, by Stephen Jauregui, Jr., Bethesda, Maryland.

Two-person zero sum games can be solved by a variety of methods, the common methods being graphical, matrix, linear programming and the Brown approximation method. The Brown method of fictitious play is investigated for two factors; speed of obtaining a solution and the ability to extract more than one solution, in case multiple solutions exist. The Brown method is found to be quite slow and to give only one solution for a single computer run. A variation on the method which is both faster and often able to produce more than one solution on a single computer run is discussed.

3. *Existence and uniqueness properties of subfunctions of Boolean functions*, by R. E. Schwartz, Institute for Defense Analyses, Arlington, Virginia.

A Boolean function $f(x_1, \cdots, x_n)$ is said to be of order n if it is dependent upon each of the variables $x_1, \cdots, x_n$. A subfunction of lower order is obtained by assigning the value 0 or 1 to one or more of the variables. Two questions arise: (1) Does $f(x_1, \cdots, x_n)$ have at least one subfunction of order r, for every integer r in the range $1 \leqq r < n$? (2) Is $f(x_1, \cdots, x_n)$ uniquely determined by its subfunctions? The first answer is yes. The second answer is also yes, with one curious exception called the "alternating symmetric function."

4. *A very fast integrator for orbital computation*, by E. H. Weiss, IBM Federal Systems Division, Gaithersburg, Maryland.

Professor K. Stumpff of Goettingen has conceived a new method for computing orbits of spacecraft. The method uses several osculating conic sections to approximate the orbit over large spans of time. If desired, numerical integration may be used to improve the solution. The integrator has been programmed in FORTRAN IV and is much faster than methods in current use. Typical cases are examined for accuracy and machine execution times.

The afternoon session was opened with an invited address by Professor A. G. Mackie of the Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, who offered *Some Comments on Existence and Uniqueness Theorems in Applied Mathematics with an Application to Thin Airfoil Theory.*

Following Professor Mackie's address, three papers were presented:

5. *Two point boundary value problems and their application to celestial mechanics*, by R. L. Stallard, IBM Federal Systems Division, Gaithersburg, Maryland.

6. *The geometric configurations associated with the osculating quadrics at a point of an analytic surface*, by P. D. Thomas, Naval Research Laboratory, Washington, D.C.

By considering certain polar reciprocal relations between the osculating quadrics of Darboux and of Bompiani at a given point of an analytic surface, the resulting geometric configuration was studied. Plane loci discovered included conics, cubics, quartics, and sextics. Space loci discovered included twisted cubics, quadric cones, osculating planes, ruled cubic surfaces, tangent developable quartic surfaces, osculating linear complexes and null systems of twisted cubics, a skew involution, and a linear congruence.

7. *New methods of calculating Bessel functions of integral order and complex argument*, by A. S. Elder, Aberdeen, Maryland.

Bessel functions of the first kind, both ordinary and modified, are calculated for an index which is large compared to the argument. The standard recurrence relation is used to obtain functions of lower order. Bessel functions of the second kind are decomposed into (1) a finite series satisfying a nonhomogeneous recurrence relation stable for increasing index, (2) a logarithmic part depending on functions of the first kind, and (3) an infinite series satisfying a nonhomogeneous recurrence relation stable for decreasing index. Results obtained by combining these elements are generally accurate to about 14 significant figures.

A short business meeting followed during which reports were tendered and routine business was conducted. The following officers were elected for one-year terms beginning 1 July, 1967: Chairman: Avron Douglis, University of Maryland; Vice-chairmen: Mary A. Lee, Sweet Briar College, and S. H. Schot, American University; Secretary: S. L. Gulick, University of Maryland; Treasurer: Joseph Milkman, U.S. Naval Academy.

The meeting concluded with the showing of two MAA films, *Theory of limits* (*Part I*), with E. J. McShane and *Let us teach guessing*, with George Polya.

G. N. TRYTTEN, *Secretary*

APRIL MEETING OF THE MISSOURI SECTION

The spring meeting of the Missouri Section of the MAA was held at Northeast Missouri State Teachers College, Kirksville, Missouri, on April 29, 1967, Professor Dale Woods, Chairman of the Section, presiding. Sixty persons were present of whom fifty-one were members.

At the business meeting the following officers were elected for 1967–68: Chairman, Professor Louise Beasley, Lindenwood College; Vice-Chairman, Professor Guido Weiss, Washington University; Secretary-Treasurer, Professor R. J. Mihalek, University of Missouri at St. Louis.

Professor John Andrews of St. Louis University reported on the high school testing program. Professor Paul Burcham of the University of Missouri at Columbia reported on the progress his committee is making in writing the history of the Missouri Section.

During the morning session Professor Neil Foland of Southern Illinois University gave the main address, "Convex Sets."

In addition to Professor Foland's address, three papers were presented during the morning session. The afternoon session consisted of three papers followed by a number of films. The following papers were presented during the two sessions:

1. *Nontopological character of completeness*, by Ralph Jones, University of Missouri at Kansas City.

2. *Geometry and composite functions*, by J. J. Andrews, St. Louis University, St. Louis.

3. *Optimization and elementary calculus*, by J. R. Foote, University of Missouri at Rolla.

4. *Bivariate probability distributions satisfying a certain summability condition*, A. G. Haddock, University of Missouri at Rolla and V. Seshadri, McGill University. (Presented by A. G. Haddock.)

5. *A mathematical model for musical composition*, by H. A. Padberg, R.S.C.J., Maryville College of the Sacred Heart, St. Louis.

6. *A brief biography of Professor George H. Jamison*, by Mary Jane Kohlenberg, Northeast Missouri State Teachers College.

E. C. PRINGLE, *Secretary-Treasurer*

APRIL MEETING OF THE OHIO SECTION

The fifty-first annual meeting of the Ohio Section of the MAA was held at The Ohio State University, Columbus, Ohio, on Saturday, April 22, 1967. Professor David Lipsich, Chairman of the Section, presided at the business meeting and Professors D. H. Staley and B. J. Yozwiak presided at the program sessions. One hundred seventy-five persons registered in attendance including one hundred forty-three members of the Association.

The following officers were elected: Chairman, Professor Daniel Finkbeiner, Kenyon College; Chairman Elect, Professor Arnold Ross, The Ohio State University; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Program Committee: Professor B. J. Yozwiak, Youngstown University, Chairman; Professor R. A. Clark, Case Institute of Technology; Professor David Brown, Hiram College.

Regular reports were made by the Chairman and the Secretary-Treasurer; there were also special reports by Professor C. E. Capel for the Committee on Curriculum (CONCUR) and by Professor J. L. Smith for the Committee on Teacher Training and Certification (CONTTAC).

The following program was presented:

1. *Non-measurable subsets of the unit interval* $[0, 1]$, by Nand Kishore, University of Toledo. Let λ^* be the Lebesgue outer measure on subsets of the real numbers. Let P be a Lebesgue

nonmeasurable subset of $[0, 1]$ such that the difference of any two elements of P is an irrational number. Then it can be shown that if P_r is a translation of P by r modulo one, where r is a rational number in $[0, 1]$, then P_r is non-measurable and that $\lambda^*(P_r) = \lambda^*(P)$. This leads to the following THEOREM: There exists a sequence $(P_n)_{n=1}^{\infty}$ of disjoint, Lebesgue nonmeasurable subsets P_n of $(0, 1]$ such that $[0, 1] = \mathsf{U}_{n=1}^{\infty} P_n$ and $\sum_{n=1}^{\infty} \lambda^*(P_n) = \infty$.

2. *A consequence of the non-existence of certain generalized polygons*, by S. E. Payne, Miami University.

3. *Euclidean completeness—metric versus synthetic*, by J. L. Smith and Joyce Heick, Muskingum College.

Metric and synthetic (Hilbert) formulations of completeness are stated and used in the proofs of theorems to help demonstrate their difference. The particular contribution of this paper is the presentation of Hilbert's completeness theorem which was translated from the 9-th edition of Hilbert's *Grundlagen der Geometrie*. The paper concludes by presenting a set which is synthetically but not metrically complete.

4. *Some results on triangular numbers*, by Syed Asadulla, Miami University.

5. *Affine transformations in a complex three-space interpreted in a real Argand six-space*, by Hunter Hardman, Marshall University.

A method for the representation of complex three-space in a real six-space is first given. Two types of linear four-spaces in the six-space are defined as "regular" and "anti-regular" four-spaces. Eighteen transformations of the complex three-space are given and when interpreted in the real six-space, they transform "anti-regular" four-spaces into "anti-regular" four-spaces. These transformations are all orthogonal in the six-space and each is a direct rotation or a rotation followed by a reflection upon a subspace of the six-space.

6. *Generalized implication and the Russell Paradox*, by J. G. Maxwell, Kent State University.

A relation called *implicative accompaniment* is defined for propositional functions on differing domains. This implicative accompaniment is shown to be a generalization of each of material, formal and strict implication. Some properties of this implicative accompaniment are outlined and the barber version as well as the original version of Russell's so-called paradox is examined and seen to involve a double implication (if and only if) which, because of circumstances now easily identified, must be false and hence not useful in answering the standard question occurring in the statement of the "paradox."

7. *Sufficient conditions for strong differentiation in infinite dimensional space*, by Thyrsa A. Frazier, Central State University.

Several writers have shown that if f is a bounded measurable function on Euclidean m-space, then the indefinite integral of f is strongly differentiable. Examples have been given of bounded measurable functions defined on infinite dimensional torus-space Q_w whose indefinite integrals are not strongly differentiable. In the present paper it is shown that for f in Q_w a sufficient condition for the strong differentiability of the indefinite integral is that f be the limit of a uniformly convergent sequence of bounded measurable functions, each depending on a finite number of the variables.

8. *An underworked example for linear algebra*, by C. A. Long, Bowling Green State University.

Vector spaces whose elements are polynomials over the real field provide illustrative and easy to obtain examples which are useful in the study of linear algebra. Such polynomial space examples lead to questions, conjectures and generalizations which are not so apparent from the usual n-tuple space or geometric space examples.

9. *The area of a Pythagorean triangle*, by I. A. Barnett, Ohio University.

It is well known that the area of a primitive Pythagorean triangle can never be a square or

twice a square. It is shown here that the area of such a triangle cannot be of the form t^p, $2^{p-2}t^p$, or $2^{p-1}t^p$, provided the Fermat conjecture is true for such a p. For example, if $p=5$, the area of a Pythagorean triangle cannot be of the form t^5, $8t^5$ or $16t^5$, and this leaves undecided the forms $2t^5$ and $4t^5$. From these results other consequences are derived on the area of a Pythagorean triangle.

10. *Some new directions for the Ohio Section*, by David Lipsich, Chairman of the Section, (Chairman's address).

11. *Topics in mathematical logic*, by Jan Mycielski, Visiting Professor, Case Institute of Technology, on leave from the Mathematical Institute, Polish Academy of Science in Warsaw (by invitation).

This lecture pointed the role of first order logic in modern mathematics and described a procedure for checking the tautologies of the propositional calculus essentially contained in A. Ehrenfeucht and E. Orlowska, *Mechanical proof procedure for propositional calculus*, Bull. Acad. Polon. Sci., Série Sci. Math. Astr. et Phys., 15 (1967) 25–30.

FOSTER BROOKS, *Secretary-Treasurer*

APRIL MEETING OF THE SOUTHEASTERN SECTION

Florida Presbyterian College was host to the 46th annual meeting of the Southeastern Section of the MAA, March 31–April 1, 1967. Professor C. V. Aucoin, retiring Chairman of the Section, and Professor R. C. Meacham, Chairman of the Mathematics Department of Florida Presbyterian College, presided at the general sessions. Two one-hour invited addresses were scheduled for the general sessions: "A Nonlinear Parabolic Problem," by F. A. Ficken (New York University) and "Functional Equations and Binary Topological Algebras," by A. D. Wallace (University of Miami). The program included a panel discussion: "The mathematics preparation expected of students transferring as juniors from a Junior College to a Senior College." Members of the panel were: H. E. Taylor (Florida State University), Waldemar Olson (University of Florida), William Tessin (Florida Atlantic University), E. P. Miles (Florida State University). Among the contributed papers, two sections were of special interest—one devoted to partially ordered algebraic systems, the other devoted exclusively to presentations by students. A banquet was scheduled Friday evening, and the program which followed it included a talk by Professor G. K. Reid (Department of Biology, Florida Presbyterian College) and two films supplied by Modern Learning Aids: John von Neumann, and Challenge in the Classroom (R. L. Moore).

Total registration for this meeting was 278, including 82 nonmembers of the Association. The following officers were elected: Chairman, R. E. Wheeler (Samford University); Vice-Chairman, Trevor Evans (Emory University). The invitation from East Carolina College to act as host to the 1968 meeting was reaffirmed, and an invitation from Winthrop College, Rock Hill, South Carolina, to act as host to the 1969 meeting was accepted.

Upon the adjournment of the business meeting, members from the state of Florida met to take initial steps in the formation of a Florida Section of the MAA. Professor Howard Taylor discussed the present status of the petition to form a separate Florida Section. Approval of the idea of a separate Section was voted unanimously by those present. Professor Taylor was elected Chairman *pro tem* and a committee was elected to nominate a slate of officers in anticipation of the establishment of this Section.

The following contributed papers were presented:

1. *Concerning metric intervals*, by E. L. Bethel, Clemson University.

Let (X, ρ) denote a metric space. The point c of (X, ρ) is said to be between points a and b

of (X, ρ) iff $\rho(a, c)+\rho(c, b)=\rho(a, b)$. We denote this in brief by acb. $I(a, b)$ denotes a subset of (X, ρ) such that $x \in I(a, b)$ iff $a \times b$. $I(a, b)$ is called an interval.

Let $M \subset I(a, b)$. Then the point $c \in M$ is said to be an extremal point of M iff there is no other point $z \in M$ such that acz or there is no other point $z \in M$ such that zcb.

THEOREM. *If (X, ρ) is complete and M is a closed subset of $I(a, b)$, then M contains an extremal point.*

2. *A new look at an old problem*, by R. G. Blake, University of Florida.

One sometimes sees the problem "Solve x raised to the x power raised to the x power $\cdots=2$ for x." Another way to state the problem is "Find x such that as $n \to \infty$ the limit of $a_n=2$ where $a_1=x$ and $a_{n+1}=x^{a_n}$. This is an example of the problem of finding u where $u=f(u)$ for a given function f.

3. *On doubly autographic groups*, by L. M. Chawla, University of Florida.

A group G is called autographic, if it can be embedded isomorphically into $A(G)$, the group of all automorphisms of G. A group G is defined to be doubly autographic, if both G and $A(G)$ are autographic.

THEOREM I. *(a) A group G having the identity as its centre is doubly autographic. (b) A group G which is a direct product of groups each of which has identity as its centre is doubly autographic. In particular a direct product of nonabelian simple groups is doubly autographic.*

THEOREM II. *(a) If $G=\Pi_{i \in A} H_i$, A an index set, is a direct product of its subgroups H_i, each of which is autographic, then G is autographic. (b) If in $G=\Pi_{i \in A} H_i$, each H_i is doubly autographic and characteristic, then G is doubly autographic. (c) If G be the direct product of its centre C and a subgroup H, then G is doubly autographic if C is doubly autographic and H is characteristic.*

4. *The Krull-Schmidt theorem for loops*, by David Clark, Emory University.

A new proof is given of the uniqueness of direct decomposition for loops satisfying ascending and descending chain conditions on normal subloops. The theorem, of course, is already known, not only for loops, but for more general algebraic systems. The main virtue of this proof is the closeness with which it parallels the standard proof for the group case. A particular point of interest is the development of a substitute for the binomial theorem in a neo-ring.

5. *Products of points—some simple algebras and their identities*, by Trevor Evans, Emory University.

A study is made of the identities satisfied by algebras of ordered pairs of elements of a set S with products defined by $(a, b) \cdot (c, d)=(\ ,\)$ where the product pair is chosen in some fixed manner from a, b, c, d. The product $(a, b) \cdot (c, d)=(b, c)$ satisfies the identity $xy \cdot yz=y$ which has the interesting property that any finite model of it has a square number of elements.

6. *Cyclically vertex-symmetric graphs*, by Forrest Dristy, Robert Ferguson and Alexander Hunter, Florida Presbyterian College.

A vertex-symmetric graph is called cyclically vertex-symmetric (CVS) if its vertices can be numbered (modulo n) in a manner such that one of its automorphisms maps any vertex, having number k, onto the vertex numbered $k1$. This type of graph can be completely described in terms of the vertices incident with the vertex numbered zero. If $t_1, t_2, \cdots, t_d$ are the vertices incident with zero on a graph with n vertices, then a graph with vertices $s_1, s_2, \cdots, s_d$ incident with zero is isomorphic if and only if there is a number relatively prime to n such that $mt_1, \cdots, mt_d\, s_1, \cdots, s_d$. The paper also contains a theorem which gives the index of imprimitivity for a CVS graph.

7. *Some transformations that reduce complex boundaries in every face of a parallelepiped to zero*, by R. M. Fiterre, Alabama University in Huntsville.

8. *The Riesz interpolation property in ordered groups*, by Laszlo Fuchs, University of Miami. (30 minutes)

Let A be an additively written abelian group with a partial order $\leqq$ such that $a \leqq b$ implies $a+c \leqq b+c$, and the elements $\geqq 0$ generate A. The Riesz interpolation property asserts that if both a_1, a_2 are $\leqq$ both b_1, b_2, then some $c \in A$ satisfies $a_i \leqq c \leqq b_j$ for $i, j = 1, 2$. Lattices share this property with $c = \sup(a_1, a_2)$, and there are lots of important groups which are not lattice-ordered and have the Riesz property, e.g., real polynomials or differentiable functions in the unit interval. The simplest types are the antilattices; here $\sup(a_1, a_2)$ exists only when either $a_1 \leqq a_2$ or $a_2 \leqq a_1$. Every A with the Riesz property is a subgroup of a cartesian product of antilattices.

9. *The substitution sign*, by B. A. Fusaro, University of South Florida.

The operation of substitution would seem to warrant a standard notation. Many texts have a list of notations but they fall short of including a notation for substitution. Similarly, mathematics dictionaries do not appear to recognize a special sign. . . . In practice, devices such as parentheses or brackets are used, and the constants to be substituted are written in various places. About a century ago, F. Sarrus introduced a notation to symbolize this operation, and mathematicians conceded the value and utility of having a special sign to indicate substitution. Sarrus' sign was modified by A. Cauchy, and others, and what emerged was a vertical bar written to the left of the function. The "limits" of substitution were placed like the limits of integration. This serviceable notation, only occasionally used today, allows a simplicity and even an elegance in expressing certain relations as, e.g., the fundamental theorem of the calculus and integration-by-parts.

10. *Mathematics by television and graduate student teachers at the University of Florida*, by W. A. Gager, University of Florida.

This paper relates the $3\frac{1}{2}$ years experience of teaching elementary college mathematics by television and using graduate student teaching assistants to conduct the discussion in the small section groups and to give individual aid as needed.

11. *A stability theorem for closed linear operators*, by Leland Graber, Florida Presbyterian College.

Let T and S be linear operators with domains in a Banach space X and ranges in a Banach space Y. Let T be closed with closed range and let S be bounded with respect to T. Suppose that the image of the null space of T under S has only the zero element in common with the range of T. The author introduces a condition which is sufficient to insure that the range of $T+\lambda S$ is closed and the null space of $T+\lambda S$ is fixed for all complex numbers λ in a deleted neighborhood of 0.

12. *On the construction of a ternary quadratic form*, by E. H. Hadlock and Amin Muwafi, University of Florida.

In this paper positive values are given to a, and d, the determinant associated with the form $f = ax^2+by^2+cz^2+2ryz+2xz$. Formulas are obtained by which integral values of b, r and c are obtained in that order. This method of construction is a generalization of the method given by L. E. Dickson.

13. *Maximal starshaped sets*, by W. R. Hare, Jr. and J. W. Kenelly, Clemson University.

While any set S in a linear space has maximal convex subsets, this is not true for starshaped sets in general. However, if S is a compact subset of an n-dimensional linear topological space, there exist maximal starshaped subsets and all such maximal sets are closed.

14. *The use of Cartesian tensors in development of complex equations of motion*, by W. G. Heffron, Bellcomm, Inc.; W. H. Land, Jr., IBM Federal Systems Division.

Part I. The purpose of Part I, and Part II to follow, is to demonstrate the use of Cartesian tensors in the development of equations of motion of particles and rigid bodies. It is shown that Cartesian tensors have many advantageous features; in particular, (a) ω^* (crossproduct) term

arises in a simple manner, (b) development of the inertia tensor is straightforward, and (c) the resulting equations are unambiguous and can be reduced to computable form with relative ease. Part I covers the following: brief review of the vector approach; discussion of pertinent properties of direction cosines; solution of simple rigid body motion by this method; and particle motion about the earth, referred to inertial, geocentric ecliptic, and earth-fixed axes.

Part II. Part II of this presentation reviews the classical tensor approach and concentrates on the application of Cartesian tensors to development of the equations of motion of a helicopter. The development considers the helicopter as a group of rigid bodies constrained to move together in a useful fashion. The forces and torques of constraint are, however, eliminated in the useful versions of the equations.

15. *Rational inequalities—a simplified approach*, by C. W. Huff, Winthrop College.

Let $q(x)=\Pi_{i=1}^{m}(x+p_i)$, $p_1<p_2<\cdots<p_m$, be a polynomial over the reals; so that $x+p_1<x+p_2<\cdots<x+p_m$ for any real number x. Then $\{x:q(x)<0\}$ is found by inspection in preference of the detailed case studies that are usually advocated in undergraduate texts. From this result one can easily ascertain the solution set $\{x:g(x)/h(x)<0\}=\{x:f(x)=g(x)h(x)<0\}$, where $g(x)$ and $h(x)$ are polynomials over the reals.

16. *A property related to absolute pure for modules*, by B. H. Maddox, Florida Presbyterian College.

In this paper, all rings have unity and all modules are unitary. If A is a ring, a left A-module E is said to be *pure with respect to* a right A-module F if for any left A-module H containing E (as a submodule) the canonical map $F\otimes E\to F\otimes H$ is an injection. The paper consists of proving four immediate results from this definition using isomorphisms that exist between certain tensor products.

17. *On oscillatory matrices*, by T. L. Markham, Auburn University.

The definition and basic structure of oscillatory matrices were given by F. R. Gantmacher and M. G. Krein in *Compositio Math.*, v. 4 (1937) 445–476. A canonical representation for an oscillatory matrix is determined, and this form is used to derive upper bounds for the determinant of an oscillatory matrix which improves a bound of Gantmacher-Krein. Special classes of oscillatory matrices are investigated and determinantal bounds are discussed relative to these classes. Finally, if A is an oscillatory matrix, a necessary and sufficient condition for A^{-1} to be an M-matrix is proved.

18. *On a form representing a large collection of primes*, by L. M. Chawla, J. E. Maxfield, and R. G. Selfridge, University of Florida.

Let p be an odd prime. It is determined which odd prime p can be represented by the form $p=|x^l\pm m(x\pm 1)|$ where l and $m\leqq\frac{1}{2}(x-1)$ are positive integers, and x an odd prime, $p\leqq 80,413$. It is determined that less than 3% of the primes in the range considered cannot be represented in this way.

19. *Linear programming a generalized traveling salesman problem*, by J. G. May, Wake Forest College.

Given n cities, in what order should they be visited so that no city is encountered twice and the distance traveled is a minimum?

We assume the existence of n points in m-dim. space such that the "Manhattan distance" between points equals the specified distance between corresponding cities.

Using these points, we formulate a linear programming solution to the above problem. The number of constraints is $2n+(n-1)m$ and the number of variables is $n^2+2(n-1)m$. The description easily submits to a generalization of this problem.

20. *On the proximate fixed-point property for multifunctions*, by T. B. Muenzenberger, University of Florida.

A multifunction F on a space X into a metric space Y is ϵ-continuous provided each point x admits a neighborhood U such that $F(x)$ is a subset of $S_\epsilon(F(x'))$ and $F(x')$ is a subset of $S_\epsilon(F(x))$ for each x' in U. A metric space X has the proximate fixed-point property for multifunctions (pFpp) provided for each $\eta>0$ there exists an $\epsilon>0$ such that for every ϵ-continuous multifunction F on X into itself there exists a point x in X such that $d(x, F(x))<\eta$.

THEOREM 1. If a metric space X has the pFpp, then every compact m-retract of X has the pFpp.

THEOREM 2. If a compact metric space X has the pFpp, then every metric homeomorph of X has the pFpp. Theorems 1 and 2 are extensions of theorems of Klee (*Stability of the fixed-point property*, Colloq. Math., 8 (1961) 43–46).

21. *On the equivalence of a ternary quadratic form*, by E. H. Hadlock and Amin Muwafi, University of Florida.

If g and its reciprocal G are two properly primitive, positive or indefinite, ternary quadratic forms, then g is equivalent to a properly primitive form, $f=ax^2+by^2+cz^2+2ryz+2sxz+2txy$ with $(a, s, t)=1$. There exists a linear transformation T of determinant 1 which takes f into an equivalent form, $f_1=a_1x^2+b_1y^2+c_1z^2+2r_1yz+2s_1xz+2t_1xy$ such that $a_1=a$, and $(s_1, t_1)=1$. Finally, every properly primitive, positive or indefinite, ternary quadratic form, with a properly primitive reciprocal, is equivalent to a properly primitive form in which the coefficients of $2xz$ and $2xy$ are one and zero respectively.

22. *The pseudo-join in implicative semilattices, by* W. C. Nemitz, Southwestern at Memphis.

Let $\langle L, \leqq, \wedge, * \rangle$ be an implicative semilattice. If a and b are elements of L, the pseudo-join of a and b is defined to be $((a*b)*b)\wedge((b*a)*a)$. The pseudo-join is an upper bound of a and b, is the larger of a and b if they are ordered, is idempotent, commutative, and absorbtive with respect to meet. Furthermore, the following conditions are all equivalent: (i) that the pseudo-join be a join, (ii) that it be associative, (iii) that it be monotone, (iv) that it satisfy either distributive law with respect to meet, (v) that it satisfy the generalized De Morgan Law.

23. *An application of Abel's identity*, by G. B. Parrish, Army Research Office, Durham, N. C.

One often asks if the solution of a boundary value problem is continuously differentiable in a closed region containing the boundary. Consider the problem: Find a function u which is harmonic in the interior R of a right circular cylinder of unit radius and which vanishes on the boundary of the cylinder except at one end where $u=f(r)$. Here r is the distance from the cylinder's axis and $f\in C''$ $[0, 1]$, $f'(0)=f(1)=0$. Using simple relations, such as Schwarz' inequality and Abel's summation identity, it is shown that the Fourier-Bessel series solution of this problem is continuously differentiable throughout $\overline{R}$.

24. *Some game-theoretic problems*, by L. T. Ratner, Vanderbilt University.

A game-theoretic approach is used in connection with the solution of certain problems arising in deterministic mathematical models.

25. *A semigroup of convex sets*, by Harold Reiter, Clemson University.

It is shown that the class C of compact convex subsets of a real normed linear space X with vector addition, is a commutative semigroup with an identity. C is made into a topological semigroup with continuous real operators. G. K. Sheppard has recently shown that certain classes of convex sets form a Hilbert space.

The operator $S: C'\to C'(C'=\{[A]=\{A+b \mid b\in X\} \mid A\in C\})$ defined by $S([A])=[(A-A)/2]$ is a continuous idempotent. If P is a parallelogram in E_2, $S^{-1}([P])$ is a singleton.

26. *Report on the activities of the Two Year College Panel of the CUPM*, by W. R. Rice, St. Petersburg Junior College.

The charge to the panel is as follows: to establish the minimum levels of course content acceptable for the Associate Degree or for transfer credit; to study computer methods as they relate

to teaching; to determine the level of mathematical training required of two-year college instructors; to investigate the use of programmed materials; to recommend materials for the library; to determine what should be included in the "General Education" mathematics program; to study the mathematics appropriate for technical and occupational training programs; to develop placement criteria; to investigate how CUPM recommendations could be utilized.

This panel is subdivided into three subpanels as follows: 1. Mathematics for university-parallel students; 2. Mathematics for technical and occupational training; 3. General mathematics for terminal students.

27. *Abnormal semantic matrices—a timely state of the art survey*, by E. P. Miles, Jr., Florida State University.

Abnormal semantic matrices, first defined by Miles at a meeting of this Section of the Association over twelve years ago, are subsets of the class of literate matrices, i.e., those with non-numeric alpha-numeric characters as elements, which are optically or phonetically comprehensible in their basic form and remain comprehensible when subjected to one or more of such operations as reflections, inversions, reversions or orthogonal transformations. Additional canonical forms or under-the-counter examples will be displayed which are multiply decomposable, phonetically implausible or punitively ill-conditioned. Applications to psychotherapy and mirror writing will be given.

28. *On the admissibility of a mean on certain continua*, by K. N. Sigmon, University of Florida.

A *mean* on a space is a continuous commutative, idempotent binary operation on the space. A *mean* is called *autodistributive* if $(xy)z=(xz)(yz)$ for all x, y, z in the space. It is not known whether the subspace S of the plane consisting of the set $W=\{(r, s)\,|\,s=\sin(1/r),\ 0<r\leqq 1\}$ together with its limit continuum C admits a mean. Some relevant results follow. Let $D=S\cup\{(r, x)\,|\,(2-r, x)\in S\}$, a subset of the plane.

PROPOSITION 1. If S admits a mean, then $xC=C$ for all $x\in C$.

PROPOSITION 2. S admits no mean which is internal (i.e., $\min\{x, y\}\leqq xy\leqq\max\{xy\}$) on W.

PROPOSITION 3. If S admits an autodistributive mean, then C is an ideal of S (i.e., $SC\subset C$).

PROPOSITION 4. D admits no autodistributive mean.

29. *A permanent inequality*, by P. B. Sigrest, Auburn University.

The following theorem is proved:

Suppose B is an $n\times n$ matrix with root λ and corresponding normalized column characteristic vector $x=(x_1, x_2, \cdots, x_n)$. Then per $(BB^*)\geqq|\lambda|^{2n}\Pi_{i=1}^{n}|x_i|^2 n!$ When B is stochastic, we obtain per $(BB^*)\geqq n!/n^n$. In order for this bound to be significant, λ must be nonzero and x must contain no zero elements. However, this generalizes some of the results obtained by M. Marcus and H. Minc on bounds for permanents of positive semidefinite matrices.

30. *Torsional stresses in a cyclindrically aeolotropic material*, by C. B. Smith, University of Florida.

A mathematical method of investigating the torsional stresses in a cylindrically aeolotropic material is presented. The method is applied to obtain the system of stresses in a hollow cylinder produced by couples acting on the ends. Several types of hollow cylinders are discussed including a cylinder with a varying diameter.

31. *Fully and partially ordered extensions of the real field*, by Andrew Sobczyk, Clemson University.

Any algebra $D(S)$ of real functions on a set S, with order defined by nonnegative functions as nonnegative elements, contains an order-isomorphic copy of the real field R for each scalar-multiple class of such a function $x\in D(S)$. For the continuous function space $C(S)$ of a nonpseudocompact completely regular topological space S, it is known (see, e.g., the book of Gillman and Jerison) that $C(S)/M$ for a nonfixed maximal ideal M is a fully ordered hyper-real field, whose transcendence degree over R is at least the power of the continuum. Theorems are obtained on representa-

tion of a partially ordered commutative algebra as a $D(S)$, as a quotient algebra $D(S)/I$, where I is an ideal of $D(S)$, and on transcendence degrees in the more general situation.

32. *The importance of certain hypotheses in theorems on L-sets*, by A. G. Sparks, Clemson University.

In a paper by Alfred Horn and F. A. Valentine, "Some properties of L-sets in the plane," Duke Math. J. 16 (1949) 131–140, we will refer to two particular theorems. They are:

THEOREM 2.9 *Let S be a simply connected, compact L-set, and suppose each point of S lies in a convex subset of S which is visible from all points of S. Then S is a sum of convex sets every two of which have a point in common.*

THEOREM 2.10 *If S is a simply connected, compact L-set, then S is a sum of convex sets every two of which have a point in common.*

An example was given to show that simple connectedness is needed in Theorem 2.10. This example does not serve for Theorem 2.9 since it does not have the stated visibility property. The following example shows that simple connectedness is needed in Theorem 2.9. Let $P_1=(-2, 2)$, $P_2=(0, 4/3)$, $P_3=(2, 2)$, $P_4=(1, 1)$, $P_5=(1, -1)$, $P_6=(-1, -1)$, and $P_7=(-1, 1)$ be the points with indicated coordinates, and let S be the set $\Delta P_1P_2P_7+\Delta P_2P_3P_4+\Delta P_4P_5P_6+\Delta P_5P_6P_7$. Then S is a compact L-set which is not simply connected, but which does have the desired visibility property. However, S is not a sum of convex sets every two of which have a point in common.

33. *A generalization of the Stone-Weierstrass Theorem to multivalued functions*, by W. T. Spears, University of Florida.

A generalization of the Stone-Weierstrass Theorem is obtained for multivalued functions which gives the classical theorem when the functions are taken as single valued.

34. *On the domain of attraction of the sum of independent random variables*, by C. V. Stanojevic, University of Detroit.

35. *Relations and bigraphs*, by Jagbir Singh and W. A. Thompson, Jr., Florida State University.

This paper generalizes some concepts of directed graphs to configurations which have directed as well as undirected lines. The subject is motivated by paired comparison problems which arise in statistics and by tournaments in which ties may occur. Both direct and indirect comparisons are employed to determine a partial ordering of the players. Conditions are determined under which such an order will exist and the uniqueness question is treated. If no such ordering exists then one may determine a "nearest adjoining order" by deleting and changing the orientation of a minimum number of lines.

36. *The number of solutions of certain systems of equations in a finite field*, R. G. Van Meter, Florida Presbyterian College.

HENRY SHARP, JR., *Secretary-Treasurer*

APRIL MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section of the MAA was held on the campus of Austin College, Sherman, Texas, on April 14–15, 1967. There were 279 persons registered including 176 members of the Association.

Papers were presented Friday in two concurrent sessions chaired by Professors A. D. Stewart of Prairie View A & M, D. H. Martin of Grayson College, T. G. Newman of Southern Methodist University, and B. B. Williams of the University of Texas at Arlington.

The welcoming address was given by President J. D. Moseley of Austin College at a dinner meeting on April 14, 1967 which was presided over by Professor Dale Maness of Austin College.

The general session on Saturday, April 15, 1967 was chaired by Professor C. J. Pipes

of Southern Methodist University and included reports of all committees and of the secretary-treasurer. At this session the following officers were elected for the coming year: Chairman, Dale Maness of Austin College; Vice-Chairman, H. A. Luther of Texas A & M University; Secretary-Treasurer, B. T. Goldbeck, Jr., of Texas Christian University.

The invited speakers for the meeting were Professor E. E. Moise, Harvard University, who spoke on "How to Tell That a Simple Over-hand Knot is Really Knotted" and Dr. T. A. Atchison, Ling-Temco-Vought, who spoke on "Nonlinear Transformations Related to the Evaluation of Improper Integrals."

The following papers were presented:

1. *Periodic solutions of linear difference equations*, by Louis Brand, University of Houston.

The linear difference equation $P(E)\times(t)=f(t)$, in which $P(E)$ is a polynomial of degree n in $E=1+\Delta$ and $f(t)$ is a sequence of period p, has the operational form $P(r)X(r)=Q(r)+F(r)$ (this MONTHLY, 71 (1964) 719–728) where $r=1/s$, the inverse of the shift operator $\{0, 1, 0, 0, \cdots\}$. It has a unique solution $g(t)$ of period p iff the zeros r_j of $P(r)$ are not poles of $F(r)$. If r_j is of multiplicity m_j, the initial values of $g(t)$ are uniquely determined by the n equations $Q^{(i)}(r_j)+F^{(i)}(r_j)=0$, $i=0, 1, \cdots, m_j-1$. Sequences $g(t)$ and $f(t)$ satisfy $P(1)\sum_{t=0}^{p-1} g(t)=\sum_{t=0}^{p-1} f(t)$.

2. *On the weight of a metric space*, by J. C. Bradford, Abilene Christian College.

Since the cardinal numbers are well ordered, there is in every topological space a dense set of smallest cardinal. This is the *weight* of the space. It is shown that in a metric space, there is always an isolated set of largest cardinal and that this cardinal is the weight of the space. Thus the weight has the dual characterization as the largest cardinal of an isolated set and the smallest cardinal of a dense set.

3. *Metrizability conditions in a Moore space*, by Geraldine Fuller Daunis, Texas Christian University and University of Texas at Arlington.

A Moore space is a space satisfying axiom 0 and parts 1, 2, and 3 of axiom 1 of Moore's *Foundations of Point Set Theory*. A Moore space is not necessarily metrizable but whether or not a normal Moore space is metrizable is still unanswered. A normal Moore space which is not metrizable will be completely normal, perfectly normal, and countably paracompact but not paracompact nor pseudo-metrizable.

The author compares a Moore space to some other generalizations of metrizable spaces. Some metrizability conditions are the results. Some of these are conditions that can be put on the basis that they are similar to the equivalent conditions for paracompactness.

4. *Breadth, connectedness, modularity*, by D. E. Edmondson, University of Texas at Austin.

Let L be a topological lattice; the theorem is proved that if L is connected and has breadth less than or equal to two, then L is modular.

5. *Schauder decompositions of some product spaces*, by Jean B. Richmond, Southern Methodist University.

A Schauder decomposition which is both shrinking and boundedly complete is given for the product space $X=\pi_2 X_i$ of a sequence $\{X_i\}_1^\infty$ of nontrivial Banach spaces with respect to l_2. Then X is reflexive if and only if each X_i is reflexive. If each X_i is a certain quasi-reflexive space, then X is isometric with X^{**} but X is not quasi-reflexive. For another choice of the X_i's, X is not isomorphic to any conjugate space.

6. *Restriction of zeros of polynomials to sectors of the complex plane*, by Ruth Goodman, Texas A & M University.

It is possible to write an nth degree polynomial $f(z)$ as a sum of $n+1-k$ definitely defined polynomials $f_{k,j}(n)$ of degree $k<n$. If a sector S contains all zeros of $f(z)$, it is easily shown that S also contains those of all the $f_{k,j}(z)$ and that S is "best possible." The converse problem is more difficult: For given k, let S contain all the zeros of the $f_{k,j}(z)$; determine the minimal sector $S'=S'(S, n, k)$ which must also contain those of $f(z)$. These questions are discussed. Some results are proved, and some problems which remain unsolved are pointed out.

7. *Some set properties of relative extreme points*, by W. J. Matthews, Southwest Texas State College.

For subsets A and S of a linear space L, a point p of S is an extreme point of S relative to A if p is not contained in the relative interior of any line segment having one end point in A and the other end point in S. The purpose of this paper is to present some set properties of relative extreme points.

8. *Some properties of generalized Orlicz spaces*, by Richard O'Neil, Rice University.

9. *On a generalization of separability, the Lindelof property and countable compactness*, by Margaret R. Wiscamb, University of St. Thomas.

A topological space satisfies the discrete countable chain condition (DCCC) if every discrete collection of open sets is countable. This property is a generalization of separability, the Lindelof property and of countable compactness. A regular space satisfies the DCCC if and only if every locally finite collection of open sets is countable. A T_4 space is countably compact if and only if every discrete collection of open subsets is finite.

10. *Fourier series and convolution*, by A. F. Kleiner, Jr., Texas A. & M University.

If f satisfies a countable number of "suitable" conditions then f is the convolution $g*h$ of functions g and h where g is in $L(0, 2\pi)$, even, positive and continuous on $(0, \pi)$ and h satisfies the same conditions that f does. Specifically, the conditions (1) $f \in L(0, 2\pi)$ and $S[f]$ converges to f on a countable set E and (2) f is in $Lp(0, 2\pi)$ for $1 \leqq p < \infty$, are shown to be suitable.

11. *Doctoral programs in mathematics and education as related to instructional needs of junior colleges and four year colleges*, by Wingo Hamilton, University of Texas at Arlington.

The problem of this study was to analyze doctoral programs in mathematics and education designed for the preparation of teachers of undergraduate mathematics. Data were secured, through use of questionnaires, from samples of officials of junior colleges, senior colleges, graduate schools, and from a panel of authorities in mathematics education. Based on these data and a review of the literature, tentative conclusions were drawn as to: (1) the need for such programs, (2) the attitude of college and university officials toward them, (3) the composition of present offerings, and (4) recommendations as to the future course of their development.

12. *Extensions of topologies*, by C. D. Tabor, Texas Christian University and Texas Woman's University.

Let $(X, \mathfrak{I})$ denote a topological space where $\mathfrak{I}$ is some topology on a set X. Norman Levine's definition of a simple extension of $\mathfrak{I}$ by a subset A of S, denoted $\mathfrak{I}(A)$, is given, and relationships concerned with $\mathfrak{I}$-boundary points and $\mathfrak{I}(A)$-boundary points of a subset of X are discussed. A concept of (simultaneously) extending $\mathfrak{I}$ by an arbitrary number of subsets of X is introduced, and a necessary and sufficient condition for an extension of this type to be the simple extension of $\mathfrak{I}$ by the union of the subsets is given. Some theorems concerned with the idea of extending one topology by the members of an open basis for another topology are mentioned.

13. *Mapping induced topologies and everywhere disconnected spaces*, by M. G. Monzingo, Southern Methodist University.

A topology can be induced on a set by a map either to or from a given topological space. In the following, the basic ingredients have been reduced from a set, a map, and a topological space to two sets and a map. Two topologies will be induced by a given map, and a number of results will be obtained. The notion of an everywhere disconnected space will be introduced, and these spaces will be shown to be mapping induced topological spaces.

14. *On convolution and moduli of continuity*, by Jack Bryant, Texas A and M University.

For convolution $f=g*h$ of functions g, h in $L_1[0, 2\pi)$, one usually obtains $w(f, t) \leqq Aw(h, t)$, where w is a "modulus of continuity" $w(f, t)=\sup \{\|f(\cdot+h)-f(\cdot)\| : 0 \leqq |h| \leqq t\}$ with respect to

some Banach space B, $\|\cdot\|$. Under very general conditions on B, one finds this result is best possible in the following sense: if $w(f, t) = 0(w(t))$ then $f = g * h$ where $g \in L$ and $w(h; t) = 0(w(t))$.

15. *Symbols of parabolic singular integrals*, by Max Jodeit, Jr., Rice University.

A certain class of parabolic singular integral operators is characterized in terms of symbols (which reduce to Fourier transforms for operators of convolution type).

16. *Some remarks on the pointwise convergence of singular integrals with nonhomogeneous kernels*, by C. C. Tu, Rice University.

Pointwise convergence of some singular integrals with nonhomogeneous kernels can be established without using J. Peetre's general theorem.

17. *A multi-ordered iteration process for solution of nonlinear simultaneous equations*, by A. R. Crawley and H. A. Luther, Texas A and M University.

An extension of the Newton-Raphson method for the solution of a nonlinear system of equations is presented in a form such that by varying certain parameters the iterative process can take on all orders of convergence up through four. To assure convergence it is required that the initial approximation be close enough to the solution desired, the Jacobian of the system exist at this solution point, and the system has continuous fourth order partial derivatives in a neighborhood of this solution point. The scheme employs derivatives through order three rather than extra function evaluations and thus is appropriate only when derivatives are easily found.

18. *Matrix powers as solutions of a difference equation*, by Louis Brand, University of Houston.

The minimum equation (order m) of an $n \times n$ matrix A can be regarded as a difference equation whose characteristic roots are the eigenvalues λ_i of A. When all $\lambda_i \neq 0$, its solution gives all integral powers A^x in terms of m matrices C_i determined by $A^0 = I, A, \cdots, A^{m-1}$. If all λ_i are distinct, $m = n$, $A^x = \sum_{i=1}^{n} C_i \lambda_i^x$ and $C_i^2 = C_i$, $C_i C_j = 0$ $(i \neq j)$. When A is singular and $\lambda_n = 0$ is simple, $A^{x+1} = \sum_{i=1}^{n-1} C_i \lambda_i^x$; then the set of *singular* matrices A^x form an Abelian group with $A^0 (\neq I)$ as unit and $A^x A^{-x} = A^0$; now $C_i^2 = \lambda_i C_i$, $C_i C_j = 0$ $(i \neq j)$ and A^{-x} is the "inverse" of A^x relative to A^0.

19. *n-Abelian varieties of groups*, by J. R. Durbin, University of Texas at Austin.

The problem discussed is that of generalizing the well-known fact that a group is Abelian if it satisfies the identity $(xy)^n = x^n y^n$ for three consecutive integers n. This leads naturally to questions about certain reduced free groups and to a problem in number theory.

20. *Pseudo involutary linear transformations*, by Shannon Smyrl, Texas Technological College.

A linear transformation A on a vector space V over an algebraically closed field is called pseudo-involutary of order n if and only if $A^n = P$, where P is a fixed projection on the range of A and n is the least positive integer for which $A^n = P$. It can be shown that a linear transformation A on V is pseudo-involutary of order n if and only if $A = m_1 P_1 + \cdots + m_k P_k$, where m_i, $i = 1, \cdots, k$ is an nth root of unity and P_i, $i = 1, \cdots, k$ is a projection such that $P_i P_j = 0$, $i \neq j$, and $P_1 + \cdots + P_k = P$, a projection on the range of A. If $P = I$ then A will be a linear involution.

21. *A note on numerical differentiation*, by S. A. Sims, Texas A and M University, and Sallie Nelson, Lockheed Electronics Company.

From a prescribed discrete sampling of a differentiable function an approximate best fit, in the sense of least squares, to its derivative is obtained.

22. *Junior college mathematics—objectives and curriculum revision—as viewed by a junior college teacher*, by F. L. Ermis, Jr., Wharton County Junior College.

Objectives in mathematics education as related to the general objectives in junior college education and curriculum revision are viewed from a junior college teacher's standpoint. Recommendations are made, along with an appeal for the Texas Section to start a special session at the yearly meeting for junior college teachers.

BEN T. GOLDBECK, JR., *Secretary-Treasurer*

ANNOUNCEMENT OF L. R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1967 recipients of these awards, selected by a committee consisting of R. P. Boas, Chairman; C. W. Curtis, and R. P. Dilworth, were announced by President Moise at the Business Meeting of the Association on August 29, 1967, at the University of Toronto. The recipients of the Ford Awards for articles published in 1966 were the following:

Wai-Kai Chen, Boolean Matrices and Switching Nets, MATH. MAG., 39 (1966), 1–8.

D. R. Fulkerson, Flow Networks and Combinatorial Operations Research, MONTHLY, 73 (1966), 115–138.

Mark Kac, Can One Hear the Shape of a Drum?, MONTHLY, 73 (1966), Part II (Slaught Paper No. 11), 1–23.

M. Z. Nashed, Some Remarks on Variations and Differentials, MONTHLY, 73 (1966), Part II (Slaught Paper No. 11), 63–76.

P. B. Yale, Automorphisms of the Complex Numbers, MATH. MAG., 39 (1966), 135–141.

HENRY L. ALDER, *Secretary*

THE 1967 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twenty-eighth annual William Lowell Putnam Mathematical Competition will be held on Saturday, December 2, 1967. This competition, which is supported by the William Lowell Putnam Intercollegiate Memorial Fund, is under the sponsorship of the Mathematical Association of America. Colleges and universities in the United States and Canada are eligible to register undergraduates in the competition.

Application blanks will be mailed about October 1 to the mathematics department chairmen of the schools on the regular mailing list and also to those who supervised the competition in 1966. If an application blank is not received by October 15, one may be secured by writing the director, James H. McKay, Department of Mathematics, Oakland University, Rochester, Michigan 48063. Your application should be filed with the director not later than November 3, 1967. Further details are provided in the Announcement Brochure which is mailed with the registration forms.

Reports of the previous competitions and the examination questions may be found in the MONTHLY for May 1938, 1939, 1940, 1941, 1942; October 1946; August-September 1947; December 1948; August–September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; January 1957; August–September (announcement of winners) and November (questions and solutions) 1957; August–September 1958, 1959; January (questions and solutions for the eighteenth, nineteenth, and twentieth competitions) 1961; August–September 1961; October 1962; August–September 1963; June–July 1964; August–September 1965, 1966, and in this issue page 771.

QUALIFICATIONS FOR A COLLEGE FACULTY IN MATHEMATICS

In December, 1965, Professor William L. Duren, Jr., then chairman of CUPM, appointed an *ad hoc* committee to prepare a report identifying suitable qualifications for a college faculty in mathematics. Although some members of this committee, including its chairman, were at the time members of CUPM, several were drawn from the ranks of mathematicians and teachers outside CUPM:

L. J. Paige (Chairman), University of California, Los Angeles
A. A. Albert, University of Chicago
R. C. Buck, University of Wisconsin
Bernard Friedman (deceased), University of California, Berkeley
M. G. Humphreys, Randolph-Macon Woman's College
K. O. May, University of Toronto
C. E. Rickart, Yale University
A. B. Willcox, Amherst College
Leo Zippin, City University of New York

Upon succeeding Professor Duren as chairman of CUPM, Professor R. D. Anderson of Louisiana State University became an *ex officio* member of the committee.

At its meeting on January 23, 1967, CUPM approved for publication the committee's report, *Qualifications for a College Faculty in Mathematics*. Taking note of the continuing critical shortage of Ph.D. mathematicians, the report identifies four important components in the formal education of college teachers:

(1) a strong undergraduate mathematics major;
(2) a "first graduate component" which introduces the student to serious graduate work in the principal disciplines of mathematics;
(3) an "advanced graduate component" which includes advanced graduate work and research seminars; and
(4) the doctoral dissertation.

It then describes teaching duties suitable for individuals with academic attainment equivalent to that represented by a given component, and it makes suggestions concerning the composition of a small undergraduate department. The committee believes that, in addition to aptitude for mathematics and interest in teaching, a good college teacher must possess both the willingness and the ability to continue his own education in mathematics indefinitely. The report suggests several activities besides publication which indicate that a teacher is continuing his mathematical growth, and it calls upon college administrations to provide the environment and the opportunity for college teachers to engage in such activities.

CUPM hopes the report will be of interest to college mathematics teachers, to department chairmen and other college administrative officials, to boards of education, and to graduate departments of mathematics at universities offering doctoral programs.

Copies of the report may be obtained without charge by writing to CUPM, Box 1024, Berkeley, California 94701.

PROFESSIONAL OPPORTUNITIES IN MATHEMATICS: NEW EDITION

A new edition of *Professional Opportunities in Mathematics* was published by the Association in September 1967, the seventh edition of this popular booklet. Including the present edition, over 85,000 copies of *Professional Opportunities* have been printed. Annual sales average about 5000 copies.

Professional Opportunities appeared originally in the MONTHLY for January 1951 as the report of a committee of the Association. Successive editions have been revised and rewritten in order to keep the material up to date. The current edition has been prepared by the MAA Committee on Advisement and Personnel, consisting of A. B. Willcox, Chairman, K. J. Arnold, E. A. Davis, J. S. Frame, Mrs. F. J. MacWilliams, C. R. Phelps, and W. H. Schmidt.

CALENDAR OF FUTURE MEETINGS

Fifty-first Annual Meeting, San Francisco, California, January 25–27, 1968.

Forty-ninth Summer Meeting, University of Wisconsin, Madison, Wisconsin, August 26–28, 1968.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Indiana University of Pennsylvania, Indiana, April 27, 1968.

FLORIDA

ILLINOIS, Southern Illinois University, Edwardsville Campus, May 10–11, 1968.

INDIANA, Marian College, Indianapolis, November 11, 1967.

IOWA, Wartburg College, Waverly, April 19, 1968.

KANSAS, Marymount College, Salina, Spring 1968.

KENTUCKY, University of Kentucky, Lexington, Spring 1968.

LOUISIANA-MISSISSIPPI, Broadwater Beach Hotel, Biloxi, Mississippi, February 16–17, 1968.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Morgan State College, Baltimore, December 9, 1967.

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, South Dakota State University, Brookings, November 11, 1967.

MISSOURI

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 26–27, 1968.

NEW JERSEY, St. Peter's College, Jersey City, November 4, 1967.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO

OKLAHOMA-ARKANSAS, Federal Aviation Agency, Oklahoma City, March 29–30, 1968.

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 14–15, 1968.

PHILADELPHIA, University of Delaware, Newark, November 18, 1967.

ROCKY MOUNTAIN, University of Denver, Colorado, May 1968.

SOUTHEASTERN, East Carolina College, Greenville, North Carolina, March 29–30, 1968.

SOUTHERN CALIFORNIA, Loyola University of Los Angeles, Los Angeles, March 9, 1968.

SOUTHWESTERN, New Mexico State University, University Park, Spring 1968.

TEXAS, Texas A and M University, College Station, April 19–20, 1968.

UPPER NEW YORK STATE, State University College at Buffalo, November 4, 1967.

WISCONSIN, Wisconsin State University, La Crosse, May 4, 1968.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, New York, New York, December 26–31, 1967.

AMERICAN MATHEMATICAL SOCIETY, San Francisco, January 23–27, 1968.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of California, Los Angeles, June 17–20, 1968.

ASSOCIATION FOR COMPUTING MACHINERY, New York Hilton Hotel, November 10, 1967.

ASSOCIATION FOR SYMBOLIC LOGIC, Sheraton-Boston Hotel, December 27–28, 1967.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 23–25, 1967.

INSTITUTE OF MATHEMATICAL STATISTICS

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Convention Hall, Philadelphia, Pennsylvania, April 17–20, 1968.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Sheraton-Chicago Hotel, November 1–3, 1967.

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SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, University of California at Santa Barbara, November 29–30, December 1–2, 1967. (Symposium on combinatorial mathematics.)



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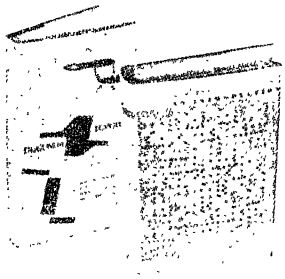
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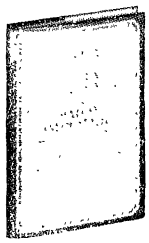
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OCTOBER 1967

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $6.00 ($8.00 in 1968). For nonmembers the subscription price is $10.00 ($15.00 in 1968).

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

FUNCTIONAL DEPENDENCE

W. F. NEWNS, The University, Liverpool

1. Introduction. It has been the fashion for some time for textbooks on Advanced Calculus or Analysis to include a section on Functional Dependence, the general idea being as follows. Real-valued functions $f_1, f_2, \cdots, f_m$ of n real variables are functionally dependent if they satisfy a relation of the form $F(f_1(x), f_2(x), \cdots, f_m(x)) = 0$ identically. Assuming the f_i to be continuously differentiable, it is claimed that the functions are dependent if and only if the rank of their Jacobian is everywhere less than m. It is clearly not intended that every sequence of m functions shall be regarded as dependent (by taking F identically zero) but just what *is* intended by the definition is usually obscure. In the proof of 'only if' in the above result, it is usually supposed that F is continuously differentiable and that its partial derivatives nowhere vanish simultaneously. The 'if' statement is usually deduced from the implicit function theorem on the assumption that the Jacobian has constant rank, and appears to be a local result. From what is assumed about F in the proof of 'only if,' and proved about F in the converse result, one can arrive at an appropriate meaning of 'dependence' for this particular theorem. This meaning, however, differs from the definition given by R. C. Buck [2] in one of the few textbooks where the definition is precise. It also differs from the definition used by A. B. Brown [1], on the basis of which he proves a result similar to the textbook result, but without the hypothesis that the Jacobian has constant rank. (In Brown's theorem there was an extra differentiability hypothesis on the f_i, a hypothesis which was later weakened by A. Sard [9].)

In view of this, it seems worth while to attempt an exposition of the subject, starting with an investigation into the relationship between various possible meanings of 'dependence.'

2. Definitions. We first make

DEFINITION 1. *Let X, I be sets, $(Y_\iota)_{\iota\in\mathrm{I}}$ a family of sets, and for each $\iota\in\mathrm{I}$ let $f_\iota\colon X\to Y_\iota$. Let F be a real-valued function with domain $\mathfrak{D}(F)\subset\prod_{\iota\in\mathrm{I}} Y_\iota$. Then we say that the f_ι are F-related iff $F(f(x))=0$ for all $x\in X$, where $f\colon X\to\prod_{\iota\in\mathrm{I}} Y_\iota$ is the function having (f_ι) as components.*

It is implied here that if (f_ι) is F-related then $\mathfrak{D}(F)$ contains the range $\mathfrak{R}(f)$ of f.

To obtain a useful definition of dependent (as F-related for some suitable F) we must not merely exclude the case where F is identically zero: we need something like the other extreme. The obvious other extreme, namely that F has no zeros, is clearly inappropriate. In a topological space, however, a function is identically zero iff its support is empty. We recall the definition.

DEFINITION 2. *Let Y be a topological space and F a real-valued function with domain Y. The support of F is the subset*

$$\text{supp } F = \overline{\{y \in Y: F(y) \neq 0\}}$$

of Y, where the bar denotes closure in Y.

The support of F is the smallest closed set outside which F vanishes identically. The other extreme from supp $F=\varnothing$ is supp $F=\mathfrak{D}(F)$, but since all we require of $\mathfrak{D}(F)$ is that it should contain $\mathfrak{R}(f)$, it is equivalent for our purposes to make the following definition, which specializes essentially to that given by R. C. Buck ([2] p. 226).

DEFINITION 3. *Let X, I be sets, (Y_ι) a family of topological spaces and for each $\iota \in$ I let $f_\iota: X \to Y_\iota$. We say that the f_ι are functionally dependent iff they are F-related by some F such that supp $F \supset \mathfrak{R}(f)$.*

Buck's requirement on F, namely that it should vanish identically on no nonempty open set, is clearly equivalent.

Having ensured by our requirement on F that the relation satisfied by the dependent functions is as nontrivial as possible, it is natural to impose further restrictions on F.

DEFINITION 4. *Let X, I, (Y_ι), (f_ι) be as in Definition 3. Then we say that the f_ι are C^0-dependent iff they are F-related by some continuous F such that* supp $F \supset \mathfrak{R}(f)$.

DEFINITION 5. *Let X be a set, let m be a positive integer and for each integer i $(1 \leqq i \leqq m)$ let f_i be a real-valued function with domain X. Let p denote ∞ or a positive integer. Then the f_i are C^p-dependent iff they are F-related by some F of class C^p such that* supp $F \supset \mathfrak{R}(f)$. *They are analytically dependent iff they are F-related by some analytic F such that* supp $F \supset \mathfrak{R}(f)$.

3. Topological characterizations of dependence.

PROPOSITION 1. *Let X, I, (Y_ι), (f_ι), f be as in Definition 3. Then the f_ι are functionally dependent iff $\mathfrak{R}(f)$ has no interior point (in $Y=\pi Y_\iota$).*

Proof. Suppose the f_ι are functionally dependent and choose F relating them and such that supp $F \supset \mathfrak{R}(f)$. If y_0 were an interior point of $\mathfrak{R}(f)$, then $\mathfrak{R}(f)$ would be a neighborhood of y_0 on which F vanishes, hence not meeting $\{y \in \mathfrak{D}(F): F(y) \neq 0\}$. This would mean $y_0 \notin$ supp F contrary to $y_0 \in \mathfrak{R}(f) \subset$ supp F. We conclude that no such point exists.

Conversely, suppose that $\mathfrak{R}(f)$ has no interior point and take F to be the characteristic function of $Y \setminus \mathfrak{R}(f)$. Then the f_ι are F-related, and the support of F is the closure of $Y \setminus \mathfrak{R}(f)$, whose complement is an open subset of $\mathfrak{R}(f)$, and hence, is empty. Thus supp $F = Y$ and the f_ι are functionally dependent.

PROPOSITION 2. *If the f_ι are C^0-dependent then $\mathfrak{R}(f)$ is nowhere dense in Y.*

Proof. Let F be a continuous function relating the f_ι and satisfying supp $F \supset \mathfrak{R}(f)$. Then F vanishes on $\overline{\mathfrak{R}(f)} \cap \mathfrak{D}(F)$ and *a fortiori* on $U \cap \mathfrak{D}(F)$, where U is

the interior of $\overline{\mathfrak{R}(f)}$: thus $U \cap \text{supp } F = \varnothing$. If U were nonempty, it would meet $\overline{\mathfrak{R}(f)}$ (it is a subset of it) and hence meet $\mathfrak{R}(f)$ contrary to $\mathfrak{R}(f) \subset \text{supp } F$. We conclude that $U = \varnothing$.

A converse of Proposition 2 is not to be expected without some restriction on the topologies of the Y_ι. We prove only an easy result.

PROPOSITION 3. *Suppose that* I *is countable, that each* Y_ι *is metrizable and that* $\mathfrak{R}(f)$ *is nowhere dense in* Y. *Then the components of* f *are* C^0*-dependent.*

Proof. The space Y being metrizable, we choose a metric for it and let $F(y)$ be the distance of y from $\mathfrak{R}(f)$. Then F is continuous and $F(y) = 0$ iff $y \in \overline{\mathfrak{R}(f)}$. Thus the components of f are F-related and since $\overline{\mathfrak{R}(f)}$ has empty interior, supp $F = Y$.

For the case of C^p-dependence, since this implies C^0-dependence we know from Proposition 2 that a nowhere dense range is necessary for C^p-dependence. Conversely:

PROPOSITION 4. *Let* X *be a set,* m *a positive integer and* $f\colon X \to \boldsymbol{R}^m$ *a function whose range is nowhere dense in* $\boldsymbol{R}^m$. *Then the components of* f *are* C^∞*-dependent.*

Proof. For any integers n, k $(1 \leqq k \leqq m)$, the equation $y_k = n$ defines a hyperplane in $\boldsymbol{R}^m$ (y_k denoting the kth coordinate of $y \in \boldsymbol{R}^m$). All such hyperplanes divide $\boldsymbol{R}^m$ into open hypercubes, and we denote by Q_1 the set of all those hypercubes not meeting $\mathfrak{R}(f)$. For any integer $s > 1$, we divide $\boldsymbol{R}^m$ similarly into hypercubes by means of hyperplanes with equations

$$y_k = n2^{-s} (n \in \boldsymbol{Z},\ 1 \leqq k \leqq m),$$

and denote by Q_s the set of those hypercubes which meet neither $\mathfrak{R}(f)$ nor any element of Q_r for $r < s$. For any positive integer s, we put

$$F_s(y) = s^{-s} \exp\left(-(\sin(2^s \pi y_1) \sin(2^s \pi y_2) \cdots \sin(2^s \pi y_m))^{-2}\right)$$

whenever y belongs to some element of Q_s, and $F_s(y) = 0$ for all other y in $\boldsymbol{R}^m$. It can be shown that F_s is of class C^∞, hence so also is $F = \sum_{s=1}^{\infty} F_s$. It is clear that F vanishes on $\mathfrak{R}(f)$, and hence that the components of f are F-related; also, the support of F contains the union of the Q_s, hence also the complement of $\overline{\mathfrak{R}(f)}$, and since $\overline{\mathfrak{R}(f)}$ has empty interior we see that supp $F = \boldsymbol{R}^m$.

REMARK. By a modified construction it is possible to prove: *Given any closed subset* S *of* $\boldsymbol{R}^m$, *there is an* $F\colon \boldsymbol{R}^m \to \boldsymbol{R}$ *of class* C^∞ *such that* $F(y) = 0$ *iff* $y \in S$. See [1] for details.

For the case of m real-valued functions we see that, for any p, C^p-dependence is equivalent to C^0-dependence and to C^∞-dependence. We thus have only three distinct kinds of dependence. (That the first two are distinct can be seen by considering the projection onto the (x, y)-plane of a well-known dense curve on a torus. Defining

$$f\colon \boldsymbol{R} \to \boldsymbol{R}^2 \text{ by } f(t) = e^{it}(2 + \cos(t\sqrt{2})),$$

we can show that $\mathfrak{R}(f)$ has no interior point but is dense in the annulus $\{z \in \mathbf{C}\colon 1 \leqq |z| \leqq 3\}$. For analytic dependence, an extra topological condition on $\mathfrak{R}(f)$ is necessary [1].)

4. Local dependence. We now suppose that X is a topological space. A family (f_ι) is *locally C^0-dependent* iff each point of X has a fundamental system of neighborhoods such that the restrictions of the f_ι to each of the neighborhoods are C^0-dependent. It is clearly enough that each point of X should possess one such neighborhood. Moreover,

PROPOSITION 5. *If Y is metrizable and the components of f are locally C^0-dependent, then for every compact $K \subset X$, the components of $f|_K$ are C^0-dependent.*

Proof. Each point of K has a neighborhood U such that $f|_U$ has C^0-dependent components, hence such that $f[U]$ is nowhere dense in Y (Prop. 2). We can cover K by a finite sequence (U_i) of such U. Thus $f[K]$, as subset of $\bigcup f[U_i]$, is nowhere dense, and consequently (Prop. 3) the components of $f|_K$ are C^0-dependent.

We next show that, at least in the most usual case, this kind of dependence is not yet another kind, but is equivalent to the first kind. The generality is retained mainly to indicate what is involved.

PROPOSITION 6. *Suppose that X is locally compact and σ-compact, that Y is a complete metric space and that f is continuous. Then the components of f are functionally dependent iff they are locally C^0-dependent.*

REMARK. The definition of functional dependence used by A. B. Brown [1] is essentially (i.e., modulo the irrelevance of p in C^p-dependence) the property given in Proposition 5, and so (since $\boldsymbol{R}^n$ is locally compact) is equivalent to our local C^0-dependence. Proposition 6 asserts the equivalence of Brown's definition with Buck's.

Proof of Proposition 6. Suppose that the components of f are functionally dependent. Then $\mathfrak{R}(f)$ has no interior point. Let $x \in X$ and choose a compact neighborhood K of x. *A fortiori*, $f[K]$ has no interior point, and being closed (as compact subset of a Hausdorff space) is nowhere dense. By Proposition 3, the components of $f|_K$ are C^0-dependent. Thus f has locally C^0-dependent components.

Conversely, supposing that f has locally C^0-dependent components, let (K_n) be a sequence of compact sets covering X. By Proposition 5, the components of $f|_{K_n}$ are C^0-dependent, hence (Proposition 2) $f[K_n]$ is nowhere dense, for every n. The sets $f[K_n]$ cover $\mathfrak{R}(f)$, so that $\mathfrak{R}(f)$ is meager (1st. category). Since Y is a Baire space (2nd. category), $\mathfrak{R}(f)$ has no interior point. The result follows from Proposition 1.

COROLLARY. *Let m, n be positive integers, $X \subset \boldsymbol{R}^n$ and let $f\colon X \to \boldsymbol{R}^m$ be continuous and have functionally dependent components. Then for any compact $K \subset X$ the components of $f|_K$ are C^∞-dependent.*

5. Functional dependence and vanishing Jacobians. The basic result of Brown and Sard mentioned in the introduction is as follows:

THEOREM 1. *Let m, n be positive integers, X an open subset of $\boldsymbol{R}^n$ and $f: X \to \boldsymbol{R}^m$ of class C^p for some $p \geqq 1$. Then*

(i) *if the components of f are functionally dependent, their Jacobian has rank less than m at every point of X;*

(ii) *the converse of* (i) *is true whenever $p \geqq \frac{1}{2}(n-m+2)$ (hence without class restrictions if $m \geqq n$).*

Before considering the proof of this theorem, we recall that the *differential* df_a of f is the linear function $u: \boldsymbol{R}^n \to \boldsymbol{R}^m$ (unique when it exists) such that

$$\lim_{x \to a} \frac{f(x) - f(a) - u(x-a)}{\|x-a\|} = 0,$$

where $\|\ \|$ denotes some norm on $\boldsymbol{R}^n$. Which norm is used is irrelevant since all norms on $\boldsymbol{R}^n$ are equivalent. The one most frequently used is the *Euclidean norm*, determined by $\|x\|^2 = \sum_{j=1}^n x_j^2$, where the x_j are the coordinates of x. We shall also use the *Cartesian norm*, defined by

$$\|x\| = \sup \{ |x_j| : 1 \leqq j \leqq n \}.$$

The continuity of df_a (which is automatic for a linear mapping between finite-dimensional spaces) is expressed by the existence of $M \in \boldsymbol{R}$ such that $\|df_a(x)\| \leqq M\|x\|$ for all $x \in \boldsymbol{R}^n$. The infimum of all such M has the same property and is denoted by $\|df_a\|$. The partial derivatives of the components of f at a are the elements of the Jacobian matrix of f at a, which is the matrix of the linear function df_a relative to the natural bases for $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$. The rank of this Jacobian matrix is equal to the rank of the differential.

In the sequel, we use the following result, called by Dieudonné [3, p. 273] the

RANK THEOREM. *Let m, n, X, f, p be as in Theorem* 1, *and suppose given a point $a \in \mathfrak{D}(f)$ such that df_x is of constant rank r for all x in some neighborhood of a. Then there exist neighborhoods U_1, U_2 of* 0, *a in $\boldsymbol{R}^n$, neighborhoods V_1, V_2 of $f(a)$,* 0 *in $\boldsymbol{R}^m$ and functions $g: U_1 \to U_2$, $h: V_1 \to V_2$ with the following properties:*

(i) *$g(0) = a$, $h(f(a)) = 0$ and g, h are one-one and have ranges U_2, V_2 respectively,*

(ii) *g, g^{-1}, h, h^{-1} are of class C^p,*

(iii) *$h \circ f \circ g = df_{a|U_1}$,*

Moreover,

(iv) *if $r = n$ (in which case it is enough to be given that df_a has rank n, since then df_x will have rank n on some neighborhood of a: equivalently if df_a is one-one) then g can be chosen to be a translation in $\boldsymbol{R}^n$,*

(v) *if $r = m$ (equivalently, if df_a has range $\boldsymbol{R}^m$), then h can be chosen to be a translation in $\boldsymbol{R}^m$.*

Readers unfamiliar with this coordinate-free treatment are recommended to consult chapter 10 of [4].

Proof of (i). Suppose that, for some point $a \in X$, the differential df_a of f at a has rank m. Then we find from the rank theorem that $f[U_2] = h^{-1}[df_a[U_1]]$ is a neighborhood of $f(a)$, so that $f(a)$ is an interior point of $\mathfrak{R}(f)$. The assertion now follows from Proposition 1.

The proof of (ii) will not be given in full. We shall indicate what is involved, and supply references.

It is enough to prove that if K is a closed cube in X with sides parallel to the axes, then $f[K]$ has no interior point. The technique is to show that $f[K]$ lies in a set of arbitrarily small volume, from which it is obvious that it cannot have an interior point.

In fact, this proves more, namely that $f[K]$ has measure zero, where 'measure' means either Lebesgue measure in $\boldsymbol{R}^m$ or m-dimensional Hausdorff measure. We use a cube, because this can be subdivided into nonoverlapping cubes. For convenience, we use the Cartesian norm on $\boldsymbol{R}^n$, so that the cubes are 'balls' in the sense of this metric. On $\boldsymbol{R}^m$ we use the Euclidean norm since we wish to estimate volume.

According to the Mean Value theorem,

$$\|f(x) - f(a)\| \leqq M\|x - a\| \tag{1}$$

for all $x \in K$, $a \in K$, where $M = \sup\{\|df_x\| : x \in K\}$ is finite. In other words, distances are increased by a factor of at most M. Let Q be a cube in K with sides of length l parallel to the axes. Then $f[Q]$ lies inside a ball of radius $\frac{1}{2}Ml$, hence of volume $M_1 l^m$, where M_1 is independent of Q. Letting L denote the length of the sides of K and dividing K into N^n cubes with sides of length L/N, we see that $f[K]$ lies in a set of volume not exceeding $N^n M_1 (L/N)^m$ which is arbitrarily small for large enough N provided $m > n$. Thus the result is proved for this case.

If $m \leqq n$ we make more specific use of the hypothesis that the rank of the Jacobian is everywhere less than m. From the compactness of K, it follows that there exists an increasing function $b: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ with $\lim_{\epsilon \to 0} b(\epsilon) = 0$ such that

$$\|f(x) - f(a) - df_a(x - a)\| \leqq b(\|x - a\|)\|x - a\| \tag{2}$$

whenever $x \in K$ and $a \in K$. For fixed a, the points $f(a) + df_a(x - a)$ all lie in a hyperplane H in $\boldsymbol{R}^m$ since $\mathfrak{R}(df_a)$ has dimension less than m. The above inequality shows that a cube of side l maps to a set of points no further than $lb(l)$ from H, hence into a hypercylinder of volume $M_2 l^{m-1} lb(l)$. Subdividing into N^n cubes of side L/N as above, we see that $f[K]$ lies in a set of volume $M_2 L^m N^{n-m} b(L/N)$, which is now arbitrarily small for large N when $m = n$.

The way to cope with the case $m < n$ is to modify the factor N^{n-m} in the estimate. We need to improve (1) to

$$\|f(x) - f(a)\| \leqq M\|x - a\|^q \tag{3}$$

so that a cube of side l maps into a ball of radius $M(l/2)^q$ and (with account taken of the rank of df) into a hypercylinder of volume $M_3 l^{q(m-1)} lb(l)$. This would

give the estimate $M_3L^{qm-q+1}N^{n-q(m-1)-1}b(L/N)$ which is good enough provided $n-q(m-1)-1\leqq 0$, i.e., $q\geqq(n-1)/(m-1)$, but an inequality of the form (3) for $q>1$ cannot be expected to hold very widely. Clearly, if it holds for all x in some neighborhood of a point a, we must have $df_a=0$. Moreover, when q is an integer, it suggests that the Taylor expansion of f about a has all terms of orders $1, 2, \cdots, q-1$ equal to 0, and that $M\|x-a\|^q$ estimates the remainder term. We expect therefore to need to assume f to be of class C^q as well as to consider points a at which $df_a=0$. Of course, if $df_x=0$ for all x in a ball, then f will be constant on that ball, and the image of the ball gives no trouble: our problem is to cope with $\{x\in X: df_x=0\}$, a closed set A which may be extremely complicated. Before stating the precise result, we observe that the isolated points of any subset of $\boldsymbol{R}^n$ are countable (and so have countable image), whilst the nonisolated points of A are accumulation points of points where df vanishes, and we may therefore expect a zero of higher order at such points.

LEMMA 1. *Let $A\subset X\subset\boldsymbol{R}^n$, X being open, and let p be a positive integer. Then there exists a sequence $(A_i)_{i\in N}$ of subsets of X such that*: (i) $A\subset\bigcup_{i\in N}A_i$, (ii) A_0 *is countable,* (iii) *For any $f: X\to\boldsymbol{R}$ of class C^p whose differential vanishes on A and any $i>0$, there exists an increasing function $b_i: \boldsymbol{R}_+\to\boldsymbol{R}_+$ such that $\lim_{\epsilon\to 0}b_i(\epsilon)=0$ and*

$$|f(x)-f(a)|\leqq b_i(\|x-a\|)\|x-a\|^p$$

whenever $x\in A_i$ and $a\in A_i$.

From this, as already indicated, we deduce

LEMMA 2. *Let X be an open subset of $\boldsymbol{R}^n$, let $f: X\to\boldsymbol{R}^m$ be of class C^p, and let $A=\{x\in X: df_x=0\}$. Then $f[A]$ is of measure zero, provided $p\geqq n/m$.*

The slight reduction (from $(n-1)/(m-1)$ to n/m) comes about because the M in (3) has become a $b_i(\|x-a\|)$ and we can therefore consider balls rather than hypercylinders. From this we obtain:

LEMMA 3. *Let X be an open subset of $\boldsymbol{R}^n$, let $f: X\to\boldsymbol{R}^m$ be of class C^p and let A_r be the set of points $x\in X$ at which the rank of df_x is $r(<m)$. Then $f[A_r]$ has measure zero provided that $p\geqq(n-r)/(m-r)$.*

For $r=0$ this is Lemma 2. For a point a where df_a has rank $r>0$, we let X_2 be the kernel of df_a, Y_1 the range of df_a and choose supplementary subspaces X_1, Y_2 in $\boldsymbol{R}^n$, $\boldsymbol{R}^m$ respectively. After a C^p change of coordinates (by applying the rank theorem to $\pi_1\circ f$, where π_1 is projection onto Y_1) we find that the restriction of the new function f^* (which is defined on a neighborhood N of 0) to any slice of N (formed by taking the intersection with N of some translate of X_2) has two properties: its range lies in the appropriate translate of Y_2 and its differential vanishes at any point corresponding (by the coordinate change) to a point of A_r. The dimensions of X_2, Y_2 being $n-r$, $m-r$ respectively, Lemma 2

applies to such restrictions to show that $f[A_r]$ meets every translate of Y_2 in a set of $(m-r)$-dimensional measure zero, and the result follows from Fubini's Theorem.

When the rank of df_x is less than m for all $x \in X$, we have $\mathfrak{R}(f) = \bigcup_{r=0}^{m-1} f[A_r]$ and so is of measure 0 (hence without interior point) provided $p \geqq (n-r)/(m-r)$ for $r = 0, 1, \cdots, m-1$, i.e., provided $p \geqq n-m+1$. The slight improvement in part (ii) of Theorem 1 comes from the observation that Lemma 3 is needed only for $r \leqq m-2$. That $f[A_{m-1}]$ has no interior point is immediate from the rank theorem, A_{m-1} being an open set on which the differential of f has constant rank $m-1$.

References. Lemma 1 was given in [6] Theorem 4.2. Lemmas 2 and 3 were given in [9]. There is also a full discussion of Sard's theorem, including Morse's lemma, in [10]. Sard's theorem is the corollary of Lemma 3 (without supposing the rank of the differential to be everywhere less than m) that the set $\bigcup_{r=2}^{m-1} f[A_r]$ has measure 0 provided $p \geqq n-m+1$. It is used in the study of immersions of manifolds and in the case $m = n$ is relevant to the change of variable in a multiple integral (cf. [5]).

REMARKS. 1. Although part (ii) of Theorem 1 is deduced from Sard's results, these results were designed to prove Sard's Theorem and do not use the hypothesis of Theorem 1 (ii) that every point is a critical point. Thus, although the lower bound on p in Sard's theorem cannot be improved (as shown by an example in [11]), that in Theorem 1 (ii) is obviously not the best possible for the (admittedly trivial) case $m = 1$. Presumably a new approach is needed.

2. Let m, n, X, f and p be as in Theorem 1. Suppose that $a \in X$ and $r < m$ are such that df_x has rank r for all x in some neighborhood of a. Then we can find neighbourhoods U, V of a, $b(=f(a))$ respectively and $F: V \to \boldsymbol{R}^{m-r}$ of class C^p such that $f[U] \subset V \subset \boldsymbol{R}^m$, $F \circ f|_U = 0$ and dF_a has rank $m-r$. (We have only to let π_2 be projection onto a supplement of the range of df_a and take $U = U_2$, $V = V_1$, $F = \pi_2 \circ h$ in the notation of the rank theorem.) The components of F therefore constitute $m-r$ independent relations between the components of $f|_U$: and these relations can be solved to give $m-r$ of the components of f in terms of the others. This is the usual textbook result, and it is used in the discussion of total differential equations (cf. [8] p. 141, for example).

It should be noted that without the hypothesis of constant rank these results may fail. For example, let $f = (f_1, f_2): \boldsymbol{R} \to \boldsymbol{R}^2$ be defined by

$$f_1(t) = \begin{cases} e^{-1/t^2} & (t > 0) \\ 0 & (t \leqq 0), \end{cases}$$

$f_2(t) = f_1(-t)$. Then f_1 and f_2 are analytically dependent (take $F(x, y) = xy$), but are not F-related on a neighborhood of the origin by any F of class C^1 for which dF_0 has rank 1. Nor is there any ϕ such that $f_1(t) = \phi(f_2(t))$ on some neighborhood of 0.

6. Analytic dependence. There is no general analogue of Theorem 1 above for analytic dependence, as is shown by the following example (cf. [7]).

Define $f: \mathbf{C}^2 \to \mathbf{C}^m$ for $m \geqq 3$ by

$$f(u, v) = (u, uv, uve_1(v), uve_2(u), \cdots, uve_{m-2}(v)),$$

where $e_1(v) = e^v$ and $e_k(v) = e^v e_{k-1}(e^v)$. Let F be a complex-valued analytic function on some neighborhood of 0 in $\mathbf{C}^m$ such that $F \circ f$ vanishes on some neighborhood of 0 in $\mathbf{C}^2$. We shall prove that F vanishes identically.

Collecting terms in the power series expansion of F about the origin, we can write

$$F = \sum_{n=0}^{\infty} F_n,$$

where F_n is a homogeneous polynomial function of total degree n in m variables. Composing with f, we have

$$\begin{aligned} F(f(u, v)) &= \sum_{n=0}^{\infty} F_n(u, uv, uve_1(v), \cdots, uve_{m-2}(v)) \\ &= \sum_{n=0}^{\infty} u^n F_n(1, v, ve_1(v), \cdots, ve_{m-2}(v)), \end{aligned}$$

a power series in u with coefficients depending on v. Since this vanishes on a neighborhood of 0 in $\mathbf{C}^2$, we conclude that the coefficient of each power of u vanishes on a neighborhood of 0 in $\mathbf{C}$, hence everywhere in $\mathbf{C}$ (since it is an entire function). Writing

$$F_n(y_1, y_2, \cdots, y_m) = \sum_{k=0}^{n} F_{nk}(y_2, y_3, \cdots, y_m) y_1^{n-k},$$

the F_{nk} are homogeneous polynomial functions of total degree k in $m-1$ variables and $\sum_{k=0}^{n} F_{nk}(v, ve_1(v), \cdots, ve_{m-2}(v)) = 0$ for all $v \in \mathbf{C}$. Hence

$$\sum_{k=0}^{n} v^k F_{nk}(1, e_1(v), \cdots, e_{m-2}(v)) = 0$$

for all $v \in \mathbf{C}$. Let $v_0 \in \mathbf{C}$. Then the polynomial function

$$v \longmapsto \sum_{k=0}^{n} v^k F_{nk}(1, e_1(v_0), \cdots, e_{m-2}(v_0))$$

vanishes for $v = v_0 + 2q\pi i$ for all $q \in \mathbf{Z}$, since e_k takes the same value at all such v. We conclude that it vanishes identically and since v_0 is arbitrary, that

$$F_{nk}(1, e_1(v), \cdots, e_{m-2}(v)) = 0$$

for all $v \in \mathbf{C}$. If we write $w = e^v$, this means that

$$F_{nk}(1, w, we_1(w), \cdots, we_{m-3}(w)) = 0$$

for all $w \in C \setminus \{0\}$, hence for all $w \in C$. But this is precisely what we had earlier, but with $m-1$ for m. It is therefore enough to consider the case $m=3$, for which we have

$$F_{nk}(1, w) = 0$$

for all $w \in C$. This gives $F_{nk}=0$ immediately, and completes the proof that $F=0$.

By defining g: $C^n \to C^m$ by $g(x_1, x_2, \cdots, x_n) = f(x_1, x_2)$ we see that:

If $n \geqq 2$ and $m \geqq 3$, the components of an analytic f: $C^n \to C^m$ whose differential has rank everywhere less than m may fail to be locally analytically dependent.

For the remaining cases, the positive result does hold. The case $m=1$ is trivial. The case $m=2$, was proved by A. B. Brown in [1]. The case $n=1$, $m>2$ follows from the case $m=2$ by using the first two components.

For the real analytic case, the same discussion as above gives the analogous negative results. The positive results follow from the complex case.

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BOUNDS ON FUNCTIONS OF MATRICES

P. C. ROSENBLOOM, University of Minnesota and Minnesota State Department of Education

1. Introduction. If f is a function on a set S of scalars (real or complex) to the scalars, then the postulate, if $Ax=\lambda x$ and $\lambda \in S$, then

$$f(A)x = f(\lambda)x \tag{1}$$

uniquely defines the extension of f to the set S_n of all $n \times n$ matrices with distinct eigenvalues belonging to S. For various classes of functions f and sets S formulas

have been given for extending f to the set $\overline{S}_n$ of all matrices A such that the spectrum $\sigma(A) \subset S$. We investigate here the conditions on f that f have a bounded or a continuous extension to $\overline{S}_n$. In finite dimensional spaces all norms on the spaces of vectors or matrices are, of course, topologically equivalent.

Let $\| \cdot \cdot \cdot \|$ be any norm on the space of $n \times n$ matrices, and for $R > 0$ define

$$N_n(f; S, R) = \sup \|f(A)\| \quad \text{for } A \in S_n, \|A\| \leqq R.$$

Our first question is: when is the norm $N_n(f; S, R)$ finite?

For simplicity of exposition, we work out the cases $n = 2$ and $n = 3$, and then extend to general $n > 1$.

2. Case $n = 2$. Consider triangular matrices:

$$A = \begin{pmatrix} \lambda & a \\ 0 & \mu \end{pmatrix},$$

where $\lambda \neq \mu$, and $\lambda, \mu \in S$. If $\delta_1 = (1, 0)$ and $\delta_2 = (0, 1)$, then $A\delta_1 = \lambda\delta_1$, $A\delta_2 = a\delta_1 + \mu\delta_2$. Therefore we have $f(A)\delta_1 = f(\lambda)\delta_1$. Furthermore we find that

$$A(t\delta_1 + \delta_2) = \mu(t\delta_1 + \delta_2)$$

for $t = a/(\mu - \lambda)$, so that

$$\begin{aligned} f(A)\delta_2 &= f(A)(t\delta_1 + \delta_2 - t\delta_1) \\ &= f(\mu)(t\delta_1 + \delta_2) - tf(\lambda)\delta_1 \\ &= t(f(\mu) - f(\lambda))\delta_1 + f(\mu)\delta_2. \end{aligned}$$

Consequently, we obtain

$$f(A) = \begin{pmatrix} f(\lambda) & a[\lambda, \mu] \\ 0 & f(\mu) \end{pmatrix}, \quad \text{where} \quad [\lambda, \mu] = \frac{f(\lambda) - f(\mu)}{\lambda - \mu}$$

is the divided difference of f. If

$$C = \sup |a_{jk}| \tag{2}$$

for $\|A\| \leqq 1$, $A = (a_{jk})$, $1 \leqq j, k \leqq 2$, then

$$|f(\lambda)| \leqq C\|f(A)\|, \qquad |a||[\lambda, \mu]| \leqq C\|f(A)\|.$$

If $\|A\| \leqq R$, then $|f(\lambda)| \leqq CN_2(f; S, R)$, $|a||[\lambda, \mu]| \leqq CN_2(f; S, R)$.

Let

$$B = \sup \|A\| \tag{3}$$

for $A = (a_{jk})$, $\sum |a_{jk}| \leqq 1$.

The norms $m(A) = \sup |a_{jk}|$ and $n(A) = \sum |a_{jk}|$ are often computationally convenient. The relations $m(A) \leqq C\|A\|$ and $\|A\| \leqq Bn(A)$, where B and C depend only on n, are special cases of the topological equivalence of all norms. Then

$$\|A\| \leqq B(|\lambda| + |\mu| + |a|) \leqq R$$

if $|\lambda|, |\mu|, |a| \leqq R/(3B)$. We set $a = R/(3B)$, and obtain

$$|f(\lambda)| \leqq CN_2(f; S, R)$$

and $|[\lambda, \mu]| \leqq 3BC\, N_2(f; S, R)/R$ for $\lambda, \mu \in S, \lambda \neq \mu$, and $|\lambda|, |\mu| \leqq R/(3B)$.

THEOREM 1. *If $N_2(f; S, R) < +\infty$ for all $R > 0$, then f is bounded and Lipschitzian on every bounded subset of S. If*

$$M_1(f; S, r) = \sup(|f(\lambda)| + |[\lambda, \mu]|)$$

for $\lambda, \mu \in S, \lambda \neq \mu, |\lambda|, |\mu| \leqq r$, then

$$M_1(f; S, r) \leqq C(1 + r^{-1})N_2(f; S, 3Br). \tag{4}$$

COROLLARY 1a. *If f can be extended to a continuous function on $\overline{S}_2$, and S is a domain in the complex plane, then f is analytic in S. If $N_2(f; S, R) < \infty$ for all $R > 0$, then f' is bounded on every bounded subset of S.*

In order to obtain an estimate of $N_2(f; S, R)$ in terms of $M_1(f; S, R)$ we use, for the sake of simplicity, the Euclidean (unitary) length of 2-dimensional vectors

$$\|(x_1, x_2)\| = (|x_1|^2 + |x_2|^2)^{1/2},$$

and the corresponding norm $\|A\| = \sup \|Ax\|$ for $\|x\| \leqq 1$ for 2×2 matrices.

For a triangular matrix A, as above, this norm reduces to

$$\|A\|^2 = \frac{s^2 + \sqrt{s^4 - 4|\lambda|^2|\mu|^2}}{2},$$

where $s^2 = |\lambda|^2 + |\mu|^2 + |a|^2$, so that $s/\sqrt{2} \leqq \|A\| \leqq s$, $|\lambda| \leqq \|A\|$, $|\mu| \leqq \|A\|$.

If $A \in S_2$ and $\|A\| \leqq R$, where $R > 0$, then let λ and μ be the eigenvalues of A. Let x be an eigenvector belonging to λ, and let y be a unit vector perpendicular to x. Then in the coordinate system with base vectors x and y, the matrix A has the above triangular form.

We obtain therefore

$$\|f(A)\|^2 \leqq |f(\lambda)|^2 + |f(\mu)|^2 + |a|^2|[\lambda, \mu]|^2 \leqq (2 + 2R^2)M_1(f; S, R)^2.$$

THEOREM 2. *If $M_1(f; S, r) < +\infty$ for all $r > 0$, then $N_2(f; S, R) < +\infty$ for all $R > 0$, and $N_2(f; S, R) \leqq (2 + 2R^2)^{1/2} M_1(f; S, R)$.*

COROLLARY 2a. *For $N_2(f; S, R)$ to be finite for every $R > 0$ it is necessary and sufficient that f be bounded and Lipschitzian on every bounded subset of S.*

COROLLARY 2b. *For f to have a continuous extension to $\overline{S}_2$ it is necessary and sufficient that f be continuously differentiable on S. If f and f' are bounded on every bounded subset of S, then $N_2(f; S, R) < +\infty$ for all $R > 0$.*

3. Case $n=3$. In the same way, we find that if

$$A = \begin{pmatrix} \lambda & a & b \\ 0 & \mu & c \\ 0 & 0 & \nu \end{pmatrix},$$

where λ, μ, and ν are distinct elements of S, then

$$f(A) = \begin{pmatrix} f(\lambda) & a[\lambda, \mu] & b[\lambda, \nu] + ac[\lambda, \mu, \nu] \\ 0 & f(\mu) & c[\mu, \nu] \\ 0 & 0 & f(\nu) \end{pmatrix}.$$

We note that the second-order divided difference is defined by

$$[\lambda, \mu, \nu] = \frac{[\lambda, \mu] - [\lambda, \nu]}{\mu - \nu}.$$

If we define B and C analogously to (2) and (3) above, then we obtain

$$\begin{aligned} |f(\lambda)| &\leqq C\|f(A)\|, \\ |a||[\lambda, \mu]| &\leqq C\|f(A)\|, \\ |b[\lambda, \nu] + ac[\lambda, \mu, \nu]| &\leqq C\|f(A)\|. \end{aligned}$$

If $|\lambda|, |\mu|, |\nu|, |a|, |b|, |c| \leqq R/(6B)$, then $\|A\| \leqq R$, so that

$$|f(\lambda)| \leqq CN_3(f; S, R).$$

If we set $a=c=R/(4B)$ and $b=0$, we obtain

$$|[\lambda, \mu]| \leqq 4BCN_3(f; S, R)/R, \quad \text{and} \quad |[\lambda, \mu, \nu]| \leqq 16B^2CN_3(f; S, R)/R^2.$$

Let us define $M_2(f; S, r) = \sup (|f(\lambda)| + |[\lambda, \mu]| + |[\lambda, \mu, \nu]|)$ for λ, μ, and ν distinct elements of S and $|\lambda|, |\mu|, |\nu| \leqq r$.

THEOREM 3. *If $N_3(f; S, R) < \infty$ for all $R>0$, then f is continuously differentiable and f' is Lipschitzian on every bounded subset of S, and*

$$M_2(f; S, r) \leqq \left(1 + \frac{2}{3r} + \frac{4}{9r^2}\right) CN_3(z; S, 6Br) \tag{5}$$

for $r>0$.

If the second-order divided differences are bounded, then $[\lambda, \nu]$ is continuous in λ, and $f'(\lambda) = \lim_{\mu \to \lambda} [\lambda, \mu]$ exists for $\lambda \in S$.

If, in the identity $[\lambda, \mu] - [\nu, \sigma] = [\lambda, \mu] - [\nu, \mu] + [\nu, \mu] - [\nu, \sigma]$, we let $\mu \to \lambda$ and $\nu \to \sigma$, we obtain

$$f'(\lambda) - f'(\sigma) = f'(\lambda) - [\sigma, \lambda] + [\sigma, \lambda] - f'(\sigma).$$

If $M_2(f; S, r) < +\infty$ and $|\lambda|, |\mu|, |\nu|, |\sigma| \leqq r$, then we infer that

$$|[\lambda, \mu] - [\nu, \sigma]| \leqq (|\lambda - \nu| + |\mu - \sigma|) M_2(f; S, r)$$

and

$$|f'(\lambda) - f'(\sigma)| \leqq 2|\lambda - \sigma| M_2(f; S, r).$$

In order to prove the converse of Theorem 3, we first note that there is an orthonormal basis x_1, x_2, x_3 (mutually orthogonal unit vectors) with respect to which A takes on the above triangular form:

$$Ax_1 = \lambda x_1$$
$$Ax_2 = ax_1 + \mu x_2,$$
$$Ax_3 = bx_1 + cx_2 + \nu x_3.$$

Indeed, if $\det(A - \lambda I) = 0$, then there is a unit vector x_1 such that $Ax_1 = \lambda x_1$. Let $A_1 x = Ax - (Ax, x_1)x_1$, where (x, y) is the scalar product of x and y. Then A_1 is a 2×2 matrix representing a linear transformation of the 2-dimensional space of vectors orthogonal to x_1 into itself. If an orthonormal base is chosen for this subspace so that A_1 has triangular form, then we obtain the desired form for A.

If we again use the Euclidean norms for vectors and matrices, we obtain

$$s/\sqrt{3} \leqq \|A\| \leqq s, \qquad |\lambda| \leqq \|A\|, \qquad |\mu| \leqq \|A\|, \qquad |\nu| \leqq \|A\|,$$

where $s^2 = |\lambda|^2 + |\mu|^2 + |\nu|^2 + |a|^2 + |b|^2 + |c|^2$.

Suppose that $M_2(f; S, r) < +\infty$ for all $r > 0$, that $A \in S_3$, and that $\|A\| \leqq R$, where $R > 0$. Then

$$\|f(A)\|^2 \leqq M_2(f; S, R)^2(3 + |a|^2 + |c|^2 + (|b| + |ac|)^2)$$
$$\leqq M_2(f; S, R)^2(3 + s^2 + 2s^3 + s^4),$$

so that $\|f(A) \leqq M_2(f; S, R)(3 + 6R^2 + 18R^4)^{1/2}$.

THEOREM 4. *If $M_2(f; S, R) < +\infty$, then so is $N_3(f; S, R)$, and $N_3(f; S, R) \leqq (3 + 6R^2 + 18R^4)^{1/2} M_2(f; S, R)$.*

COROLLARY 4a. *For the extension of f to S_3 to be bounded on every bounded subset of S_3, it is necessary for f and its divided differences of order $\leqq 2$ to be bounded on every bounded subset of S.*

COROLLARY 4b. *If $N_3(f; S, R) < +_1\infty$ for all $R > 0$, then f can be extended to a continuous function on $\overline{S}_2$. If S is a domain in the complex plane and $N_3(f; S, R) < +\infty$ for all $R > 0$, then f is analytic in S.*

4. Any $n > 1$. The same methods can be applied to any $n > 1$. If $x_1, \cdots, x_n$ is a set of linearly independent vectors in n-space, and

$$Ax_k = \sum_{j<\kappa} a_{jk} x_j + \lambda_\kappa x_k, \qquad 1 \leqq k \leqq n, \tag{6}$$

where $\lambda_k \in S (1 \leqq k \leqq n)$, and the λ's are distinct, then

$$f(A)x_k = \sum_{j<k} b_{jk}x_k + f(\lambda_k)x_k, \qquad 1 \leqq k \leqq n, \tag{7}$$

where for $1 \leqq k < j$

$$b_{jk} = \sum a_{r_1r_2}a_{r_2r_3} \cdots a_{r_{m-1}r_m}[\lambda_{r_1}, \lambda_{r_2}, \cdots, \lambda_{r_m}],$$

the sum being taken over all sequences $(r_1, \cdots, r_m)$ such that

$$j = r_1 < r_2 < \cdots < r_{m-1} < r_m = k.$$

The nth order divided difference may be defined by

$$[\lambda_1, \cdots, \lambda_n] = \frac{[\lambda_1, \cdots, \lambda_{n-2}, \lambda_{n-1}] - [\lambda_1, \cdots, \lambda_{n-2}\lambda_n]}{\lambda_{n-1} - \lambda_n}.$$

Equation (7) can be proved by induction. If B and C are defined as before, then

$$\|A\| \leqq B(\sum |a_{jk}| + \sum |\lambda_k|) \leqq R$$

if $|a_{jk}| \leqq 2R/n(n+1)B = r$, $|\lambda_k| \leqq r$, for $1 \leqq j < k \leqq n$. Then

$$|f(\lambda_k)| \leqq C\|f(A)\|, \qquad |b_{jk}| \leqq C\|f(A)\|,$$

for $1 \leqq j < k \leqq n$. Set $a_{pq} = r$ if $q = p+1$, $1 \leqq p < n$, and $a_{pq} = 0$ otherwise. Then $b_{1m} = r^{m-1}[\lambda_1, \cdots, \lambda_m]$, so that we infer $|[\lambda_1, \cdots, \lambda_m]| \leqq C\|f(A)\|/r^{m-1}$ if $|\lambda_k| \leqq r$ for $1 \leqq k \leqq m$.

If we define

$$M_{n-1}(f; S, r) = \sup (|f(\lambda_1)| + |[\lambda_1, \lambda_2]| + \cdots + |[\lambda_1, \cdots, \lambda_n]|)$$

for $\lambda_1, \cdots, \lambda_n$ distinct elements of S and $|\lambda_k| \leqq r$ for $1 \leqq k \leqq n$, then

$$M_{n-1}(f; S, r) \leqq \left(\sum_0^{n-1} r^{-k}\right) CN_n(f; S, Bn(n+1)r/2).$$

We also prove by induction that an orthonormal base $x_1, \cdots, x_n$ of vectors can be chosen so that A satisfies (6). Then $f(A)$ is given by (7).

Now we have

$$|a_{jk}| = |(x_j, Ax_k)| \leqq \|x_j\|\|Ax_k\| \leqq \|A\|,$$
$$|\lambda_k| = |(x_k, Ax_k)| \leqq \|A\|.$$

Therefore if $\|A\| \leqq R$, then $|b_{jk}| \leqq (1+R^{n-1})M_{n-1}(f; S, R)$, and it follows that

$$\|f(A)\| \leqq n(1 + R^{n-1})M_{n-1}(f; S, R).$$

THEOREM 5. *The function f can be extended to a bounded function on every bounded part of $S_n (n \geqq 3)$ if and only if f and all divided differences of order $\leqq n$ are bounded on every bounded part of S.*

Then f is of class C^{n-2} on S and $f^{(n-2)}$ is Lipschitzian on every bounded part of S.

If S is convex and $f^{(k)}$ exists and is bounded for $1 \leqq k \leqq n-2$ and $f^{(n-2)}$ is Lipschitzian on every bounded part of S, then f can be extended to a continuous function on every bounded part of $\overline{S}_n$.

A MIXED DOUBLES TOURNAMENT PROBLEM

C. C. YALAVIGI, Government College, Mercara, India

Introduction. This paper concerns the design of mixed tournaments which are open to an equal number of men and women. In each match of a tournament two teams each containing 2^s men players and 2^s women players (where s denotes an integer $\geqq 0$) compete with one another. The available literature on designing mixed tournaments (see E. N. Gilbert [2]) shows that the added requirement of a team to contain 2^s men and 2^s women (for $s \neq 0$) does not seem to have been considered.

For scoring purposes ordinary tournaments in which each match consists of 2^{s+1} players (where $s \neq 0$) are designed for a high degree of symmetry e.g. each player appears with every other player 2^s-1 times as a teammate and 2^s times as an opponent. In a mixed tournament these symmetries hold separately for men players and women players. Moreover, if men players are denoted by 1, 2, $\cdots$, v and women players by $1^{(a)}$, $2^{(a)}$, $\cdots$, $v^{(a)}$, players of the same number never meet. To be more precise, a high degree constraint of a mixed tournament will be

I. (a) each player shall appear with every other player of the same sex exactly 2^s-1 times as a teammate and 2^s times as an opponent.
 (b) each player shall appear with every player of the opposite sex (except the player with the same number) exactly 2^s times as a teammate and 2^s times as an opponent.

In the design of a tournament subject to I, the total number of matches b is given by

$$b = v(v-1)/(2^{s+1}). \tag{1.1}$$

To prove (1.1) it is sufficient to note that the total number of appearances of men players is 2^{s+1} in a single match and $v(v-1)$ in all the matches.

Clearly the equation (1.1) restricts the number of players admissible to a tournament and designs are possible only for some forms of v.

2. Design for I with v a power of an odd prime in the form $4t+1$ and $s=0$. Let $v=4t+1=p^n$ where p denotes an odd prime and n an integer $\geqq 1$. In

constructing the designs of this section we will use a technique involving incomplete balanced block designs which is due to R. C. Bose [1].

We identify the players with the elements of GF $(4t+1)$. Construct a set of $2t$ generators

$$G_j = (A_j, B_j^{(a)}; X_j, Y_j^{(a)}) \qquad j = 0, 1, \cdots, 2t-1, \tag{2.1}$$

where A_j, X_j, denote men players, $B_j^{(a)}$, $Y_j^{(a)}$, denote women players, and $A_jB_j^{(a)}$ and $X_jY_j^{(a)}$, form two teams which play against one another.

From these generators, we may derive all the matches of a tournament, i.e.,

$$G_j(\alpha) = (A_j+\alpha, B_j^{(a)}+\alpha; X_j+\alpha, Y_j^{(a)}+\alpha) \qquad j = 0, 1, \cdots, 2t-1, \tag{2.2}$$

where α runs through all elements of GF$(4t+1)$.

These matches will not satisfy I unless certain conditions are fulfilled by four sets of numbers called the MM, WW, MW_t and MW_0 differences. They are

MM differences $= A_j - X_j$
WW differences $= B_j^{(a)} - Y_j^{(a)}$
MW_t differences $= A_j - B_j^{(a)}$ and $X_j - Y_j^{(a)}$
MW_0 differences $= Y_j^{(a)} - A_j$ and $B_j^{(a)} - X_j$.

These differences must satisfy the hypothesis of the following theorem.

THEOREM 1. *A mixed doubles tournament of this section when generated by a set of $2t$ generators $G_0, G_1 \cdots, G_{2t-1}$, in the manner prescribed above, satisfies* I *if and only if:*

(i) *the MM_t differences are in $2t$ distinct sets of 2^s-1 equal differences except $s=0$ and 1 where no two differences are equal for $s=1$,*
(ii) *the WW_t differences are in $2t$ distinct sets of 2^s-1 equal differences except $s=0$ and 1 where no two differences are equal for $s=1$,*
(iii) *no two of the $4t$ MW_t differences are equal,*
(iv) *no two of the $4t$ MW_0 differences are equal,*
(v) *none of the MM, WW, MW_t and MW_0 differences is zero.*

The proof is left to the reader. We proceed to construct a system of generators satisfying the conditions of Theorem 1.

Let

$$G_j = (x^{0+j}, x^{3t+j}; x^{2t+j}, x^{t+j}) \qquad j = 0, 1, \cdots, 2t-1, \tag{2.3}$$

where x denotes a primitive root in GF$(4t+1)$.

We have the following differences.

$$\begin{aligned} &A_j - B_j^{(a)} = x^{0+j} - x^{3t+j}, \quad A_j - X_j = x^{0+j} - x^{2t+j}, \quad X_j - Y_j^{(a)} = x^{2t+j} - x^{t+j}, \\ &B_j^{(a)} - Y_j^{(a)} = x^{3t+j} - x^{t+j}, \quad Y_j^{(a)} - A_j = x^{t+j} - x^{0+j}, \quad B_j^{(a)} - X_j = x^{3t+j} - x^{2t+j}. \end{aligned} \tag{2.4}$$

These differences must satisfy

$$
\begin{aligned}
&\text{(i)}' && x^{0+i}-x^{2t+i}\neq x^{0+j}-x^{2t+j} && \text{for } i\neq j\\
&\text{(ii)}' && x^{3t+i}-x^{t+i}\neq x^{3t+j}-x^{t+j} && \text{for } i\neq j\\
&\text{(iii)}'\text{A}. && x^{0+i}-x^{3t+i}\neq x^{0+j}-x^{3t+j},\quad x^{2t+i}-x^{t+i}\neq x^{2t+j}-x^{t+j}, && \text{for } i\neq j\\
&\text{(2.5)} \qquad \text{B}. && x^{0+i}-x^{3t+i}\neq x^{2t+j}-x^{t+j} && \text{for } i=j \text{ and } i\neq j\\
&\text{(iv)}'\text{A}. && x^{t+i}-x^{0+i}\neq x^{t+j}-x^{0+j},\ x^{3t+i}-x^{2t+i}\neq x^{3t+j}-x^{2t+j}, && \text{for } i\neq j\\
&\qquad\quad \text{B}. && x^{t+i}-x^{0+i}\neq x^{3t+j}-x^{2t+j} && \text{for } i=j \text{ and } i\neq j\\
&\text{(v)}' && x^{0+j}-x^{3t+j}\neq 0,\quad x^{0+j}-x^{2t+j}\neq 0,\quad x^{2t+j}-x^{t+j}\neq 0,\\
& && x^{3t+j}-x^{t+j}\neq 0,\quad x^{t+j}-x^{0+j}\neq 0,\quad x^{3t+j}-x^{2t+j}\neq 0,
\end{aligned}
$$

where $i=0, 1, \cdots, 2t-1$.

The proof of these inequalities is as follows.

Let the inequalities (i)', (ii)', (iii)'A and (iv)'A be expressed as

$$Ax^i \neq Ax^j, \qquad i \neq j, \tag{2.6}$$

where A denotes a related constant.

The proof of (2.6) requires $x^{i-j}\neq 1$ for $i\neq j$. This is true because $i-j\not\equiv 0 \bmod(4t)$.

Consider the inequalities (iii)'B and (iv)'B. It is easy to rewrite (iii)'B as

$$(x^{4t+i} - x^{3t+i}) \neq (x^{2t+j} - x^{t+j}). \tag{2.7}$$

This is true since $x^{2t}=-1$ and $x^{i-j}\neq -1$. The proof of (iv)'B is similar.

For the proof of (v)', note that $x, x^2, x^3, \cdots, x^{4t}=1$ are all different.

Hence (2.3) satisfies the conditions of Theorem 1. As illustrations, three designs follow.

Set of generators of mixed doubles tournaments for $v=4t+1=p$

v Generators

13 $(1, 5^{(a)}; 12, 8^{(a)})$, $(2, 10^{(a)}; 11, 3^{(a)})$, $(4, 7^{(a)}; 9, 6^{(a)})$, $(8, 1^{(a)}; 5, 12^{(a)})$, $(3, 2^{(a)}; 10, 11^{(a)})$, $(6, 4^{(a)}; 7, 9^{(a)})$

17 $(1, 3^{(a)}; 16, 14^{(a)})$, $(3, 9^{(a)}; 14, 8^{(a)})$, $(9, 10^{(a)}; 8, 7^{(a)})$, $(10, 13^{(a)}; 7, 4^{(a)})$, $(13, 5^{(a)}; 4, 12^{(a)})$, $(5, 15^{(a)}; 12, 2^{(a)})$, $(15, 11^{(a)}; 2, 6^{(a)})$, $(11, 16^{(a)}; 6, 1^{(a)})$.

For $v=3^2$ we represent GF(9) by the residues mod $(3, x^2+x+2)$. The set of generators is therefore

$(1, (x+2)^{(a)}; 2, (2x+1)^{(a)})$, $(x, (x+1)^{(a)}; 2x, (2x+2)^{(a)})$, $(2x+1, 1^{(a)}; x+2, 2^{(a)})$, $(2x+2, x^{(a)}; x+1, (2x)^{(a)})$.

This set of generators is developed by adding x, $2x+1$, $2x+2$, 2, $2x$, $x+2$, $x+1$, 1.

We may note that the designs of this section are also produced by another set of generators, viz.,

$$G_j = (x^{0+j}, x^{1+j}; x^{2t+j}, x^{2t+1+j}) \qquad j = 0, 1, \cdots, 2t-1.$$

This set of generators satisfies Theorem 1. The proof is left to the reader.

3. Design for I with v a power of an odd prime in the form $4t-1$ and $s=0$.

This section will develop designs of mixed doubles tournaments for $v=4t-1=p^n$. These designs will use the method of Section 2.

We give the following theorem.

THEOREM 2. *A mixed doubles tournament of this section when generated by a set of* $2t-1$ *generators* $G_0, G_1 \cdots, G_{2t-2}$ *in the manner prescribed as in Section* 2 *satisfies* I *if and only if:*

(i) *no two of the* $2t-1$ MM *differences are equal,*
(ii) *no two of the* $2t-1$ WW *differences are equal,*
(iii) *no two of the* $4t-2$ MW_t *differences are equal,*
(iv) *no two of the* $4t-2$ MW_0 *differences are equal,*
(v) *none of the* MM, WW, MW_t *and* MW_0 *differences is zero.*

The proof is left to the reader. A system of $2t-1$ generators satisfying the conditions of Theorem 2 may be represented as

$$G_j = (x^{0+j}, x^{1+j}; x^{2t-1+j}, x^{2t+j}) \qquad j = 0, 1, \cdots, 2t-2 \tag{3.1}$$

where x denotes a primitive root in GF$(4t-1)$.

For these generators the differences are

$$\begin{aligned} &A_j-B_j^{(a)}=x^{0+j}-x^{1+j}, \quad X_j-Y_j^{(a)}=x^{2t-1+j}-x^{2t+j}, \quad A_j-X_j=x^{0+j}-x^{2t-1+j}, \\ &B_j^{(a)}-Y_j^{(a)}=x^{1+j}-x^{2t+j}, \quad B_j^{(a)}-X_j=x^{1+j}-x^{2t-1+j}, \quad Y_j^{(a)}-A_j=x^{2t+j}-x^{0+j}. \end{aligned} \tag{3.2}$$

These differences must satisfy

$$\begin{aligned}
&\text{(i)}' \quad x^{0+i}-x^{2t-1+i}\neq x^{0+j}-x^{2t-1+j}, \quad \text{for } i\neq j \\
&\text{(ii)}' \quad x^{1+i}-x^{2t+i}\neq x^{1+j}-x^{2t+j}, \quad \text{for } i\neq j \\
&\text{(iii)}'\text{A.} \quad x^{0+i}-x^{1+i}\neq x^{0+j}-x^{1+j}, \quad x^{2t-1+i}-x^{2t+i}\neq x^{2t-1+j}-x^{2t+j}, \quad \text{for } i\neq j \\
&\quad\text{B.} \quad x^{0+i}-x^{1+i}\neq x^{2t-1+j}-x^{2t+j}, \quad \text{for } i=j \text{ and } i\neq j \\
&\text{(iv)}'\text{A.} \quad x^{1+i}-x^{2t-1+i}\neq x^{1+j}-x^{2t-1+j}, \quad x^{2t+i}-x^{0+i}\neq x^{2t+j}-x^{0+j}, \quad \text{for } i\neq j \\
&\quad\text{B.} \quad x^{1+i}-x^{2t-1+i}\neq x^{2t+j}-x^{0+j}, \quad \text{for } i=j \text{ and } i\neq j \\
&\text{(v)}' \quad x^{0+j}-x^{1+j}\neq 0, \quad x^{2t-1+j}-x^{2t+j}\neq 0, \quad x^{0+j}-x^{2t-1+j}\neq 0, \quad x^{1+j}-x^{2t+j}\neq 0, \\
&\qquad x^{1+j}-x^{2t-1+j}\neq 0, \quad x^{0+j}-x^{2t+j}\neq 0,
\end{aligned} \tag{3.3}$$

where $i=0, 1, \cdots, 2t-2$.

The proof of these inequalities is as follows: We express (i)', (ii)', (iii)'A and (iv)'A in the form $Ax^i\neq Ax^j$, $i\neq j$, where A denotes a related constant.

This inequality holds true if $x^{i-j}\neq 1$ for $i\neq j$ which is easily verifiable.

The inequalities (iii)'B and (iv)'B are similar. We may obtain (iii)'B from $x^{i-j}\cdot x^{2t-1}\neq 1$ which is true since $x^{i-j}\neq -1$ and $x^{2t-1}=-1$. The proof of (iv)'B is left to the reader.

For proving (v)', we note that $x, x^2, \cdots, x^{4t-2}=1$ are all different.

Hence the considered set of generators satisfies the conditions of Theorem 2. We give the following illustrations.

Set of generators of mixed doubles tournaments for $v=4t-1=p$

v	Generators
7	$(1, 3^{(a)}; 6, 4^{(a)}), (3, 2^{(a)}; 4, 5^{(a)}), (2, 6^{(a)}; 5, 1^{(a)})$
11	$(1, 2^{(a)}; 10, 9^{(a)}), (2, 4^{(a)}; 9, 7^{(a)}), (4, 8^{(a)}; 7, 3^{(a)}),$ $(8, 5^{(a)}; 3, 6^{(a)}), (5, 10^{(a)}; 6, 1^{(a)}).$

The designs of Sections 2 and 3 may also use the sets of generators for $v=6t\pm1$. For $v=6t+1$ they are

$$G_j = (x^{0+j}, x^{1+j}; x^{3t+j}, x^{3t+1+j}) \qquad j = 0, 1, \cdots, 3t-1,$$

and for $v=6t-1$ they are

$$G_j = (x^{0+j}, x^{1+j}; x^{3t-1+j}, x^{3t+j}) \qquad j = 0, 1, \cdots, 3t-2.$$

The proofs for these sets of generators are similar to those described in Sections 2 and 3.

This alternate method gives all the solutions of Sections 2 and 3 except $v=3^n$.

4. Design for I with v a power of an odd prime in the form $4t+1$ and $s=1$.

This section will develop designs of mixed tournaments for $v=4t+1=p^n$ where p denotes an odd prime and n an integer $\geqq 1$. In each match four men players and four women players participate.

In the design of these tournaments, a generator is an eight-tuple $G'_{j'}=(A_{j'}, B_{j'}^{(a)}, C_{j'}, D_{j'}^{(a)}; X_{j'}, Y_{j'}^{(a)}, Z_{j'}, W_{j'}^{(a)})$ which is a suitable combination of two quadruple generators. Clearly a set of t generators specifies all the matches of a tournament. It therefore remains to characterize the set of generators which will produce the design of this section.

We associate with each generator six sets of numbers called the MM_t, MM_0, WW_t, WW_0, MW_t and MW_0 differences as follows:

MM_t differences $=A_{j'}-C_{j'}$ and $X_{j'}-Z_{j'}$

MM_0 differences $=A_{j'}-X_{j'}$, $A_{j'}-Z_{j'}$, $X_{j'}-C_{j'}$ and $Z_{j'}-C_{j'}$

WW_t differences $=B_{j'}^{(a)}-D_{j'}^{(a)}$ and $Y_{j'}^{(a)}-W_{j'}^{(a)}$

WW_0 differences $=B_{j'}^{(a)}-Y_{j'}^{(a)}$, $B_{j'}^{(a)}-W_{j'}^{(a)}$, $Y_{j'}^{(a)}-D_{j'}^{(a)}$ and $W_{j'}^{(a)}-D_{j'}^{(a)}$

MW_t differences $=A_{j'}-B_{j'}^{(a)}$, $D_{j'}^{(a)}-C_{j'}$, $X_{j'}-Y_{j'}^{(a)}$, $W_{j'}^{(a)}-Z_{j'}$, $A_{j'}-D_{j'}^{(a)}$, $B_{j'}^{(a)}-C_{j'}$, $X_{j'}-W_{j'}^{(a)}$ and $Y_{j'}^{(a)}-Z_{j'}$

MW_0 differences $=A_{j'}-Y_{j'}^{(a)}$, $A_{j'}-W_{j'}^{(a)}$, $Y_{j'}^{(a)}-C_{j'}$, $W_{j'}^{(a)}-C_{j'}$, $B_{j'}^{(a)}-Z_{j'}$, $X_{j'}-D_{j'}^{(a)}$, $B_{j'}^{(a)}-X_{j'}$ and $Z_{j'}-D_{j'}^{(a)}$.

These differences must satisfy the hypothesis of the following theorem.

THEOREM 3. *A mixed tournament of this section when generated by a set of t generators $G'_0, G'_1, \cdots, G'_{t-1}$ in the manner prescribed above satisfies* I, *if and only if:*

(i) *no two of the* $2t$ MM_t *differences are equal,*

(ii) *no two of the* $2t$ WW_t *differences are equal,*

(iii) *the MW_t differences are equal in $4t$ distinct pairs,*
(iv) *the MW_0 differences are equal in $4t$ distinct pairs,*
(v) *the MM_0 differences are equal in $2t$ distinct pairs,*
(vi) *the WW_0 differences are equal in $2t$ distinct pairs,*
(vii) *none of the MM_t, MM_0, WW_t, WW_0, MW_t and MW_0 differences is zero.*

The proof is left to the reader. We proceed to construct a system of generators satisfying Theorem 3. Let

$$G_{j'}' = (x^{0+j'}, x^{1+j'}, x^{2t+j'}, x^{2t+1+j'}; x^{t+j'}, x^{t+1+j'}, x^{3t+j'}, x^{3t+1+j'}), \tag{4.1}$$

where $j'=0, 1, \cdots, t-1$ and x denotes a primitive root of GF$(4t+1)$. Clearly

$$\begin{aligned}
&A_{j'} - X_{j'} = x^{0+j'} - x^{t+j'}, && A_{j'} - Y_{j'}^{(a)} = x^{0+j'} - x^{t+1+j'},\\
&A_{j'} - Z_{j'} = x^{0+j'} - x^{3t+j'}, && A_{j'} - W_{j'}^{(a)} = x^{0+j'} - x^{3t+1+j'},\\
&B_{j'}^{(a)} - X_{j'} = x^{1+j'} - x^{t+j'}, && B_{j'}^{(a)} - Y_{j'}^{(a)} = x^{1+j'} - x^{t+1+j'},\\
&B_{j'}^{(a)} - Z_{j'} = x^{1+j'} - x^{3t+j'}, && B_{j'}^{(a)} - W_{j'}^{(a)} = x^{1+j'} - x^{3t+1+j'},\\
&X_{j'} - C_{j'} = x^{t+j'} - x^{2t+j'}, && Y_{j'}^{(a)} - C_{j'} = x^{t+1+j'} - x^{2t+j'},\\
&Z_{j'} - C_{j'} = x^{3t+j'} - x^{2t+j'}, && W_{j'}^{(a)} - C_{j'} = x^{3t+1+j'} - x^{2t+j'},\\
&X_{j'} - D_{j'}^{(a)} = x^{t+j'} - x^{2t+1+j'}, && Y_{j'}^{(a)} - D_{j'}^{(a)} = x^{t+1+j'} - x^{2t+1+j'},\\
&Z_{j'} - D_{j'}^{(a)} = x^{3t+j'} - x^{2t+1+j'}, && W_{j'}^{(a)} - D_{j'}^{(a)} = x^{3t+1+j'} - x^{2t+1+j'},\\
&X_{j'} - Y_{j'}^{(a)} = x^{t+j'} - x^{t+1+j'}, && X_{j'} - Z_{j'} = x^{t+j'} - x^{3t+j'},\\
&X_{j'} - W_{j'}^{(a)} = x^{t+j'} - x^{3t+1+j'}, && Y_{j'}^{(a)} - Z_{j'} = x^{t+1+j'} - x^{3t+j'},\\
&Y_{j'}^{(a)} - W_{j'}^{(a)} = x^{t+1+j'} - x^{3t+1+j'}, && W_{j'}^{(a)} - Z_{j'} = x^{3t+1+j'} - x^{3t+j'},\\
&A_{j'} - B_{j'}^{(a)} = x^{0+j'} - x^{1+j'}, && A_{j'} - C_{j'} = x^{0+j'} - x^{2t+j'},\\
&A_{j'} - D_{j'}^{(a)} = x^{0+j'} - x^{2t+1+j'}, && B_{j'}^{(a)} - C_{j'} = x^{0+j'} - x^{2t+j'},\\
&B_{j'}^{(a)} - D_{j'}^{(a)} = x^{1+j'} - x^{2t+1+j'}, && D_{j'}^{(a)} - C_{j'} = x^{2t+1+j'} - x^{2t+j'}.
\end{aligned} \tag{4.2}$$

The conditions of Theorem 3 therefore imply that

$$\begin{aligned}
&\text{(i)'a.}\quad x^{0+i'} - x^{2t+i'} \neq x^{0+j'} - x^{2t+j'}, && x^{t+i'} - x^{3t+i'} \neq x^{t+j'} - x^{3t+j'},\\
&\text{b.}\quad x^{0+i'} - x^{t+i'} \neq x^{0+j'} - x^{t+j'}, && x^{0+i'} - x^{3t+i'} \neq x^{0+j'} - x^{3t+j'},\\
&\quad x^{2t+i'} - x^{t+i'} \neq x^{2t+j'} - x^{t+j'}, && x^{2t+i'} - x^{3t+i'} \neq x^{2t+j'} - x^{3t+j'},\\
&\text{c.}\quad x^{1+i'} - x^{2t+1+i'} \neq x^{1+j'} - x^{2t+1+j'}, && x^{t+1+i'} - x^{3t+1+i'} \neq x^{t+1+j'} - x^{3t+1+j'},\\
&\text{d.}\quad x^{1+i'} - x^{t+1+i'} \neq x^{1+j'} - x^{t+1+j'}, && x^{1+i'} - x^{3t+1+i'} \neq x^{1+j'} - x^{3t+1+j'},\\
&\quad x^{2t+1+i'} - x^{t+1+i'} \neq x^{2t+1+j'} - x^{t+1+j'},\\
&\quad x^{2t+1+i'} - x^{3t+1+i'} \neq x^{2t+1+j'} - x^{3t+1+j'},\\
&\quad \text{for } i' \neq j' \text{ and } i' = 0, 1, \cdots, t-1
\end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&\text{(ii)}' \quad x^{0+j'}-x^{1+j'}=x^{2t+1+j'}-x^{2t+j'}, && x^{0+j'}-x^{2t+1+j'}=x^{1+j'}-x^{2t+j'},\\
&\qquad x^{t+j'}-x^{t+1+j'}=x^{3t+1+j'}-x^{3t+j'}, && x^{t+j'}-x^{3t+1+j'}=x^{t+1+j'}-x^{3t+j'},\\
&\text{(iii)}' \quad x^{0+j'}-x^{t+1+j'}=x^{3t+1+j'}-x^{2t+j'}, && x^{0+j'}-x^{3t+1+j'}=x^{t+1+j'}-x^{2t+j'},\\
(4.3)\qquad & x^{1+j'}-x^{3t+j'}=x^{t+j'}-x^{2t+1+j'}, && x^{1+j'}-x^{t+j'}=x^{3t+j'}-x^{2t+1+j'},\\
&\text{(iv)}' \quad x^{0+j'}-x^{t+j'}=x^{3t+j'}-x^{2t+j'}, && x^{0+j'}-x^{3t+j'}=x^{t+j'}-x^{2t+j'},\\
&\text{(v)}' \quad x^{1+j'}-x^{t+1+j'}=x^{3t+1+j'}-x^{2t+1+j'}, && x^{1+j'}-x^{3t+1+j'}=x^{t+1+j'}-x^{2t+1+j'},
\end{aligned}
$$

(vi)′ none of the differences (4.2) is zero.

The proof of (4.3) is as follows: Let the inequalities (i)′ be of the form

$$A'x^{i'} \neq A'x^{j'}, \qquad i' = j' \tag{4.4}$$

where A' denotes a related constant.

The proof of this inequality requires $x^{i'-j'} \neq \pm 1$ for $i' \neq j'$. This is true since $i'-j' \not\equiv 0 \bmod(2t)$.

For the proof of the equations (ii)′, (iii)′, (iv)′ and (v)′, we may use $x^0 = x^{4t} = 1$, $x^{2t} = -1$. The proof is left to the reader.

The inequalities (vi)′ are true because $x, x^2, \cdots, x^{4t}=1$ are all different.

Hence (4.1) satisfies the conditions of Theorem 3. As illustrations two designs follow.

Set of generators of mixed tournaments for $v = 4t+1 = p$ and $s = 1$

v Generators

13 $(1, 2^{(a)}, 12, 11^{(a)}; 8, 3^{(a)}, 5, 10^{(a)})$, $(2, 4^{(a)}, 11, 9^{(a)}; 3, 6^{(a)}, 10, 7^{(a)})$, $(4, 8^{(a)}, 9, 5^{(a)}; 6, 12^{(a)}, 7, 1^{(a)})$

17 $(1, 3^{(a)}, 16, 14^{(a)}; 13, 5^{(a)}, 4, 12^{(a)})$, $(3, 9^{(a)}, 14, 8^{(a)}; 5, 15^{(a)}, 12, 2^{(a)})$, $(9, 10^{(a)}, 8, 7^{(a)}; 15, 11^{(a)}, 2, 6^{(a)})$, $(10, 13^{(a)}, 7, 4^{(a)}; 11, 16^{(a)}, 6, 1^{(a)})$.

An alternate development of the designs of this section is possible by sets of generators for v in the forms $6t \pm 1$. The generators of these forms of v are

$$G'_{j'} = (x^{j'+0}, x^{j'+1}, x^{j'+3t}, x^{j'+3t+1}; x^{j'+(3t/2)}, x^{j'+(3t/2)+1}, x^{j'+(9t/2)}, x^{j'+(9t/2)+1})$$
$$j' = 0, 1, \cdots, (3t/2) - 1, \quad \text{for } v = 6t + 1$$

$$G'_{j'} = (x^{j'+0}, x^{j'+1}, x^{j'+3t-1}, x^{j'+3t}; x^{j'+(3t-1)/2}, x^{j'+(3t+1)/2}, x^{j'+(9t-3)/2}, x^{j'+(9t-1)/2})$$
$$j' = 0, 1, \cdots, (3t - 3)/2, \quad \text{for } v = 6t - 1.$$

The proof that these generators satisfy the conditions of Theorem 3 is similar to the proof for the generators of this section. This alternate method gives all the solutions of Section 4 except $v = 3^n$.

5. In this section the theorems of Sections 2 and 4 will be generalized as follows.

GENERALIZED THEOREM 1. *A mixed tournament for $v = 4t+1 = p^n$ and s an integer $\geqq 0$ (where p denotes an odd prime and n an integer $\geqq 1$), when generated by*

a set of $(2t/2^{s+1})$ *generators* $G_0, G_1, \cdots, G_{(2t/2^{s+1})-1}$, *in the manner prescribed either in Section* 2 *or* 4, *satisfies* I *if and only if:*

(i) *no two of the* $2^s \cdot t$ MM_t *differences (except* $s=0$) *are equal,*

(ii) *no two of the* $2^s \cdot t$ WW_t *differences (except* $s=0$) *are equal,*

(iii) *the* MW_t *differences are in* $4t$ *distinct sets of* 2^s *equal differences except* $s=0$ *where no two differences are equal,*

(iv) *the* MW_0 *differences are in* $4t$ *distinct sets of* 2^s *equal differences except* $s=0$ *where no two differences are equal,*

(v) *the* MM_0 *differences are in* $2t$ *distinct sets of* 2^s *equal differences except* $s=0$ *where no two differences are equal,*

(vi) *the* WW_0 *differences are in* $2t$ *distinct sets of* 2^s *equal differences except* $s=0$ *where no two differences are equal,*

(vii) *none of the* MM_t, MM_0, WW_t, WW_0, MW_t *and* MW_0 *differences is zero.*

REMARK. The theorems of Sections 2 and 4 can also be generalized in a similar manner for $v=6t\pm 1=p^n$ and s any integer $\geqq 1$.

Acknowledgement. I am grateful to the referee for his helpful suggestions in improving the earlier version of this paper.

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PRIME SEMIGROUPS AND SEMIRINGS

AZRIEL ROSENFELD, University of Maryland

1. Introduction. A group has no proper subgroups if and only if it is trivial or cyclic of prime order, i.e. isomorphic to the additive group of integers modulo 1 or modulo a prime p. We shall refer to such a group as a *prime group*. A field has no proper subfields if and only if it is isomorphic to the field of rational numbers or to the field of integers modulo a prime p; these fields are called the *prime fields*. The purpose of this note is to generalize these well-known results by characterizing the semigroups and semirings which have no proper subsemigroups or subsemirings.

2. Prime semigroups. We call a semigroup (=a set closed under an associative composition) *prime* if it has only the empty set, trivial (=one-element) sets, and itself as subsemigroups. (By allowing trivial subsemigroups, we make it possible for a nontrivial group to be a prime semigroup, even though it has the trivial subgroup $\{e\}$.)

Our main result about prime semigroups is

THEOREM 1. *A semigroup is prime if and only if it is a prime group or has at most two elements.*

Proof. If S has at most two elements, it has only the empty set, trivial sets, and itself as subsets, and so is certainly prime. If S is a prime group, it is finite so that any nonempty subsemigroup S' of S is a finite cancellation semigroup, hence a group; thus since S is prime, S' must be $\{e\}$ or S, making S a prime semigroup. Conversely, if S is a group and a prime semigroup, it is certainly a prime group. It thus remains only to show that if S is a prime semigroup and not a group, it can have at most two elements. We prove this by considering several possible cases:

(a) If S has both left and right absorbent elements, it has a unique two-sided absorbent element a, and can have no other left or right absorbent element. Hence $\{a\}$ is the only trivial left or right ideal of S. Since a left or right ideal is a subsemigroup, we thus have $xS=\{a\}$ or S, and $Sx=\{a\}$ or S, for all $x\in S$.

(a1) If $xS=Sx=\{a\}$ for all x, the composition on S is defined by $xy=a$ for all $x, y\in S$; but for this composition any subset containing a is a subsemigroup, so that S can have at most two elements.

(a2) Otherwise, there exists $b\in S$, evidently $\neq a$, such that (say) $Sb=S$. (The proof if there exists b such that $bS=S$ is exactly analogous.) In particular, there exists $e\in S$, $\neq a$ such that $eb=b$. Now $\{x \mid x\in S; ex=x\}$ is a right ideal and contains a and b, so cannot be trivial; hence it must be all of S, making e a left identity. Furthermore, for all $y\neq a\in S$, Sy is a left ideal and contains $ey=y$, so cannot be trivial and must be all of S; thus there exists $y'\in S$ such that $y'y=e$. It follows readily that $S-\{a\}$ is a subgroup of S; hence this subgroup must be trivial, so that S has just two elements.

(b) Suppose now that S has no right absorbent elements (the proof if it has no left absorbent elements is exactly analogous). Then S has no trivial left ideals, so that $Sx=S$ for all $x\in S$, and in particular, for any $x\in S$ there exists $e_x\in S$ such that $e_xx=x$.

(b1) If S has an element z which is not left absorbent, $\{z\}$ cannot be a right ideal; hence the right ideal $\{w \mid w\in S; e_zw=w\}$, which contains z, must be all of S, making e_z a left identity. Moreover, for all $x\in S$ we have $Sx=S$, so that there exists $x'\in S$ such that $x'x=e_z$, and S is a group, contradiction.

(b2) Hence every element of S is left absorbent, so that the composition on S is defined by $xy=x$ for all $x, y\in S$; but for this composition, any subset is a subsemigroup, so that S can have at most two elements.

A shorter proof of Theorem 1 can be given along the following lines:

If every element of S is idempotent, it is not difficult to show that S has at most two elements. If $x\in S$ is not idempotent, the nontrivial subsemigroup (x) generated by x must be all of S. If all the powers of x are distinct, (x^2) is a nontrivial proper subsemigroup; let m be the smallest positive integer such that $x^m=x^r$ for some $r>m$. If $r=m+1$, then $x^s=x^m$ for all $s>m$ and $\{x^{m-1}, x^m\}$ is a nontrivial subsemigroup, so that S has at most two elements. Otherwise, readily

$\{x^m, \cdots, x^{r-1}\}$ is a nontrivial subsemigroup, so that we must have $m=1$; in this case it is easy to show that S must be a cyclic group of prime order.

The proof given above, though somewhat longer, is needed as a basis for proving Theorem 2. (I am indebted to the referee for suggesting that an outline of the shorter proof of Theorem 1 be included.)

COROLLARY. *A semigroup has no subsemigroups other than itself and $\varnothing$ if and only if it is trivial.*

Proof. If S is nontrivial, cases (a) cannot hold, since if S had an absorbent element, $\{a\}$ would be a subsemigroup; case (b1) cannot hold, since if S were a group, $\{e\}$ would be a subsemigroup; and case (b2) cannot hold, since $\{x\}$ would be a subsemigroup for each $x \in S$.

Note that in the course of proving Theorem 1 we have established

PROPOSITION *1. The semigroup S has no proper, nontrivial left or right ideals if and only if one of the following is true:*

(a) *S is a group,*

(b) *S has an absorbent element a, and $S-\{a\}$ is a group,*

(c) *S has at most two elements.*

Proof. By the proof of Theorem 1, if S has no proper nontrivial left or right ideals, one of the following must hold:

(a) S is a group (Case b1). In this case, $\varnothing$ and S are the only ideals.

(b) $S-\{a\}$ is a group, where a is absorbent (Case a2). In this case the ideals are $\varnothing$, $\{a\}$ and S.

(c1) $xy=a$ for all $x, y \in S$, where a is absorbent (Case a1). In this case any subset containing a is a two-sided ideal, so that S can have at most two elements.

(c2) $xy=x$ for all $x, y \in S$, or $xy=y$ for all $x, y \in S$ (Cases b2). In the former case any subset is a right ideal, in the latter case a left ideal, so that S can have at most two elements.

COROLLARY. *A semigroup has no proper left or right ideals if and only if it is a group or empty.*

3. Prime semirings. By a *semiring* we mean a set closed under two compositions, $+$ and $\cdot$, where $+$ is associative and commutative; $\cdot$ is associative; and $\cdot$ distributes over $+$. We call a semiring *prime* if it has only the empty set, trivial sets, and itself as subsemirings. (Here again, by allowing trivial subsemirings, we make it possible for a nontrivial ring to be a prime semiring even though it has the trivial subring $\{0\}$.)

LEMMA. *A multiplicatively absorbent element of a semiring is additively idempotent.*

Proof. An absorbent element 0 must be unique; but for all x we have $x(0+0) = x0+x0 = 0+0$ and $(0+0)x = 0x+0x = 0+0$, so that $0+0$ is also absorbent and so must be the same as 0.

THEOREM 2. *A semiring is prime if and only if its additive semigroup is prime.*

Proof. Since a subsemiring is an additive subsemigroup, "if" is clear. To prove "only if," we again consider various possible cases:

(a) If S has a unique multiplicatively absorbent element 0, then as in the proof of Theorem 1, the only trivial left or right ideal of S is $\{0\}$ (it is an ideal since by the Lemma 0 is additively idempotent). Hence if S is prime, we have $xS=\{0\}$ or S, and $Sx=\{0\}$ or S, for all $x\in S$.

(a1) If $xS=Sx=\{0\}$ for all $x\in S$, the multiplicative composition on S is defined by $xy=0$ for all $x, y\in S$. For this composition, a nonempty subset of S is a subsemiring if and only if it contains 0 and is an additive subsemigroup. Thus if every additive subsemigroup of S contains 0, the Theorem is proved. If not, 0 must fail to be in some monogenic additive subsemigroup $A=\{na \mid n\in Z^+\}$, $a\in S$. Hence the sup of A and $\{0\}$ in the lattice of additive subsemigroups must be all of S. But readily this sup is $A\cup\{0\}\cup\{na+0 \mid n\in Z^+\}=A\cup\{0\}\cup(A+\{0\})$, and this has $\{0\}\cup(A+\{0\})$ as a subsemiring (since it is an additive subsemigroup and contains 0); hence this subsemiring must be $\{0\}$ or all of S.

(a1.1) If $A+\{0\}=\{0\}$, we have $A\cup\{0\}=S$, and if A' is any subsemigroup of A, then $A'\cup\{0\}$ is a subsemiring of S. Hence for S to be prime, A can have no subsemigroups but itself and $\varnothing$, which implies (Corollary to Theorem 1) that $A=\{a\}$ is trivial, so that $S=\{0, a\}$.

(a1.2) If $(A+\{0\})\cup\{0\}=S$, and $0\notin A+\{0\}$, we can apply the argument of (a1.1) to $A+\{0\}$ to prove it trivial, so that $S=\{0, a+0\}$.

(a1.3) Otherwise, we have $A+\{0\}=S$, so that $0\in A+\{0\}$, say $0=ma+0$; but then $S=\{a+0, 2a+0, \cdots, (m-1)a+0, 0\}$ is an additive group with identity 0, and since it can have no proper subsemigroups which contain 0, in particular it can have no proper subgroups, making it a prime group.

(a2) If (a1) does not hold, then just as in the proof of Theorem 1, $S-\{0\}$ is a multiplicative group. Let its identity be 1; then $\{n1 \mid n\in Z^+\}$ is a subsemiring, so is trivial ($=\{1\}$) or is all of S.

(a2.1) If $\{n1\}$ is trivial, 1 is additively idempotent, so that $\{0, 1+0\}$ is a subsemiring. Hence if $1+0\neq 0$ we must have $S=\{0, 1+0\}$, while if $1+0=0$, then $\{0, 1\}$ is a subsemiring, so that in any case $S=\{0, 1\}$.

(a2.2) But if $\{n1\}=S$, note that any additive subsemigroup of $\{n1\}$ is an ideal, so that the additive semigroup $\{n1\}$ must be prime.

(b) If (a) does not hold, then just as in the proof of Theorem 1, either (b1) or (b2) must hold:

(b1) S is a multiplicative group, say with identity 1, and as in case (a2), we have $\{n1\}=\{1\}$ or S.

(b1.1) If $\{n1\}=\{1\}$, every element of S is additively idempotent (indeed, $x+x=x1+x1=x(1+1)=x1=x$ for all $x\in S$), but no $x\neq 1$ can be multiplicatively idempotent, so that for any $a\neq 1$ the subsemiring generated by either a or a^{-1} is all of S. Readily, for a^{-1} this subsemiring is

$$\left\{\sum_{i=1}^{k}\delta_i a^{-i} \;\middle|\; k\in Z^+;\ \delta_i=0 \text{ or } 1,\ 1\leqq i\leqq k\right\},$$

and since 1 must be in it, we have $1=\sum_{i=1}^{n}\delta_i a^{-i}$ for some $n\in Z^+$ and some choice of δ's. Hence $a^n=\sum_{i=1}^{n}\delta_i a^{n-i}$ is a sum of lower (nonnegative) powers of a, so that any power of a is such a sum, and since a generates S, we have $S=\{\sum_{i=0}^{n}\delta_i a^i \mid \delta_i=0 \text{ or } 1,\ 1\leqq i\leqq n\}$. It follows that $b=\sum_{i=0}^{n}a^i$ is additively absorbent. But then for all $x, y\in S$ we have

$$x+y=xy^{-1}bb^{-1}y+bb^{-1}y=(xy^{-1}b+b)b^{-1}y=(b)b^{-1}y=y,$$

so that any $y\in S$ is additively absorbent, and since an absorbent element must be unique, S is trivial.

(b1.2) If $\{n1\}=S$, then as in case (a2.2), the additive semigroup of S must be prime. [Note, however, that in this case if S is nontrivial, 1 is not additively idempotent, so that as in the first part of case (b1.1), no element of S can be additively idempotent. But if the additive semigroup of S is prime, it is finite, hence does contain an idempotent (see [1]). Thus this case cannot arise unless S is trivial.]

(b2) The composition on S is given by $xy=x$ (or by $xy=y$) for all $x, y\in S$. In this case any subset is a multiplicative subsemigroup, so that any additive subsemigroup is a subsemiring; hence the additive semigroup of S must be prime. [Note, however, that in this case, if S has more than two elements, its additive semigroup is a prime group $\{0, a, \cdots, (p-1)a\}$, and since $(ma)(na)=(mn)a=(na)(ma)$, the multiplicative composition on S must be commutative, so that $x=xy=yx=y$ for all $x, y\in S$, making S trivial, contradiction. Hence this case cannot arise if S has more than two elements.]

Using Theorem 2 we can immediately prove

THEOREM 3. *The semiring S is prime if and only if one of the following is true:*

(1) *S has at most two elements,*

(2) *S is isomorphic to the zero-ring whose additive group is the additive group of integers modulo a prime p,*

(3) *S is isomorphic to the field of integers modulo a prime p.*

Proof: Each of (1—3) is a prime additive semigroup, so that "if" is assured. Conversely, by Theorems 1 and 2, if S is prime it either has at most two elements, or its additive semigroup is a prime group. Moreover, if S has more than two elements, one of cases (a1.3) or (a2.2) of Theorem 2 must hold. In the former case, S is a zero-ring; while in the latter case, $S-\{0\}$ is a multiplicative group, so that S is a division ring, and readily must be isomorphic to a field of integers modulo p.

We also have

PROPOSITION 2. *A ring has no proper left or right ideals if and only if it is a division ring or is isomorphic to the zero-ring whose additive group is the integers modulo p, where $p=1$ or is prime.*

Proof. Since a ring R has a multiplicatively absorbent element 0, case (b) of Theorem 2 cannot arise. In case (a1), R is a zero-ring, and in case (a2) $R-\{0\}$ is a multiplicative group, so that R is a division ring. Moreover, in case (a1) any

additive subgroup of R is an ideal (since it contains 0), so that the additive group of R must be prime. Conversely, if the additive group is prime, there are no proper subgroups and *a fortiori* no proper left or right ideals; and as is well known, a division ring can have no proper left or right ideals.

COROLLARY. *A ring has no proper subrings if and only if it is a prime semiring.*

Proof. If R has no proper subrings, it has no proper left or right ideals, so that by the Proposition it is a prime semiring (case (2) of Theorem 3) or a division ring. But in the latter case it must be a prime field, and it cannot be the field of rational numbers since this has proper subrings; hence it is a prime semiring (case (3) of Theorem 3) in this case too.

Reference

1. Solution to Problem 3133, this MONTHLY, 72 (1965) 324.

ON GROUPS, QUASI- AND OTHERWISE

T. S. FRANK, Le Moyne College

1. Introduction. It is traditional, whenever the group concept is discussed for the first time, to drag forth as examples the familiar numerical sets and their operations. The integers under addition, the nonzero rational numbers under multiplication, and so forth are customarily given as examples of systems which satisfy the group axioms, while modifications of these, e.g. the integers under multiplication, serve as bad examples, that is systems which do not quite satisfy the group axioms. Other examples can be obtained by constructing from one of the familiar groups $(G, \cdot)$ a system $(G, \circ)$, where the operation is defined by: $a \circ b = a \cdot b^{-1}$. The integers under subtraction, the nonzero rationals under division, etc. are examples of such systems. But the operation is usually so poorly behaved (only the closure axiom holds in general) that any further discussion of such systems is usually abandoned.

Thus if we replace the group operation by its inverse, we do not in general any longer have a group. In fact it is an easy consequence of the group axioms that a necessary and sufficient condition that such a system again be a group is that every element in the original group be of order 2. The question remains, however: For the other groups, how poorly behaved and how weak is the structure formed when the operations of these groups are replaced by their inverse operations?

The degree to which these systems are "nongroups" is investigated in the present paper by considering certain quasigroups, of which these systems are examples. Although there are many ways of measuring the degree to which a

given system resembles or fails to resemble a group, the approach taken here is that of mapping the system homomorphically into a group and then examining the kernel of this homomorphism. Intuitively, the smaller the kernel the more grouplike the system is. In view of the preceding paragraph it should not be surprising that the group most closely resembled (in the present sense) by a system of this sort is one in which each element is of order 2. Further, we shall see that this group is easily constructible.

2. Preliminaries.

DEFINITION. *By an inversive quasigroup* $(Q, \circ)$ *is meant a nonempty set* Q *along with a closed, single-valued binary operation* $\circ$ *on* Q, *satisfying the following axioms*:

(1) *For* $a, b \in Q$ *there is a unique* $d \in Q$ *such that* $d \circ a = b$;

(2) *For any* $c \in Q$ *there is* c' *in* Q *such that for each* $a \in Q$, $(a \circ c) \circ c' = a$. (*This element* c' *is called a pseudo-inverse for* c.)

Example. As mentioned in the introduction we have a convenient class of examples of inversive quasigroups. Let $(G, \cdot)$ be a group, and construct a quasigroup $(G, \circ)$ by defining $a \circ b = a \cdot b^{-1}$. It is easy to see that $d \circ a = b$ is solved by taking $d = b \cdot a$, and that c^{-1} will play the role of a pseudo-inverse c' for c.

REMARK. Although the inversive quasigroups constructed in the example above form a large class, it is not difficult to give examples of inversive quasigroups which cannot be so constructed from groups.

The following useful theorems are immediate consequences of the definition.

THEOREM 1. *Every inversive quasigroup has right cancellation.*

THEOREM 2. *If* c' *is a pseudo-inverse for* c, *then* c *is a pseudo-inverse for* c'.

THEOREM 3. *If* c' *and* c'' *are pseudo-inverses for* c, *then for any* $a \in Q$, $a \circ c' = a \circ c''$.

4. Homomorphisms.

DEFINITION. *A homomorphism from an inversive quasigroup* $(Q, \circ)$ *to a group* $(G, \cdot)$ *is a mapping* h *from* Q *to* G *such that for each* $a, b \in Q$, $h(a \circ b) = h(a) \cdot h(b)$. *The kernel* K_h *of* h *is* $\{x \in Q : h(x) = \text{identity of } (G, \cdot)\}$.

DEFINITION. *The nucleus* $N(Q)$ *of a quasigroup* $(Q, \circ)$ *is* $\{x \circ x' : x \in Q\}$.

It follows from Theorem 3 that each product $x \circ x'$ in $N(Q)$ is uniquely defined, although the pseudo-inverse x' may not be. Further, from Theorem 2, $N(Q) = \{x' \circ x : x \in Q\}$.

THEOREM 4. *If* h *is a homomorphism from a quasigroup* $(Q, \circ)$ *to a group* $(G, \cdot)$, *then* $(K_h, \circ)$ *is an inversive sub-quasigroup of* $(Q, \circ)$.

The proof is straightforward and will not be included.

DEFINITION. *For any ordered triple* (a, b, c) *of elements of* Q *there is, by solvabil-*

ity of equations in $(Q, \circ)$, *an element* $x \in Q$ *such that* $a \circ (b \circ c) = x \circ [(a \circ b) \circ c]$. *The element* x *is called an associator and is denoted* $\langle abc \rangle$. *The set of all associators of* $(Q, \circ)$ *is denoted* $A(Q)$.

THEOREM 5. *If* h *is a homomorphism from an inversive quasigroup* $(Q, \circ)$ *to a group* $(G, \cdot)$, *then* $N(Q) \cup A(Q) \subseteq K_h$.

Proof. Let e be the identity of $(G, \cdot)$. Then it is immediate that $h(x \circ x') = e$ for any $x \in Q$. Furthermore, applying h to both sides of the equation $a \circ (b \circ c) = \langle abc \rangle \circ [(a \circ b) \circ c]$, we obtain

$$h(a) \cdot h(b) \cdot h(c) = h(\langle abc \rangle) \cdot h(a) \cdot h(b) \cdot h(c), \quad \text{so} \quad h(\langle abc \rangle) = e.$$

DEFINITION. *A subset* S *of* Q *is said to partition the quasigroup* $(Q, \circ)$ *if ana only if:* (1) *For any* $a, b \in Q$, *the sets* $S \circ a$ *and* $S \circ b$ *are either identical or disjoint, and* (2) *if* $x \in S \circ a$ *and* $y \in S \circ b$, *then* $x \circ y \in S \circ (a \circ b)$.

THEOREM 6. *Let* h *be a homomorphism from* $(Q, \circ)$ *to* $(G, \cdot)$. *Then* K_h *partitions* $(Q, \circ)$.

Proof. Denoting K_h simply by K and letting e be the identity of $(G, \cdot)$, we suppose first that for some $a, b \in Q$, $K \circ a \cap K \circ b \neq \varnothing$. Then there are $k_1, k_2 \in K$ such that $k_1 \circ a = k_2 \circ b$. Hence $h(a) = h(k_1 \circ a) = h(k_2 \circ b) = h(b)$. Now

$$(K \circ b) \circ a' = K, \text{ for } h[(K \circ b) \circ a'] = h(K) \cdot h(b) \cdot h(a') = h(b) \cdot h(a)^{-1} = e.$$

Thus $(K \circ b) \circ a' \subseteq K$ which, by left-solvability implies $(K \circ b) \circ a' = K$. By Theorem 2, $K \circ b = K \circ a$. Thus the first condition of the preceding definition is satisfied. Next assume $x \in K \circ a$, $y \in K \circ b$. Then $h(x) = h(a)$, $h(y) = h(b)$, so $h(a \circ b) = h(x \circ y)$. Thus $(x \circ y) \circ (a \circ b)' \in K$, or equivalently, $x \circ y \in K \circ (a \circ b)$.

DEFINITION. *Let* Σ *denote the intersection of all inversive sub-quasigroups of* $(Q, \circ)$ *which partition* $(Q, \circ)$ *and contain* $N(Q) \cup A(Q)$.

Now since the kernel of any homomorphism from $(Q, \circ)$ into any group must contain $(\Sigma, \circ)$ as an inversive sub-quasigroup, Σ is the smallest possible candidate for the kernel of a homomorphism. The next two theorems show that there is a homomorphism of $(Q, \circ)$ into a group with Σ as its kernel.

THEOREM 7. *The collection* $\{\Sigma \circ a : a \in Q\}$ *of right-translates of* Σ *form a group under the operation:* $\Sigma \circ a \sim \Sigma \circ b = \Sigma \circ (a \circ b)$.

Proof. (1) The operation is well-defined, since Σ partitions $(Q, \circ)$. For if $\Sigma \circ a_1 = \Sigma \circ a_2$, and $\Sigma \circ b_1 = \Sigma \circ b_2$, then $a_2 \in \Sigma \circ a_1$ and $b_2 \in \Sigma \circ b_1$, so $a_2 \circ b_2 \in \Sigma \circ (a_1 \circ b_1)$. But since $a_2 \circ b_2 \in \Sigma \circ (a_2 \circ b_2)$, we have $\Sigma \circ (a_1 \circ b_1) = \Sigma \circ (a_2 \circ b_2)$.

(2) Σ is a left identity. For any $a \in Q$, $a \circ a' \in \Sigma$, so $\Sigma \circ (a \circ a') = \Sigma$. But then

$$\Sigma \sim \Sigma \circ a = \Sigma \circ (a \circ a') \sim \Sigma \circ a = \Sigma \circ [(a \circ a') \circ a] = \Sigma \circ a.$$

(3) The operation is associative, for since $\langle abc \rangle \in \Sigma$, $\Sigma = \Sigma \circ \langle abc \rangle$. Thus

$$(\Sigma \circ a \sim \Sigma \circ b) \sim \Sigma \circ c = \Sigma \circ \langle abc\rangle \sim [(\Sigma \circ a \sim \Sigma \circ b) \sim \Sigma \circ c]$$
$$= \Sigma \circ [\langle abc\rangle \circ [(a \circ b) \circ c]] = \Sigma \circ [a \circ (b \circ c)]$$
$$= \Sigma \circ a \sim (\Sigma \circ b \sim \Sigma \circ c).$$

(4) $\Sigma \circ a'$ is a left inverse for $\Sigma \circ a$.

The group constructed in the preceding theorem will be denoted $(\Gamma, \sim)$.

THEOREM 8. *There is a homomorphism γ from $(Q, \circ)$ to $(\Gamma, \sim)$ such that $K_\gamma = \Sigma$.*

Proof. We need only take $\gamma(a) = \Sigma \circ a$ and observe that $\gamma(a \circ b) = \Sigma \circ (a \circ b) = \Sigma \circ a \sim \Sigma \circ b = \gamma(a) \sim \gamma(b)$.

This group $(\Gamma, \sim)$ is then the group most closely resembled by $(Q, \circ)$ (in this particular homomorphism sense) and Σ, as a subset of Q, gives us a measure of the degree to which $(Q, \circ)$ is grouplike.

5. The special case. We return now to the topic mentioned in the introduction, namely inversive quasigroups constructed from groups. If we begin with a group $(G, \cdot)$ and construct from it the quasigroup $(G, \circ)$ by letting $a \circ b = a \cdot b^{-1}$, then as we have seen the inversive sub-quasigroup $(\Sigma, \circ)$ is the smallest which partitions $(G, \circ)$ and contains $N(Q) \cup A(Q)$. The group $(\Gamma, \sim)$ was seen to be

$$\{\Sigma \circ a : a \in G, \Sigma \circ a \sim \Sigma \circ b = \Sigma \circ (a \circ b)\}.$$

In terms of the operation of $(G, \cdot)$ this is the group

$$\{\Sigma \cdot a : a \in G, Z \cdot a \sim \Sigma \cdot b = \Sigma \cdot b^{-1} \cdot a\}.$$

The following lemma, which seems to be not particularly well known, is of some use in the proof of Theorem 9.

LEMMA. *Let S be a nonempty subset of a group, with the property that any two right-translates of S either coincide or are disjoint. Then S is a coset of some subgroup.*

Proof. Without loss of generality we can assume that the group identity e is in S. Then an element a is in S if and only if $Sa = S$. (We are denoting the group operation here by juxtaposition.) For if $a \in S$, then $a \in Sa$ since $e \in S$. The condition in the statement of the lemma then insures that $Sa = S$. Conversely, if $Sa = S$, then since $a \in Sa$, $a \in S$. Now let $x, y \in S$. Then $S = Sx = Sy$, or $S = Sxy^{-1}$. Hence $xy^{-1} \in S$, so S is a subgroup.

THEOREM 9. *$(\Sigma, \cdot)$ is the normal subgroup of $(G, \cdot)$ generated by the squares and commutators of $(G, \cdot)$.*

Proof. That $(\Sigma, \cdot)$ is a subgroup follows from the lemma and the fact that the identity of $(G, \cdot)$ is in Σ. Now since $A(G) \subseteq \Sigma$, the associator $\langle xyz\rangle \in \Sigma$. It is a computational exercise to show that this element is of the form $x \cdot z \cdot y^{-1} \cdot x \cdot y^{-1} \cdot z^{-1}$. With $y = x$ this becomes the commutator element $x \cdot z \cdot x^{-1} \cdot z^{-1}$, while

letting $y=z=$identity of $(G, \cdot)$ we obtain the square x^2. Conversely the associator $x\cdot z\cdot y^{-1}\cdot x\cdot y^{-1}\cdot z^{-1}$ can be written in the form

$$[x\cdot z\cdot y^{-1}\cdot x^{-1}\cdot y\cdot z^{-1}]\cdot[z\cdot y^{-1}\cdot x^2\cdot y\cdot z^{-1}\cdot x^{-2}]\cdot[x^2]\cdot[z^2]\cdot[z^{-1}\cdot y^{-1}]^2\cdot[y\cdot z\cdot y^{-1}\cdot z^{-1}].$$

Since each term in brackets is either a square or a commutator, the proof will be complete as soon as Σ is shown to be normal. To establish normality, let $b\cdot\sigma\in b\cdot\Sigma$. Now $b\in\Sigma\circ b$ and $\sigma^{-1}\in\Sigma\circ\sigma^{-1}$, so $b\cdot\sigma=b\circ\sigma^{-1}\in\Sigma\circ(b\circ\sigma^{-1})=\Sigma\cdot\sigma^{-1}\cdot b^{-1}$ $=\Sigma\cdot b^{-1}=\Sigma\cdot b^2\cdot b^{-1}=\Sigma\cdot b$. Hence $b\cdot\Sigma\subseteq\Sigma\cdot b$, the reverse inclusion follows similarly, and $(\Sigma, \cdot)$ is normal in $(G, \cdot)$.

THEOREM 10. $(\Gamma, \sim)\cong(G, \cdot)/(\Sigma, \cdot)$.

Proof. As mentioned above, $(\Gamma, \sim)=\{\Sigma\cdot a\colon a\in G,\ \Sigma\cdot a\sim\Sigma\cdot b=\Sigma\cdot b^{-1}\cdot a\}$. But $\Sigma\cdot b^{-1}\cdot a=\Sigma\cdot(a\cdot b\cdot a^{-1}\cdot b^{-1})\cdot b^2\cdot b^{-1}\cdot a=\Sigma\cdot a\cdot b$. Thus $(\Gamma, \sim)=\{\Sigma\cdot a\colon a\in G,$ $\Sigma\cdot a\sim\Sigma\cdot b=\Sigma\cdot a\cdot b\}$.

We can now sum up the past few results in

THEOREM 11. *If $(G, \circ)$ is the inversive quasigroup constructed from the group $(G, \cdot)$, then the group most closely resembling $(G, \circ)$ is $(G, \cdot)/(\Sigma, \cdot)$.*

REMARK. Since Σ contains the squares of $(G, \cdot)$, the elements of $(\Gamma, \sim)$ are of order 2. In the event $(G, \cdot)$ is abelian, the situation is particularly nice. For then $(\Sigma, \cdot)$ reduces to the subgroup of squares of $(G, \cdot)$. Thus the inversive quasigroup of integers under the operation of subtraction is most like the group of integers modulo 2, while the rationals under the operation of subtraction is completely nongrouplike (since every rational is a "square," more accurately a double). As perhaps a more interesting example, a rather tedious calculation shows that the quasigroup constructed from the group of symmetries of the square most closely resembles the Klein four-group.

Portions of the material presented here are taken from the author's doctoral dissertation at Syracuse University, directed by Professor Walter R. Baum.

ON THE GROUP OF LINEAR TRANSFORMATIONS LEAVING A POLYNOMIAL INVARIANT

L. FLATTO, Belfer Graduate School of Science and B. GROSS, Rutgers, The State University

1. It is well known that with every nonsingular quadratic form $Q(x)$ in n-variables with coefficients from a given field F, one may associate the group $O(n)$ of linear transformations leaving the polynomial invariant. The question naturally arises of characterizing those groups of linear transformations leaving invariant other polynomials. For the case $n=2$, we will show that with certain specified exceptions, the groups are finite. We obtain bounds for the order of these groups. The bounds depend on the order of the polynomial. If F is re-

stricted to be the real field R or the complex field C, then we can describe the structure of the various groups which may arise. That the groups are in general finite is mentioned in Burnside's book ([1] pp. 369–371), but no clear proof is indicated. For $n>2$ the complexity of the problem is much greater. As shown in 3 the case $n=3$ and F the complex field is closely related to a theorem of Hurwitz [3] concerning the group of conformal maps of compact Riemann Surfaces. For arbitrary n and F it is difficult to obtain any general insight and we limit our discussion to the polynomials $\sum_{i=1}^{n} x_i^k$ $(k>2)$, in which case the problem is completely solved.

Since for finite fields it is possible for two distinct polynomials $P_1(x)$ and $P_2(x)$ to have the property that $P_1(a)=P_2(a)$ for all $a\in F$, there is some ambiguity attached to the statement $P(Tx)=P(x)$. We therefore specify that $P(Tx)=P(x)$ shall mean that the two polynomials are identical as polynomials.

2. We state two preliminary lemmas. Both of these are known, but we include the proofs for the sake of completeness.

LEMMA 1. *Suppose T is a linear transformation leaving P invariant, i.e., $P(Tx)=P(x)$. Let*

$$H(x) = \det\left(\frac{\partial^2 P}{\partial x_i \partial x_j}\right).$$

Then $H(x)=(\det T)^2 H(Tx)$.

Proof. Let $T=(a_{ij})$ so that $x'=Tx$ becomes $x_i' = \sum_{j=1}^{n} a_{ij}x_j$, $(1\leqq i\leqq n)$. Differentiating $P(Tx)=P(x)$ twice we obtain

$$\frac{\partial^2 P}{\partial x_i \partial x_j} = \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial^2 P}{\partial x_k' \partial x_l'} \frac{\partial x_k'}{\partial x_i} \frac{\partial x_l}{\partial x_j} = \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial^2 P}{\partial x_k' \mu x_l'} a_{lj}a_{ki},$$

so that $(\partial^2 P/\partial x_i \partial x_j) = (a_{ij})'(\partial^2 P/\partial x_i' \partial x_j')(a_{ij})$, where $(a_{ij})'$ denotes the transpose of (a_{ij}). Thus $H(x)=(\det T)^2 H(Tx)$.

LEMMA 2. *Let $x_i y_i$ $(1\leqq i\leqq 3)$ be two-dimensional vectors over the field F. Suppose that any two of the x_i's and any two of the y_i's are linearly independent. Then there exists a linear transformation T (necessarily nonsingular), such that $Tx_i=\lambda_i y_i$ where $\lambda_i\neq 0$ $(1\leqq i\leqq 3)$. If T_1 is another such linear transformation, then $T_1=cT$ for some $c\neq 0$.*

REMARK. This is just the algebraic formulation of the fundamental theorem of projective geometry.

Proof. Using x_1, x_2 as basis for the 2-dimensional vector space over F, we have $x_1=(1, 0)$, $x_2=(0, 1)$, $x_3=(a_1, a_2)$ where $a_1a_2\neq 0$ and $y_i'=(b_{i1}, b_{i2})$ $(1\leqq i\leqq 3)$. Since $Tx_i=\lambda_i y_i$ we conclude that

$$T = \begin{pmatrix} \lambda_1 b_{11} & \lambda_2 b_{21} \\ \lambda_1 b_{12} & \lambda_2 b_{22} \end{pmatrix}.$$

Since $Tx_3=\lambda_3 y_3$ we have

$$\lambda_3 b_{31} = a_1 b_{11}\lambda_1 + a_2 b_{21}\lambda_2$$
$$\lambda_3 b_{32} = a_1 b_{12}\lambda_1 + a_2 b_{22}\lambda_2 .$$

Since

$$\begin{vmatrix} a_1 b_{11} & a_2 b_{21} \\ a_1 b_{12} & a_2 b_{22} \end{vmatrix} = a_1 a_2 \begin{vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{vmatrix} \neq 0$$

we can solve the above system for λ_1, λ_2 and obtain $\lambda_1 = A_1\lambda_3$, $\lambda_2 = A_2\lambda_3$ where A_1, A_2 depend on a_1, a_2, b_{i1}, b_{i2}. A_1, A_2 are different from zero, for otherwise we would read off from (2) that y_3, y_2 or y_3, y_1 are linearly dependent. Since $\lambda_3 \neq 0$ is arbitrarily chosen, we obtain the desired result.

3. We now treat the case $n=2$. Let K denote the algebraic closure of F. Let $P=\sum_{j=0}^{n} P_j$ where P_j is homogeneous of degree j. If G denotes the group of nonsingular transformations leaving P invariant, then $G=\bigcap_{j=1}^{n} G_j$ where G_j denotes the group of nonsingular transformations leaving P_j invariant. We may henceforth assume that P is a homogeneous polynomial. Now

$$P(x) = \prod_{i=1}^{m} L_i^{k_i}(x)$$

where the L_i's are distinct homogeneous linear factors with coefficients from K (we call two linear factors distinct if they are linearly independent). We write

$$L_i(x) = a_i \cdot x, \quad \text{where} \quad a_i = (a_{i1}, a_{i2}), \quad x = (x_1, x_2), \quad a_i \cdot x = a_{i1}x_1 + a_{i2}x_2.$$

Let $k=\deg P$ so that $k=\sum_{i=1}^{m} k_i$. We state the following

THEOREM 3.1. *If $m>2$, then G is a finite group whose order $\leqq k^2(k-1)(k-2)$. If $m=1$ or 2 and F is algebraically closed then G is an infinite group. For $m=1$ G is conjugate to*

$$\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| \, a^{k_1} d^{k_2} = 1\right\}$$

if $k_1 \neq k_2$ and to

$$\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| \, a^{k_1} d^{k_1} = 1\right\} \cup \left\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \middle| \, b^{k_1} c^{k_1} = 1\right\}$$

if $k_1 = k_2$.

Proof. Suppose that $m>2$ and that $P(Tx)=P(x)$. It follows from the unique factorization theorem that $a_i \cdot Tx = \lambda_i a_{\sigma(i)} \cdot x$, where $\lambda_i \in K$ and $\neq 0$ and $\sigma(i)$ is a permutation of 1 to m. Let T' denote the transpose of T. Then

$$T'(a_i) = \lambda_i a_{\sigma(i)} \qquad (1 \leqq i \leqq 3) \tag{1}$$

where we have $N=m(m-1)(m-2)$ choices for $\sigma(i)$. It follows from Lemma 2 that for each of the N choices (1) has a solution. Suppose that for exactly m of the choices we can find a solution T such that $P(Tx)=P(x)$ and for each of these pick a T. We label these $T_1, T_2, \cdots, T_m$. By Lemma 2, $T=c_jT_j$ where $c_j\neq 0$ for some j, $(1\leqq j\leqq m)$. Since $P(T_jx)=P(x)$ and $P(Tx)=P(x)$ we conclude $P(x)=c_j^kP(x)$ so that $c_j^k=1$. Thus c_j can take on at most k distinct values and G is finite, its order $\leqq k(m-1)(m-2)\leqq k^2(k-1)(k-2)$.

We consider next the case where $m=1$, 2 and F is algebraically closed (so that $F=K$). By a linear change of variables, we may assume

$$\text{(a) } P=x_1^{k_1} \quad \text{or} \quad \text{(b) } P=x_1^{k_1}x_2^{k_2}.$$

A direct calculation yields in case (a)

$$T=\left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \middle| \, a^{k_1}=1,\ a\ d\neq 0\right\}.$$

Now $P(x)=0\Rightarrow P(Tx)=0$. In case (b) $P=0$ along the lines $x_1=0$, $x_2=0$ so that T takes these lines into themselves. Thus

$$T=\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad \text{where} \quad ad\neq 0 \text{ and } a^{k_1}d^{k_2}=1.$$

If $k_1=k_2$ we also have the possibility

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \quad \text{where} \quad bc\neq 0 \text{ and } b^{k_1}c^{k_1}=1.$$

We remark that for $m\geqq 2$, $P(Tx)=P(x)$ implies that T is automatically nonsingular (as two linearly independent a_i's go over into two linear independent a_i's).

If $F=R$ or C we have the following theorem which yields more precise information about the structure of G.

THEOREM 3.2. (a) *Let $F=C$. For $m=1$, 2 G is described as in Theorem* 1. *For $m>2$ let G_0 denote the subgroup of G consisting of the diagonal matrices $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right)$, where $\lambda^k=1$. Then G/G_0 is isomorphic to one of the following groups: the cyclic groups, the dihedral groups, the groups of the regular solids.*

(b) *Let $F=R$. If $m=1$, then G is described as above. If $m=2$, and P factors over R then G is described as above and if P does not factor over R, then G is isomorphic to the full orthogonal group $O(2)$. If $m>2$, G is isomorphic to a cyclic or dihedral group.*

REMARK. One can exhibit polynomials for which each of the above cases is realized.

Proof. The cases $m=1$, 2 follow as in Theorem 3.1. If $F=R$, $m=2$, and P does not factor over R, then $L_1=\alpha\overline{L}_2$ where $\alpha\neq 0$, and $\overline{L}=\bar{a}_1x+\bar{a}_2y$, if $L=a_1x$

$+a_2y$. In this case $P(x)=\alpha Q^{k_1}(x)$ where $Q(x)$ is a positive definite quadratic form. After a linear change of variables $Q=x_1^2+x_2^2$. It is then clear that G is the full orthogonal group.

Let $m>2$. Theorem 3.1 shows that G is a finite group. If $F=R$, then G leaves a positive definite quadratic form invariant ([1], p. 257). After a linear change of variables we may assume that the form is $x_1^2+x_2^2$. Thus G (after the change of variables) consists of orthogonal matrices. It is well known that a finite subgroup of $O(2)$ is either cyclic or dihedral. One sees quite easily that if G is cyclic its order is $\leqq$ the degree of $P(x)$, and if G is dihedral then its order is $\leqq$ twice degree of $P(x)$. If $F=C$ we argue as follows. Let T be represented by

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2. \end{aligned}$$

Then we can map G into the group of linear fractional transformation by letting $T\to T'$ where T' is the fractional linear transformation

$$z' = \frac{az+b}{cz+d}.$$

Let G_1 denote the image of G. The kernel of this map is obviously given by G_0 so that $G/G_0\cong G_1$. Now G_1 is a finite group of linear fractional transformations. These groups are known ([2], p. 133) and are precisely those listed in Theorem 3.2.

As mentioned in the introductory paragraph, it is difficult to obtain a general theory for $n>2$ and we obtain results of a fragmentary nature.

If $n=3$ and F is the complex field then we may reason as follows. If $P(x_1, x_2, x_3)$ is an irreducible polynomial and $P(x_1, x_2, x_3)=0$ is free of singularities, then the projective variety $P(x_1, x_2, x_3)=0$ is a compact Riemann Surface. By a theorem of Hurwitz [3] the order of G^* (the group of conformal maps of the surface onto itself) $\leqq 84(g-1)$ provided $g>1$, g denoting the genus of the surface. We may map the group G into G^* in the obvious fashion and, if the kernel of the map is finite, we have a bound on the order of G.

We illustrate these ideas with the following examples. Let

$$P(x) = x_1^k + x_2^k + x_3^k.$$

The genus in this case is given by $g=\frac{1}{2}\sum_{i=1}^{b}(\alpha_i-1)-n+1$ ([2], p. 227) where the α's are the orders of the branch points, n is the number of sheets, and b the number of branch points. These numbers can be computed as follows. The above mentioned surface is identical with the algebraic curve S: $y=(-x^k-1)^{1/k}$. S has k sheets, k branch points at

$$x = \exp\frac{(2j+1)\pi i}{k} \qquad (0 \leqq j \leqq k-1),$$

each branch point being of order k. Thus

$$g = \tfrac{1}{2}k(k-1) - k + 1 = \frac{(k-1)(k-2)}{2}$$

and so for $k>3$, $g>1$. Hence order of $G^* \leqq 84(g-1) = 42k(k-3)$.

We now show that the kernel k of the mapping of G into G^* is the identity so that G is a finite group of order $\leqq 42k(k-3)(k>3)$. Suppose that $T \in K$ so that $TX = \lambda X$ whenever $x_1^k + x_2^k + x_3^k = 0$. Here $X = (x_1, x_2, x_3)$ and $\lambda = \lambda(X)$ is real valued and an eigenvalue of T. Thus $\lambda(X)$ is constant and we may write $\lambda(X) = \lambda$. It is easily checked that the three vectors $(0, 1, -1)$, $(1, 0, -1)$, $(1, 1, [-2]^{1/k})$ are linearly independent and that they lie in the surface $x_1^k + x_2^k + x_3^k = 0$. Thus $TX = \lambda X$ for all X. Since $P(X) = P(TX) = P(\lambda X) = \lambda^k P(X)$ and k is odd, it follows that $\lambda = 1$. Thus K consists of the identity and G is finite, its order $\leqq 42k(k-3)$.

The above results for $P(x) = x_1^k + x_2^k + x_3^k$ $(k>3)$ were worked out to illustrate the general procedure indicated for the case $n=3$ and F the complex field. Actually we obtain for the polynomials $x_1^k + x_2^k + x_3^k$ a stronger result by more elementary methods. We state the following

THEOREM 3.3. *Let* $P(x) = \sum_{i=1}^n x_i^k$ $(k>2)$. *Let* F *be either of infinite characteristic or of finite characteristic* p *where* $p \nmid k(k-1)$. *Let* G *be the set of linear transformations* T *for which* $P(Tx) = P(x)$. *Then* G *is the group of linear transformations* $x_i' = \epsilon_i x_{\sigma(i)}$ *where* $\epsilon_i^k = 1$ $(1 \leqq i \leqq n)$ *and* $\sigma(i)$ *is an arbitrary permutation of* 1 *to* n.

Proof. The transformations $x_i' = \epsilon_i x_{\sigma(i)}$ $(1<i<n)$ clearly belong to G and we show conversely that every transformation is of this form. Let $T = (a_{ij})$ where $x_i' = \sum_{j=1}^n a_{ij}x_j$ $(1 \leqq i \leqq n)$. A direct calculation shows that $H(x) = [k(k-1)]^n x_1^{k-2} \cdots x_n^{k-2}$. Since we assume that $p \nmid k(k-1)$, $H(x) \neq 0$ and it follows from Lemma 2 of section 2 that $\det T \neq 0$. Hence G is a group. Employing again Lemma 2 we obtain

$$x_1^{k-2} \cdots x_n^{k-2} = (\det T)^2 \prod_{i=1}^n \left(\sum_{j=1}^n a_{ij}x_j \right)^{k-2}.$$

It follows from the unique factorization theorem that $x_i' = \lambda_i x_{\sigma(i)}$ where $\lambda_i \in F$ $(1 \leqq i \leqq n)$. Now $P(Tx) = P(x)$ implies that

$$\sum_{i=1}^n x_i^k = \sum_{i=1}^n \lambda_i^k x_{\sigma(i)}^k = \sum_{i=1}^n \lambda_{\sigma^{-1}(i)}^k x_i^k$$

where σ^{-1} is the permutation inverse to σ. Thus $\lambda_{\sigma^{-1}(i)} = 1$ for $1 \leqq i \leqq n$ which means that $\lambda_i^k = 1$ $(1 \leqq i \leqq n)$ and the proof is complete.

This work was sponsored in part by NSF grant GP-4444.

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UNIQUENESS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

D. V. V. WEND, Montana State University

Introduction. It is not hard to prove that the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0, \tag{1}$$

has a unique solution on some interval containing x_0 if $f(x, y)$ is continuous in both x and y on some rectangle R centered at (x_0, y_0) and non-**increasing** in y for each fixed x on R. (See, for example, [1, p. 384, Theorem 7.5] and [3, p. 34, Corollary 6.3].) The question naturally arises whether (1) has a unique solution if $f(x, y)$ is non-**decreasing** in y. That the answer is "no" in general is shown by the example $y' = y^{2/3}$, $y(0) = 0$, which has the solutions $y(x) \equiv 0$ and $y(x) = x^3/27$. However, we will prove in this note, under suitable restrictions, a "one-sided" uniqueness theorem when $f(x, y)$ is nondecreasing in **both** x *and* y as a special case of a uniqueness theorem for a general nth order initial value problem.

The uniqueness theorem. We prove the uniqueness theorem for the general nth order initial value problem

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}), \; y^{(k)}(0) = y_k, \qquad k = 0, 1, \cdots, n-1 \; (y^{(0)} = y). \tag{2}$$

THEOREM. *Let R be the region defined by the inequalities $0 \leqq x - x_0 < a$, $|s_k - y_k| < b_k$, $k = 0, 1, \cdots, n-1$, where $y_k \geqq 0$ for $k > 0$. Suppose the function $f(x, s_0, s_1, \cdots, s_{n-1})$ in (2) is nonnegative, continuous and nondecreasing in x, and continuous and nondecreasing in s_k for each $k = 0, 1, \cdots, n-1$ in the region R. If in addition*

$$f(x, y_0, \cdots, y_{n-1}) \not\equiv 0 \quad \text{in } R \text{ for } x > x_0, \tag{3}$$

then the initial value problem (2) has at most one solution in R.

Proof. We may assume without loss of generality that $x_0 = 0$ since otherwise a translation would give us this case. Then any solution $y(x)$ of (2) satisfies the integral equation [1, p. 366, Ex. 2]

$$y(x) = \sum_{j=0}^{n-1} \frac{x^j}{j!} y_j + \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f(t, y(t), \cdots, y^{(n-1)}(t))\, dt,$$

so

$$y^{(k)}(x) = \sum_{j=k}^{n-1} \frac{x^{j-k}}{(j-k)!} y_j + \int_0^x \frac{(x-t)^{n-k-1}}{(n-k-1)!} f(t, y, \cdots, y^{(n-1)})\, dt, \tag{4}$$

$$k = 0, 1, \cdots, n-1,$$

and $y^{(n)}(x) = f(x, y(x), \cdots, y^{(n-1)}(x))$. Condition (3) and the hypotheses that f is nonnegative and $y_k \geqq 0$ for $k > 0$ then imply that $y(x)$ and its first $n-1$ derivatives are strictly increasing functions of x for $x > 0$.

Suppose the solution $y(x)$ is defined in R for $0 \leqq x \leqq b$, $0 < b < a$. Since y and

its derivatives are continuous and increasing, for $\delta>0$ sufficiently small there exists for each $k=0, 1, \cdots, n-1$ a smallest η_k such that $0<\eta_k<b$ and $y^{(k)}(\eta_k)=y_k+\delta$. Let $\eta=\max(\eta_0, \cdots, \eta_{n-1})$ and define $y_\delta(x)=y(x+\eta)$, $0\leqq x\leqq b-\eta$. Then $y_\delta^{(k)}(0)=y^{(k)}(\eta)\geqq y^{(k)}(\eta_k)=y_k+\delta$, and using the hypothesis $f(x, s_0, \cdots, s_{n-1})$ is nondecreasing in x and the fact that $y_\delta^{(n)}(x)=y^{(n)}(x+\eta)=f(x+\eta, y_\delta(x), \cdots, y_\delta^{(n-1)}(x))$, it follows from (4) that

$$y_\delta^{(k)}(x)-y^{(k)}(x)\geqq\delta\sum_{j=k}^{n-1}\frac{x^{j-k}}{(j-k)!}+\int_0^x\frac{(x-t)^{n-k-1}}{(n-k-1)!}\{f(t, y_\delta, \cdots, y_\delta^{(n-1)})-f(t, y, \cdots, y^{(n-1)})\}dt. \tag{5}$$

Since $y_\delta^{(k)}(0)>y^{(k)}(0)$, $y_\delta^{(k)}(x)>y^{(k)}(x)$ for each k for nonnegative x sufficiently close to 0. Suppose $y_\delta^{(k)}(t)>y^{(k)}(t)$ for $0\leqq t<x$ for each k, but for some k, $y_\delta^{(k)}(x)=y^{(k)}(x)$. Then it follows from (5) that $f(t, y_\delta, \cdots, y_\delta^{(n-1)})-f(t, y, \cdots, y^{(n-1)})<0$ for some t, $0<t<x$, a contradiction, since f is nondecreasing in the last n variables. Hence $y_\delta(x)>y(x)$ for $0\leqq x<b-\eta$.

Let $v(x)$ be the maximal solution of (2) [3, p. 28, Exercise 4.3]. Then by the same argument, if $0<\delta'<\delta$, $y(x)\leqq v(x)<y_{\delta'}(x)<y_\delta(x)$ for $0<x<b-\eta$. But $y_\delta(x)\to y(x)$ and $\eta\to 0$ as $\delta\to 0$, so $y(x)=v(x)$ and (2) has a unique solution to the right of x_0.

REMARKS.

1. If $f(x, y)$ in (1) is also nondecreasing for $x\leqq x_0$, then the above theorem, together with [3, p. 34, remark below Theorem 6.2] gives a two-sided uniqueness theorem.

2. In the special case when $n=1$ and $f(x, y)$ is a function of y alone, the condition $f(y)$ is nondecreasing in y can be dropped [2].

3. The proof of the theorem depends on perturbing the given solution and its derivatives in a special way. The hypotheses of the theorem guarantee that if the initial values in (2) are perturbed by a sufficiently small $\delta>0$, then there exists a corresponding solution u_δ. Furthermore, inequality (5) and the subsequent argument remain valid to show that $u_\delta(x)>y(x)$ for $0\leqq x$. However, as $\delta\to 0$, $u_\delta(x)\to v(x)$, the maximal solution of (2), so that uniqueness cannot be guaranteed.

Some Examples. The initial value problem

$$y^{(n)}=(y^{(n-1)})^{2/3}, \qquad y^{(k)}(0)=0, \qquad k=0, 1, \cdots, n-1, \tag{6}$$

has the solution $y(x)=0$ for all x. On the other hand, letting $u=y^{(n-1)}$, then $u'=u^{2/3}$, $u(0)=0$, which has the solution $u(x)=x^3/27$, and so

$$y(x)=\frac{2x^{n+2}}{9(n+2)!}$$

is also a solution of (6) for all x. Condition (3) in this case is $(y^{(n-1)}(0))^{2/3}\neq 0$, but from (6), $y^{(n-1)}(0)=0$.

As another example consider

$$(7)\quad y^{(n)} = (y^{(n-1)})^{2/3}\exp(y^{(n-1)}), \qquad y^{(k)}(0) = 0, \qquad k = 0, 1, \cdots, n-1.$$

Again $y(x)=0$ is a solution. Letting $u=y^{(n-1)}$, $u(x)$ is defined by the equation

$$x = \int_0^u t^{-2/3}e^{-t}\,dt.$$

As $u\to\infty$, $x\to r_0=\int_0^\infty t^{-2/3}e^{-t}dt<\infty$. Then a second solution is given by

$$y(x) = \int_0^x \frac{(x-t)^{n-2}}{(n-2)!}\,u(t)\,dt,$$

defined on $0\leqq x<r_0<\infty$.

The functions on the right in the previous two examples can be multiplied by arbitrary continuous, nonnegative, nondecreasing functions of x to give other examples.

As a final example consider $y''=(y-x)^{2/3}\exp(y-x)$, $y(0)=0$, $y'(0)=1$. Letting $u=y-x$, $u''=u^{2/3}e^u$, $u(0)=0$, $u'(0)=0$, and a nontrivial solution is defined by

$$x = \int_0^{y-x} \frac{ds}{\sqrt{2\int_0^s t^{2/3}e^t\,dt}}.$$

Again as $y\to\infty$, $x\to r_0<\infty$, while another solution $y(x)=x$ is defined for all x. Here $f(x, y, y')$ is decreasing in x for all $x\neq y$, but $f(x, 0, 1)>0$ for all $x\neq 0$.

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ON A CHARACTERIZATION OF DISTRIBUTIONS BY EXPECTED VALUES OF EXTREME ORDER STATISTICS

L. K. CHAN, University of Western Ontario

The expected values of the smallest and largest order statistics X_{1n} and X_{nn}, respectively, in a sample of size n from a distribution with distribution function $u=F(x)$ is defined to be

$$EX_{1n} = n\int_{-\infty}^{\infty} x(1-F(x))^{n-1}dF(x) = n\int_0^1 F^{-1}(u)(1-u)^{n-1}du$$

and

$$EX_{nn} = n\int_{-\infty}^{\infty} x(F(x))^{n-1}dF(x) = n\int_0^1 F^{-1}(u)u^{n-1}du,$$

where $F^{-1}(u)=\inf\{x|\ F(x)\geqq u\}$ (cf. [1], p. 523). Since $0\leqq F(x)\leqq 1$, EX_{1n} and EX_{nn} exist for $n\geqq 2$ if $EX=\int_0^1 F^{-1}(u)du$ exists.

Let $F(x)$ and $G(y)$ be the distribution functions of two distributions such that EX and EY exist. In this note we show that a necessary and sufficient condition that these two distributions are identical is that $EX_{1n}=EY_{1n}$ (or $EX_{nn}=EY_{nn}$) for $n\geqq 1$. The necessary condition is trivial. For the sufficient condition, suppose $EX_{nn}=EY_{nn}$, $n\geqq 1$. Since for every real number t and every positive integer k

$$\left|F^{-1}(u)\sum_{r=0}^{k}\frac{(iut)^r}{r!}\right| \leqq \sum_{r=0}^{\infty}\left|F^{-1}(u)\frac{(ut)^r}{r!}\right| \leqq e^{|t|}\left|F^{-1}(u)\right|$$

and $\int_0^1 e^{|t|}\left|F^{-1}(u)\right|du=e^{|t|}E|X|<\infty$, by the Lebesgue dominated convergence theorem we have

$$\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^1 F^{-1}(u)\cdot u^r du = \int_0^1 F^{-1}(u)e^{iut}du.$$

Similarly

$$\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^1 G^{-1}(u)\cdot u^r du = \int_0^1 G^{-1}(u)e^{iut}du.$$

Then by assumption for every real number t

$$\int_0^1 F^{-1}(u)e^{iut}du = \int_0^1 G^{-1}(u)e^{iut}du.$$

By the uniqueness of the Fourier transform this implies that $F^{-1}(u)=G^{-1}(u)$ almost everywhere. The fact that both $F(x)$ and $G(y)$ are nondecreasing and continuous from the right implies that both $F^{-1}(u)$ and $G^{-1}(u)$ are nondecreasing and continuous from the left; hence $F^{-1}(u)=G^{-1}(u)$ for every $0\leqq u\leqq 1$. Therefore we conclude that $F(x)$ and $G(y)$ are identical. The case where $EX_{1n}=EY_{1n}$, $n\geqq 1$ can be similarly proved.

Work done at the Summer Research Institute of The Canadian Mathematical Congress.

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CORRECTION FOR "LATTICE POINT COVERINGS BY PLANE FIGURES"

IVAN NIVEN AND H. S. ZUCKERMAN

The last two sentences in the proof of Lemma 3 in the paper (this MONTHLY, 74 (1967) 353–362) are incorrect. Replacing them with the following will make the argument correct.

We may presume that the angles α_j and α_k (at A_j and A_k) satisfy $0<\alpha_j<\pi/2$, $0<\alpha_k<\pi/2$, for if there is an angle $\theta\geqq\pi/2$ among α_i, α_j, α_k then one of the four half lines emanating from the vertex at θ would intersect the opposite side, so we could take A_i at that vertex. We can presume angle $A_iKA_j\geqq\pi/2$ since the other case follows by symmetry.

In the triangle A_jRP_i choose P_i^* on A_jP_i and R^* on A_jR so that $P_i^*R^*$ is parallel to P_iR and $\overline{P_i^*R^*}=1$. Draw $P_i^*Q_i^*$, with Q_i^* on A_iA_k, parallel to A_jA_k, then $Q_i^*S^*$ parallel to $P_i^*R^*$ with S^* on A_kA_j. Then $P_i^*Q_i^*S^*R^*$ is a parallelogram. We will prove that area $P_i^*Q_i^*S^*R^*\geqq 1$. This will imply that the distance between $P_i^*R^*$ and $Q_i^*S^*$ is at least 1 and hence that a line of lattice points crosses $P_i^*Q_i^*$ and R^*S^*. Since $\overline{P_i^*R^*}=1$ this will imply the existence of a lattice point in the parallelogram and hence in the triangle.

To prove that area $P_i^*Q_i^*S^*R^*\geqq 1$ we write x for the length $\overline{P_iP_i^*}$, and observe that $\overline{Q_iQ_i^*}=x\sin\alpha_j/\sin\alpha_k$. Then we see that

$$\overline{P_i^*Q_i^*}=w_i+x\cos\alpha_j+x\frac{\sin\alpha_j}{\sin\alpha_k}\cos\alpha_k=w_i+x\frac{\sin(\alpha_j+\alpha_k)}{\sin\alpha_k}.$$

Also the distance between the parallel lines P_iQ_i and A_jA_k is 1, so the distance between the parallel lines $P_i^*Q_i^*$ and A_jA_k is $1-x\sin\alpha_j$. Letting $A(x)$ denote the area of $P_i^*Q_i^*S^*R^*$ we have

$$A(x)=\left(w_i+x\frac{\sin(\alpha_j+\alpha_k)}{\sin\alpha_k}\right)(1-x\sin\alpha_j). \tag{1}$$

Now what are the possible values of x? If angle $A_iKA_j=\pi/2$ then P_i^* coincides with P_i and $x=0$. As angle A_iKA_j increases K moves along A_kA_j towards A_j and x increases. Now x increases continuously as K approaches its limiting position, A_j. At the same time R^* approaches A_j and P_i^* moves along A_iA_j, in the direction of A_j, towards some limiting position P_i' where $\overline{P_i'A_j}=1$. Thus x increases to the limiting value $\overline{P_iA_j}-\overline{P_i'A_j}=\operatorname{cosec}\alpha_j-1$. Let Q_i' and S' be the corresponding limiting positions of Q_i^* and S^*. Then $\overline{Q_i'S'}=1$ and $Q_i'S'$ is parallel to A_iA_j. By the hypothesis of the lemma we have $w_k\geqq 1$ and the line of length w_k is at a distance 1 from A_iA_j. Since $\overline{Q_i'S'}=1\leqq w_k$ we see that the distance of $Q_i'S'$ from A_iA_j is at least 1. From this we see that the area of $P_i'A_jS'Q_i'$ is at least 1, that is $A(\operatorname{cosec}\alpha_j-1)\geqq 1$. We also have $A(0)=w_i\geqq 1$ from (1).

Now we know that $A(x)\geqq 1$ at the two endpoints $x=0$ and $x=\operatorname{cosec}\alpha_j-1$. But from (1) we see that $A(x)$ is a quadratic polynomial in x, with negative coefficient of x^2. Hence $A(x)\geqq 1$ for all x satisfying $0\leqq x\leqq\operatorname{cosec}\alpha_j-1$.

MATHEMATICAL NOTES

THE PROBLEM OF REFLECTING QUEENS

D. A. KLARNER, McMaster University

The following problem was posed in [2] as a research problem: Is it possible for $n \geqq 3$ to group the set $\{1, 2, \cdots, 2n\}$ into n pairs (a_i, b_i) with $b_i > a_i$ so that all of the numbers $b_i - a_i$, $b_i + a_i$, $i = 1, 2, \cdots, n$ are distinct? In this note we convert a special form of this problem into a combinatorial game which is directly related to the problem of placing n queens on an $n \times n$ chessboard so that no two queens attack one another.

We will study a more restricted version of the problem posed: For which n is it possible to form pairs $(1, a_1)$, $(2, a_2)$, $\cdots$, (n, a_n) with $\{a_1, a_2, \cdots, a_n\} = \{n+1, n+2, \cdots, 2n\}$, so that all of the numbers $a_i - i$, $a_i + i$, $i = 1, 2, \cdots, n$ are distinct? To see this modified problem in terms of the problem of the queens, we start with an $n \times n$ chessboard with a $1 \times n$ strip of squares added along one side which we call a *reflecting strip*; next, we define a *reflecting queen* to be a standard chess queen with the added power of being able to reflect her path via the reflecting strip. In Fig. 1, we show how two queens normally at peace attack one another when this convention is made.

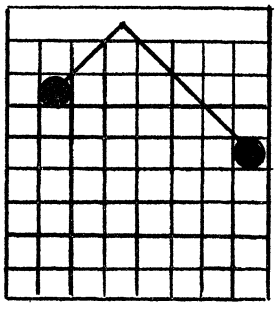


FIG. 1

The problem posed in the preceding paragraph has a solution for a given n if, and only if, it is possible to place n reflecting queens on an $n \times n$ checker board with a reflecting strip so that no pair of queens attack each other. To prove this, number the rows of the $n \times n$ square $1, 2, \cdots, n$ beginning with the first row below the reflecting strip, and number the columns from left to right $n+1, n+2, \cdots, 2n$; the intersection of the ith row and jth column is the (i, j) square. It is easy to see that a reflecting queen on the (i, j) square attacks the following squares provided the coordinates designate squares on the board:

Column squares:	$(i \pm 1, j), (i \pm 2, j), \cdots$
Row squares:	$(i, j \pm 1), (i, j \pm 2), \cdots$

Diagonal squares: $(i \pm 1, j \pm 1), (i \pm 2, j \pm 2), \cdots$

$(i \pm 1, j \mp 1), (i \pm 2, j \mp 2), \cdots$

Reflected diagonal squares: $(1, j \pm (i+1)), (2, j \pm (i+2)), \cdots$

Now it is easy to verify the following statement and its converse: if $(a_1, a_2, \cdots, a_n)$ is a permutation of the numbers $1, 2, \cdots, n$ such that all of the numbers $n+i \pm a_i$, $i=1, 2, \cdots, n$ are distinct, then no pair of reflecting queens placed on the squares $(a_1, n+1), (a_2, n+2), \cdots, (a_n, 2n)$ attack one another. If $(a_1, n+1), (a_2, n+2), \cdots, (a_n, 2n)$ corresponds to a reflecting queens solution, then so does $(a_n, n+1), (a_{n-1}, n+2), \cdots, (a_1, 2n)$, since the latter corresponds to a reflection about the center column of the first solution. Rotations and reflections of solutions of the queens problem are regarded as the same; in the reflecting queens problem the situation is altered. Here the board has only one symmetry (i.e., the one just mentioned) since a rotation shifts the reflecting strip to another edge. Fig. 2 depicts a solution of the queens problem which corresponds to two essentially different solutions of the reflecting queens problem; namely, those corresponding to the sets of pairs (2, 9), (7, 10), (5, 11), (8, 12), (1, 13), (4, 14), (6, 15), (3, 16) and (4, 9), (2, 10), (7, 11), (3, 12), (6, 13), (8, 14), (1, 15), (5, 16).

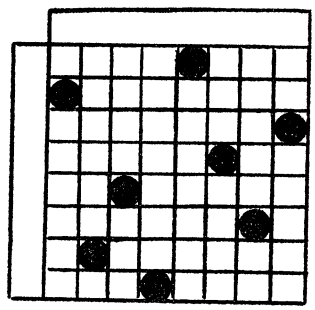


FIG. 2

Kraitchik listed in [1] all of the essentially different solutions of the queens problem for $n=4$, 5, 6 and 8 and gave the number of these solutions, $Q(n)$, for $n \leqq 12$. We list the solutions of the queens problem for $n \leqq 8$ below using the notation for the reflecting queens solutions with $(a_1, a_2, \cdots, a_n)$ an abbreviation of $\{(a_1, n+1), (a_2, n+2), \cdots\}$.

SOLUTIONS OF THE QUEENS PROBLEM

$n=4$: $A=(3, 1, 4, 2)$

$n=5$: $B_1=(3, 1, 4, 2, 5)$, $B_2=(2, 5, 3, 1, 4)$

$n=6$: $C=(4, 1, 5, 2, 6, 3)$

$n=7$: $D_1=(5, 1, 4, 7, 3, 6, 2)$, $D_2=(1, 4, 7, 3, 6, 2, 5)$

$D_3=(4, 7, 3, 6, 2, 5, 1)$, $D_4=(2, 5, 1, 4, 7, 3, 6)$

$D_5=(2, 5, 7, 4, 1, 3, 6)$, $D_6=(3, 1, 6, 2, 5, 7, 4)$

$n=8$: $E_1=(2, 5, 7, 4, 1, 8, 6, 3)$, $E_2=(5, 1, 4, 6, 8, 2, 7, 3)$
$E_3=(5, 7, 1, 4, 2, 8, 6, 3)$, $E_4=(2, 7, 5, 8, 1, 4, 6, 3)$
$E_5=(1, 5, 8, 6, 3, 7, 2, 4)$, $E_6=(1, 6, 8, 3, 7, 4, 2, 5)$
$E_7=(2, 4, 6, 8, 3, 1, 7, 5)$, $E_8=(2, 5, 7, 1, 3, 8, 6, 4)$
$E_9=(2, 6, 1, 7, 4, 8, 3, 5)$, $E_{10}=(2, 7, 3, 6, 8, 5, 1, 4)$
$E_{11}=(3, 6, 2, 5, 8, 1, 7, 4)$, $E_{12}=(3, 5, 2, 8, 1, 7, 4, 6)$

Using this list it is not difficult to check each of the solutions of the queens problem for $n \leqq 8$ and their rotations to find that the reflecting queens problem for $n \leqq 8$ has only the following solutions: $n=4$, A; $n=5$, B_1; $n=6$, none; $n=7$, D_1; $n=8$, E_1, E_2, E_3, E_4, and a rotation of E_4 shown in Figure 2. The number of solutions of the queens problem is 46, 92, 351, and 1787 for $n=9$, 10, 11 and 12 respectively. It seems likely that some of these solutions correspond to reflecting queens solutions, but nothing is known for $n>8$. We are led to conjecture on the basis of this evidence that the reflecting queens problem has at least one solution for every $n>6$; if this were the case, the Shen question would consequently be answered in the affirmative.

References

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CALCULATING THE GENERAL SOLUTION OF A LINEAR DIOPHANTINE EQUATION

JAMES BOND, Indiana University

Weinstock [1] observed the usual proof for the existence of the g.c.d. (greatest common divisor) d of nonzero integers $a_1, \cdots, a_n$ by showing that d is the minimal positive element of the set of all integral combinations of $a_1, \cdots, a_n$ had a computational aspect. He developed an algorithm for determining d which also permitted the determination of integers $x_1, \cdots, x_n$ such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$. The paper ends with the remark that the algorithm gives only particular solutions of the above equation. We show how to obtain a general solution of such an equation by using the algorithm $n-1$ times.

We begin with a short presentation of Weinstock's algorithm based on the division algorithm for least positive remainders. See [1] for the often more effective algorithm based on least absolute value remainders. Suppose $a_1, \cdots, a_n$ are given nonzero integers with g.c.d. d. Choose integers $l_1^{(0)}, \cdots, l_n^{(0)}$ such that $0 < r_0 = a_1l_1^{(0)} + \cdots + a_nl_n^{(0)}$. If $r_0 \mid a_1, r_0 \mid a_2, \cdots, r_0 \mid a_n$ then $r_0 = d$ and we have finished. If not, suppose that $r_0 \nmid a_i$ for some i, $1 \leqq i \leqq n$. Then $a_i = r_0q_1 + r_1$, $0 < r_1 < r_0$, with q_1, r_1 integers. We have $r_1 = a_i - r_0q_1 = a_1l_1^{(1)} + \cdots + a_nl_n^{(1)}$, where $l_1^{(1)} = -q_1l_1^{(0)}, \cdots, l_{i-1}^{(1)} = -q_1l_{i-1}^{(0)}$, $l_i^{(1)} = 1 - q_1l_i^{(0)}l_{i+1}^{(0)} = -q_1l_{i+1}^{(0)}, \cdots, l_n^{(1)} = -q_1l_n^{(0)}$. After a finite number of steps we may suppose an index k reached for the first time for which $r_k \mid a_1, \cdots, r_k \mid a_n$, hence $d = r_k = a_1l_1^{(k)} + \cdots + a_nl_n^{(k)}$.

We now exhibit the general solution of a linear Diophantine equation in a form suitable for the purpose of this note. Given nonzero integers $a_1, \cdots, a_n$,

let $d_1=a_1$, $d_2=$g.c.d. of a_1 and a_2, $\cdots$, $d_n=$g.c.d. of $a_1, \cdots, a_n$. Then the general solution in integers of the equation

$a_1x_1+a_2x_2+\cdots+a_nx_n=t_nd_n$, t_n an integer, $n\geqq 2$, is given by

$$\begin{aligned}
x_1 &= t_nx_1^{(n)} + t_{n-1}\frac{a_n}{d_n}\,x_1^{(n-1)} + t_{n-2}\frac{a_{n-1}}{d_{n-1}}\,x_1^{(n-2)} + \cdots + t_1\frac{a_2}{d_2}\\
x_2 &= t_nx_2^{(n)} + t_{n-1}\frac{a_n}{d_n}\,x_2^{(n-1)} + t_{n-2}\frac{a_{n-1}}{d_{n-1}}\,x_2^{(n-2)} + \cdots - t_1\frac{d_1}{d_2}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_{n-1} &= t_nx_{n-1}^{(n)} + t_{n-1}\frac{a_n}{d_n}\,x_{n-1}^{(n-1)} - t_{n-2}\frac{d_{n-2}}{d_{n-1}}\\
x_n &= t_nx_n^{(n)} - t_{n-1}\frac{d_{n-1}}{d_n}
\end{aligned}$$

where $t_1, \cdots, t_{n-1}$ are arbitrary integers, $d_2=a_1x_1^{(2)}+a_2x_2^{(2)}$, $d_3=a_1x_1^{(3)}+a_2x_2^{(3)}+a_3x_3^{(3)}$, $\cdots$, and $d_n=a_1x_1^{(n)}+\cdots+a_nx_n^{(n)}$.

We give only the inductive step necessary for proving this result. Suppose the result proved for $n=k$, $k\geqq 2$. Let $x_1^{(k+1)}, \cdots, x_{k+1}^{(k+1)}$ be a solution of $a_1x_1+\cdots+a_{k+1}x_{k+1}=d_{k+1}$ and suppose that $x_1, \cdots, x_{k+1}$ is a solution of $a_1x_1+\cdots+a_{k+1}x_{k+1}=t_{k+1}d_{k+1}$, with $a_1, \cdots, a_{k+1}, t_{k+1}$ nonzero integers. Then

$$a_1(x_1 - t_{k+1}x_1^{(k+1)}) + \cdots + a_{k+1}(x_{k+1} - t_{k+1}x_{k+1}^{(k+1)}) = 0$$

which we rewrite as

$$\frac{a_1}{d_{k+1}}(x_1 - t_{k+1}x_1^{(k+1)}) + \cdots + \frac{a_k}{d_{k+1}}(x_k - t_{k+1}x_k^{(k+1)}) = -\frac{a_{k+1}}{d_{k+1}}(x_{k+1} - t_{k+1}x_{k+1}^{(k+1)}).$$

Since $a_1/d_{k+1}, \cdots, a_{k+1}/d_{k+1}$ are relatively prime, this equation holds if and only if

$$x_{k+1} - t_{k+1}x_{k+1}^{(k+1)} = (-t_k)\frac{d_k}{d_{k+1}}.$$

For each choice of an integer t_k, by induction

$$\begin{aligned}
x_1 - t_{k+1}x_1^{(k+1)} &= t_k\frac{a_{k+1}}{d_{k+1}}\,x_1^{(k)} + t_{k-1}\frac{a_k/d_{k+1}}{d_k/d_{k+1}}\,x_1^{(k-1)} + \cdots + t_1\frac{a_2/d_{k+1}}{d_2/d_{k+1}}\\
x_2 - t_{k+1}x_2^{(k+1)} &= t_k\frac{a_{k+1}}{d_{k+1}}\,x_2^{(k)} + t_{k-1}\frac{a_k/d_{k+1}}{d_k/d_{k+1}}\,x_2^{(k-1)} + \cdots - t_1\frac{d_1/d_{k+1}}{d_2/d_{k+1}}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_k - t_{k+1}x_k^{(k+1)} &= t_k\frac{a_{k+1}}{d_{k+1}}\,x_k^{(k)} - t_{k-1}\frac{d_{k-1}/d_{k+1}}{d_k/d_{k+1}},
\end{aligned}$$

where $t_1, \cdots, t_{k-1}$ are arbitrary integers and

$$\frac{a_1}{d_{k+1}} x_1^{(i)} + \cdots + \frac{a_i}{d_{k+1}} x_i^{(i)} = \frac{d_i}{d_{k+1}}, \qquad \text{for } i = 2, \cdots, k.$$

Transposing the terms involving t_{k+1} to the right side and simplifying, we obtain the desired form for $x_1, \cdots, x_{k+1}$. It is easy to verify that each choice of integers $t_1, \cdots, t_k$ does lead to a solution in integers of $a_1x_1 + \cdots + a_{k+1}x_k = t_{k+1}d_{k+1}$.

Once the general solution of a linear Diophantine equation is seen to be in the above form it remains only to determine $d_2, \cdots, d_n$, and $x_1^{(2)}, x_2^{(2)}, \cdots, x_1^{(n)}, \cdots, x_n^{(n)}$. Apply Weinstock's algorithm to $\{a_1, a_2\}$ to obtain $r_{k_2}, l_1^{(k_2)}, l_2^{(k_2)}$ such that $d_2 = r_{k_2} = a_1l_1^{(k_2)} + a_2l_2^{(k_2)}$. Next apply Weinstock's algorithm to $\{a_1, a_2, a_3\}$ to obtain $r_{k_3}, l_1^{(k_3)}, l_2^{(k_3)}, l_3^{(k_3)}$ such that $d_3 = r_{k_3} = a_1l_1^{(k_3)} + a_2l_2^{(k_3)} + a_3l_3^{(k_3)}$. We may continue in this manner finally obtaining $r_{k_n}, l_1^{(k_n)}, \cdots, l_n^{(k_n)}$ such that $d_n = r_{k_n} = a_1l_1^{(k_n)} + \cdots + a_nl_n^{(k_n)}$. Set $x_1^{(2)} = l_1^{(k_2)}, x_2^{(2)} = l_2^{(k_2)}, \cdots, x_1^{(n)} = l_1^{(k_n)}, \cdots, x_n^{(n)} = l_n^{(k_n)}$.

Reference

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SUBDIRECTLY IRREDUCIBLE UNARY ALGEBRAS

MICHAEL YOELI, Technion, Haifa, Israel

I. Introduction. In this paper we are concerned with systems $\boldsymbol{A} = (A, \Gamma)$, where A is a finite set (of *points*) and Γ is a transformation of A, i.e., a function $\Gamma: A \to A$. Such a system is called a (finite) *unary algebra* [1], i.e., an abstract algebra with a single unary operation. Alternatively, it can be considered as complete transition graph [2], i.e., a directed graph with exactly one edge issuing from each vertex. Representations of arbitrary abstract algebras as subsystems of direct products have been studied extensively by means of lattices of congruences [3]. In this paper we consider such *subdirect* representations of unary algebras, applying simple structural properties of their graphs. Our main result gives necessary and sufficient conditions for unary algebras represented by a connected graph to be irreducible, i.e., to have only trivial subdirect representations.

The results obtained are applicable to the technically significant problem of synthesizing single-input sequential machines (automata) by parallel composition [4]. For more information on sequential machine compositions (automata products) the reader is referred to [5] and the further bibliography given there.

II. Statement of main theorem. Henceforth let $\boldsymbol{A} = (A, \Gamma)$, $\boldsymbol{A}_1 = (A_1, \Gamma_1)$, etc., denote unary algebras. The image of $a \in A$ under Γ will be denoted by $a\Gamma$.

DIRECT PRODUCT. $\boldsymbol{A} = \boldsymbol{A}_1 \times \boldsymbol{A}_2$ means that $A = A_1 \times A_2$ and that Γ is defined componentwise: $(a_1, a_2)\Gamma = (a_1\Gamma_1, a_2\Gamma_2)$.

SUBALGEBRA. $A_1 \subseteq A_2$ means that $A_1 \subseteq A_2$, and that Γ_1 is the restriction of Γ_2 to A_1. Thus, A_1 has to be closed under Γ_2. $\boldsymbol{A}_1 \leqq \boldsymbol{A}_2$ will mean that $\boldsymbol{A}_1$ is isomorphic to some subalgebra of $\boldsymbol{A}_2$.

$\boldsymbol{A}$ is *irreducible* if $\boldsymbol{A} \leqq \boldsymbol{A}_1 \times \boldsymbol{A}_2$ always implies that either $\boldsymbol{A} \leqq \boldsymbol{A}_1$ or $\boldsymbol{A} \leqq \boldsymbol{A}_2$. $\boldsymbol{A}$ is a *cycle* of length n, if its points can be ordered to form a sequence $(x_0, x_1, \cdots, x_{n-1})$ such that $x_{i-1}\Gamma = x_i (i = 1, \cdots, n-1)$ and $x_{n-1}\Gamma = x_0$.

$\boldsymbol{A}$ is *connected*, if it contains only one cycle as subgraph. One easily verifies that $\boldsymbol{A}$ is connected iff the corresponding graph (considered as a nonoriented graph) is connected.

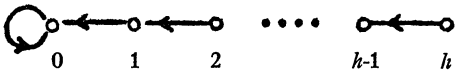


FIG. 1. Graph of unary algebra J_h

For any positive integer h we denote by J_h the unary algebra indicated in Fig. 1. We have:

THEOREM 1. *A connected unary algebra A is irreducible, iff $\boldsymbol{A}$ is either a cycle of length n, where n is a power of a prime number, or $\boldsymbol{A} \cong J_h$ for some $h \geqq 1$.*

III. Proof of main theorem. To prove Theorem 1, we shall derive some lemmas, and introduce some additional notation and terminology (cf. [1]).

For any positive integers n_1 and n_2 we denote by (n_1, n_2) and $[n_1, n_2]$ their g.c.d. and l.c.m., respectively. The following result is well known and easily verified.

LEMMA 1. *If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are connected unary algebras containing cycles of lengths n_1 and n_2 respectively, then $\boldsymbol{A}_1 \times \boldsymbol{A}_2$ contains (n_1, n_2) cycles, each of length $[n_1, n_2]$.*

If $A_1 \cap A_2 = \varnothing$, we set $\boldsymbol{A} = \boldsymbol{A}_1 + \boldsymbol{A}_2$, where $A = A_1 \cup A_2$ and $a_i\Gamma = a_i\Gamma_i$ for every $a_i \in A_i (i = 1, 2)$.

Clearly, every unary algebra is the sum of connected unary algebras, and the operation $\times$ is distributive with respect to $+$. Hence

LEMMA 2. *If $\boldsymbol{A}$ is a connected unary algebra, and $\boldsymbol{A} \leqq \boldsymbol{A}_1 \times \boldsymbol{A}_2$, then there exist connected unary algebras $\boldsymbol{A}_1'$ and $\boldsymbol{A}_2'$ such that $\boldsymbol{A}_i' \subseteq \boldsymbol{A}_i$ $(i = 1, 2)$ and $\boldsymbol{A} \leqq \boldsymbol{A}_1' \times \boldsymbol{A}_2'$.*

A further, rather evident result is the following:

LEMMA 3. *A cycle of length n is irreducible, iff n is some power of a prime number.*

If Γ is a transformation of the set A, we define Γ^r, $r \geqq 0$, as usual: $a\Gamma^0 = a$, $a\Gamma^{r+1} = (a\Gamma^r)\Gamma$, for every $a \in A$.

If a and b are points of $\boldsymbol{A}$, such that $a\Gamma^r = b$, for some $r \geqq 0$, then a is a *predecessor* of b. $P(a)$ will denote the set of all predecessors of a. The height $|a|$ of a point a is the least integer $r \geqq 0$, for which $a\Gamma^r$ is *cyclic* (i.e., belongs to a cycle). We set $|\boldsymbol{A}| = \max_{a \in A} |a|$. Let $\boldsymbol{A} = \boldsymbol{A}_1 \times \boldsymbol{A}_2$.

If $a=(a_1, a_2)$ is a point of $\boldsymbol{A}$, then $a\Gamma^r=(a_1\Gamma_1^r, a_2\Gamma_2^r)$ is cyclic, iff both $a_1\Gamma_1^r$ and $a_2\Gamma_2^r$ are cyclic. Hence

LEMMA 4. $|\boldsymbol{A}_1\times\boldsymbol{A}_2| = \max(|\boldsymbol{A}_1|, |\boldsymbol{A}_2|)$.

A simple consequence is the following:

LEMMA 5. *For every $h\geqq 1$, the unary algebra J_h (Fig. 1) is irreducible.*

A unary algebra is *basic* [1], if it is connected and its cycle has length 1. We shall now prove

LEMMA 6. *If $\boldsymbol{A}$ is a basic unary algebra which contains at least two points of height 1, then $\boldsymbol{A}$ is reducible.*

Proof. Let x be the unique cyclic point of $\boldsymbol{A}$, and $a\neq b$ two points of height 1. Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be the subalgebras of $\boldsymbol{A}$, such that $A_1=\{x\}\cup P(a)$ and $A_2=A-P(a)$. Since $b\notin A_1$ and $a\notin A_2$, the $\boldsymbol{A}_i$ are *proper* subalgebras of $\boldsymbol{A}$. One easily verifies that $\boldsymbol{A}\leqq\boldsymbol{A}_1\times\boldsymbol{A}_2$. Thus, $\boldsymbol{A}$ is reducible.

Let $\boldsymbol{A}=(A, \Gamma)$ and $A_1\subseteq A$. We define $\boldsymbol{A}/A_1$ to be the unary algebra (A_1, Γ_1), where $a_1\Gamma_1=a_1\Gamma$ if both a_1 and $a_1\Gamma$ belong to A_1, whereas $a_1\Gamma_1=a_1$ if $a_1\in A_1$, but $a_1\Gamma\notin A_1$.

Next, we prove the following

LEMMA 7. *Let $\boldsymbol{A}$ be a basic unary algebra and let $|\boldsymbol{A}|=h\geqq 1$. Then $\boldsymbol{A}$ is irreducible, iff $\boldsymbol{A}\cong J_h$.*

Proof. If $\boldsymbol{A}\cong J_h$, then A is irreducible, by Lemma 5. Assume now that $\boldsymbol{A}$ is irreducible. By Lemma 6, and since $|\boldsymbol{A}|\geqq 1$, there exists exactly one point a_1 of height 1.

We set $\boldsymbol{A}_1=\boldsymbol{A}/P(a_1)$. Then $\boldsymbol{A}\leqq\boldsymbol{A}_1\times J_h$. Indeed, one easily verifies that $\boldsymbol{A}$ is isomorphic to $(\boldsymbol{A}_1\times J_h)/A_2$, where

$$A_2=\{(a, |a|)\mid a\in P(a_1)\}\cup\{(a_1, 0)\}.$$

$\boldsymbol{A}$ was assumed to be irreducible. Since $P(a_1)$, the set of points of $\boldsymbol{A}_1$, is a proper subset of A, we must have $\boldsymbol{A}\leqq J_h$, where $|\boldsymbol{A}|=h$. Hence $\boldsymbol{A}\cong J_h$.

Let $\boldsymbol{A}=(A, \Gamma)$ be an arbitrary unary algebra and x a cyclic point of $\boldsymbol{A}$. We set $T(x)=\{a\mid a\in A \text{ and } a\Gamma^{|a|}=x\}$. Then $\boldsymbol{A}/T(x)$ is basic and we have

LEMMA 8. *Let $\boldsymbol{A}_1$ be a basic unary algebra with cyclic point x_1, $\boldsymbol{A}_2$ a cycle, and x_2 a point of $\boldsymbol{A}_2$. Then $(\boldsymbol{A}_1\times\boldsymbol{A}_2)/T(x_1, x_2)\cong\boldsymbol{A}_1$.*

Proof. Consider the mapping $\eta: a_1\to(a_1, a_2)$ of A_1 into $A_1\times A_2$, where a_2 is the point of $\boldsymbol{A}_2$ determined by $a_2\Gamma^{|a_1|}=x_2$. It is easily seen that this mapping η is an isomorphism of $\boldsymbol{A}_1$ onto $(\boldsymbol{A}_1\times\boldsymbol{A}_2)/T(x_1, x_2)$.

LEMMA 9. *Let $\boldsymbol{A}$ be a connected unary algebra with a cycle of length $n>1$, such that $|\boldsymbol{A}|\geqq 1$. Then $\boldsymbol{A}$ is reducible.*

Proof. Let $\boldsymbol{A}_1$ be the basic algebra obtained from $\boldsymbol{A}$ by identifying all its

cyclic points, and A_2 the cycle of A. Then $A_1 \times A_2$ is a connected unary algebra with cycle length n. Let x_1 be the cyclic point of A_1 and x, x_2 arbitrary cycle points of A and A_2, respectively. Clearly $A/T(x) \leqq A_1$. Hence, by Lemma 8, $A/T(x) \leqq (A_1 \times A_2)/T(x_1, x_2)$. This implies that $A \leqq A_1 \times A_2$. Since both A_1 and A_2 have fewer points than A, A is reducible. We are now in a position to prove Theorem 1.

Proof of Theorem 1. The "if" part follows immediately from Lemmas 3 and 5. Let now A be a connected unary algebra which is irreducible. Let n be the cycle length of A. If $n=1$, i.e., A is basic, and $|A| = h \geqq 1$, then by Lemma 7, $A \cong J_h$. If $n=1$ and $|A| = 0$, then A is a cycle of length 1. If $n>1$, then by Lemma 9, $|A| = 0$, i.e., A is a cycle of length n. By Lemma 3, n is a power of a prime number. This completes the proof.

Acknowledgments. The author is grateful to Dr. C. M. Ablow, Stanford Research Institute, for helpful discussions and comments.

This research was done at the Stanford Research Institute, Menlo Park, California, under Contract No. AF 19(628)-5092 with the Air Force Cambridge Research Laboratories, Office of Aerospace Research, United States Air Force, Bedford, Massachusetts.

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ON A THEOREM OF JENSEN

R. S. KULKARNI, Harvard University

Let K be a finite algebraic number field, L a finite extension of K, A (resp. B) ring of integers in K (resp. L). Fröhlich [1] has completely determined the structure of B as an A-module. Jensen [2] has obtained corresponding results when L is an infinite algebraic extension of K, in particular, he proved that in this case, there always exist relative integral bases; he also obtained necessary and sufficient conditions for B-submodules of L to have a free A-basis.

In this note, we give a characterization of countably generated projective modules over a Dedekind domain. Use of condition (2) in the theorem immediately yields Jensen's results. Jensen's proofs seem to be somewhat complicated.

Jensen pointed out to me that this formulation may be regarded as a generalization of Pontryagin's theorem; cf. L. Fuchs [3, Section 14].

THEOREM. *Let A be a Dedekind domain, and K its quotient field. Let P be a torsion-free A-module of countable rank, i.e.,* $\dim_K P \otimes_A K \leqq \aleph_0$.

Then the following conditions are equivalent:

(1) *P is projective.*

NO GRAPH IS PERFECT

MEHDI BEHZAD, Wayne State University and Pahlavi University, Iran, and
GARY CHARTRAND, University of Michigan and Western Michigan University

Given a finite sequence of nonnegative integers $s_1 \leqq s_2 \leqq \cdots \leqq s_p$, it is not an easy matter, in general, to determine whether these numbers are the degrees of the points of some graph with p points. Certainly, if such a graph G exists, then $0 \leqq s_i \leqq p-1$ for each i and Σs_i is even (for this is twice the number of lines of G), but these conditions fall far short of being sufficient. Methods have been devised to test whether a given sequence of integers is "graphical" (see [1], [2]), but the procedures involved are often tedious.

It is the object of this note to show that with certain sequences, the question of whether they are graphical can be answered quickly.

The only graphs under consideration here are ordinary graphs, i.e., finite, undirected graphs without loops or multiple lines. We denote the degree of a point v in G (the number of lines of G incident with v) by $\deg v$.

A graph with two or more points is called *perfect* if for each pair of distinct points u and v, $\deg u \neq \deg v$, i.e., no two points have the same degree.

THEOREM 1. *No graph is perfect.*

Proof. If G is perfect and has p points $v_1, v_2, \cdots, v_p$, where $\deg v_i < \deg v_{i+1}$, then since $0 \leqq \deg v_i \leqq p-1$, it follows that $\deg v_i = i-1$ for $i = 1, 2, \cdots, p$. This implies, however, that v_1 is adjacent to no points of G while v_p is adjacent to all other points of G, which is clearly impossible. Hence, no graph is perfect.

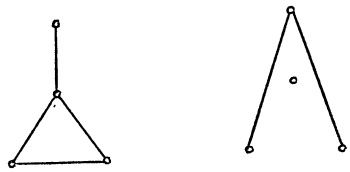


FIG. 1

Since no sequence of distinct integers can be graphical, we consider those sequences in which exactly two integers are equal. A graph with at least two points is *quasiperfect* if there are precisely two points with the same degree. Contrary to the situation with perfect graphs, quasiperfect graphs do exist. Each of the graphs of Figure 1 has 4 points and is quasiperfect.

The *complement* $\overline{G}$ of a graph G is the graph having the same points as G and in which two points are adjacent if and only if these two points are not adjacent in G.

The following two observations concern quasiperfect graphs with p points,

the first of which is a direct result of the fact that if a point v of G has degree s, then v has degree $p-s-1$ in $\overline{G}$.

Remark 1. $\overline{G}$ is quasiperfect if and only if G is.

Remark 2. A quasiperfect graph with $p \geqq 3$ points has either exactly one point of degree $p-1$ or exactly one of degree 0.

Concerning Remark 2, since all degrees lie among the p integers $0, 1, \cdots, p-1$, the degrees of the points of a quasiperfect graph G take on all these values except one. Hence, at least one of 0 and $p-1$ is the degree of a point of G. We have already seen that no graph can contain both a point of degree 0 and one of degree $p-1$. If a quasiperfect graph with at least 3 points contains two points of degree 0 (or, equivalently, two of degree $p-1$) then the deletion of one of these results in a perfect graph, contrary to Theorem 1.

THEOREM 2. *For every integer $p \geqq 2$, there are exactly two nonisomorphic quasiperfect graphs with p points. Furthermore, these two graphs are complementary and the equal degrees are $[p/2]$ in the connected graph and $[(p-1)/2]$ in the disconnected graph.*

Proof. We proceed by induction on p, with the theorem being obvious for $p=2$. Assume that the result holds for $p=n$, $n \geqq 2$, the connected graph being G_1 and its disconnected complement G_2. Two quasiperfect graphs H_1 and H_2 with $n+1$ points can be constructed by adding a new point to G_1 and adding a new point adjacent to all points of G_2, respectively. Clearly, H_1 and H_2 are nonisomorphic complementary graphs. That these are the only quasiperfect graphs with $n+1$ points is readily observed, for if G is any quasiperfect graph with $n+1$ points, the removal of the point having degree 0 or n results in either G_1 or G_2, implying that G is H_1 or H_2, respectively.

By hypothesis, the equal degrees in G_1 are $[n/2]$. These points with the equal degrees have the same degrees in the disconnected graph H_1, and the result follows by observing that $[n/2]=[((n+1)-1)/2]$. The equal degrees in G_2 are $[(n-1)/2]$, and so the equal degrees in H_2 are $[(n-1)/2]+1=[(n+1)/2]$.

Theorem 2 thus implies that a sequence of p integers having exactly two equal terms is graphical if and only if the sequence is either 0 through $p-2$ with $[(p-1)/2]$ appearing twice or 1 through $p-1$ with $[p/2]$ appearing twice, and in each case, the corresponding graph is unique up to isomorphism.

Work of the second author was supported by the U. S. Air Force Office of Scientific Research under grant AF-AFOSR-754-65.

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AN ELEMENTARY AND SIMPLE PROOF OF THE CONNECTEDNESS OF THE CLASSICAL GROUPS

YUNG-CHOW WONG AND YIK-HOI AU-YEUNG, University of Hong Kong

In his book [1], C. Chevalley, whose notations we follow, considers the connected classical groups $GL(n, C)$, $SL(n, C)$, $U(n)$, $SU(n)$, $SO(n)$, $SO(n, C)$, $SL(n, R)$, $Sp(n)$ and $Sp(n, C)$, and proves by topological methods the connectedness of $GL(n, C)$, $U(n)$, $SU(n)$, $SO(n)$ and $Sp(n)(n \geqq 1)$ ([1], p. 16 and p. 37). On the other hand H. Weyl has proved by algebraic methods that $U(n)$ is arcwise connected ([2], p. 194) and $SU(n)$ is simply connected ([2], p. 268). Furthermore, E. Sperner has proved by algebraic methods that the set of all real matrices with positive (resp. negative) determinant is arcwise connected ([3], p. 160).

In this note, we give an elementary and simple proof of the arcwise connectedness of all the classical groups. (Since the classical groups are all manifolds, their connectedness is equivalent to their arcwise connectedness.) The idea of the proof is: Let Γ be the group under consideration. We shall find in Γ a fixed element E such that for any $A \in \Gamma$, there is an arc $A(t)$, $0 \leqq t \leqq 1$, in Γ with $A(0) = E$ and $A(1) = A$. In fact, in the case of the group $Sp(n, C)$, we shall take

$$E = \begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

where O, I are respectively the $n \times n$ zero matrix and the $n \times n$ identity matrix. In all the other cases we shall take $E = I$. The construction of the arc $A(t)$ is based entirely on known theorems in matrix theory.

THEOREM. *Each of the classical groups $GL(n, C)$, $SL(n, C)$, $U(n)$, $SU(n)$, $SO(n)$, $SO(n, C)$, $SL(n, R)$, $Sp(n)$ and $Sp(n, C)$ is arcwise connected.*

Proof. In the proof all the matrices considered are of order $n(\geqq 1)$ unless stated otherwise. We shall denote by $(K_1, \cdots, K_p)$ a diagonal block matrix with the square matrix $K_1, \cdots, K_p$ lying on its diagonal.

(a) *The connectedness of $GL(n, C)$.* Let $A_1 \in GL(n, C)$. Then there exists a complex matrix P such that

$$A_1 = \exp P$$

([4], p. 18). The arc required is $A_1(t) = \exp(tP)$.

(b) *The connectedness of $SL(n, C)$.* Let $A_2 \in SL(n, C)$. Then there exists a complex matrix P such that

$$A_2 = \exp P$$

and trace $P = 2k\pi\sqrt{-1}$ for some integer k. The arc required is

$$A_2(t) = \exp(tP) \exp(tQ),$$

where $Q = (0, \cdots, 0, -2k\pi\sqrt{-1})$.

(c) *The connectedness of* $U(n)$. Let $A_3 \in U(n)$. Then we can write

$$A_3 = U(\exp(a_1\sqrt{-1}), \cdots, \exp(a_n\sqrt{-1}))U^{-1},$$

where $U \in U(n)$ and $a_1, \cdots, a_n$ are real numbers ([5], p. 274). The arc required is

$$A_3(t) = U(\exp(ta_1\sqrt{-1}), \cdots, \exp(ta_n\sqrt{-1}))U^{-1}.$$

(d) *The connectedness of* $SU(n)$. Let $A_4 \in SU(n)$. Then A_4 can be expressed in the same form as the A_3 in (c) but with the additional condition that $\sum_{i=1}^{n} a_i = 2k\pi$ for some integer k. The arc required is

$$A_4(t) = U(\exp(ta_1\sqrt{-1}), \cdots, \exp(ta_{n-1}\sqrt{-1}), \exp(t(a_n - 2k\pi)\sqrt{-1}))U^{-1}.$$

(e) *The connectedness of* $SO(n)$. Let $A_5 \in SO(n)$. Then we can write

$$A_5 = V(K_1, \cdots, K_q, 1, \cdots, 1)V^{-1},$$

where $V \in O(n)$, q is zero or some positive integer $\leqq n/2$, and

$$K_\lambda = \begin{pmatrix} \cos\theta_\lambda & \sin\theta_\lambda \\ -\sin\theta_\lambda & \cos\theta_\lambda \end{pmatrix} \qquad (\lambda = 1, \cdots, q)$$

([5], p. 285). If we put

$$K_\lambda(t) = \begin{pmatrix} \cos(t\theta_\lambda) & \sin(t\theta_\lambda) \\ -\sin(t\theta_\lambda) & \cos(t\theta_\lambda) \end{pmatrix} \qquad (\lambda = 1, \cdots, q),$$

then the arc required is

$$A_5(t) = V(K_1(t), \cdots, K_q(t), 1, \cdots, 1)V^{-1}.$$

(f) *The connectedness of* $SO(n, C)$. Let $A_6 \in SO(n, C)$. Then we can write

$$A_6 = A_5 \exp(\sqrt{-1}\, D),$$

where $A_5 \in SO(n)$ and D is a real skew-symmetric matrix ([6], p. 4). The arc required is

$$A_6(t) = A_5(t)\exp(t\sqrt{-1}D),$$

where $A_5(t)$ is constructed from A_5 as in (e).

(g) *The connectedness of* $SL(n, R)$. Let $A_7 \in SL(n, R)$. Then we can write

$$A_7 = FA_5,$$

where $A_5 \in SO(n)$ and F is a real positive definite symmetric matrix with determinant 1 ([5], p. 286 and p. 274). Let

$$F = V(\exp a_1, \cdots, \exp a_n)V^{-1},$$

where $V \in O(n)$ and $a_1, \cdots, a_n$ are real numbers such that $\sum_{i=1}^{n} a_i = 0$ ([5], p. 285). Then the arc required is

$$A_7(t) = V(\exp(ta_1), \cdots, \exp(ta_n))V^{-1}A_5(t),$$

where $A_5(t)$ is constructed from A_5 as in (e).

(h) *The connectedness of* $Sp(n)$. Let $A_8 \in Sp(n)$. Then we can write

$$A_8 = W(\exp(a_1\sqrt{-1}), \cdots, \exp(a_n\sqrt{-1}))W^{-1},$$

where $W \in Sp(n)$ and $a_1, \cdots, a_n$ are real numbers [7]. The arc required is

$$A_8(t) = W(\exp(ta_1\sqrt{-1}), \cdots, \exp(ta_n\sqrt{-1}))W^{-1}.$$

(i) *The connectedness of* $Sp(n, C)$. Let $A_9 \in Sp(n, C)$. Then we can write

$$A_9 = \begin{pmatrix} J & I-J \\ -(I-J) & J \end{pmatrix}\begin{pmatrix} I & O \\ G & I \end{pmatrix}\begin{pmatrix} I & H \\ O & I \end{pmatrix}\begin{pmatrix} A & O \\ O & A'^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} I & O \\ 2J & I \end{pmatrix}\begin{pmatrix} J & I-J \\ -(I+J) & J \end{pmatrix}\begin{pmatrix} I & O \\ G & I \end{pmatrix}\begin{pmatrix} I & H \\ O & I \end{pmatrix}\begin{pmatrix} A & O \\ O & A'^{-1} \end{pmatrix}$$

where each of the factors belongs to $Sp(n, C)$, A is nonsingular, A' is the transpose of A, G and H are both symmetric, and J is a diagonal matrix such that $J^2 = J$ [8]. Let P be a complex matrix such that $A = \exp P$, and let P' be the transpose of P. Then the arc required is

$$A_9(t) = \begin{pmatrix} I & O \\ 2tJ & I \end{pmatrix}\begin{pmatrix} tJ & I-tJ \\ -(I+tJ) & tJ \end{pmatrix}\begin{pmatrix} I & O \\ tG & I \end{pmatrix}\begin{pmatrix} I & tH \\ O & I \end{pmatrix}\begin{pmatrix} \exp(tP) & O \\ O & \exp(-tP') \end{pmatrix}.$$

REMARK. By the methods used in (e), (f) and (g), one can also prove respectively that the set of all matrices in $O(n)$ with determinant equal to -1, the set of all matrices in $O(n, C)$ with determinant equal to -1, and the set of all real matrices with positive (resp. negative) determinant are all arcwise connected.

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ON ITERATES OF EULER'S ϕ-FUNCTION

J. C. PARNAMI, Panjab University, India

1. Let $\phi_n(m)$ be the iterated ϕ-function such that $\phi_1(m)=\phi(m)$, $\phi_n(m)=\phi(\phi_{n-1}(m))$, and let $w(m)=n$ denote the smallest n for which $\phi_n(m)=1$.

In a problem in this MONTHLY, 71 (1964) p. 1047, Douglas Lind has stated that if

$$m \leqq 2^{\alpha} \quad \text{then} \quad w(m) \leqq \alpha.$$

A proof of this has been given by Satish Kumar Gupta. Here I prove the following two theorems, suggested by Professor Gupta.

THEOREM 1. *If* $m>2\cdot 3^j$, *then* $w(m)>j+1$.

THEOREM 2. $[\log_3 (3m)] \leqq w(m) \leqq [\log_2 (2m)]$.

2. LEMMA 1. *If* $m=2^{\alpha}\cdot m_1$; *where* $m_1 \geqq 2$, *then* $w(m)\geqq \alpha+1$.

Proof. For $\alpha=1$, we have $m\geqq 4$; and $w(m)\geqq 2$. Therefore the lemma is true for $\alpha=1$.

Let the lemma be true for $\alpha\leqq\beta$. To show that the lemma holds for $\alpha=\beta+1$, let $m=2^{\beta+1}\cdot m_1$, where $m_1\geqq 2$.

CASE 1. When m_1 is odd, then $\phi(m)=2^{\beta}\cdot\phi(m_1)$. Since $\phi(m_1)\geqq 2$; by the induction hypothesis, $w(\phi(m))\geqq\beta+1$. Hence $w(m)\geqq\beta+2$.

CASE 2. When m_1 is even, let $m_1=2^{\gamma}\cdot m_2$, where m_2 is odd and $\gamma\geqq 1$; then $m=2^{\beta+\gamma+1}\cdot m_2$. Therefore, $\phi(m)=2^{\beta+\gamma}\cdot\phi(m_2)=2^{\beta}\cdot(2^{\gamma}\cdot\phi(m_2))$. Hence $w(\phi(m))\geqq\beta+1$, so that $w(m)\geqq\beta+2$. This proves the lemma.

LEMMA 2. *If* $m=2^{\alpha}\cdot m_1$, *where* $m_1>3^{\delta}$ *and* $\alpha\geqq 1$, *then* $w(m)>\alpha+\delta$.

Proof. For $\delta=0$, the result is a direct consequence of Lemma 1. For $\delta=1$, let $m_1=2^{\beta}\cdot\gamma$, where γ is odd.

If $\beta\geqq 1$, then we have $m=2^{\alpha+1}\cdot(2^{\beta-1}\cdot\gamma)$, and $2^{\beta-1}\cdot\gamma=m_1/2\geqq 2$. Hence, by Lemma 1,

$$w(m) \geqq \alpha+2>\alpha+1.$$

If $\beta=0$, then $\phi(m)=2^{\alpha-1}\cdot\phi(m_1)$, since m_1 is odd and >3, $\phi(m)$ is even and $\geqq 4$. Therefore $\phi(m)=2^{\alpha}\cdot(\phi(m_1)/2)$.

Now by Lemma 1, $w(\phi(m))\geqq\alpha+1$, i.e., $w(m)\geqq\alpha+2>\alpha+1$, so the lemma is true for $\delta=1$.

Suppose that the lemma holds for $\delta\leqq j-1$. To prove the lemma for $\delta=j$, we proceed as follows:

CASE 1. When m_1 is even, we have $m=2^{\alpha+1}\cdot(m_1/2)$. Here $m_1/2>3^j/2>3^{j-1}$, therefore

$$w(m)>\alpha+1+j-1=\alpha+j.$$

CASE 2. When m_1 is odd, let $m_1 = p_{11}^{\beta} \cdot p_{22}^{\beta} \cdot \cdots \cdot p_{rr}^{\beta}$, $r \geqq 1$, where p_i's are distinct prime $\geqq 3$, and $\beta_i \geqq 1$, for $1 \leqq i \leqq r$. Then

$$\phi(m) = 2^{\alpha-1} \cdot \prod_{i=1}^{r} p_i^{\beta_i - 1}(p_i - 1),$$

$$= 2^{\alpha-1+r} \cdot 3^{-r} \left(\prod_{i=1}^{r} \frac{p_i^{\beta_i - 1} \cdot 3(p_i - 1)}{2} \right).$$

(i) When $r > j$, then, by Lemma 1, $w(\phi(m)) \geqq \alpha + r - 1 \geqq \alpha + j$, and $w(m) > \alpha + j$.

(ii) When $r \leqq j$, then

$$\phi(m) = 2^{\alpha+r-1} \cdot 3^{-r} \prod_{i=1}^{r} \frac{p_i^{\beta_i - 1} \cdot 3(p_i - 1)}{2}.$$

Now

$$\frac{3(p_i - 1)}{2} = \frac{2p_i + p_i - 3}{2} \geqq p_i \geqq 3.$$

Therefore $\phi(m) = 2^{\alpha+r-1} \cdot m_3$, where $m_3 > 3^{j-r}$. Hence $w(\phi(m)) > \alpha + r - 1 + j - r = \alpha + j - 1$. Hence $w(m) > \alpha + j$, which proves Lemma 2.

3. *Proof of Theorem* 1.

CASE 1. When m is even, let $m = 2 \cdot m_1$, where $m_1 > 3^j$. Then, by Lemma 2, $w(m) > j + 1$.

CASE 2. When $m > 2$, is odd, we have $\phi(m) = \phi(2m)$, and the result follows from Case 1.

4. *Proof of Theorem* 2.

CASE 1. When $m = 2 \cdot 3^j$, for some nonnegative integer j, then

$$[\log_3 (3m)] = j + 1 = w(m).$$

CASE 2. When $m \neq 2 \cdot 3^j$ for any nonnegative integer j, let $t \geqq 0$ be selected, so that

$$2 \cdot 3^{t+1} > m > 2 \cdot 3^t.$$

Then $w(m) > t + 1 > \log_3 m - \log_3 2$, i.e., $w(m) > [\log_3 m - \log_3 2] + 1$. Hence,

$$w(m) \geqq [\log_3 m - \log_3 2] + 2 \geqq [\log_3 (3m)],$$

which proves the left side of the inequality. Now choose r, a positive integer, such that $2^r < m \leqq 2^{r+1}$ then

$$w(m) \leqq r + 1 < \log_2 m + 1,$$

therefore $w(m) \leqq [\log_2 m] + 1 = [\log_2 (2m)]$, which proves the right side of the inequality. This completes the proof.

AMICABLE NUMBERS WITH OPPOSITE PARITY

A. A. GIOIA, Western Michigan University and A. M. VAIDYA, Gujarat University, Ahmedabad, India

1. Introduction. If $\sigma(n)$ denotes the sum of the positive divisors of the integer n, then n_1 and n_2 are called *amicable* if

$$\sigma(n_1) = n_2 \text{ and } \sigma(n_2) = n_1. \tag{1.1}$$

A well-known example of a pair of amicable numbers is 220 and 284. It is also known that if $n = \prod_{i=1}^{t} p_i^{a_i}$, then

$$\sigma(n) = \prod_{i=1}^{t} (1 + p_i + p_i^2 + \cdots + p_i^{a_i}). \tag{1.2}$$

Escott [1] gave a list of 290 pairs of amicable numbers. Recently Rolf [2] announced the new pair 79750, 88730. It is remarkable that in each of the 391 pairs, both numbers have the same parity. In this paper we consider necessary conditions for the existence of a pair of amicable numbers with opposite parity.

2. The first theorem introduces the notation which will be used throughout the paper.

THEOREM 2.1. *Suppose n_1 and n_2 are amicable, n_1 is even and n_2 is odd. Then $n_1 = 2^r M^2$, $r \geqq 1$, and $n_2 = N^2$, where M and N are odd integers larger than* 1.

Proof. Since n_1 and n_2 satisfy (1.1), both $\sigma(n_1)$ and $\sigma(n_2)$ are odd. But $\sigma(n)$ is odd if and only if n is a square or the double of a square; this follows directly from (1.2) since each factor of $\sigma(n)$ must be odd. Thus $n_1 = 2^r M^2$, $r \geqq 1$, M odd, and $n_2 = N^2$, N odd. From (1.1) it is obvious that $N \neq 1$.

If $M = 1$, we have $2^r - 1 = N^2$ from (1.1). If $r \geqq 2$, then $2^r - 1 \equiv 3 \pmod 4$, but $N^2 \equiv 1 \pmod 4$; also, $r = 1$ implies $N = 1$, which is impossible. Thus $M \neq 1$ and the proof is complete.

THEOREM 2.2. *If $n_1 = 2^r M^2$ and $n_2 = N^2$ are amicable numbers with opposite parity, then N must be composite.*

Proof. Suppose $M^2 = \prod_{i=1}^{t} p_i^{2\alpha_i}$, p_i odd primes, $\alpha_i > 0$, and assume N is a prime. Let a_i, b be the unique positive integers satisfying

$$2^{a_i - 1} < p_i < 2^{a_i}, \qquad i = 1, \cdots, t$$
$$2^{b-1} < N < 2^b.$$

If n_1 and n_2 are amicable then

$$(2^{r+1} - 1) \prod_{i=1}^{t} (1 + p_i + \cdots + p_i^{2\alpha_i}) = 1 + N + N^2,$$

$$(2^{r+1} - 1) \prod (1 + 2^{a_i} + \cdots + 2^{2\alpha_i a_i}) > 1 + 2^{b-1} + 2^{2(b-1)};$$

but the left side is less than $(2^{r+1}-1)\prod 2^{2\alpha_i a_i+1}$, and the right side is $2^{3(b-1)}-1 \geqq 2^{2(b-1)+1}-1$, since $b \geqq 2$. Thus,

$$(2^{r+1}-1)\prod 2^{2\alpha_i a_i+1} > 2^{2(b-1)+1}-1,$$

$$(2^{r+1}-1)\prod 2^{2\alpha_i a_i+1} \geqq 2^{2b-1},$$

$$2^r \prod 2^{2\alpha_i a_i+1} \geqq 2^{2b-1}.$$

From this it easily follows that

$$\frac{r+t+1}{2} + \sum_{i=1}^{t} \alpha_i a_i \geqq b. \tag{2.1}$$

Also, since n_1, n_2 are amicable

$$2^r \prod_{i=1}^{t} p_i^{2\alpha_i} + N^2 = 1 + N + N^2,$$

$$2^r \prod 2^{2\alpha_i(a_i-1)} \leqq 2^b$$

$$r + \sum 2\alpha_i(a_i-1) \leqq b.$$

Combining this with (2.1) gives

$$\sum_{i=1}^{t} \alpha_i a_i - 2\sum_{i=1}^{t} \alpha_i \leqq \frac{t-r+1}{2}. \tag{2.2}$$

We now consider two cases:

CASE I: $(p_i, 3)=1$, $i=1, \cdots, t$. Since $p_i \neq 3$, then $p_i \geqq 5$ and $a_i \geqq 3$, so

$$t \leqq \sum_{i=1}^{t} \alpha_i \leqq \sum \alpha_i a_i - 2\sum \alpha_i \leqq \frac{t-r+1}{2}.$$

Then $r+t \leqq 1$ which is impossible since r and t are positive integers.

CASE II: $p_1=3$. Then $a_1=2$. If $t>1$, (2.2) gives

$$\sum_{i=2}^{t} \alpha_i a_i - 2\sum_{i=2}^{t} \alpha_i \leqq \frac{t-r+1}{2}.$$

But $a_i \geqq 3 (i=2, \cdots, t)$; hence

$$t-1 \leqq \sum_{i=2}^{t} \alpha_i \leqq \frac{t-r+1}{2},$$

and $t+r \leqq 3$. Since $r>0$ and $t>1$, we have $t=2$ and $r=1$. Thus $n_1 = 2\cdot 3^{2\alpha_1} p_2^{2\alpha_2}$ and $\sigma(n_1)=\sigma(N^2)$ implies

$$3\sigma(3^{2\alpha_1} p_2^{2\alpha_2}) = 1 + N + N^2$$

from which it follows that $(N, 3)=1$. Now from $\sigma(n_1)=n_1+N^2$ we have

$$3\sigma(3^{2\alpha_1}p_2^{2\alpha_2}) = 2\cdot 3^{2\alpha_1}p^{2\alpha_2} + N^2$$

which implies $3 \mid N$. This proves $t \geqslant 1$.

With $t=1$ and $a_1=2$, equation (2.2) gives $r \leqq 2$. Assume that $r=2$. Then $n_1 = 4\cdot 3^{2\alpha}$. Condition (1.1) gives

$$7(1 + 3 + \cdots + 3^{2\alpha}) = 4\cdot 3^{2\alpha} + N^2, \text{ and} \tag{2.3}$$

$$1 + N + N^2 = 4\cdot 3^{2\alpha} + N^2. \tag{2.4}$$

Equating the expressions for N^2 obtained from (2.3) and (2.4) yields

$$7(1 + 3 + \cdots + 3^{2\alpha-1}) + 11\cdot 3^{2\alpha} = 16\cdot 3^{4\alpha} + 1.$$

But this is impossible; hence, $r=1$.

Finally, the remaining possibility that $t=r=1$ is easily dismissed. In fact this can be done without assuming that $p_1=3$ or that N is a prime. Say $n_1 = 2p^2$, prime p. Then we have $\sigma(2p^2) = 3(1+p+p^2) = 2p^2+N^2$, so

$$(p + 1)(p + 2) = (N + 1)(N - 1). \tag{2.5}$$

If $p \leqq N$, then $p+1 \leqq N+1$ and $p+2 \geqq N-1$, so that $N-3 \leqq p \leqq N$. But for any of the four possibilities $p = N-i$ $(i=0, 1, 2, 3)$, equation (2.5) gives a contradiction. If $p > N$, then $p+1 > N+1$ and $p+2 < N-1$, which is absurd. This completes the proof of Theorem 2.2; we have also proved the

COROLLARY 2.2. *If M is a prime and if N is odd, then $2M^2$ and N^2 cannot be amicable.*

Our next theorem gives necessary conditions on amicable pairs with opposite parity in case r is odd.

THEOREM 2.3. *Suppose that 2^rM^2 and N^2 are amicable with opposite parity. If r is odd, then*

(2.6) $(M, 3) = (N, 3)$, *and*

(2.7) *there exists a prime q and a positive integer γ such that $q^\gamma \| N$ and $q \equiv \gamma \equiv 1 \pmod 3$;*

if $r \equiv 3 \pmod 4$ then

(2.8) *there exists a prime p (not necessarily distinct from q) and a positive integer δ such that $p^\delta \| N$ and $2p \equiv \delta \equiv 2 \pmod 5$, and*

$$M \equiv N \equiv \frac{r+1}{4}\sigma(M^2) \equiv 0 \pmod 5. \tag{2.9}$$

Proof. Let $r = 2a-1$, $a>0$; the usual conditions for amicability give

$$(2^{2a} - 1)\sigma(M^2) = 2^{2a-1}M^2 + N^2, \text{ and} \tag{2.10}$$

$$(2^{2a} - 1)\sigma(M^2) = \sigma(N^2). \tag{2.11}$$

Since $2^{2a}-1 \equiv 0 \pmod 3$, (2.10) shows that 3 cannot divide exactly one of M, N.

This proves (2.6). From (2.11) we see that $3 \mid \sigma(N^2)$, so $\sigma(N^2)$ contains a factor $1+q+\cdots+q^{2\gamma}$ which is divisible by 3. Evidently $q \not\equiv 0 \pmod 3$. If $q \equiv 1 \pmod 3$, then $1+q+\cdots+q^{2\gamma} \equiv 1+1+\cdots+1 = 2\gamma+1 \equiv 0 \pmod 3$, and $\gamma \equiv 1 \pmod 3$. If $q \equiv 2 \pmod 3$, then $1+q+\cdots+q^{2\gamma} \equiv 1+(2+1)+\cdots+(2+1) \not\equiv 0$. This proves (2.7).

If $r = 4b-1$ and $b > 0$ then

$$(2^{4b}-1)\sigma(M^2) = 2^{4b-1}M^2 + N^2, \text{ and} \tag{2.12}$$

$$(2^{4b}-1)\sigma(M^2) = \sigma(N^2). \tag{2.13}$$

Now $2^{4b}-1 \equiv 0 \pmod 5$; from (2.12) we see that either both of M and N are divisible by 5 or neither is. Suppose that $M \not\equiv 0 \pmod 5$ and $N \not\equiv 0 \pmod 5$. Then M^2 is either 1 or 4 (mod 5) and similarly for N^2. Also $2^{4b-1} \equiv 3 \pmod 5$. Thus, the right side of (2.12) cannot be 0 (mod 5), while the left side is always 0 (mod 5). Hence

$$M \equiv N \equiv 0 \pmod 5. \tag{2.14}$$

Now consider equation (2.13). Since $5 \mid (2^{4b}-1)$, there is a prime p such that $1+p+\cdots+p^{2\delta}$ is a factor of $\sigma(N^2)$ and this factor is divisible by 5. Clearly $p \equiv 1$ and $\delta \equiv 2 \pmod 5$ is the only solution of

$$1 + p + \cdots + p^{2\delta} \equiv 0 \pmod 5,$$

which proves (2.8).

To complete the proof of (2.9), assume $\sigma(M^2) \not\equiv 0 \pmod 5$. By (2.14) and (2.12) we have $2^{4b}-1 \equiv 0 \pmod{25}$. Now

$$\begin{aligned} 0 \equiv 2^{4b} - 1 &\equiv (1-10)^b - 1 \pmod{25} \\ &= \sum_{j=0}^{b} \binom{b}{j} (-10)^j - 1 \equiv -10b \pmod{25}. \end{aligned}$$

From this we see that $b \equiv 0 \pmod 5$; but $b = (r+1)/4$ and the proof is complete.

3. Let $t(n)$ denote the number of positive divisors of n. It is easy to see that

$$\sigma(n^2) - t(n^2) \equiv n - 1 \pmod 4, \quad \text{if } n \text{ is odd.} \tag{3.1}$$

For, if n is odd and $d \mid n$, then d is odd and $d^2 \equiv n^2 \equiv 1 \pmod 4$. Hence $d \equiv n^2/d \pmod 4$ and so $d + (n^2/d) \equiv 2 \pmod 4$. Summing over $d \mid n^2$, $d < n$ gives $\sigma(n^2) - n \equiv t(n^2) - 1$, which evidently proves (3.1). An immediate consequence of (3.1) is given in the following

THEOREM 3.1. *Suppose $n_1 = 2^r M^2$ and $n_2 = N^2$ are amicable, where $r \geqq 1$, M and N are odd. If $r = 1$, then N cannot be a square; if $r > 1$, then M is neither a square nor a prime of the type $4k+3$.*

Proof. If $r = 1$, (1.1) shows that $\sigma(N^2) \equiv 3 \pmod 4$, and from (3.1) we have $t(N^2) \equiv -N \pmod 4$. If $N = m^2$, $m = \prod p^{\delta}$, then $N^2 = \prod p^{4\delta}$ and $t(N^2)$

$= \prod(4\delta+1) \equiv 1 \pmod 4$, but $-N = -m^2 \equiv 3 \pmod 4$. Hence N cannot be a square.

If $r>1$, (1.1) shows that $\sigma(M^2) \equiv 3 \pmod 4$. Then (3.1) gives $t(M^2) \equiv -M \pmod 4$. As above, M cannot be a square; also if M is a prime of the type $4k+3$, then $t(M^2)=3$, but $-M \equiv 1 \pmod 4$. The proof is complete.

In some cases we can extend the above results to show that M is composite. For example, if $r \equiv 3 \pmod 4$ and M is prime, by Theorem 2.3 we must have $M=5$. But then

$$(2^{r+1}-1)\sigma(5^2) = 2^r 5^2 + N^2$$

yields $N^2 = 2^r(37) - 31$. This says that -31 is a quadratic residue of 37, which is a contradiction. Hence if $r \equiv 3 \pmod 4$, then M is composite.

The first author's work was supported in part by Texas Technological College Research Grant No. 191–4735.

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A THEOREM ON ARBITRARY FUNCTIONS

SEYMOUR HABER, National Bureau of Standards, Washington, D.C.

A well-known theorem states that an "arbitrary" real function—that is, a function f such that $f(x)$ is a real number for each real x, with no further conditions—can have at most enumerably many strict maxima [1]. (A real function f is said to have a strict maximum at a point p. If there is a neighborhood N of p_0 such that $f(p) < f(p_0)$ for all $p \neq p_0$ in N.) We show here, as a consequence of this theorem, that a combination $a_1(x)b_1(y) + \cdots + a_n(x)b_n(y)$ of arbitrary real functions a_i and b_i has a certain partial continuity property.

LEMMA 1. *If $f(x)$ is defined and real for all x in E^1, then f has at most enumerably many strict maxima.*

LEMMA 2. *If f is defined and real on E^1 and for each x either $f(x)=0$ or $\lim_{x' \to x} f(x')=0$, then f is zero at all points of E^1, with at most enumerably many exceptions.*

Proof. It is sufficient to show that the set of points at which f is positive is at most enumerable; but at each such point f has a strict maximum, and the result follows from Lemma 1.

LEMMA 3. *If $f(x, y) = a_1(x)b_1(y) + \cdots + a_n(x)b_n(y)$, then for any numbers $x_1, x_2, \cdots, x_{n+1}$ and $y_1, y_2, \cdots, y_{n+1}$, the matrix (a_{ij}) with $a_{ij} = f(x_i, y_j)$ is singular.*

For, this $(n+1)\times(n+1)$ matrix is the sum of n matrices $(a_k(x_i)b_k(y_j))$, each of which is of rank at most one, so that its rank is at most n.

THEOREM. *Let* $f(x, y)=a_1(x)b_1(y)+\cdots+a_n(x)b_n(y)$, *and let* D *be the set of points* (x, x) *in the plane. If* g *is a function defined and continuous on* D, *and for each point* $p_0\in D, f(p)\to g(p_0)$ *as* p *approaches* p_0 *through the complement of* D, *then* $f(p)=g(p)$ *at all points of* D *except possibly for a set which is at most enumerable.*

Proof. We first consider the case in which g is identically zero on D. For each positive integer k let S_k be the set of all real numbers x such that $|a_i(x)|>k$ or $|b_i(x)|>k$ for some i. For $i=1, 2, \cdots, n$ let $a_{i,k}(x)$ be equal to $a_i(x)$ for $x\notin S_k$, and equal to zero on S_k; define $b_{i,k}$ similarly, and set $f_k(x, y)=\sum_{i=1}^{n}a_{i,k}(x)b_{i,k}(y)$. If $p_1=(x_1, y_1)\in D$ and $p\to p_1$ through the complement of D, then $f_k(p)\to 0$ just as does $f(p)$, since at each point p in E^2, $f_k(p)$ is equal either to $f(p)$ or to zero. Now let $x_2, x_3, \cdots, x_{n+1}$ be numbers distinct from each other and from x_1, and let $y_i=x_i (i=1, \cdots, n+1)$, and set $a_{i,j}=f_k(x_i, y_j)$. Letting $x_2, \cdots, x_{n+1}$ approach x_1, we see that the points (x_i, y_j) all approach p_1. For $i\neq j$ these points lie in the complement of D, and so the off-diagonal entries of the matrix $(a_{i,j})$ approach zero. Since f_k is bounded, it follows that all terms of the determinant of $(a_{i,j})$, other than the single term which is the product of the diagonal entries, go to zero. However, by Lemma 3, the determinant of $(a_{i,j})$ is itself zero, and so the product

$$f_k(x_1, x_1)\cdot f_k(x_2, x_2)\cdots f_k(x_{n+1}, x_{n+1})$$

goes to zero.

Set $h(x)=f_k(x, x)$. We have shown that, for every real x_1,

$$h(x_1)\cdot h(x_2)\cdots h(x_{n+1})\to 0$$

as the distinct numbers $x_2, \cdots, x_{n+1}$ approach x_1. This implies that h satisfies the hypotheses of Lemma 2. For if there were an x_1 such that $h(x_1)=a>0$ and $h(x)$ does not approach zero as $x\to x_1$, there would be an $\epsilon>0$ such that in every neighborhood of x_1 there would be at least one point at which $|h(x)|>\epsilon$. By the usual reasoning, there would then be infinitely many such points in each neighborhood of x_1, and, taking n of them as $x_2, x_3, \cdots, x_{n+1}$, we would have

$$|h(x_1)\cdot h(x_2)\cdots h(x_{n+1})|>\epsilon^n a.$$

It now follows from Lemma 2 that $f_k(x, x)=0$ for all, except at most enumerably many, real numbers x. Call the set of exceptional numbers T_k. Then $f(x, x)=0$ unless $x\in T_k\cup S_k$, for each positive integer k. The set of x for which $f(x, x)\neq 0$ is thus contained in $\bigcap_{k=1}^{\infty}(T_k\cup S_k)$, and so in $(\bigcup_{k=1}^{\infty}T_k)\cup(\bigcap_{k=1}^{\infty}S_k)$. The second of these last two sets is void, since if $x_0\in\bigcap_{k=1}^{\infty}S_k$, then for each positive integer k some one of the a_i and b_i takes on a value greater than k at x_0. Since there are only finitely many a_i and b_i, there must be one of them (call it c) such that $|c(x_0)|>k$ for infinitely many k; which is impossible. $\bigcup_{k=1}^{\infty}T_k$ is at most enumerable, so the theorem is proved for the case of g identically zero on D.

For general g, we apply this result to the function $F(x, y)=f(x, y)-g(x, x)$.

REMARKS. (a) The set D could be taken to be any subset of the plane having the property that each point of D has a neighborhood N such that $D \cap N$ meets each horizontal line and each vertical line at most once.

(b) These results hold, more generally, whenever the a_i and b_i are real valued functions on a space X with a countable base. The proof is the same.

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SOME REMARKS ON TRANSFORMATION GROUPS AND SEMIGROUPS

W. J. GRAY, University of Alabama

In this note we establish some elementary properties of compact connected semigroups with cut points. Our results lead to a simplified proof of a theorem of Faucett. We indicate the analogues of our results in topological transformation groups.

THEOREM 1. *Let S be a compact connected Hausdorff semigroup and e be a left (right) identity of S. If G denotes the maximal connected subgroup of S containing e, then $Gx = x$ ($xG = x$) for every cut point $x \in S$.*

Proof. Let e be a right identity. Now G is compact, and if $g \in G$, the correspondence $x \to xg$ is a homeomorphism. Let z be a cut point of S. Then

$$S - z = U \cup V, \qquad U \neq \varnothing, \qquad V \neq \varnothing, \qquad U \cap V = \varnothing,$$

where U and V are open in S. Let x_1 and x_2 be noncut points of S contained in U and V, respectively (cf. [4]) and set $A = x_1G$, $B = x_2G$, $C = zG$. Then A and B are connected sets consisting of noncut points of S, and each point of C is a cut point which separates A and B in S.

If x and y are two points of C, let $x \leqq y$ if $x = y$ or $S - \{y\} = U' \cup V'$, where $A \cup \{x\} \subset U'$ and $B \subset V'$, where U' and V' are separated in S. $\leqq$ satisfies the axioms of a partial order, and is linear on C. Furthermore, the intersection of the closed set $U' \cup \{y\}$ with the closed set C is the set

$$\{q;\ q \in C \text{ and } q \leqq y\}. \tag{1}$$

Since C is linearly ordered by $\leqq$, the collection of all sets of the form (1) have the finite intersection property so that C contains a least element m. Since $CG = C$, G acts as a group of order preserving homeomorphisms on C and hence $mG = m$. Thus $m = z$. Likewise, if e is a left identity, $Gz = z$.

COROLLARY 1. *If S is a clan, let G be the identity component of the maximal subgroup of S which contains the identity. Then $xG = x = Gx$ for every cut point $x \in S$.*

COROLLARY 2. *If S is a clan, let $H(e)$ denote the maximal subgroup of S which contains the identity e, and C be the set of cut points of S. Either $\overline{C}\cap H(e)=\varnothing$ or else $H(e)$ is a totally disconnected subset of the boundary of C.*

Proof. Assume that $\overline{C}\cap H(e)\neq\varnothing$. By Theorems 1 and 4 of [3], $C\cap H(e)\neq\varnothing$ and evidently $H(e)\subset C$. Thus if G is the identity component of $H(e)$, then $tG=t=Gt$ for every $t\in H(e)$ and G must consist of e alone.

COROLLARY 3. *If S is a compact connected Hausdorff semigroup which contains a cut point and no proper left ideal, then multiplication is trivial, i.e., if $x, y\in S$, $xy=x$, and S is isomorphic to the set of constant functions from S to itself. The right dual of this corollary holds.*

Proof. By [1], S is the union of its minimal right ideals, and each minimal right ideal is a group. Let R be a minimal right ideal and e be the identity of R. Then e is a right identity of S. Hence $zR=z$ for a cut point $z\in S$. Since $Sz=S$ we have $xR=x$ for every $x\in S$. Thus R consists of e alone, and $eS=e$.

COROLLARY 4. *If S is a compact connected Hausdorff semigroup and L is a minimal left ideal of S, then if L is separated by one of its points, each element of L is an idempotent of S. The right dual of this corollary holds.*

Proof. L satisfies the hypothesis of Corollary 3, hence multiplication is trivial on L and $x^2=x$ for every $x\in L$.

THEOREM (Faucett [2]). *Let S be a compact connected Hausdorff semigroup, and K be the minimal ideal of S. If K is separated by one of its points, exactly one of the following must occur:*

(i) *K is a minimal left ideal of S and each of its elements is a right zero of S.*

(ii) *K is a minimal right ideal of S and each of its elements is a left zero of S.*

Proof. Assume K is not a minimal right ideal of S. Let R be a minimal right ideal containing a cut point z of K, and $K-\{z\}=U\cup V$, where U and V are disjoint nonempty open subsets of K. If R_1 is any other minimal right ideal, then R_1 is connected and disjoint from R, hence must lie in, say V. If L is a minimal left ideal and $L\cap U\neq\varnothing$, then L must intersect R_1, hence L is separated by z. Then by Corollary 4, the group $R\cap L$ must be trivial. Since minimal left ideals are disjoint, we also must have $U\cup\{z\}\subset L$, hence $R\subset V\cup\{z\}$. Then U must contain a minimal right ideal R_2, so that no minimal left ideal other than L can intersect R_2. Hence $K=L$, and by Corollary 4, each minimal right ideal of S is trivial (because it is a group). Thus if $x\in K$, we have $xS=x$, and (i) holds. (ii) will result if we assume that K is not a minimal left ideal of S.

Theorem 1 has an analogy in topological transformation groups. In fact,

THEOREM 2. *If (S, G, π) is a transformation group, where S is a Hausdorff continuum and G is a compact connected group, then G leaves every cut point of S fixed.*

THEOREM 3. *If (S, G, π) is a transformation group, where S is a locally connected Hausdorff continuum with a cut point, and G is a connected group, then G has a fixed point.*

Proof. In the proof of Theorem 1, replace C by the set

$$Q = \cap \{S(a, b);\ \ a \in A \text{ and } b \in B\},$$

where $S(a, b) = \{a, b\} \cup \{x;\ x \text{ separates } a \text{ and } b \text{ in } S\}$. Q is invariant under G and $Q \neq \varnothing$ since $z \in Q$. Further, Q is closed since S is locally connected. If no point of $A \cup B$ is fixed under G, then Q consists of cut points only, so that we may complete the proof as in Theorem 1.

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A COMMENT ON A RECENT NOTE ON THE FAREY SERIES

ALAN ZAME, University of Miami

In a recent note Jean Blake [1] proves the following two theorems:

THEOREM 1. *The sums of the numerators of the fractions of a Farey series F_n of order n is equal to one-half the sum of the denominators of these fractions.*

THEOREM 2. *In F_n the denominator of the immediate predecessor and immediate successor of one-half is the greatest odd integer $\leqq n$.*

The proofs in the reference seem much too long. To prove the first, note that

$$\sum_{\substack{d \leqq n \\ (d,n)=1}} d = \tfrac{1}{2} n \phi(n),$$

so that the sum of the numerators corresponding to a denominator k is just $\frac{1}{2}k\phi(k)$, which is just one half of the sum of these denominators. To prove the second part, just notice that the appropriate fractions are given by $\frac{1}{2} \pm 1/2n$ if n is odd and $\frac{1}{2} \pm 1/(2n-2)$ if n is even: these are readily seen to be in the sequence F_n and are clearly the terms in that sequence nearest to $\frac{1}{2}$.

Reference

1. Jean Blake, Some characteristic properties of the Farey sequences, this MONTHLY, 73 (1966) 50–52.

A SUM OF FRACTIONAL PARTS

K. S. WILLIAMS, University of Manchester, England

Let L denote the set of points $x=(x_1, \cdots, x_n)$ with integral coordinates in Euclidean n-space. For any prime $p\geqq 3$, let $C=C(p)$ be the set of points of L in the cube $0\leqq x_i<p$ $(i=1, 2, \cdots, n)$. Suppose that $f(x)$ is any polynomial of degree d in $x_1, \cdots, x_n$ with integral coefficients, which does not vanish identically (mod p). We let $\{a\}$ denote the fractional part of the real number a and consider the problem of estimating

$$\sum_{x\in C}\{f(x)/p\}$$

for large primes p. When $d=1$ this sum is just $\frac{1}{2}(p-1)p^{n-1}=\frac{1}{2}p^n+O(p^{n-1})$. For $d>1$ we prove, using a result of Uchiyama [1], the following theorem.

THEOREM 1.

$$\sum_{x\in C}\{f(x)/p\} = \tfrac{1}{2}p^n + O(p^{n-1/2}\log p), \tag{1}$$

as $p\to\infty$, where the constant implied in the O-symbol depends only upon n and d.

Proof.

$$\sum_{x\in C}\{f(x)/p\} = \sum_{r=0}^{p-1}\sum_{\substack{x\in C\\ f(x)\equiv r \pmod p}}\{r/p\} = 1/p\sum_{r=0}^{p-1} r\sum_{\substack{x\in C\\ f(x)\equiv r \pmod p}} 1. \tag{2}$$

Now let $e(t)$ denote $\exp(2\pi i t p^{-1})$ for any real t. Then as

$$(1/p)\sum_{v=0}^{p-1} e(av) = \begin{cases} 1 & \text{if } a\equiv 0 \pmod p \\ 0 & \text{if } a\not\equiv 0 \pmod p \end{cases}$$

we have

$$\begin{aligned}\sum_{x\in C}\{f(x)/p\} &= (1/p^2)\sum_{r=0}^{p-1} r\sum_{x\in C}\sum_{v=0}^{p-1} e(v(f(x)-r)) \\ &= (1/p^2)\sum_{v=0}^{p-1}\sum_{x\in C} e(vf(x))\sum_{r=0}^{p-1} re(-rv).\end{aligned}$$

The term corresponding to $v=0$ is just

$$(1/p^2)\sum_{x\in C}\sum_{r=0}^{p-1} r = \tfrac{1}{2}(p-1)p^{n-1}.$$

For $v\not\equiv 0$ we have

$$\sum_{r=0}^{p-1} re(-rv) = -p/(1-e(-v)),$$

whence

$$\left|\sum_{x\in C}\{f(x)/p\} - \tfrac{1}{2}p^{n-1}(p-1)\right| = \left|(1/p)\sum_{v=1}^{p-1} 1/(1-e(-v))\sum_{x\in C} e(vf(x))\right|$$

$$\leqq (1/p)\sum_{v=1}^{p-1} 1/\left|1-e(-v)\right|\left|\sum_{x\in C} e(vf(x))\right|.$$

By a recent result of Uchiyama [1]

$$\left|\sum_{x\in C} e(vf(x))\right| \leqq kp^{n-1/2}$$

where the constant k depends only upon n and d. Hence we have

$$\left|\sum_{x\in C}\left\{\frac{f(x)}{p}\right\} - \frac{1}{2}p^{n-1}(p-1)\right| \leqq \frac{k}{2}p^{n-3/2}\sum_{v=1}^{p-1}\frac{1}{\sin(\pi v/p)},$$

from which the theorem follows in view of the well-known result

$$\sum_{v=1}^{p-1}\frac{1}{\sin(\pi v/p)} \leqq p\sum_{v=1}^{(p-1)/2}\frac{1}{v} \leqq p\log p.$$

We now show that the error term in (1) can be improved for polynomials $f(x)$ of certain special types, namely diagonal and quadratic polynomials.

THEOREM 2. *If* $f(x)=a_1x_1^{k_1}+\cdots+a_nx_n^{k_n}+a_0$, *where each* $k_i\geqq 1$ *and* $p\nmid a_1\cdots a_n$, *then, provided* $n\geqq 3$,

$$\sum_{x\in C}\{f(x)/p\} = (p^n/2) + O(p^{n-1}),$$

as $p\to\infty$, *where the constant implied in the O-symbol depends only upon* $k_1, \cdots, k_n$.

THEOREM 3. *Let* $f(x_1, x_2)$ *be a quadratic polynomial which is not a function of only one integral variable and which has a discriminant not divisible by* p. *Suppose further that the discriminant of* $f_1(x_1, x_2, 0)$ *is also not divisible by* p, *where* $f_1(x_1, x_2, x_3)\equiv x_3^2 f(x_1/x_3, x_2/x_3)$. *Then*

$$\sum_{x_1=0}^{p-1}\sum_{x_2=0}^{p-1}\{f(x_1, x_2)/p\} = \tfrac{1}{2}p^2 + O(p),$$

as $p\to\infty$.

Proof of Theorem 2. By (2)

$$\sum_{x\in C}\{f(x)/p\} = (1/p)\sum_{r=1}^{p-1} rN_r, \tag{3}$$

where N_r denotes the number of $x\in C$ with $f(x)\equiv r \pmod p$. Weil has shown [2] that

$$N_r = p^{n-1} + O(p^{n/2}),$$

as $p \to \infty$, where the constant in the O-symbol depends only on $k_1, \cdots, k_n$. The theorem follows at once.

Proof of Theorem 3. As is well known, under the conditions stated in the theorem, we may reduce $f(x_1, x_2)$ by a linear nonsingular transformation (mod p) to the canonical form

$$ay_1^2 + by_2^2 + c \qquad (p \nmid abc).$$

This transformation will not affect the sum under consideration and so we need only consider

$$\sum_{y_1=0}^{p-1} \sum_{y_2=0}^{p-1} \left\{\frac{ay_1^2 + by_2^2 + c}{p}\right\} \qquad (p \nmid abc).$$

The theorem then follows from (3), as in this case

$$N_r = \begin{cases} p - (-ab/p) & \text{if } r \not\equiv c \\ p + (p-1)(-ab/p) & \text{if } r \equiv c. \end{cases}$$

CONJECTURE: (1) may be replaced by

$$\sum_{x \in C} \{f(x)/p\} = \tfrac{1}{2}p^n + O(p^{n-1/2}), \quad \text{as } p \to \infty. \tag{1'}$$

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TAYLOR'S FORMULA AND THE EXISTENCE OF nTH DERIVATIVES

P. R. BEESACK, Carleton University, Ottawa

A common form of Taylor's theorem states sufficient conditions for a function f to have an expansion of the form

$$f(a+h) = a_0 + a_1 h + a_2 h^2 + \cdots + a_{n-1}h^{n-1} + R_n(h), \tag{1}$$

valid in some one- or two-sided neighborhood of $h=0$, the remainder term R_n involving a factor of the form $f^{(n)}(X)$, where X is strictly between a and $a+h$. The existence (finite or infinite) of $f^{(n)}$ *throughout* at least some deleted neighborhood (or one-sided neighborhood) of a obviously must be postulated in order to prove any theorem of this kind. On the other hand, $f^{(n)}(a)$ need not exist for the validity of such an expansion.

A less well-known form of Taylor's theorem is that due to de la Vallée Poussin [1, pp. 78–80], which we state here in the following form:

If $f^{(n)}(a)$ exists, and if the function M_n is defined in a suitable deleted neighborhood, $\mathring{N}(0)$, by

$$(2)\quad f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}h^{n-1} + M_n(h)h^n,$$

then $\lim_{h\to 0} M_n(h) = f^{(n)}(a)/n!$.

This result is valid whether $f^{(n)}(a)$ is finite or properly infinite; that is provided only the limit

$$\lim_{h\to 0} \frac{f^{(n-1)}(a+h) - f^{(n-1)}(a)}{h}$$

exists, finite or infinite. In case $f^{(n)}(a)$ is finite, this result is equivalent to that given in [2], although the proofs are quite different. The theorem is also valid in case one-sided neighborhoods, and corresponding one-sided derivatives are used throughout. Moreover, in case $f^{(n)}(a)$ is finite, f has a *unique* expansion of the form (2). That is, if there are constants $a_0, a_1, \cdots, a_{n-1}$ such that

$$(3)\qquad f(a+h) = a_0 + a_1h + a_2h^2 + \cdots + a_{n-1}h^{n-1} + M_n(h)h^n,$$

where M_n is bounded in some $\mathring{N}(0)$, then $a_k = f^{(k)}(a)/k!$ for $0 \leqq k \leqq n-1$.

The purpose of this note is to find additional conditions on M_n which will ensure the existence of $f^{(n)}(a)$ in case f has an expansion of the form (3). The mere boundedness of M_n in $\mathring{N}(0)$ is clearly not sufficient even in the case $n=1$. In view of the expansion (2) of de la Vallée Poussin, one is tempted to replace the assumption that M_n is bounded in $\mathring{N}(0)$ by the assumption that $\lim_{h\to 0} M_n$ exists. Although this condition is sufficient to guarantee the existence of $f^{(n)}(a)$ in case $n=1$ (if f is assumed continuous at a), it is not sufficient for $n>1$. A simple counterexample for $n=2$ is given by the function f defined by

$$f(x) = \begin{cases} x^3 \cos(x^{-1}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Here, $f(h) = 0 + 0\cdot h + M_2(h)h^2$ for $h \neq 0$, and

$$\lim_{h\to 0} M_2(h) = \lim_{h\to 0} h\cos(h^{-1}) = 0$$

exists, while $f''(0)$ does not exist.

Returning to the expansion (3) it is clear that if $f^{(n)}(a)$ exists, then M_n', M_n'', $\cdots$, $M_n^{(n-1)}$ must all exist throughout some $\mathring{N}(0)$. The preceding example shows, however, that even adding these conditions will not guarantee the existence of $f^{(n)}(a)$. The necessary and sufficient condition for the existence of

$f^{(n)}(a)$ (finite) is given in the following theorem. For convenience we state only the two-sided form of the theorem.

THEOREM. *If f is continuous at a, then $f^{(n)}(a)$ exists (finite)$\Leftrightarrow\exists$ constants $a_0, a_1, \cdots, a_{n-1}$ and a function M_n defined in some $\tilde{N}(0)$ such that*

$$f(a+h) = a_0 + a_1h + a_2h^2 + \cdots + a_{n-1}h^{n-1} + M_n(h)h^n, \tag{4}$$

for $h\in\tilde{N}(0)$, where $M_n^{(n-1)}$ exists in $\tilde{N}(0)$, and

$$\lim_{h\to 0} M_n(h) \text{ exists, and } \lim_{h\to 0} h^j M_n^{(j)}(h) = 0 \quad \text{for } j = 1, 2, \cdots, n-1. \tag{5}$$

Moreover, in this case, $a_0, a_1, \cdots, a_{n-1}$ and M_n are uniquely determined and $a_k = f^{(k)}(a)/k!$ for $k = 0, 1, \cdots, n-1$, while

$$\lim_{h\to 0} M_n(h) = \frac{f^{(n)}(a)}{n!}.$$

Proof. We prove both parts by induction, proving first the sufficiency of the conditions. If $n=1$ we thus assume that f is continuous at a, and that $\exists$ a constant a_0 and a function M_1 defined in a $\tilde{N}(0)$ such that $f(a+h)=a_0+M_1(h)h$, for $h\in\tilde{N}(0)$, and $\lim_{h\to 0} M_1(h)$ exists. Letting $h\to 0$, we obtain $f(a)=a_0$. Hence

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-a_0}{h} = \lim_{h\to 0} M_1(h) \text{ exists},$$

and the conditions are sufficient in the case $n=1$.

Now, suppose that the conditions are sufficient for $n=k\geqq 1$. Let f be continuous at a, and suppose that $\exists$ constants $a_0, a_1, \cdots, a_k$, and a function M_{k+1} defined in a $\tilde{N}(0)$ such that

$$f(a+h) = a_0 + a_1h + a_2h^2 + \cdots + a_kh^k + M_{k+1}(h)h^{k+1}, \quad \text{for } h \in \tilde{N}(0), \tag{6}$$

where $M_{k+1}^{(k)}$ exists in $\tilde{N}(0)$, $\lim M_{k+1}(h)=A$ exists, and

$$\lim h^j M_{k+1}^{(j)}(h) = 0 \qquad \text{for } j = 1, 2, \cdots, k.$$

Then for $h\in\tilde{N}(0)$ we obtain, by differentiating (6),

$$f'(a+h) = a_1 + 2a_2h + \cdots + (ka_k)h^{k-1} + M'_{k+1}h^{k+1} + (k+1)M_{k+1}h^k,$$

or

$$f'(a+h) = a_1 + (2a_2)h + \cdots + (ka_k)h^{k-1} + \overline{M}_k(h)h^k, \qquad \text{for } h \in \tilde{N}(0), \tag{7}$$

where $\overline{M}_k(h)\equiv hM'_{k+1}(h)+(k+1)M_{k+1}(h)$ is defined in $\tilde{N}(0)$.

We now show that the induction assumption can be applied to the function f'. First we note that $f(a)=a_0$ follows directly from (6) by letting $h\to 0$. Hence

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-a_0}{h} = \lim_{h\to 0}(a_1 + a_2h + \cdots + a_kh^{k-1} + M_{k+1}(h)h^k) = a_1;$$

moreover, by (7) it follows that $\lim f'(a+h)=a_1=f'(a)$, so that f' is continuous at a. Also,

$$h^j\overline{M}_k^{(j)}(h) = h^{j+1}M_{k+1}^{(j+1)}(h) + (k+j+1)h^jM_{k+1}^{(j)}(h) \to 0$$

as $h\to 0$ for $j=1, 2, \cdots, (k-1)$ by the hypotheses on M_{k+1}, and similarly, $\lim \overline{M}_k(h)=0+(k+1)A$ exists.

Hence the induction assumption applies to f' as given by (7). It follows that $(f')^{(k)}(a)=f^{(k+1)}(a)$ exists, and that

$$ma_m = \frac{f^{(m)}(a)}{(m-1)!} \qquad \text{for } m = 1, 2, \cdots, k,$$

$$\lim \overline{M}_k(h) = \frac{f^{(k+1)}(a)}{k!} = (k+1)A.$$

Since $a_0=f(a)$ was proved, this completes the induction and proves the sufficiency of the conditions.

To prove the necessity in the case $n=1$, we assume that $f'(a)$ exists (so that f is automatically continuous at a). We are to prove that $\exists$ a constant a_0 and a function M_1 defined in a $\tilde{N}(0)$ such that

$$f(a+h) = a_0 + M_1(h)h,$$

for $h\in\tilde{N}(0)$, and $\lim M_1(h)=f'(a)$. It clearly suffices to define $a_0=f(a)$, and to define M_1 by the equation $f(a+h)=f(a)+M_1(h)h$, for $h\in\tilde{N}_\delta(0)$, where $\delta>0$ is chosen so that $f(a+h)$ is defined in $N_\delta(0)$. Such a δ exists because $f'(a)$ exists. But then

$$\lim M_1(h) = \lim \frac{f(a+h)-f(a)}{h} = f'(a) \text{ exists},$$

and the proof in the case $n=1$ is complete.

Now assume that the conditions are necessary for $n=k\geqq 1$. Suppose that $f^{(k+1)}(a)$ exists. Then $f^{(k)}$ exists in some $N_\delta(a)$ (and is even continuous at a), and $f^{(k-1)}, \cdots, f', f$ are all continuous in $N_\delta(a)$. In particular, the constants $a_0=f(a), \cdots, a_k=f^{(k)}(a)/k!$ are all defined, and so is the function M_{k+1} defined in $\tilde{N}_\delta(0)$ by the equation

$$f(a+h) = f(a) + f'(a)h + \cdots + \frac{f^{(k)}(a)}{k!}h^k + M_{k+1}(h)h^{k+1}. \tag{8}$$

Clearly $M_{k+1}^{(k)}$ exists in $\tilde{N}_\delta(0)$ by (8). As in the sufficiency proof, we obtain

$$f'(a+h) = f'(a) + f''(a)h + \cdots + \frac{f^{(k)}(a)}{(k-1)!}h^{k-1} + \overline{M}_k(h)h^k, \tag{9}$$

for $h\in\tilde{N}_\delta(0)$, where $\overline{M}_k(h)\equiv hM'_{k+1}(h)+(k+1)M_{k+1}(h)$ is also defined in $\tilde{N}_\delta(0)$.

On the other hand, since $(f')^{(k)}(a)$ exists, the induction assumption applies to the function f', so $\exists$ constants $A_0, \cdots, A_{k-1}$ and a function R_k defined in some $\tilde{N}_\epsilon(0)$ such that

$$(10) \quad f'(a+h) = A_0 + A_1 h + \cdots + A_{k-1}h^{k-1} + R_k(h)h^k, \qquad h \in \tilde{N}_\epsilon(0),$$

where $R_k^{(k-1)}$ exists in $\tilde{N}_\epsilon(0)$, $\lim R_k(h)$ exists, and $\lim h^j R_k^{(j)}(h) = 0$ for $j = 1, 2, \cdots, k-1$. By the uniqueness, shown in the sufficiency part of the proof, it follows that

$$A_0 = f'(a), \qquad A_1 = \frac{f''(a)}{1!}, \cdots, A_{k-1} = \frac{f^{(k)}(a)}{(k-1)!}.$$

Hence, from comparison of (9) and (10), it also follows that $\overline{M}_k(h) = R_k(h)$ for all $h \in \tilde{N}_\gamma(0)$, where $\gamma = \min(\delta, \epsilon)$. Thus, $\lim_{h\to 0} \overline{M}_k(h) = \lim_{h\to 0} R_k(h) = B$ exists, and $\lim_{h\to 0} h^j \overline{M}_k^{(j)}(h) = 0$, $j = 1, 2, \cdots, k-1$.

To complete the induction (and the proof of the theorem) it remains only to prove that $\lim M_{k+1}(h)$ exists, and $\lim h^j M_{k+1}^{(j)}(h) = 0$ for $j = 1, 2, \cdots, k$. The first of these follows from (8) and (9), which give

$$\begin{aligned}
\lim M_{k+1}(h) &= \lim \left[f(a+h) - \left\{ f(a) + f'(a)h + \cdots + \frac{f^{(k)}(a)}{k!} h^k \right\} \right] \Big/ h^{k+1} \\
&= \frac{1}{k+1} \lim \left[f'(a+h) - \left\{ f'(a) + \cdots + \frac{f^{(k)}(a)}{(k-1)!} h^{k-1} \right\} \right] \Big/ h^k \\
&= \frac{1}{k+1} \lim \overline{M}_k(h) = \frac{B}{k+1}.
\end{aligned}$$

Using this result and the definition

$$\overline{M}_k(h) = hM'_{k+1}(h) + (k+1)M_{k+1}(h),$$

it now follows that $\lim hM'_{k+1}(h) = 0$, proving the second limit relation for $j = 1$, and any $k \geqq 1$. In particular, the induction is complete in case $k = 1$.

For $k \geqq 2$, the remaining limit relation follows for $j = 1, 2, \cdots, k-1$ by writing

$$\begin{aligned}
\lim h^j M_{k+1}^{(j)}(h) &= \lim \frac{h^{k+j+1} M_{k+1}^{(j)}(h)}{h^{k+1}} \\
&= \lim \frac{h^{k+j+1} M_{k+1}^{(j+1)}(h) + (k+j+1)h^{k+j} M_{k+1}^{(j)}(h)}{(k+1)h^k} \\
&= \frac{1}{k+1} \lim h^j \{ hM_{k+1}^{(j+1)}(h) + (k+j+1)M_{k+1}^{(j)}(h) \} \\
&= \frac{1}{k+1} \lim h^j \overline{M}_k^{(j)}(h) = 0.
\end{aligned}$$

Taking $j=k-1$ in the preceding, we have

$$\lim h^{k-1}\{hM_{k+1}^{(k)}(h) + 2kM_{k+1}^{(k-1)}(h)\} = 0,$$

and since $\lim h^{k-1}M_{k+1}^{(k-1)}(h)=0$ has been proved, it also follows that $\lim h^k M_{k+1}^{(k)}(h)=0$, completing the proof.

REMARK 1. An examination of the proof of the theorem shows that the sufficiency half of the theorem remains valid in the form "*$f^{(n)}(a)$ exists, finite or properly infinite*," provided conditions (5) are replaced by the two conditions:

(5_1) $\lim_{h\to 0} M_n(h)$ exists, finite or infinite; $\lim h^j M_n^{(j)}(h) = 0, \quad 1 \leqq j \leqq n-1;$

(5_2) $\lim_{h\to 0} hM_n(h) = 0.$

In case $\lim M_n(h)=+\infty$, then $f^{(n)}(a)=+\infty$, and similarly for $-\infty$. On the other hand (as pointed out by the referee), these conditions are *not necessary* in case $f^{(n)}(a)$ is infinite. That this is the case for each $n\geqq 1$ can be seen by the example

$$f(x) = x^{(2n-1)/2} \text{ if } x \geqq 0; \qquad f(x) = (-1)^n f(-x) \text{ if } x < 0.$$

For this function, $f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0$ and $f^{(n)}(0)=+\infty$. Taking $a=0$ in (4), and $a_0=a_1=\cdots=a_{n-1}=0$, we have $M_n(h)=|h|^{-1/2}$ for $h\neq 0$. Although it is true that

$$\lim M_n(h) = +\infty, \qquad \lim hM_n(h) = 0,$$

one easily verifies that $\lim |h^j M_n^{(j)}(h)| = +\infty$ (and not 0) for $1\leqq j\leqq n-1$.

By almost the same method of proof used in proving necessity above, however, one can prove that *if $f^{(n)}(a)$ exists, finite or properly infinite* (and f is continuous at a if $n=1$, or f' is continuous at a for $n\geqq 2$) then conditions (5) of the theorem may be replaced by the two (necessary) conditions:

(5_3) $\lim_{h\to 0} M_n(h)$ *exists, finite or infinite*;

(5_4) $\lim_{h\to 0} h^{j+1} M_n^{(j)}(h) = 0, \quad 0 \leqq j \leqq n-1.$

On the other hand, that these conditions are *not sufficient* for the existence of $f^{(n)}(a)$ can be seen using the example preceding the statement of the main theorem. In this example, $n=2$ and $\lim M_2(h)=0$; whether conditions (5_3) and (5_4) are sufficient in case $\lim M_n(h)=\infty$ seems unlikely, but we have no counterexample for this case.

REMARK 2. $f^{(n)}$ exists throughout $N(a)\Leftrightarrow$the conditions stated in the theorem hold, and in addition $M_n^{(n)}$ exists in $\tilde{N}(0)$. This follows at once from the representation (4).

REMARK 3. $f^{(n)}$ is continuous at $a \Leftrightarrow$ the conditions stated in the theorem hold, and in addition,

$$(5') \qquad M_n^{(n)} \text{ exists in } \tilde{N}(0), \quad \text{and} \quad \lim h^n M_n^{(n)}(h) = 0.$$

No difficulty occurs in completing the induction steps in the proof, and the argument in the case $n=1$ is trivial, as usual.

REMARK 4. If f is a complex-valued function of a real variable, the theorem remains valid; indeed the *proof* of the sufficiency part of the theorem remains valid. The necessity (as well as the sufficiency) follows by applying the theorem separately to the real and imaginary parts of f.

Similarly, if f is a complex-valued function of a complex variable the proof of the sufficiency part of the theorem remains valid, while the necessity part of the theorem follows from the Cauchy-Taylor expansion of the function for $n>1$, and from the definition of a derivative when $n=1$.

The author thanks the referee for his helpful suggestions.

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CLASSROOM NOTES

EDITED BY GEORGE RANEY, University of Connecticut

Material for this department should be sent to George Raney, Department of Mathematics, University of Connecticut, Storrs, CT06268.

SESQUIREGULAR MEASURES

S. K. BERBERIAN, The University of Iowa

In the theory of measure in locally compact spaces, the term "regular measure" has been defined in various ways. In this note several of the commonly used definitions will be compared, and simplifications will be observed in the case of finite measures.

Throughout, X denotes a locally compact Hausdorff space. We follow the terminology of [3]. In particular a weakly Borel measure ρ is said to be *inner regular* if

$$(1) \qquad \rho(A) = \text{LUB}\,\{\rho(C): C \subset A,\ C \text{ compact}\}$$

for all weakly Borel sets A, *outer regular* if

$$(2) \qquad \rho(A) = \text{GLB}\,\{\rho(U): U \supset A,\ U \text{ open}\}$$

for all weakly Borel sets A, and *biregular* if it is both inner and outer regular. If a weakly Borel measure is either inner regular or outer regular, its restriction to the class of Borel sets is a regular Borel measure (see [5, p. 228, Theorem F] or [1, p. 199, Theorem 5]; the only delicate point is that if $C \subset U$, where C is compact and U is open, then there exists an open Borel set V such that $C \subset V \subset U$ [5, p. 219, Theorem A]).

A weakly Borel measure τ will be called *sesquiregular* if it is outer regular, and if

$$\tau(U) = \text{LUB}\{\tau(C): C \subset U, C \text{ compact}\} \tag{3}$$

for all open sets U. Sesquiregular measures occur in the theory of integration developed in Hewitt-Ross [7, p. 123]; the main aim of this note is to exhibit the simple relation between sesquiregularity and inner regularity.

THEOREM 1. *If τ is a sesquiregular weakly Borel measure on X, there exists one and only one inner regular weakly Borel measure ρ on X such that $\rho(E) = \tau(E)$ for all Borel sets E. Specifically,*

$$\rho(A) = \text{LUB}\{\tau(C): C \subset A, C \text{ compact}\} \tag{4}$$

for all weakly Borel sets A.

Proof. Since the restriction of τ to the class of Borel sets is a regular Borel measure, formula (4) defines an inner regular weakly Borel measure ρ [2, Theorem 2]. As to uniqueness, it is obvious that any two inner regular weakly Borel measures that agree on compact sets are identical (in fact, agreement on compact G_δ's is sufficient [1, p. 200, Theorem 1]).

Conversely, it is shown in the next theorem that each inner regular ρ determines a unique sesquiregular τ.

LEMMA 1. *Let ρ be an inner regular weakly Borel measure. If A_0 is a weakly Borel set such that $A_0 \subset U_0$ for some open set U_0 of finite measure, then*

$$\rho(A_0) = \text{GLB}\{\rho(U): U \supset A_0, U \text{ open}\}.$$

Proof. If C_n is a sequence of compact subsets of $U_0 - A_0$ whose measure approximates that of $U_0 - A_0$, then $U_0 - C_n$ is a sequence of open supersets of A_0 whose measure approximates that of A_0.

THEOREM 2. *If ρ is an inner regular weakly Borel measure on X, there exists one and only one sesquiregular weakly Borel measure τ on X such that $\tau(E) = \rho(E)$ for all Borel sets E. Specifically,*

$$\tau(A) = \text{GLB}\{\rho(U): U \supset A, U \text{ open}\} \tag{5}$$

for all weakly Borel sets A.

Proof. It is routine to verify that the set function τ defined by (5) is monotone and countably subadditive, that it agrees with ρ on open sets and Borel

sets [5, p. 228, Theorem F], and that it satisfies the conditions (2) and (3). The proof that τ is additive can be reduced to the observation that $\tau(A)=\rho(A)$ whenever $\tau(A)$ is finite (Lemma 1). This proves existence. It is easy to see that a sesquiregular weakly Borel measure is uniquely determined by its values on Borel sets (or merely on compact G_δ's).

The sesquiregular analogue of [2, Theorem 2] is as follows (cf. [6]):

COROLLARY. *If μ is a regular Borel measure on X, there exists one and only one sesquiregular weakly Borel measure τ that extends μ.*

Proof. Let ρ be the unique inner regular weakly Borel extension of μ [2, Theorem 2], and define τ as in Theorem 2.

For finite measures, the concepts of inner regularity and sesquiregularity coincide, as we shall see in the next theorem.

LEMMA 2. *Suppose ρ_1 and ρ_2 are weakly Borel measures on X such that $\rho_1(C)=\rho_2(C)$ for all compact sets C. Assume that for each $i=1, 2$ there exists a Borel set E_i such that $\rho_i(X-E_i)=0$. Then $\rho_1=\rho_2$.*

Proof. We note first that ρ_1 and ρ_2 agree on Borel sets [1, p. 185, Theorem 3]. Define $E=E_1\cup E_2$. If A is any weakly Borel set, clearly $\rho_1(A-E)=\rho_2(A-E)=0$; since $A\cap E$ is a Borel set [1, p. 181, Theorem 2], we have also $\rho_1(A\cap E)=\rho_2(A\cap E)$, and therefore $\rho_1(A)=\rho_2(A)$.

THEOREM 3. *If ρ is a finite measure on the weakly Borel sets of X, the following conditions are equivalent:*

(a) *ρ is inner regular,*

(b) *ρ is biregular,*

(c) *ρ is sesquiregular,*

(d) *ρ is outer regular, and there exists a Borel set E such that $\rho(X-E)=0$.*

Proof. By Lemma 1, (a) implies (b). It is trivial that (b) implies (c).

(c) implies (d): By (3) there is a sequence of compact sets C_n such that $\rho(X)$ is the least upper bound of the $\rho(C_n)$. The union of the C_n is a Borel set E such that $\rho(E)=\rho(X)$, and so $\rho(X-E)=0$ by finiteness.

(d) implies (a): The restriction of ρ to the class of Borel sets is a regular Borel measure μ. Let ρ' be the inner regular weakly Borel extension of μ; to prove that ρ is inner regular, it will suffice to show that $\rho=\rho'$. Since μ is bounded, it follows that ρ' is finite; by the argument in the preceding paragraph, there exists a Borel set E' such that $\rho'(X-E')=0$. Since ρ and ρ' agree on Borel sets, $\rho=\rho'$ by Lemma 2.

The analogue of Lemma 1 for sesquiregular measures is as follows [7, p. 126, Theorem 11.32]:

COROLLARY. *If τ is a sesquiregular weakly Borel measure, and A_0 is a weakly Borel set of finite measure, then*

$$\tau(A_0) = \mathrm{LUB}\{\tau(C): \quad C \subset A_0,\ C \text{ compact}\}.$$

Proof. Choose any open set U_0 such that $U_0 \supset A_0$ and $\tau(U_0)$ is finite (τ is outer regular); let E_0 be a Borel set such that $E_0 \subset U_0$ and $\tau(U_0 - E_0) = 0$ (see the proof of Theorem 3). The formula $\tau'(A) = \tau(A \cap U_0)$ defines a finite weakly Borel measure τ' that is obviously outer regular. Since $\tau'(X - E_0) = \tau(U_0 - E_0) = 0$, it follows from criterion (d) of Theorem 3 that τ' is inner regular; the conclusion of the corollary follows on applying (1) to τ' and A_0.

REMARKS. 1. If X is not σ-compact, there exists a nonzero outer regular weakly Borel measure on X that vanishes for all Borel sets [2, p. 417], and is therefore not sesquiregular.

2. An inner regular measure need not be sesquiregular. For example if G is a locally compact group which is neither discrete nor σ-compact, and if ρ is the inner regular Haar measure on the weakly Borel sets of G, then ρ is not outer regular [3, Theorem 1].

3. A sesquiregular measure need not be inner regular. For example, let G and ρ be as in Remark 2, and let τ be the sesquiregular Haar measure on the weakly Borel sets of G defined as in Theorem 2. Since ρ is not outer regular, it cannot be proportional to τ; hence τ cannot be inner regular (if it were, it would be proportional to ρ [2, Theorem 3]).

4. Let ν be a finite Baire measure on X, and let f be any bounded continuous function on X. Even though f may not be a Baire function, the "integral of f with respect to ν" may be defined as follows: (i) choose a Baire set F such that $\nu = \nu_F$ [1, p. 154, Theorem 2], (ii) observe that $\chi_F f$ is a bounded Baire function [1, p. 182, Exercise 13, and p. 48, Exercise 2], and (iii) define $\int f d\nu$ to be $\int \chi_F f d\nu$ [1, p. 125, Lemma 4].

For example, if G is a locally compact abelian group with character group X, a well-known representation theorem may be stated as follows: *A continuous positive definite function on G is the Fourier-Stieltjes transform of one and only one finite biregular weakly Borel measure on X* [8, p. 19, Theorem 1.4.3, and p.17, Theorem 1.3.6]. In view of the preceding paragraph, and the theorems of this note (especially Theorem 3), the representation theorem remains correct with "biregular weakly Borel measure" replaced by "Baire measure" or "regular Borel measure" or "inner regular weakly Borel measure" or "sesquiregular weakly Borel measure."

5. The validity of "(a) implies (b)" in Theorem 3 reveals an error in [4]; a "regular weakly Borel *PO*-measure" is indeed biregular, contrary to the assertion in [4, p. 88, Example 4]. The erroneous assertion can simply be excised; it is used nowhere in [4].

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A NEW PROOF OF LEBESGUE'S COVERING LEMMA

JOHN HUNT, University of Virginia

Introduction. Lebesgue's covering lemma states that, given an open covering $U_1, \cdots, U_n$ of a compact metric space X, ρ, there is a positive number δ such that if $\rho(x, y) < \delta$ then both x and y belong to some U_i. The purpose of this short note is to enlarge upon this conclusion and thereby provide a more interesting proof of the lemma than the usual ones.

We first explain how we arrive at the new result. Figure 1 shows a compact metric space covered by two open subsets U and V. If δ is the distance between $U - V$ and $V - U$, then any two points whose distance apart is less than δ both lie in U or V; further, no number greater than δ will ensure this. Figure 2 shows a compact metric space X, ρ covered by a finite number of open subsets $U_1, \cdots, U_n$ and one may suspect that the same idea holds. The lines of the figure divide the set X up into a number of "compartments" (those white bits crossed by no lines) and by analogy one may suspect that two of these compartments A and B, at a positive distance apart, have the properties

(i) if $\rho(x, y) < \rho(A, B)$ then both x and y belong to some U_i,

(ii) no number greater than $\rho(A, B)$ has this property. Except in a trivial case this is so, and it is the extension of Lebesgue's lemma that we shall prove.

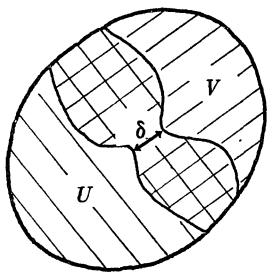


FIG. 1

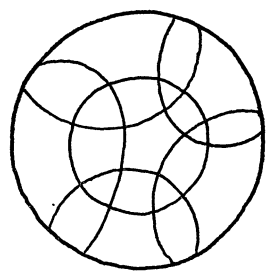


FIG. 2

The trivial exception. When each pair of points is contained in some U_i, no pair of compartments satisfies (ii), because every positive number satisfies (i). In this case, however, Lebesgue's lemma is trivial.

DEFINITION OF A COMPARTMENT. *Let X be a set covered by a finite number of subsets $X_1, \cdots, X_n$. A compartment (of the covering $X_1, \cdots, X_n$) is a nonempty set expressible as the intersection of n distinct sets, consisting of X_i's and complements of X_i's.*

It follows from the definition that the compartments of a finite covering of X form a finite, disjoint covering of X. Where no confusion arises, we simply speak of compartments, instead of compartments of a particular covering. We do this below.

THEOREM. *If $U_1, \cdots, U_n$ is an open covering of a compact metric space X, ρ, and some pair of points is contained in no U_i, then there are two compartments A and B, a positive distance apart, such that*

(i) *if $\rho(x, y) < \rho(A, B)$ then both x and y belong to some U_i,*

(ii) *no number greater than $\rho(A, B)$ has property* (i).

Proof. The two points contained in no U_i belong to a pair of compartments contained in no U_i. Thus we may define

$$\delta = \min \rho(E, F),$$

where E and F are any compartments contained in no U_i. Then δ is attained as the distance between some pair of compartments A and B, and it satisfies the requirements of the theorem.

First, $\delta > 0$. For, let E and F be compartments such that $\rho(E, F) = 0$. Then from E and F we can select sequences $\{x_i\}$ and $\{y_i\}$ such that $\rho(x_i, y_i) \to 0$. By compactness, there is a point x and a subsequence $\{x_{N_i}\}$ of $\{x_i\}$ such that $x_{N_i} \to x$. Since $y_{N_i} \to x$ as well, x belongs to both $\overline{E}$ and $\overline{F}$. But x belongs to some open U_k. Thus U_k meets both E and F and so, by the definition of compartment, contains both E and F.

δ also satisfies (i) and (ii). For, let $\rho(x, y) < \delta$. x and y belong to compartments E and F. If $E = F$ then both x and y necessarily belong to some U_i, because each compartment is contained in some U_i. If $E \neq F$ then $\rho(E, F) < \delta$ and some U_i contains both E and F. Thus some U_i contains both x and y. On the other hand, if $\delta_1 > \delta$ then there are two compartments E and F, contained in no U_i, such that $\rho(E, F) < \delta_1$. In E and F we can select points x and y such that $\rho(x, y) < \delta_1$. Then x and y belong to no U_i since otherwise U_i would contain both E and F.

AN INTERESTING DUAL GALOIS CORRESPONDENCE

W. A. LABACH, Northwestern University

Let P and Q be complete lattices and let $\sigma: P \to Q$ and $\tau: Q \to P$ be isotone (i.e., order preserving) mappings. Define for $x \in P$, $\bar{x} = x^{\sigma\tau}$ and for $y \in Q$ define $y^* = y^{\tau\sigma}$. Suppose that $\bar{x} \leqq x$ and $y \leqq y^*$ for each $x \in P$ and $y \in Q$, respectively. Then it is a standard exercise to verify that $x \to \bar{x}$ is a dual closure operation, $y \to y^*$ is a closure operation, and that σ and τ are inverse isomorphisms (after suitable restriction) between the complete lattices of closed elements of P and Q [1], [2]. The mappings σ, τ are said to establish a *dual Galois correspondence* between P and Q. It is hoped that those with a knowledge of group theory will be able to find interesting applications for the following example of a dual Galois correspondence.

Let G be a group and let $A(G)$ denote its automorphism group, $L(G)$ its lattice of subgroups, and $L(A(G))$ the lattice of subgroups of its automorphism group. Let H be a subgroup of G and let H^σ denote the set of all automorphisms α of G such that $x^{-1}(x)\alpha \in H$ for all $x \in G$.

LEMMA. *H^σ is a subgroup of $A(G)$.*

Proof. Suppose $\alpha, \beta \in H^\sigma$. Then

$$x^{-1}(x)\alpha\beta = x^{-1}(x)\alpha[(x)\alpha]^{-1}[((x)\alpha)\beta] \in H, \qquad x \in G.$$

Thus $\alpha\beta \in H^\sigma$. In a similar way we may show that the inverse of each element of H^σ is in H^σ. Thus H^σ is a group.

Suppose that K is a subgroup of $A(G)$. We define K^τ to be the subgroup generated by the set of all elements of the form $x^{-1}(x)\alpha$, $x \in G$, $\alpha \in K$.

THEOREM 1. *The pair of mappings σ, τ establish a dual Galois correspondence between $L(G)$ and $L(A(G))$.*

Proof. Obvious.

THEOREM 2. *If K is a subgroup of $A(G)$ then K^τ is a normal subgroup of G.*

Proof. It is sufficient to show that $x^{-1}a^{-1}(a)\alpha x$ is in K^τ for each $\alpha \in K$, $a \in G$, and $x \in G$. Now

$$x^{-1}a^{-1}(a)\alpha x = (ax)^{-1}(a)\alpha x = [(ax)^{-1}(ax)\alpha][x^{-1}(x)\alpha]^{-1} \in K^\tau.$$

THEOREM 3. *If K is a normal subgroup of $A(G)$ then K^τ is a characteristic subgroup of G.*

Proof. Let $x^{-1}(x)\alpha$ be a generator of K^τ where $\alpha \in K$, $x \in G$. It is sufficient to show that $(x^{-1}(x)\alpha)\gamma \in K^\tau$ for each $\gamma \in A(G)$. By hypothesis, $\gamma^{-1}\alpha\gamma$ is an element of K, say β. Now

$$(x^{-1}(x)\alpha)\gamma = (x^{-1})\gamma(x)\alpha\gamma = [(x)\gamma]^{-1}(x)\gamma\beta = [(x)\gamma]^{-1}[(x)\gamma]\beta \in K^\tau.$$

THEOREM 4. *If H is a characteristic subgroup of G then H^σ is a normal subgroup of $A(G)$.*

Proof. It is sufficient to show that $x^{-1}(x)\gamma^{-1}\alpha\gamma \in H$ for each $x \in G$, $\alpha \in H^\sigma$, and $\gamma \in A(G)$. Notice that for any $y \in G$ it follows that $(y)\alpha = yh$ for some $h \in H$ since $\alpha \in H^\sigma$. Now

$$x^{-1}(x)\gamma^{-1}\alpha\gamma = x^{-1}[((x)\gamma^{-1})\alpha]\gamma = x^{-1}[(x)\gamma^{-1}h]\gamma = x^{-1}[(x)\gamma^{-1}\gamma][h]\gamma = x^{-1}xh' = h'$$

for some h, $h' \in H$ by the previous sentence and the fact that H is a characteristic subgroup of G. Thus H^σ is a normal subgroup of $A(G)$.

THEOREM 5. *The mapping σ is a complete meet-homomorphism of $L(G)$ into $L(A(G))$ and the mapping τ is a complete join-homomorphism of $L(A(G))$ into the complete sublattice of normal subgroups of $L(G)$.*

Proof. This follows at once from Theorem 2 and well-known properties of closure operations [1], [2].

The work in this paper was done while the author held a National Science Foundation Fellowship at the University of Illinois.

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A BEST STATISTIC WITH VARIANCE NOT EQUAL TO THE CRAMÉR-RAO LOWER BOUND

W. C. GUENTHER, University of Wyoming

1. Introduction. The Cramér-Rao inequality is discussed in recent textbooks ([2], [3], and [5]). The inequality provides a nonconstructive method for finding the minimum variance unbiased statistic ("best statistic") for an unknown parameter θ which appears in a probability density function. More recent results, discussed in [2, chap. 8] and [4, chap. 8], provide a constructive procedure for finding a best statistic. For most of the distributions which are frequently studied in a course in mathematical statistics, the variance of the obvious estimate of the parameter satisfies the Cramér-Rao inequality making it unnecessary to call upon the more sophisticated procedures mentioned above. In this paper we present an example in which the best statistic has a variance greater than the Cramér-Rao lower bound and it is quite unlikely that one would uncover the estimate without a constructive approach.

2. The example. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution having p.d.f.

$$f(x;\theta) = \theta x^{\theta-1}, \qquad 0 < x < 1, \theta > 0. \tag{1}$$

Let W be an unbiased statistic for θ. Then according to Cramér-Rao

$$\sigma_W^2 \geqq \frac{1}{nE\{[\partial \ln f(X;\theta)/\partial\theta]^2\}}. \tag{2}$$

It is easy to show that

$$\frac{\partial \ln f(X;\theta)}{\partial\theta} = \ln X + \frac{1}{\theta}, \qquad E\left[\left(\ln X + \frac{1}{\theta}\right)^2\right] = \frac{1}{\theta^2},$$

so that $\sigma_W^2 \geqq \theta^2/n$.

The best estimate of θ is easily found. Since (1) is a member of the exponential class of density functions, we immediately know that $Y = -\sum_{i=1}^{n} \ln X_i$ is a sufficient statistic for θ and, according to Theorem 6 ([2] p. 224), the best statistic is a function of Y which has expectation θ. Now

$$E(e^{tY}) = \left(1 - \frac{t}{\theta}\right)^{-n} \qquad t < \theta$$

so that the p.d.f. of Y is $g(y) = (\theta^n/\Gamma(n))y^{n-1}e^{-\theta y}$, $y > 0$. Since $E[(n-1)/Y] = \theta$, we see that $(n-1)/Y = Z$, $n \geqq 2$, is the best statistic. Also it is easy to verify that $\sigma_Z^2 = \theta^2/(n-2) > \theta^2/n$.

3. The improved bound of Bhattacharyya. A special case of a result due to Bhattacharyya [1] for an improved lower bound for the variance of W yields

$$\sigma_W^2 \geqq \frac{\lambda_{22}}{\lambda_{11}\lambda_{22} - \lambda_{12}^2}, \tag{3}$$

where

$$\lambda_{ij} = E[T_iT_j], \qquad T_k = \frac{1}{P}\frac{\partial^k P}{\partial \theta^k}, \qquad P = \prod_{i=1}^{n} f(X_i, \theta).$$

It is easy to show that the Bhattacharyya bound is greater or equal to the Cramér-Rao bound and the two bounds are the same when a statistic exists which produces equality in the latter.

For the density (1) we get

$$\lambda_{11} = E(T_1^2) = E\left[\left(-Y + \frac{n}{\theta}\right)^2\right] = \frac{n}{\theta^2}$$

$$\lambda_{12} = E(T_1T_2) = E\left[\left(-Y + \frac{n}{\theta}\right)^3 - \frac{n}{\theta^2}\left(-Y + \frac{n}{\theta}\right)\right] = -\frac{2n}{\theta^3}$$

$$\lambda_{22} = E(T_2^2) = E\left[\left(-Y + \frac{n}{\theta}\right)^4 - \frac{2n}{\theta^2}\left(-Y + \frac{n}{\theta}\right)^2 + \frac{n^2}{\theta^4}\right] = \frac{2n(n+3)}{\theta^4},$$

where $Y = -\sum_{i=1}^{n} \ln X_i$, $E(Y^k) = [\Gamma(n+k)]/[\theta^k\Gamma(n)]$. Thus (3) yields

$$\sigma_W^2 \geqq \frac{\theta^2}{n}\frac{n+3}{n+1} \quad \text{and we note} \quad \frac{\theta^2}{n} < \frac{\theta^2}{n}\frac{n+3}{n+1} < \frac{\theta^2}{n-2}.$$

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A SIMPLE PROOF OF THE SCHRÖDER-BERNSTEIN THEOREM

I. I. KOLODNER, Carnegie Institute of Technology

The objective of the present note is to clarify the proof of the Schröder-Bernstein Theorem given in Suppes ([1], p. 95–96). While in principle our construction does not involve anything beyond what already appears in [1], we stress here the fact that the proof is achieved by finding a fixed point of a certain function. This function turns out to be an isotonic self-mapping of a complete lattice, and thus has a fixed point.

A clarification of this type seems not to have attracted the attention of other expositors. The proof in [1] follows an *ex-machina* pattern: an equation is written, one does this and that, and presto—the conclusion follows. The same attitude is taken by Fraenkel ([2], p. 102–103) on which the proof of [1] is based, and the same is indicated by Dieudonné ([3], p. 10, Exercise 4, and p. 13, Exercise 3) through hints.

The Schröder-Bernstein Theorem asserts: *If the functions* f_1: $X_1 \to X_2$ *and* f_2: $X_2 \to X_1$ *are one-to-one, then there exists a function* g: $X_1 \to X_2$ *which is one-to-one and onto.*

The beginning of our proof is standard: the assertion follows if one can produce subsets $A_1 \subset X_1$ and $A_2 \subset X_2$ such that

$$f_1(A_1) = A_2' \quad \text{and} \quad f_2(A_2) = A_1'. \tag{1}$$

(Primes denote complements.) Indeed, the restrictions $f_1|A_1$: $A_1 \to A_2'$ and $f_2|A_2$: $A_2 \to A_1'$ are then one-to-one and onto functions and thus g: $X_1 \to X_2$ defined by

$$g(x) = \begin{cases} f_1(x), & \text{if } x \in A_1, \\ (f_2|A_2)^{-1}(x), & \text{if } x \in A_1', \end{cases} \tag{2}$$

has the desired property.

The functions f_1 and f_2 induce associated set functions F_1: $P(X_1) \to P(X_2)$ and F_2: $P(X_2) \to P(X_1)$ on the corresponding power sets, where $F_i(U) = \{f_i(x) \mid x \in U \in P(X_i)\}$, $i = 1, 2$. Let C_i: $P(X_i) \to P(X_i)$, $i = 1, 2$ be the complement function, i.e., $C_i(U) = U' \subset X_i$. Then condition (1) implies that the pair (A_1, A_2) will be a suitable pair of sets if and only if it is a solution of the pair of conditional equations

$$F_1(x) = C_2(y), \qquad F_2(y) = C_1(x). \tag{3}$$

Thus it suffices to show that (3) has a solution.

Eliminating y from equations (3) we now get a *single fixed point equation* for $P(X_1)$,

$$x = K(x), \quad \text{where} \quad K = C_1 \circ F_2 \circ C_2 \circ F_1 : P(X_1) \to P(X_1). \tag{4}$$

It is easily checked that if U is a solution of (4) then the pair $(U, (C_2 \circ F_1)(U))$

is a solution pair of the system (3). Thus our objective is now to show that K has a fixed point.

To solve (4) observe that F_1 and F_2, being set functions induced by point functions, are isotonic (i.e., order preserving) under the orderings of $P(X_1)$ and $P(X_2)$ by set inclusion. The complements C_1 and C_2 are antitonic (i.e., order inverting) functions. Thus K, being a composition of two isotonic and two antitonic functions, is an isotonic function.

At this point we may invoke a well known fixed point theorem for isotonic functions since $(P(X_1), \subset)$ is a complete lattice; see, e.g., [4], p. 54. The matter is so simple, however, that we hardly need a reference to finish the proof. Let

$$S = \{x \in P(X_1) \mid x \subset K(x)\};$$

S is not empty, since $\varnothing \subset K(\varnothing)$. We shall show that $U = \bigcup\{x \in S\}$ is a fixed point of K. If $a \in U$, then $a \in x$ for some $x \in S$ and so $a \in K(x)$; since K is isotonic and $x \subset U$, it follows that $K(x) \subset K(U)$ whence $a \in K(U)$, or $U \subset K(U)$. Since K is isotonic, this yields $K(U) \subset K(K(U))$ showing that $K(U) \in S$. From the definition of U we now conclude that $K(U) \subset U$, and consequently $U = K(U)$, as asserted.

Coming back to the original question, we see that $A_1 = U$, $A_2 = (C_2 \circ F_1)(U)$ is a pair satisfying the conditions (1).

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ON NEWTON-RAPHSON ITERATION

J. F. TRAUB, Bell Telephone Laboratories, Murray Hill, N. J.

Let α be a zero of multiplicity m, $m \geqq 1$, of a sufficiently smooth function f and assume that f' does not vanish in the interval of interest except possibly at α. For the case of a simple zero, the Newton-Raphson iteration function ϕ is usually derived by linearizing f using the first two terms of its Taylor series about an arbitrary point. The error term is obtained from the remainder of the Taylor expansion of f or from the expansion of ϕ around α, [1, pp. 20, 75]. The error term involves an unknown parameter. The case of multiple zeros is usually handled [1, Chapter 7] separately from that of simple roots.

We give a method for deriving formulas for the iteration function and its error simultaneously. Simple and nonsimple zeros are handled together.

We then give a novel principle for the construction of iteration functions and apply it to the derivation of the Newton-Raphson and Halley iteration functions.

Let

$$f(t) = (t - \alpha)^m Q(t). \tag{1}$$

Then

$$f'(t) = m(t - \alpha)^{m-1}Q(t) + (t - \alpha)^m Q'(t) \tag{2}$$

and $(t-\alpha)f'(t) = mf(t) + (t-\alpha)^{m+1}Q'(t)$. Hence

$$\alpha = t - m\frac{f(t)}{f'(t)} - (t - \alpha)^{m+1}\frac{Q'(t)}{f'(t)}. \tag{3}$$

Let $\phi(t) = t - m[f(t)/f'(t)]$. Then

$$\phi(t) = \alpha + (t - \alpha)^{m+1}\frac{Q'(t)}{f'(t)}. \tag{4}$$

$\phi(t)$ is the Newton-Raphson iteration function for a zero of multiplicity m and (4) is an expression for the error.

From (2) and (4),

$$\frac{\phi(t) - \alpha}{(t - \alpha)^2} = \frac{Q'(t)}{mQ(t) + (t - \alpha)Q'(t)}. \tag{5}$$

Hence

$$\lim_{t\to\alpha}\frac{\phi - \alpha}{(t - \alpha)^2} = \frac{Q'(\alpha)}{mQ(\alpha)}. \tag{6}$$

By expanding $Q(t)$, $Q(t) = [f^{(m)}(\alpha)/m!] + [f^{(m+1)}(\alpha)/(m+1)!](t-\alpha) + \cdots$, we obtain

$$\lim_{t\to\alpha}\frac{\phi - \alpha}{(t - \alpha)^2} = \frac{f^{(m+1)}(\alpha)}{m(m + 1)f^{(m)}(\alpha)}, \tag{7}$$

which is a standard result.

If α is a simple zero,

$$\begin{aligned} \phi &= \alpha + (t - \alpha)^2[Q'(t)/f'(t)], \\ \phi &= \alpha + (t - \alpha)^2\frac{f''(t)}{2f'(t)} - (t - \alpha)^3\frac{Q''(t)}{2f'(t)}. \end{aligned} \tag{8}$$

From (7) or (8) we obtain the usual relation for the case of simple roots,

$$\frac{\phi - \alpha}{(t - \alpha)^2} \to \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}.$$

We now obtain the Newton-Raphson iteration function from another point of view. Assume that α is a simple zero. From (1),

$$\alpha = t - (f/Q). \tag{9}$$

Q is not known but we approximate it as well as possible from available information. From $Q(t) = f'(t) - (t-\alpha)Q'(t)$ we observe that f' differs from Q by terms of order $t-\alpha$ and hence we approximate Q by f' in (9) to obtain the Newton-Raphson iteration function.

One may show that

$$Q = T + \frac{(t-\alpha)^2}{2}\left[Q'' - f''\frac{Q'}{f'}\right], \qquad T = f' - \frac{f''}{2}\frac{f}{f'}.$$

That is, T differs from Q by terms of order $(t-\alpha)^2$. This suggests the iteration function

$$\phi = t - \frac{f}{T} = t - \frac{f}{f' - (f''f/2f')}.$$

But this is just Halley's third-order iteration. A brief historical survey of this often rediscovered iteration function may be found in [1, p. 91].

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A SIMPLE PROOF OF THE CHINESE REMAINDER THEOREM

C. J. Mozzochi, University of Connecticut

Chinese Remainder Theorem. *Let $m_1, \cdots, m_n$ be relatively prime integers. Let $a_1, \cdots, a_n$ be arbitrary integers. Then*:

1. *The congruences $x \equiv a_i \pmod{m_i}$ $i = 1, \cdots, n$ have at least one simultaneous solution.*

2. *Any two simultaneous solutions are congruent modulo $m_1 \cdots m_n$.*

The proof follows directly from the following result of number theory:

A. Suppose that $m_1, \cdots, m_n$ are pairwise relatively prime integers. Then $x \equiv y \pmod{m_i}$ $i = 1, \cdots, n$ if and only if $x \equiv y \pmod{m_1 \cdots m_n}$.

Proof. If $a \in Z$, let $[x]_a = \{y \mid x \equiv y \bmod a\}$ and let Z_a be the set of all residue classes modulo a. Define

$$\alpha: Z_{m_1 \cdots m_n} \to Z_{m_1} \times \cdots \times Z_{m_n} \text{ by } ([x]_{m_1 \cdots m_n})\alpha = ([x]_{m_1}, \cdots, [x]_{m_n})$$

for each $x \in Z$. By A, α is a well-defined, 1-1 mapping of $Z_{m_1 \cdots m_n}$ into $Z_{m_1} \times \cdots \times Z_{m_n}$. Since $\text{Card}(Z_{m_1 \cdots m_n}) = m_1 \cdots m_n = \text{Card}(Z_{m_1} \times \cdots \times Z_{m_n})$, α is onto. But then, given integers $a_1, \cdots, a_n$ there is an integer x such that $([x]_{m_1 \cdots m_n})\alpha = ([a_1]_{m_1}, \cdots, [a_n]_{m_n})$ and therefore $x \equiv a_i \bmod m_i$, $i = 1, \cdots, n$. By A, any two solutions are congruent modulo $m_1, \cdots m_n$.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington D.C. 20005

WHAT ADVANCED DEGREE WITHOUT MATHEMATICS RESEARCH?

G. R. RISING, State University of New York at Buffalo

You probably all know the story of the African explorer who lived among the lions so long that he came to know their language. (I say probably because I first read it in this journal in a delightful article by Moses Richardson.) Much to his surprise he found that the lions believed God was a lion. Compare this story with my own experience among the mathematicians. When I told how we wanted to build a doctoral program in mathematics education which would have significance for our many students who staff or will staff mathematics departments in state colleges, I was taken aback by the rejoinder that since those people are to teach mathematics, they should be in mathematics and *not* education programs.

For a great many reasons I disagree with this attitude. The staff in mathematics education does not wish to infringe upon the central role of the mathematics department producing research mathematicians—but rather to play a cooperative (and service) role in training teachers of undergraduate mathematics. It is evident that there are not now enough research-oriented mathematicians to staff undergraduate mathematics programs. I wish to defend the stand: *The degree appropriate to college mathematics teachers who do not choose to become mathematical researchers is in many cases the doctorate in mathematics education.*

The program we are establishing will require:

(1) The equivalent of more than a master's degree (30 hours) in graduate mathematics. (Those students who do not have undergraduate mathematics majors must fulfill upper division undergraduate requirements in addition to this.)

(2) A major field examination made out in cooperation with mathematics department members.

(3) A thesis (usually empirical, possibly historical or philosophical) with at least one mathematician on the committee.

(4) A strong dose of psychology and research design when the study is empirical.

(5) Course work in mathematics education involving curricular and pedagogical problems.

(6) A minimum amount (usually 8 semester hours) of additional education course work.

This program is clearly interdisciplinary. The guidance of an individual student will be undertaken in close cooperation with the departments both of mathematics and of psychology (or history or philosophy).

Here is an example of the course work a student planning an empirical dissertation might elect.

Research Methods		
F522—Statistical methods in education: inference I	3	
F523—Statistical methods in education: inference II	3	
F529—Statistical methods in education: survey of advanced techniques	3	
		9
Mathematics Education		
E551—Current mathematics curriculum: arithmetic, algebra, analysis	3	
E552—Current mathematics curriculum: geometry	3	
E547—Seminar in mathematics teaching	3	
E548 —Research seminar in mathematics education	3	
E599—Independent study and dissertation	9	
		21
Mathematics		
Math 513–14—Advanced algebra	6	
Math 529–30—Theory of numbers	6	
A course in foundations of mathematics	6	
		18
Electives		
Math. Stat. 505–6—Basic statistical analysis I, II	8	
Phil. 390—Philosophy of mathematics	3	
Psych. 547—Theories of learning	3	
		14
Total beyond Ed. M.		62

This program gives the student a good background in mathematics taught by members of the mathematics department designed to give breadth of understanding of the field as well as an educational background useful in teaching courses in mathematical pedagogy and in working with teachers in elementary and secondary schools.

It also develops his ability to produce scholarly research, not in mathematics, to be sure, but in mathematical pedagogy, in which he can and usually will carry out further studies, whereas for the Ph.D. mathematician in the same school the last piece of research is often his first. The student has adequate guidance in this program. In the mathematics department he would be low man, working on a non-research project to which his advisor, by the very nature of his work, is not deeply committed.

This program solves the problem of providing an alternate route for students who wish to teach undergraduate mathematics in reputable but not research-oriented institutions.

Instead of creating a new second-rate degree, instead of faking a dissertation, instead of turning away talented but non-research-oriented students, I suggest once again that a degree now offered be utilized and strengthened by cooperation between mathematicians and mathematics educators. In competition the mathematicians, the educators, and the students would, it seems to me, all be losers.

ALTERNATIVES TO RESEARCH

D. E. CHRISTIE, Bowdoin College and J. H. WELLS, University of Kentucky and Institute for Advanced Study

A formidable challenge to excellence is posed by two trends in collegiate mathematics education: one is the admirable flow of numerous students into fields requiring mathematics, with a consequent demand for good college teachers; the other is the lamentable tendency for undergraduate mathematics faculties to be understaffed and overworked. The two trends together generate a challenge which the CUPM Panel on College Teacher Preparation now confronts: How can undergraduate faculty members achieve and maintain high quality in mathematics teaching?

Let us assume that active enthusiasm for mathematics is one prerequisite for excellent teaching. At universities, we expect research-oriented professors to impart a sense of mathematics as a growing, beautiful subject. Mathematics students at colleges need similar insight and inspiration. In this article we consider questions which represent one facet of the challenge just cited: Can a college teacher be a truly professional mathematician without doing research? What can take the place of research in keeping mathematics teachers alive? Our intention in suggesting answers to these questions is twofold. We hope that individual teachers will find things which they may wish to do. We also hope that strong departments and officers of societies will recognize ways in which they can and should assist their neighbors and constituents.

First, there are some standard mental push-ups by which an isolated and busy teacher can strive to remain current. These activities are closely associated with courses which he gives. He may:

1. Read, in a modern treatise more advanced than the textbook he uses, chapters parallel to class assignments.
2. Scan books and articles in a search for lively examples and applications.
3. Prepare a syllabus for outside independent reading for his better students.
4. Write substitute units because of his dissatisfaction with a textbook.
5. Do minor research, puzzling over questions arising naturally in his courses.

In summary, courses taught are a first natural avenue to creative activity.

Secondly, the college teacher may take more drastic positive measures to keep his teaching fresh. He may:

6. Join with others in an interdepartmental course, seminar, or project.
7. Conduct one of his courses experimentally in developmental or seminar style, realizing that no longer will it be enough to give routine and perfunctory attention to the course.
8. Seek out a summer conference or a neighborhood coalition in which improvements for a crucial course can be worked out.
9. Find motivation for effort or change by acquiring from graduate departments precise descriptions of key first year graduate courses along with preferred or minimal undergraduate requirements. (This information, rarely found in catalogues or brochures, will be of value to his students as well as to him.)

Third, he may try to take better advantage of a fleeting but invaluable asset, the presence of students who have exceptional aptitude and energy. He may:

10. Advise interested students concerning competitive opportunities: e.g., the Putnam contest, problems in *Monthly*, *Magazine*, etc.
11. Organize a mathematics club, or meeting of majors, at which talks are given by advanced students as well as staff and visitors.
12. Urge one or more of his best students to undertake independent reading with weekly lectures to him.
13. Establish, in connection with a course at any level, a project seminar at which reports are given by students.

Fourth, if he has opportunity and incentive, he may make a major investment of time in professional activity. He may:

14. Read expository articles and books (e.g., MAA publications) particularly if it is made easy for him to find them.
15. Read review articles on recent advances in mathematics.
16. Attend a summer institute (there should be many more summer programs for college teachers).
17. Participate in a study group with occasional area conferences and exchange by mail of working papers. (The assumption here is that someone has diagnosed and called attention to a particular need.)
18. Spend his sabbatical at a university where he is really welcome and will receive advice about a program valid for him.

Additional or modified measures can easily be added to this list.

Several of the alternatives to research listed above come close to research without having new theorems as a main objective. Lifting a sentence from a recent letter to *Science* by George W. Mackey (discussing a rather different issue): "Reworking old material fades imperceptibly into discovery of new, and what starts with an attempt to understand something old from a personal point

of view often results in new progress and new discoveries," [1]. The good college teacher of mathematics, who does not consider himself a research mathematician, will often teach creatively if there is time, motivation, opportunity, and perhaps some guidance.

There is no substitute for *time*. A teacher burdened with a fifteen hour teaching load and a normal budget of committee work will find it difficult to respond to challenges such as we have outlined. Note then the need for compensating released time for the creative teacher as well as for the research man.

If time is available, what are the sources of *motivation?* Teachers tend to respond to opportunities closely related to their own courses and their own students. New programs to assist teachers should then emphasize relevance to teaching objectives. Active support by department chairmen and encouragement of colleagues are essential. Productive mathematical activity should earn both appreciation and tangible reward. Teachers and administrators should learn to recognize the distinction between routine performance of academic chores and a genuinely professional involvement in mathematics. The teacher must be convinced that beneficial mathematical activity on his part is needed and expected, and that his rewards will exceed those of less dedicated colleagues.

Finally, can the well motivated teacher of undergraduates, with time available, actually find the *means* and *opportunity* of pursuing the goals we have so freely listed? Certainly there is already an impressive reservoir of good books for parallel study by a teacher or independent reading by a good student. Many of these are listed in the *Basic Library List* of CUPM. For example, a modest but extremely worthwhile project would be to study and work problems in Courant or Apostol while teaching a course in calculus, or to study Halmos' book *Finite Dimensional Vector Spaces* while teaching a sophomore course in linear algebra. In the direction of independent work for better students, an excellent project for juniors and seniors is the preparation of a problem notebook based on Boas' book *Primer of Real Functions*. Expository articles always seem scarce, but excellent ones can be found in MAA and NCTM publications and in other sources mentioned in the *List*. There is reason to expect that mathematical publications will in the future more closely assist the needs of the professional college teacher who is not committed to research. It is well known that research-oriented professors of mathematics are a minority group just as research-minded physicians are a minority group in the medical profession. The needs of the mathematical minority are magnificently provided for by the American Mathematical Society. The interests of the majority are increasingly heeded in other organizations.

Obviously not all of the activities listed above exist for each individual teacher, and even if they do he may not know of them. College mathematics teachers are greatly in need of a means of communication which would keep all of them fully abreast of professional opportunities (e.g., 8, 16, 17, 18). Because of the existing lack of communication, many college teachers are unaware of the availability of Federal support for activities other than pure research. Both the National Science Foundation and the Department of Health, Education, and

Welfare support curriculum reform and innovation, staff training, and experimental research. We urge each college mathematics department, especially those who have not done so in the past, to become informed about these services. Direct inquiries will bring information about conferences, institutes, and other opportunities for college teachers. Furthermore, CUPM serves as a clearing house for information concerning curricular developments. The CUPM Consultants Bureau and the MAA Visiting Lecturer Program bring timely professional counsel to campuses making a request.

Mathematicians, either in or out of research, keep alive through the creative interaction of minds. Such interaction is possible even to the isolated busy teacher through books, journals, conferences, seminars, and institutes, as well as through contacts with students and colleagues. Never before have college teachers of mathematics been offered so much assistance aimed at restoring their vitality and at promoting their productivity. But never before has the responsibility for vigorous initiative lain on so many shoulders. Each teacher, as always, must seek for himself an optimal but realistic program of continuing study and exploration. Each department chairman bears a special responsibility to encourage junior members to stay alive professionally, to see that the bright young men have assignments which will challenge them, and to adjust teaching loads so that the professionally oriented teacher has free time for his work. College administrations bear similar responsibilities. Increasingly, the research departments must design and initiate programs for the strengthening of their neighbors (from whom they receive graduate students). Conferences, exchanges of staff, and sabbatical hospitality require initiative from the strong departments. The officers and leaders of professional organizations need now, more than in the past, to analyze and to fulfill the needs of the majority of their profession rather than merely to communicate with each other.

Reference

1. Science 153 (1966) 1057.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

COLLABORATING EDITORS: J. BARLAZ, Rutgers—The State University; L. CARLITZ, Duke University; HASKELL COHEN, University of Massachusetts; H. EVES, University of Maine; M. S. KLAMKIN, Ford Scientific Laboratory; R. C. LYNDON, University of Michigan; and A. WILANSKY, Lehigh University.

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P.O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before February 29, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2015. *Proposed by W. A. McWorter, University of British Columbia*

Let $\ll$ be an anti-reflexive partial order on the set $\{1, \cdots, n\}$. Let A be an $n \times n$ matrix where the (i, j)-entry is 1 or 0 according as $i \ll j$ or not. (i) Prove that there exists a permutation matrix P such that PAP^{-1} is upper triangular. (ii) Show that $A = 0$ if A is idempotent (i.e., $A^2 = A$).

E 2016. *Proposed by J. M. Khatri, Baroda, India*

Let Δ_r be the triangular number $\frac{1}{2}r(r+1)$. Solve in integers the system:

$$\Delta_a + \Delta_b = \Delta_c + \Delta_d, \qquad x^2 + y^2 = z^2 + w^2,$$
$$\Delta_{a+x} + \Delta_{b+y} = \Delta_{c+z} + \Delta_{d+w}.$$

E 2017. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Let h be the length of an altitude of an isosceles tetrahedron and suppose the orthocenter of a face divides an altitude of that face into segments of lengths h_1 and h_2. Prove that $h^2 = 4h_1h_2$.

E 2018. *Proposed by W. A. McWorter, University of British Columbia*

Let M_n be the group of all nonsingular $n \times n$ matrices over a field F of characteristic 0. Let S_n be the subgroup of M_n consisting of all permutation matrices. Prove that if two elements of S_n are conjugate in M_n, then they are already conjugate in S_n.

E 2019. *Proposed by Norman Miller, Queen's University, Kingston, Ontario*

Given four noncollinear, nonconcyclic points in a plane, show that (a) at

most three of the points can each be outside the circle that passes through the other three, and (b) at most two of the points can each be inside the circle that passes through the other three.

E 2020. *Proposed by G. F. Schumm, University of Chicago*

Given n points ($n>3$) such that no three are collinear and no four concyclic. Must there exist a circle which passes through three of them and contains the remaining points in its interior?

E 2021. *Proposed by D. S. Mitrinović, University of Belgrade*

If $|\bar{a}b-\bar{b}c| = |a\bar{a}-c\bar{c}|$, determine whether the equation $az^2+bz+c=0$ (a, b, c complex numbers, $a\neq 0$, $|a| \neq |c|$) has a root z_1 with the property $|z_1| = 1$. The converse is known. (See the proposer's book, *Functions of a Complex Variable*, Nordhoff, 1965, p. 38, #2.13.)

E 2022. *Proposed by Richard Parris, Student, Tufts University*

Ordered k-tuples are formed using the two symbols A, B. In how many ways can this be done, if it is required that "A" does not occur in adjacent positions? I.e., for $k=3$, the triples (A, A, A), (A, A, B), (B, A, A) are not counted.

E 2023. *Proposed by L. F. Meyers, The Ohio State University*

Which fields have nonzero elements a and b such that $a^2+b^2=1$? Generalize.

E 2024. *Proposed by R. B. Eggleton, Avondale College, Cooranbong, Australia*

In discussing the number $F_{73}=2^{2^{73}}+1$, W. W. Rouse Ball states: "Its digits are so numerous that, if it were printed in full with the type and number of pages used in this book, many more volumes would be required than are contained in all the public libraries in the world." (*Mathematical Recreations and Essays*, Eleventh (revised) edition, 1963.) (a) Is this an exaggeration? (b) What are the last three digits of F_{73}?

SOLUTIONS OF ELEMENTARY PROBLEMS

Additive, weakly multiplicative functions

E 1895 [1966, 664]. *Proposed by A. A. Gioia, Texas Technological College*

Let a complex-valued arithmetic function f be called

(1) additive if $f(a+b)=f(a)+f(b)$,

(2) weakly multiplicative if $f(ab)=f(a)f(b)$, whenever $(a, b)=1$,

(3) strongly multiplicative if $f(ab)=f(a)f(b)$.

Prove that if a function is additive and weakly multiplicative, then it is strongly multiplicative.

Solution by Z. Z. Uoiea, Salt Lake City, Utah. Condition (1) and induction imply that $f(n)=nf(1)$. By (2), $f(1)=f(1)^2$, so that $f(1)=0$ or 1. Hence $f(n)\equiv 0$ or $f(n)\equiv n$. In both cases, f satisfies (3).

Also solved by Harold Abelson, D. T. Adams, R. G. Archibald, Anders Bager (Denmark), T. L. Bartlow, Brother Louis Zirkel, J. L. Brown, Jr., Peter Bundschuh (Germany), T. J. Burke, J. P. Celenza, G. C. C. Chang, Mannis Charosh, P. R. Chernoff, John Christopher, R. J. Cormier, Red Cougar, J. H. Cozzens & M. S. Kaplan, Eleanor G. Dawley, D. E. Daykin (Malaysia), Bob De Vore, Stan Dick & Preston Stein, J. F. Dillon, W. G. Dotson, Jr., D. R. Duncan, E. S. Eby, H. M. W. Edgar, R. B. Eggleton (Australia), M. A. Fechter, Neal Felsinger, M. S. Fineman, Annabel Fong, E. A. Franz, F. N. Fritsch, Philip Fung, N. Ganesan (India), Richard Gisselquist, R. E. Giudici, Anton Glaser, Michael Goldberg, Frederic Gooding, Jr., M. G. Greening (Australia), G. A. Heuer, K. E. Hirst (England), J. E. Homer, Jr., J. A. Hornby & P. M. Ellis, D. G. Huffman, R. A. Jacobson, Erwin Just, Michael Kerwan; Stephen Kiely & Anthony La Bruzza, Fred Kochman, Betty Kvarda, R. N. Leggett, Jr., Joel Levy, Charles McCracken, Norman Miller, J. A. Mokoena (Zambia), H. V. Monks, Alexander Morgan, J. B. Muskat, D. E. Myers, Dave Nixon, Michael Olinick, Robert Prener, Bob Prielipp, J. R. Purdy, Stanley Rabinowitz, B. L. S. P. Rao (India), Jürg Rätz (Switzerland), C. Riley, D. C. Rine, R. N. Schneider, Michael Schulz, G. F. Schumm, Margaret B. Seay, N. E. Sexauer, David Sibley, D. L. Silverman, Stephen Soldz, Al Somayajulu, Stephen Spindler, Sidney Spital, S. T. Stern, Walter Stromquist, P. K. Subramanian, Kishore Theckedath (India), A. M. Vaidya (India), C. S. Venkataraman & R. Sivaramakrishnan (India), J. E. Vinson, A. D. Wadhwa (India), Charles Wexler, Mary M. Whiteside, J. E. Wilkins, Jr., Dale Woods, K. L. Yocom, G. J. Zaher, Jr., Joel Zinn, and the proposer.

Red Cougar, F. N. Fritsch and Alexander Morgan each show that also $f(n)=0$ or n if (2) is replaced by the weaker hypothesis:

(2′) extremely weakly multiplicative if $f(ab)=f(a)f(b)$ for some pair of nonzero integers a, b.

A Congruence

E 1896 [1966, 665]. *Proposed by David Singmaster, University of California, Berkeley*

For every prime p show that

$$\binom{np}{p} \equiv n \pmod{p^2}.$$

I. *Solution by P. R. Chernoff, Harvard University.* From the identity $(1+x)^{np} = ((1+x)^p)^n$ we obtain

$$\binom{np}{p} = \sum_{k_1+\cdots+k_n=p} \binom{p}{k_1} \cdots \binom{p}{k_n}.$$

Since $\binom{p}{k}$ is divisible by p for $0<k<p$, every term in the above sum is divisible by p^2 except the n terms, each equal to 1, in which one of the k_i is equal to p. Thus $\binom{np}{p} \equiv n \pmod{p^2}$.

II. *Solution by M. A. Fechter, Orlando, Fla.* For $p>3$, Guérin (L'intermédiare des math., 23 (1916) 174) gives, in effect, the much stronger result

$$\binom{kp^m - 1}{p-1} \equiv 1 \pmod{p^{m+2}}, \qquad p \nmid k.$$

Setting $n=p^rk(r\geqq 0)$, we get

$$\frac{1}{n}\binom{np}{p}=\binom{np-1}{p-1}=\binom{kp^{r+1}-1}{p-1}\equiv 1 \pmod{p^{r+3}}.$$

Thus $\binom{np}{p}\equiv n \pmod{p^{2r+3}}$ where $p>3$, $p^r \mid n$, $p^{r+1}\nmid n$. For $p>3$ this is a stronger result than required. The cases $p=2$ and $p=3$ can easily be taken care of.

Also solved by Anders Bager (Denmark), Sister Marion Beiter, W. J. Blundon, Robert Breusch, P. Bundschuh (Germany), Mannis Charosh, Anton Chernoff, Raymond Chu, J. H. Cozzens, J. F. Dillon, R. E. Eggleton (Australia), Mrs. A. Elek (Hungary), Udovičić Enes (Yugoslavia), Marco Ettrick, M. A. Fechter, Michael Goldberg, Jerry Goodman, A. L. Gould, M. G. Greening (Australia), Jerome Harrison, Scott Harrod, W. E. Hoff, F. T. Howard, Donald Jeffords, M. S. Kaplan & J. H. Cozzens, H. E. Lahmann (Germany), Douglas Lind, W. W. Meyer, Norman Miller, H. V. Monks, Wanda Jane Mourant, J. B. Muskat, Slamar Ninoslav (Yugoslavia), Stanton Philipp, J. R. Purdy, Stanley Rabinowitz, Jürg Rätz (Switzerland), Lois J. Reid, Marilyn Rodeen, Harry Ruderman, Robert Spira, E. W. Trost (Switzerland), J. E. Vinson, A. D. Wadhwa (India), Charles Wexler, and the proposer.

Editorial Note. Charosh also establishes $\binom{np}{p}\equiv n \pmod{p^3}$ for $p>3$.

In the solution of E 1346 [1959, 591] which was to show that $\binom{2p}{2}\equiv 2 \pmod{p}$, N. G. Gunderson established that for $p>3$, $a>0$, we have

$$\binom{ap^n}{p}\equiv ap^{n-1} \pmod{p^{2n+1}}.$$

Now let $n=1$.

Expected Distance between Vertices of a Dodecahedron

E 1897 [1966, 665]. *Proposed by F. P. Callahan, Blue Bell, Pa.*

A bug is crawling on the edges of a regular dodecahedron. Each time it comes to a vertex it chooses with equal probability one of the two edges which end in that vertex distinct from the one by which it reached that vertex. What is the expected distance it covers in order to get from a vertex A to any other vertex B? ($A=B$ is not excluded.) On starting, the bug can go in any of the three directions. (Cf. E 1752 [1966, 200].)

Solution by J. R. Kuttler and Nathan Rubinstein, The Johns Hopkins University Applied Physics Laboratory. Let the dodecahedron have unit edges and let B denote the point at which the bug terminates his journey. Let the remaining points be labeled A_1, A_2, A_3, A_4 or A_5 as the minimum number of edges the bug traverses to reach B is 1, 2, 3, 4 or 5, respectively. (See Fig. 1.)

Let $E(A_i, A_j)$ denote the expected distance the bug will travel to reach B if he is at vertex A_i and chooses the edge from A_i terminating at A_j. From Fig. 1 we immediately obtain the linear system:

$$\begin{aligned}
E(A_1, B) &= 1, \qquad E(B, A_1) = 1 + E(A_1, A_2),\\
E(A_1, A_2) &= 1 + \tfrac{1}{2}E(A_2, A_2) + \tfrac{1}{2}E(A_2, A_3),\\
E(A_2, A_1) &= 1 + \tfrac{1}{2}E(A_1, A_2) + \tfrac{1}{2}E(A_1, B),\\
E(A_2, A_2) &= 1 + \tfrac{1}{2}E(A_2, A_1) + \tfrac{1}{2}E(A_2, A_3),
\end{aligned}$$

$$E(A_2, A_3) = 1 + \tfrac{1}{2}E(A_3, A_3) + \tfrac{1}{2}E(A_3, A_4),$$
$$E(A_3, A_2) = 1 + \tfrac{1}{2}E(A_2, A_1) + \tfrac{1}{2}E(A_2, A_2),$$
$$E(A_3, A_3) = 1 + \tfrac{1}{2}E(A_3, A_2) + \tfrac{1}{2}E(A_3, A_4),$$
$$E(A_3, A_4) = 1 + \tfrac{1}{2}E(A_4, A_3) + \tfrac{1}{2}E(A_4, A_5),$$
$$E(A_4, A_3) = 1 + \tfrac{1}{2}E(A_3, A_2) + \tfrac{1}{2}E(A_3, A_3),$$
$$E(A_4, A_5) = 1 + E(A_5, A_4), \qquad E(A_5, A_4) = 1 + E(A_4, A_3).$$

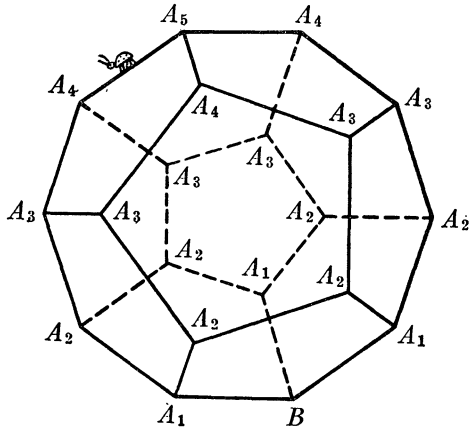


FIG. 1. Bug on dodecahedron.

Solving this system gives:

$$E(A_1, B) = 1, \quad E(B, A_1) = 20, \quad E(A_1, A_2) = 19,$$
$$E(A_2, A_1) = 11, \quad E(A_2, A_2) = \tfrac{49}{3}, \quad E(A_2, A_3) = \tfrac{59}{3},$$
$$E(A_3, A_2) = \tfrac{44}{3}, \quad E(A_3, A_3) = 18, \quad E(A_3, A_4) = \tfrac{58}{3},$$
$$E(A_4, A_3) = \tfrac{52}{3}, \quad E(A_4, A_5) = \tfrac{58}{3}, \quad E(A_5, A_4) = \tfrac{55}{3}.$$

Now it is clear that the desired expected path lengths $E(A_i)$ for the bug starting at A_i and traveling in any direction are given by

$$E(A_5) = E(A_5, A_4) = \tfrac{55}{3},$$
$$E(A_4) = \tfrac{1}{3}E(A_4, A_5) + \tfrac{2}{3}E(A_4, A_3) = 18,$$
$$E(A_3) = \tfrac{1}{3}E(A_3, A_4) + \tfrac{1}{3}E(A_3, A_3) + \tfrac{1}{3}E(A_3, A_2) = \tfrac{52}{3},$$
$$E(A_2) = \tfrac{1}{3}E(A_2, A_3) + \tfrac{1}{3}E(A_2, A_2) + \tfrac{1}{3}E(A_2, A_1) = \tfrac{47}{3},$$
$$E(A_1) = \tfrac{2}{3}E(A_1, A_2) + \tfrac{1}{3}E(A_1, B) = 13,$$
$$E(B) = E(B, A_1) = 20.$$

It is interesting to note that the longest expected path length occurs when the bug starts at the same point on which he terminates.

Also solved by Jorge Dou (Spain), and by the proposer.

A Primitive Root of Unity Identity

E 1899 [1966, 665]. *Proposed by C. R. Knapp, University of Kansas*

If a_k denotes any primitive nth root of unity, prove that

$$a_k^{k\phi(n)-1} = e^{2\pi i/n} = a_1,$$

where $\phi(n)$ is the Euler totient function.

Solution by Donna J. Seaman, Olympic College, Bremerton, Wash. By the Euler-Fermat theorem, $k^{\phi(n)} \equiv 1 \pmod{n}$ where $(k, n) = 1$. Since a_k is a primitive nth root if $(k, n) = 1$, we have at once

$$a_k^{k\phi(n)-1} = e^{2\pi i/n} = a_1.$$

Also solved by Harold Abelson, Anders Bager (Denmark), Red Cougar, J. F. Dillon, M. A. Fechter, R. E. Giudici, Michael Goldberg, Donald Jeffords, Wanda Jane Mourant, J. B. Muskat, C. B. A. Peck, J. R. Purdy, Stephen Soldz, Charles Wexler, and the proposer.

Simultaneous Diophantine Equations

E 1900 [1966, 665]. *Proposed by D. R. Rao, Secunderabad, India*

Find the solution of the system $X+Y=Z+U=W^2$, $X^2+Y^2=Z^2-U^2$, where X, Y, Z, U, W are positive integers.

Solution by N. J. Fine, the Pennsylvania State University. Let us assume, as we may, that we have a solution with $X \geqq Y$. Then

$$\frac{X^2 + Y^2}{X + Y} = \frac{Z^2 - U^2}{Z + U} = Z - U,$$

so $X+Y$ divides X^2+Y^2. Hence, $X+Y$ divides

$$2(X^2 + Y^2) - (X + Y)^2 = (X - Y)^2.$$

We may therefore write $(X-Y)^2 = t(X+Y) = tW^2$, so $t = s^2$, $s \geqq 0$. Since $X \geqq Y$, $X-Y = sW$. Because $X-Y < X+Y = W^2$, we have $s < W$. Next,

$$X^2 + Y^2 = \tfrac{1}{2}[(X + Y)^2 + (X - Y)^2] = \tfrac{1}{2}W^2(W^2 + s^2),$$

so $Z-U = (X^2+Y^2)/(X+Y) = \frac{1}{2}(W^2+s^2)$. Therefore $s \equiv W \pmod{2}$, so we may write $W = s+2v$, with $0 < v \leqq W/2$. Solving, we get

$$\begin{aligned} X &= W^2 - vW, & Y &= vW, \\ Z &= W^2 - vW + v^2, & U &= vW - v^2. \end{aligned} \tag{1}$$

Conversely, for every pair of integers v, W with $0<v\leqq W/2$, equations (1) yield a solution of the original system with $X\geqq Y$. It is easy to see that $(X, Y, Z, U)=(v, W)^2$. Thus the primitive solutions correspond to $(v, W)=1$.

Also solved by Robert Bart, W. J. Blundon, P. R. Chernoff, M. E. Christie, W. C. Daugherity, Buster Dunsmore, R. E. Edwards, R. B. Eggleton (Australia), M. A. Fechter, Michael Goodman, M. G. Greening (Australia), Michael Grost, G. A. Heuer, Donald Jeffords, Betty Kvarda & R. Killgrove, Douglas Lind, G. F. Lowerre, D. C. B. Marsh, William McKay, B. D. McLamore, Norman Miller, H. V. Monks, P. L. Montgomery, R. F. Morrow, Wanda Jane Mourant, J. B. Muskat, C. C. Oursler, C. B. A. Peck, Stanton Philipp, Jürg Rätz (Switzerland), Simeon Reich (Israel), Hammad Rizavi (Pakistan), Steven Russ, B. A. Smale, Al Somayajulu & Jim Smith, Robert Spira, Walter Stromquist, R. S. Underwood, Allan Wachs, A. D. Wadhwa (India), Charles Wexler, J. W. Wilson, K. L. Yocom, and the proposer.

On the Product of the Areas of Three Triangles

E 1901 [1966, 665]. *Proposed by Richard Laatsch, Miami University, Oxford, Ohio*

Let $\triangle ABC$ have unit area. Construct $\triangle A'B'C'$ such that each of A, B, C is an interior point on a different one of its sides. (Label so that A and A' are on opposite sides of BC, etc.) Find the supremum and infimum of the product of the areas of $\triangle A'BC$, $\triangle AB'C$, and $\triangle ABC'$ if:

1. Two sides of $\triangle ABC$ are parallel to sides of $\triangle A'B'C'$;
2. One side of $\triangle ABC$ is parallel to a side of $\triangle A'B'C'$;
3. No parallelism restriction is made.

Solution by Norman Miller, Queen's University, Kingston, Ontario.

Case 1. Take $A'B'\|AB$ and $B'C'\|BC$ (see Fig. 1). The $\triangle AB'C$, congruent to $\triangle ABC$, has unit area. The product ha (Fig. 1) is twice the area of $\triangle ABC$, which equals 2. The $\triangle A'BC$ is similar to $\triangle ABC'$, whence $x/a=h/AC'$ or $x\cdot AC'=ah$. Now the product of the three required areas is $(1)(\frac{1}{2}xa)(\frac{1}{2}h\cdot AC')$ $=\frac{1}{2}x\cdot AC'=\frac{1}{2}ah=1$. Hence as $A'C'$ rotates about B the product of the three areas remains constant.

Case 2. (See Fig. 2.) Take $B'C'\|BC$. Suppose first that $A'B'$ remains fixed in direction while $A'C'$ rotates about B. By the same argument as in Case 1 we have

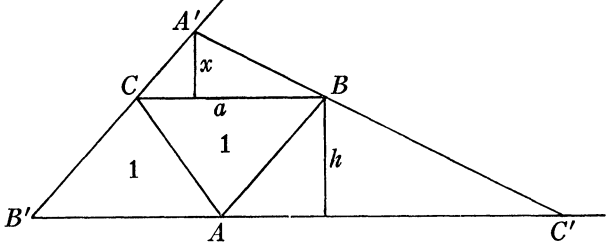


Fig. 1

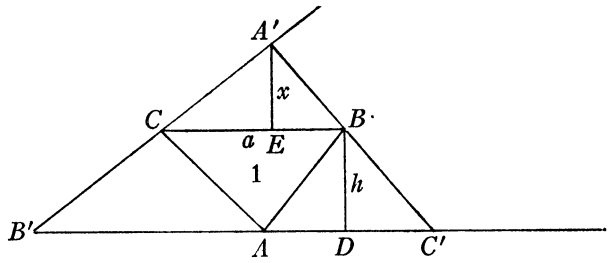


FIG. 2

the product of the areas of $\triangle A'BC$ and $\triangle ABC' = \frac{1}{2}x \cdot AC'$. Now $x/EB = h/DC'$, whence $x \cdot DC' = h \cdot EB$ or $x(AC' - AD) = h \cdot EB$. Hence $x \cdot AC' = x \cdot AD + h \cdot EB$. As $A'C'$ rotates so that C' recedes from A, $A' \to C$, AD and h are constant, $x \to 0$, and $EB \to a$. Hence $x \cdot AC' \to ha = 2$. It follows that the product of the areas of $\triangle A'BC$ and $\triangle AC'B$ approaches 1. Now when $A'B'$, through C, is allowed to take different positions, the product of the areas of the three required triangles can be made as small or as large as we please. Therefore the infimum $= 0$ and the supremum $= \infty$.

CASE 3. Since Case 2 is included in Case 3, the infimum for Case 2 cannot be less and the supremum cannot be greater than those for Case 3. Hence the required values here also are 0 and ∞.

Also solved by Yu Chang and Sidney Spital, M. A. Greening (Australia), Allan Wachs, and by the proposer.

A Known Sum

E 1902 [1966, 655]. *Proposed by David Shelupsky, the City College, New York*

Let $s(x) = \sum_{m=0}^{\infty} x^{2m+1}/1 \cdot 3 \cdot \cdots \cdot (2m+1)$. Prove that for all nonnegative x the inequality

$$e^{-x^2/2}s(x) < (\tfrac{1}{2}\pi)^{1/2}$$

holds, and that as $x \to +\infty$ the left-hand side is monotone increasing with the right-hand side as its limit.

Solution by D. C. B. Marsh, Colorado School of Mines. Since $s(x)$ satisfies $s'(x) = 1 + x \cdot s(x)$ with $s(0) = 0$, one may solve this linear equation routinely to find

$$e^{-x^2/2}s(x) = \int_0^x e^{-t^2/2}dt.$$

With positive integrand, the above integral is monotone increasing with non-negative x to the well-known limiting value of $\sqrt{\frac{1}{2}\pi}$.

Also solved by Donald Batman, M. A. Bershad, R. J. Bickel, D. A. Breault, Peter Bundschuh (Germany), P. R. Chernoff, W. O. Egerland, M. G. Greening (Australia), Eldon Hansen, J. Kuttler & N. Rubinstein & V. G. Sigillito & J. T. Stadter, Norman Miller, Senjanovic Pavao (Yugoslavia), Stanton Philipp, C. Riley, Michael Schulz, Richard Sinkhorn, Al Somayajulu, Stephen Spindler, Sidney Spital, V. K. Venkataraman, Charles Wexler, J. Ernest Wilkins, Jr., David Zeitlin, and the proposer.

Editorial Note. The sum may be found also in the following books: NSB, *Handbook of Mathematical Functions, etc.*, Eq. 26.2.11 (cited by Breault); Erdelyi, et al., *Higher Transcendental Functions*, vol. 2, sect. 9.9 (cited by Hansen); Spiegel, *Theory and Problems of Advanced Calculus*, p. 257 (cited by Spindler).

A Diophantine Equation

E 1904 [1966, 666]. *Proposed by D. L. Silverman, Beverly Hills, California*

Solve in positive integers: $x+y+z=xyz(x+2y-2z-2)$.

Solution by Michael Goldberg, Washington, D. C. From the given equation, $x+y+z$ is equal to xyz or a multiple of xyz. The only integral triplets satisfying this condition are (1, 1, 1), (1, 1, 2) and (1, 2, 3). By trial, the first two triplets yield no solution. The only permutation of the third that yields a solution is $x=1$, $y=3$, and $z=2$.

Also solved by Merrill Barneby, Robert Bart, D. A. Breault, Peter Bundschuh (Germany), P. R. Chernoff, H. M. W. Edgar, R. B. Eggleton (Australia), M. A. Fechter, W. V. Gamzon, Michael Goodman, M. G. Greening (Australia), David Hancasky, Agatha Himmelfarb, A. E. A. Hunt, R. A. Jacobson, Donald Jeffords, H. E. Lahmann (Germany), G. K. Liebschner, D. C. B. Marsh, Norman Miller, H. V. Monks, Wanda Jane Mourant, W. L. Mrozek, C. B. A. Peck, J. R. Purdy, Simeon Reich (Israel), S. L. Robinson, Steven Russ, Al Somayajulu, W. B. Stovall, Jr., G. C. Thompson, A. D. Wadhwa (India), W. B. Watson, J. W. Wilson, K. L. Yocom, and the proposer.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before April 30, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5520. *Proposed by T. J. Head, University of Alaska*

Let Λ be a ring with identity. A (left) Λ-module M is said to have a *projective cover* if there is a projective Λ-module P and a Λ-homomorphism $\phi: P\to M$ of P onto M for which the only submodule S of P satisfying $S+\ker(\phi)=P$ is $S=P$. Each projective Λ-module has a projective cover, namely the identity mapping of P onto P. For which rings Λ is it true that only the projective Λ-modules have projective covers?

5521. *Proposed by Harsh Pittie, Princeton University*

Prove that the automorphism group of D_n (the dihedral group of order $2n$) is isomorphic to the holomorph of Z_n (the integers, mod n).

5522. *Proposed by T. C. Brown, Kiev State University, U.S.S.R.*

Let S be a periodic semigroup whose idempotents lie in the center. Then the mapping $x \to x^{n(x)}$, where $x^{n(x)}$ is idempotent, is a homomorphism.

5523. *Proposed by J. T. Rosenbaum, University of Pittsburgh*

What pairs ϵ, θ, with $0 < \epsilon \leqq \theta < 1$, make the following statement true? If $\{E_n\}$ is any sequence of measurable subsets of $[0, 1]$, each with measure at least θ, there will exist a set of measure ϵ which is covered by E_n for infinitely many n. (Note. This is not implied by $|\overline{\lim} E_n| \geqq \epsilon$.)

5524. *Proposed by P. A. Schweitzer, Institute for Defense Analyses, Arlington, Va.*

Let d_n be the nth moment of the distance from a point P outside the unit circle to a randomly chosen point within the circle. Show that

$$d_n = R^n \cdot F\left(-\frac{n}{2}, -\frac{n}{2}, 2; \frac{1}{R^2}\right),$$

where $R > 1$ is the distance from P to the center of the circle, and F is the hypergeometric function.

Let e_n be the nth moment of the distance between two points chosen randomly within the unit circle. Show that

$$e_n = \frac{4}{(n+4)} F\left(-\frac{n}{2}, -\frac{n}{2}, 2; 1\right) = \frac{4(n+1)!}{(n+4)\Gamma(2+\frac{1}{2}n)^2}.$$

5525. *Proposed by Howard Kleiman, Queensborough Community College, New York*

Let p be a prime such that $2p+1$ is not a prime. Show that there is no cyclotomic polynomial of degree $2p$.

5526. *Proposed by R. O. Davies and Allan Hayes, The University, Leicester, England*

If P is due south of Q, then anyone travelling from P to Q on a Friday, without either crossing his own path or stopping, must at some instant be travelling due north. Discuss this assertion.

5527. *Proposed by F. Cunningham, Jr., Bryn Mawr College*

Prove that for every simple closed curve in 3-space there is a plane projection

of it having at least two double points (or a multiple point of order greater than two). Give an example of such a curve for which no plane projection has more than two double points (or one triple point).

5528. *Proposed by L. A. Steen, Saint Olaf College, Northfield, Minn.*

Let $\{D_i\}_1^\infty$ be a sequence of pairwise disjoint open disks contained inside the unit disk; let r_i denote the radius of D_i. Prove that

$$\sum_{i=1}^{\infty} r_i^2 < 1 \quad \text{whenever} \quad \sum_{i=1}^{\infty} r_i < \infty.$$

(In other words: Whenever you remove from the interior of the closed unit disk a sequence of pairwise disjoint open disks whose boundaries are of finite total length, the remaining set has positive Lebesgue measure.)

5529. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Evaluate

$$\int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{\sin k_j(x - a_j)}{(x - a_j)}\, dx,$$

with k_j, a_j, $j=1, 2, \cdots, n$ real numbers.

SOLUTIONS OF ADVANCED PROBLEMS

Metrizable topologies on a Hausdorff Space

5396 [1966; 547]. *Proposed by Albert Wilansky, Lehigh University*

Find a set X and two normal and Hausdorff topologies T and T', for X, such that every T compact set is T' closed, but not every T' compact set is T closed.

Solution by H. W. Berkowitz, Rutgers, The State University. Example: Let X be the subset of the real line consisting of the closed unit interval and the point 2, i.e., $X = [0, 1] \cup \{2\}$. Let T' be the topology on X given by the usual euclidean metric. Let $T = \{A \subset X: 2 \notin A\} \cup \{A \subset X: 2 \in A \text{ and } X - A \text{ is countable}\}$, i.e., T gives the discrete topology to $[0, 1]$ and makes 2 a limit point of every set with a countable complement.

The compact sets of T are the finite sets, and these are T' closed. $[0, 1]$ is T' compact and is not T closed since 2 is a T limit point of $[0, 1]$. T' is certainly normal and Hausdorff. T is Hausdorff. T is normal since if A and B are T closed, disjoint subsets of X, then either A or B does not contain 2. Assume that A does not contain 2, then A is open and $X - A$ is open, $B \subset X - A$.

THEOREM. *Let X be a set with two topologies, T and T', which are Hausdorff and first countable. Assume every T compact set is T' closed. Then every T' compact set is T closed.*

Proof. Assume the conclusion does not hold. There exists $A \subset X$, where A is T' compact and not T closed. There exists $x \notin A$ such that x is a T limit point of A. There exists a sequence $\{a_i\} \subset A$ such that $\{a_i\}$ T converges to x. Hence $\{a_i\} \cup \{x\}$ is T compact, which implies $\{a_i\} \cup \{x\}$ is T' closed.

$\{a_i\} \subset A$ and A is T' closed, $x \notin A$ so $\{a_i\}$ is T' closed. Therefore, $\{a_i\}$ is T' compact which implies $\{a_i\}$ has a T' limit point in itself, call it a. There exists $\{b_i\}$ a subsequence of $\{a_i\}$ such that $\{b_i\}$ T' converges to a and $a \notin \{b_i\}$. Now $\{b_i\}$ T converges to x, so $\{x\} \cup \{b_i\}$ is T compact and hence T' closed. But $a \in \mathrm{Cl}[\{x\} \cup \{b_i\}]$ in T', a contradiction. Q.E.D.

We can obtain a stronger theorem than the one above, the proof of which is an application of Zorn's lemma.

DEFINITION. X is a *k-space* if a subset A of X is closed iff A intersects each compact set of X in a closed set. It is known that first countable Hausdorff spaces are k-spaces (see Kelley, "General Topology"). Then

THEOREM. *Let X be a set with Hausdorff topologies T and T' satisfying:* (i) *sets compact in T are closed in T',* (ii) *(X, T) is a k-space. Then every T' compact set is T closed.*

Also solved by S. P. Franklin, D. A. Hejhal, J. R. Isbell, F. E. Siwiec and the proposer.

Notes: Hejhal, Isbell and Franklin also observed that the two topologies cannot be simultaneously metrizable. Franklin refers to his paper "Spaces in which sequences suffice," *Fundamenta Mathematica* 57 (1965), pp. 107–116 for a wider application of the arguments involved. Isbell quoted the theorem on k-spaces as having been circulated by J. deGroot in 1965.

Rings without zero divisors

5420 [1966, 895]. *Proposed by T. P. Kezlan, University of Texas*

Suppose that R is a ring with no nonzero divisors of zero and with the property that every proper subring of R is finite. Prove that R is a field of prime characteristic.

Solution by E. A. Schreiner, Western Michigan University. By the hypothesis, R satisfies the cancellation laws. Thus, for $a \neq 0$, the mapping $r \to ar$ is one-one so card $(R) =$ card (aR). Since aR is a subring and proper subrings are finite, aR can not be a proper subring. Thus $a \neq 0$ implies $aR = R$. Similarly $a \neq 0$ implies $Ra = R$. Since the equations $ax = b$ and $ya = b$ are solvable for all $a \neq 0$, $b \in R$, R is a division ring. The characteristic of a division ring is either zero or a prime. Characteristic zero would imply the existence of an infinite proper subring isomorphic to the integers so that R must have prime characteristic.

If a is not in $Z(R)$, the center of R, then $A = \{x \in R \mid ax = xa\}$ is a proper, hence finite, subring containing all powers of a. Thus, $a^k = 1$ for some positive integer k and so $a^{k+1} = a$. Hence for every $x \in R$, there exists an integer $n(x) > 1$ such that $x^{n(x)} - x \in Z(R)$, that is, R is an H-ring (N. Jacobson, *Structure of Rings*, pp. 220–221) and therefore is commutative. Therefore R is a field of prime characteristic.

Also solved by D. T. Adams, W. O. Alltop, Cornelius Baytop & J. E. Joseph, Kenneth Burke, Charles Franke & Marcelle Friedman, L. Carlitz, Charles Chouteau, D. E. Crabtree, Jay DeMars, William Forman, E. R. Gentile, Marvin Gruber, G. A. Heuer, E. C. Hook, R. A. Howland, Kwangil Koh, C. M. Joiner & Stephen Price, E. S. Langford, Dorembus Leonard (Israel), Jiang Luh, C. F. Martin, M. D. Mavinkurve (India), Ka Menehune, Harry Miller, Joseph Oppenheim, Charles Riley, W. R. Scott, G. B. Seligman, Surjeet Singh (India), M. F. Smiley, L. R. Vermain (Australia), G. P. Weller, R. Wiegandt (Hungary), and the proposer.

Ed. Note: With their solutions, Scott and Seligman show that all infinite rings of the type described may be written $R=\cup_{n=1}^{\infty}\mathrm{GF}(p^{q^n})$, p, q primes.

The absolute value of a trigonometric type kernel

5421 [1966, 896]. *Proposed by D. J. Newman, Yeshiva University*

Evaluate

$$\int_0^{\infty}\left|\int_0^{\pi/2}\left(1-\frac{\sin xy}{\tan x}\right)dx\right|dy.$$

Solution by Naoki Kimura and Tetsundo Sekiguchi, University of Arkansas. Set $I(y)=\int_0^{\pi/2}[1-(\sin xy)/(\tan x)]dx$, $y\geqq 0$. Since

$$\lim_{y\to+\infty}\int_0^{\pi/2}\frac{x}{\tan x}\,\frac{\sin xy}{x}\,dx=\pi/2$$

by Dirichlet's theorem, we have

$$\lim_{y\to+\infty} I(y)=\pi/2-\pi/2=0. \tag{1}$$

By a simple computation, we have

$$I(y)-I(y+2)=\int_0^{\pi/2}(\cos xy+\cos x(y+2))dx=\frac{2}{y(y+2)}\sin\frac{\pi}{2}y. \tag{2}$$

Therefore, it follows for any positive integer N

$$I(y)-I(y+2N)=\sum_{n=0}^{N-1}\frac{2(-1)^n}{(y+2n)(y+2n+2)}\sin\frac{\pi}{2}y.$$

Since $I(y+2N)\to 0$ as $N\to+\infty$ by (1), we have

$$I(y)=\sum_{n=0}^{\infty}\frac{2(-1)^n}{(y+2n)(y+2n+2)}\sin\left(\frac{\pi}{2}\cdot y\right). \tag{3}$$

For any fixed $y\geqq 0$, the series (3) is an alternating series such that the sequence of the absolute values of the terms is decreasing. So the sign of $I(y)$ is the same as that of $\sin(\pi y/2)$. Thus

$$\begin{aligned} &I(y)>0 \quad \text{if } 4n<y<4n+2,\\ &I(y)<0 \quad \text{if } 4n+2<y<4n+4, \end{aligned} \qquad \text{where } n=0,1,2,\cdots, \tag{4}$$

$$\int_{4n}^{4n+4} |I(y)|\,dy = \int_{4n}^{4n+2} I(y)\,dy + \int_{4n+2}^{4n+4} (-I(y))\,dy \quad \text{by (4)}$$

$$= \int_{4n}^{4n+2} (I(y) - I(y+2))\,dy$$

$$= \int_{4n}^{4n+2} \frac{2}{y(y+2)} \sin\frac{\pi}{2} y\,dy \quad \text{by (2)}$$

$$= \int_{4n}^{4n+2} \frac{\sin(\pi/2)y}{y}\,dy - \int_{4n}^{4n+2} \frac{\sin(\pi/2)y}{y+2}\,dy$$

$$= \int_{4n}^{4n+2} \frac{\sin(\pi/2)y}{y}\,dy + \int_{4n+2}^{4n+4} \frac{\sin(\pi/2)y}{y}\,dy$$

$$= \int_{4n}^{4n+4} \frac{\sin(\pi/2)y}{y}\,dy.$$

Summing from $n=0$ to $n=N-1$ and letting $N\to\infty$ it follows that

$$\int_0^\infty |I(y)|\,dy = \int_0^\infty \left| \int_0^{\pi/2} \left(1 - \frac{\sin xy}{\tan x}\right) dx \right| dy = \frac{\pi}{2}.$$

Also solved by N. J. Fine, Emil Grosswald, E. L. Whitney, and the proposer.

Inequalities for the Riemann Zeta Function

5422 [1966, 896]. *Proposed by A. A. Mullin, University of California*

Let ζ be the Riemann zeta-function restricted to the real numbers. Prove the following: $\zeta(s-1)\cdot\zeta(2s) \geqq \zeta^2(s)\cdot\zeta(2s) \geqq \zeta^2(s) \geqq \zeta(2s)$, for $s>2$. Does one have in fact strict inequalities? Which, if any, of the following inequalities hold:

$$|\zeta(s-1)\cdot\zeta(2s)| \geqq |\zeta(s)|^2 \cdot |\zeta(2s)| \geqq |\zeta(s)|^2 \geqq |\zeta(2s)|$$

for the extension of ζ to $\operatorname{Re} s>2$?

Solution by Emil Grosswald, University of Pennsylvania.

A. All three inequalities are strict for $s>2$, the second one actually holds at least for $s>\frac{1}{2}$, the third one for $s>1$. Indeed, the first one may be written as

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} > \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and is strict because the Euler function $\phi(n)$ satisfies $\phi(1)=\phi(2)=1$, $\phi(n)>1$ for $n\geqq 3$.

The second strict inequality is equivalent to the statement $\zeta(2s) = \sum_{n=1}^{\infty} n^{-2s} > 1$, which holds trivially for $s>\frac{1}{2}$.

The last inequality is

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \sum_{n=1}^{\infty} \frac{n^s d(n)}{n^{2s}} = \zeta^2(s) > \zeta(2s) = \sum_{n=1}^{\infty} \frac{1}{n^{2s}},$$

where $d(n)$ is the number of divisors of n. Trivially, $n^s d(n) \geqq n^s \geqq 1$ with strict inequality for $s>0$ and $n>1$; hence, the inequality holds in $s>1$, the region of convergence of $\sum_{n=1}^{\infty} n^{-s} d(n)$.

B. Let $s=\sigma+it$; then only the first inequality holds for the extension of $\zeta(s)$ in $\sigma>2$. Indeed,

$$\left|\frac{\zeta(s-1)}{\zeta^2(s)}\right| = \prod_p \left|\frac{(1-p^{-s})^2}{1-p^{1-s}}\right| = \prod_p \left|(1-2p^{-s}+p^{-2s}) \sum_{m=0}^{\infty} \frac{p^m}{p^{ms}}\right|$$

$$= \prod_p \left|1 + \frac{p-2}{p^s} + \sum_{m=0}^{\infty} \frac{p^m(p-1)^2}{p^{(m+2)s}}\right| = \prod_p |1+\phi(p,s)|,$$

where $\phi(p, s) = ((p-2)/p^s) + (1-p^{-1})^2(1-p^{1-s})^{-1}$. Setting $\theta = t \log p$, we find for $p=2$, that $\operatorname{Re} \phi(2, s) = \frac{1}{4}(1-(\cos\theta/2^{\sigma-1})) > 0$. For $p \geqq 3$ it is determined that $\operatorname{Re} \phi(p, s) = A\{[p(p-2)(1-2p^{1-\sigma}\cos\theta + p^{2-2\sigma}) + (p-1)^2]\cos\theta + (p-1)^2 p^{\sigma-1}\}$ where A is a trivial, positive factor. It is easy to verify that the minimum is attained for $\cos\theta = -1$, so that, for $\sigma>2$,

$$\begin{aligned}(1/A)\operatorname{Re}\phi(p,s) &> p^{\sigma-1}(p-1) - [p(p-2)(1+p^{1-\sigma})^2 + (p-1)^2] \\ &> p(p-1)^2 - p(p-2)(1+p^{-1})^2 + (p-1)^2 \\ &= (p-1)^3 - p(p-2)(1+p^{-1})^2 \\ &= (p-1)^2[p-2-(2/p)-(1/p^2)] + (1+p^{-1})^2 > 0 \quad \text{for } p \geqq 3.\end{aligned}$$

Consequently, the real part of each factor exceeds unity and so does, *a fortiori*, its modulus, hence also the modulus of the product.

The second inequality does not hold in the half plane $\sigma>2$, but has to be replaced by the weaker one $|\zeta(2s)| > 2-\zeta(4) \cong .982$ (<1).

The last inequality does not hold either for complex $s=\sigma+it$, $\sigma>2$; one has instead $|\zeta(s)|^2 > \frac{2}{5}|\zeta(2s)|$.

Also solved by Charles R. Wall.

Expectation of a function of a sequence of zero's and one's

5423 [1966, 896]. *Proposed by James Singer, Brooklyn College*

Suppose that

$$u_n = \frac{1}{2^n} \sum_{(n)} \sum_{i=1}^{n} \frac{e_i}{2^{2i-s_i}},$$

where $e_i = 0$ or 1 for all i, $s_i = e_1 + e_2 + \cdots + e_i$, and the sum $\sum_{(n)}$ is extended over all possible allowable choices of $e_1, \cdots, e_n$; find a simple expression for u_n and $\lim_{n\to+\infty} u_n$.

I. *Solution by D. Ž. Djoković, Belgrade, Yugoslavia.* The set of all sequences $(e_1, \cdots, e_n)$ of zeros and ones such that $e_i = 1$ we shall denote by S_i. The set S_i is partitioned into i subsets $S_{ij} (j = 1, \cdots, i)$. The sequence $(e_1, \cdots, e_n)$ belongs to S_{ij} if and only if $e_i = 1$ and $s_i = j$. It is easy to see that S_{ij} contains exactly $2^{n-i}\binom{i-1}{j-1}$ sequences. Hence,

$$\begin{aligned} u_n &= \frac{1}{2^n} \sum_{i=1}^{n} \sum_{j=1}^{i} 2^{n-i} \binom{i-1}{j-1} \frac{1}{2^{2i-j}} \\ &= \sum_{i=1}^{n} \frac{1}{2^{3i-1}} \sum_{j=1}^{i} \binom{i-1}{j-1} 2^{j-1} \\ &= \frac{2}{3} \sum_{i=1}^{n} \left(\frac{3}{8}\right)^i = \frac{2}{5}\left(1 - \left(\frac{3}{8}\right)^n\right) \to \frac{2}{5} \quad (n \to +\infty). \end{aligned}$$

II. *Solution by N. J. Fine, The Pennsylvania State University.* The e_i may be regarded as independent random variables taking the values 0, 1 with probabilities $\frac{1}{2}, \frac{1}{2}$. Then $E(2^{e_i}) = \frac{3}{2}$, $E(e_i 2^{e_i}) = 1$, and

$$\begin{aligned} u_n &= E\left(\sum_{i=1}^{n} \frac{1}{2^{2i}} 2^{e_1} 2^{e_2} \cdots 2^{e_{i-1}} (e_i 2^{e_i}) \right) \\ &= \sum_{i=1}^{n} \frac{1}{2^{2i}} E(2^{e_1}) E(2^{e_2}) \cdots E(2^{e_{i-1}}) E(e_i 2^{e_i}) \\ &= \sum_{i=1}^{n} \frac{1}{2^{2i}} \left(\frac{3}{2}\right)^{i-1} = \frac{2}{5}\left(1 - \left(\frac{3}{8}\right)^n\right) \\ &\to \frac{2}{5}. \end{aligned}$$

Also solved by S. T. M. Ackermans (Netherlands), Duane W. Bailey, D. M. Bloom, Paul Brock, Yu Chang & Sidney Spital, Miltiades Demos, J. G. Deutsch, Jerry Fischer, William Forman, G. A. Heuer, A. G. Konheim (Germany), D. A. Marcus, Steven Minsker, C. B. A. Peck, F. W. Steutel (Netherlands), Jeffrey Tollefson, Dieter Wolke (Germany), and the proposer.

A general relation of Ramanujan

5424 [1966, 896]. *Proposed by Van E. Wood, Battelle Memorial Institute*

The relation of Ramanujan

$$\text{(A)} \qquad \int_0^\infty \frac{e^{-yx}}{x(\pi^2 + \ln^2 x)}\, dx = e^y - \int_0^\infty \frac{y^x}{\Gamma(1+x)}\, dx, \qquad y \geqq 0$$

is readily verified by the Laplace transform method (G. H. Hardy, *Ramanujan* University Press, Cambridge, 1940, p. 196). Ramanujan, who, according to Hardy, disliked contour integration, proved (A) indirectly by discovering a generalization, the correctness of which could be shown without using integration in the complex plane. Show that (A) can be verified directly (that is, with-

out finding a more general integral of which it is a special case) by a method which does not involve contour integration.

Solution by R. E. Shafer, Lawrence Radiation Laboratory, University of California.

$$\int_0^\infty x^{\xi-1}e^{-zx}\left(\cos \pi\xi - \frac{\sin \pi\xi}{\pi}\cdot \log x\right)\cdot\frac{dx}{\pi^2 + \log^2 x} = e^z - \int_{-\xi}^\infty \frac{z^x}{\Gamma(1+x)}\,dx$$

for $\mathrm{Re}(\xi)\geqq 0$, $|\arg z| < \pi/2$. We note that

$$E_{1-\nu}(z) = \frac{\Gamma(\nu)}{z^\nu} - e^{-z}\cdot\left[\frac{1}{\nu} + \frac{z}{\nu(\nu+1)} + \frac{z^2}{\nu(\nu+1)(\nu+2)} + \cdots\right]$$

$$= \frac{e^{-z}}{\Gamma(1-\nu)}\int_0^\infty \frac{e^{-x}x^{-\nu}}{z+x}\,dx, \qquad |\arg z| < \pi,\ \mathrm{Re}\,(\nu) < 1.$$

The right hand expression may be replaced by

$$\int_{-\xi}^{-\xi+1} e^z\cdot z^\nu E_{1-\nu}(z)\,\frac{d\nu}{\Gamma(\nu)} = \int_{-\xi}^{-\xi+1}\left[e^z - \sum_{n=0}^\infty \frac{z^{\nu+n}}{\Gamma(1+\nu+n)}\right]d\nu$$

$$= \frac{1}{\pi}\int_0^\infty \frac{e^{-x}}{z+x}\,dx\cdot\int_{-\xi}^{-\xi+1} \sin \pi\nu\cdot e^{\nu \log z/x}\,d\nu$$

$$= \int_0^\infty \frac{e^{-x}}{(\log^2 z/x + \pi^2)}\,(z/x)^{-\xi+1}\left\{\cos \pi\xi + \frac{\sin \pi\xi}{\pi}\log z/x\right\}\frac{dx}{z}$$

for $|\arg z| < \pi$, $\mathrm{Re}\,(\xi)\geqq 0$ which leads to the desired result.

Also solved by the proposer.

A nonnegative definite matrix

5425 [1966, 896]. *Proposed by V. R. R. Uppuluri, Oak Ridge National Laboratory*

Let A and B be two real, symmetric, positive definite matrices. Show that $C=2A(A+B)^{-1}A+2B(A+B)^{-1}B-(A+B)$ is a nonnegative definite matrix.

I. *Solution by Wallace Givens, Oak Ridge National Laboratory.* Since $A+B>0$, $A+B=(T'T)^{-1}$ so

$$C = 2AT'TA + 2BT'TB - (T'T)^{-1} = T^{-1}[2P^2 + 2Q^2 - 1](T')^{-1},$$

where $P=TAT'$ and $Q=TBT'$ so $P+Q=1$. Hence, $TCT'=2P^2+2(1-P)^2-1 = (2P-1)^2$ is nonnegative definite so C is also nonnegative definite.

II. *Solution by Ingram Olkin, Stanford University, and A. W. Marshall, Boeing Scientific Research Laboratories.* If $X: p\times n$, $M: n\times n$, and $M\geqq 0$ (positive semi-definite), then the function $f(X)=XMX'$ is convex, i.e.,

$$(1) \qquad (\alpha X + \beta Y)M(\alpha X + \beta Y)' \leqq \alpha XMX' + \beta YMY',$$

where $0 \leqq \alpha \leqq 1$, $\alpha+\beta=1$, and $A \leqq B$ means that $B-A \geqq 0$. The proof of (1) is obtained by observing that (1) is equivalent to

$$(2) \qquad 0 \leqq \alpha\beta(X - Y)M(X - Y)'.$$

Applying (1) to $M=(A+B)^{-1}$, $X=A$, $Y=B$, $\alpha=\frac{1}{2}$, we obtain $A(A+B)^{-1}A + B(A+B)^{-1}B \geqq (A+B)/2$, which solves the problem. Note that we have used $A+B>0$ (positive definite), but do not require $A>0$, $B>0$.

An immediate generalization of (1) is $(\sum_1^k \alpha_j X_j)M(\sum_1^k \alpha_j X_j') \leqq \sum_1^k \alpha_j (X_j M X_j')$, $0 \leqq \alpha_j$, $\sum \alpha_j = 1$. Thus, for $M = \sum_1^k A_j > 0$, $X_j = A_j$, we obtain

$$\sum_1^k A_j \left(\frac{A_1 + \cdots + A_k}{k}\right)^{-1} A_j \geqq \frac{A_1 + \cdots + A_k}{k}.$$

III. *Solution by Marvin Marcus, University of California at Santa Barbara.* In a forthcoming paper (*An inequality for linear transformations*) the following result is proved: if H and K are positive definite hermitian transformations on a unitary space V and x and y are arbitrary vectors then

$$(1) \qquad (H^{-1}x, x) + (K^{-1}y, y) \geqq ((H + K)^{-1}x + y, x + y).$$

Now set $x=Au$, $y=Bu$, $H=K=A+B$ and (1) becomes

$$(A(A + B)^{-1}Au, u) + (B(A + B)^{-1}Bu, u) \geqq ((2(A + B))^{-1}(A + B)u, (A + B)u)$$

which immediately simplifies to the desired inequality. Note that A and B can be assumed positive-definite hermitian.

IV. *Solution by M. F. Neuts, Purdue University.* Clearly $A+B$ is real, symmetric and positive definite. Writing $(A+B)(A+B)^{-1}(A+B)$ for the last term in C, expanding and regrouping, we obtain

$$C = (A - B)(A + B)^{-1}(A - B)$$

and since $A-B$ is symmetric, the result is now obvious. If $A-B$ is nonsingular, then C is positive definite.

V. *Solution by Duane W. Bailey, Amherst College.* This result is a particular case of a more general theorem. A Banach *-algebra $\mathfrak{A}$ with continuous involution $x \to x^*$ is said to be *symmetric* if $\mathrm{Sp}(x^*x) \subset [0, \infty)$ for each $x \in \mathfrak{A}$. (See Rickart, *General Theory of Banach Algebras* for the facts about these algebras used below.) A linear functional f on $\mathfrak{A}$ is *positive* if $f(x^*x) \geqq 0$ for all $x \in \mathfrak{A}$. If $x \in \mathfrak{A}$ is self-adjoint ($x^*=x$) we use the notation $x \geqq 0$ to mean $\mathrm{Sp}(x) \subset [0, \infty)$, and similarly for $x>0$.

THEOREM. *If a and b are self-adjoint elements of a symmetric Banach *-algebra $\mathfrak{A}$, and if $a>0$, $b>0$, then $c=2a(a+b)^{-1}a+2b(a+b)^{-1}b-(a+b)$ satisfies $c \geqq 0$.*

Proof. Let f be a positive functional on $\mathfrak{A}$. Then

$$f(a) = f((a+b)(a+b)^{-1}a) = f(a(a+b)^{-1}a) + f(b(a+b)^{-1}a)$$
$$f(b) = f((a+b)(a+b)^{-1}b) = f(a(a+b)^{-1}b) + f(b(a+b)^{-1}b).$$

Hence,

$$\begin{aligned} f(c) &= f(2a(a+b)^{-1}a) + f(2b(a+b)^{-1}b) - f(a) - f(b) \\ &= f(a(a+b)^{-1}a) + f(b(a+b)^{-1}b) - f(b(a+b)^{-1}a) - f(a(a+b)^{-1}b) \\ &= f((a-b)(a+b)^{-1}a) - f((a-b)(a+b)^{-1}b) \\ &= f((a-b)(a+b)^{-1}(a-b)). \end{aligned}$$

Because $\mathfrak{A}$ is symmetric, $a>0$ and $b>0$ imply $(a+b)>0$ and $(a+b)^{-1}>0$. It follows that $(a+b)^{-1}$ has a self-adjoint square root $h^2=(a+b)^{-1}$. Consequently,

$$f(c) = f((a-b)(a+b)^{-1}(a-b)) = f((h(a-b))^*(h(a-b)) \geqq 0$$

from the positivity of f. Again, by the symmetry of $\mathfrak{A}$

$\mathrm{Sp}(c) \subseteq \{f(c) \mid f \text{ a positive functional on } \mathfrak{A}\} \subseteq [0, \infty)$. Thus $c \geqq 0$.

The required result is an immediate consequence of the theorem because the (Banach) algebra of $n \times n$ matrices is symmetric and the notions of positivity and nonnegativity coincide with those above.

The theorem as stated applies, in addition, to the algebra of all bounded linear operators on a Hilbert space as well as to the group algebras of many locally compact groups.

Also solved by S. T. M. Ackermans (Netherlands), F. Balatoni (Hungary), Joel Brenner D. Bzowy, Richard Datko & V. Seshadi, D. Z. Djoković (Yugoslavia), Mary Embry, G. J. Etgen, Harley Flanders, C. M. Joiner & Stephen Pierce, E. L. Koh, A. W. Marshall, M. D. Mavinkurve (India), W. L. Morris, Ingram Olkin, F. W. Pontig (England), Stanislaw Sedziwy (Poland), Barry Simon, Robert Singleton, M. F. Smiley, Olga Taussky, H. R. van der Vaart, Nolan Wallach, J. E. Wilkins, Jr., J. P. Williams and the proposer.

Simply-connecting a topological space

5426 [1966, 896]. *Proposed by Omar Khayyam, Jr., University of California, Berkeley.*

Prove or disprove: Every topological space X can be embedded in a simply-connected space X^* where X^* is obtained from X by the adjunction of one point.

Solution by W. O. Alltop, Naval Ordinance Test Station. The statement in question is true. Suppose X is a topological space with topology T. Let $X^* = X \cup \{p\}$ where $p \notin X$. Let $T^* = \{\phi\} \cup \{U \cup \{p\} : U \in T\}$. Then the space X is imbedded in the space X^* with topology T^*. We claim that X^* is simply-connected.

Suppose a and b are two points of X^*. Let $f: I \to X^*$ be defined as follows: $f(0)=a, f(1)=b, f(t)=p$ for $0<t<1$. Now f is a path in X^* from a to b. Thus, X^* is path-wise connected.

Now suppose f and g are two paths in X^* from a to b. We define $H: I \times I \to X^*$

as follows: $H(t, 0)=f(t)$, $H(t, 1)=g(t)$, $H(0, u)=a$, $H(1, u)=b$, $H(t, u)=p$ elsewhere. H is a fixed endpoint homotopy between f and g, so f and g are homotopically equivalent with respect to their endpoints. Thus, X^* is indeed simply-connected.

Also solved by S. L. Bloom, M. D. Mavinkurve (India), G. L. Porter and the proposer.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Correspondence about Reviews will be welcome.

Linear Transformations and Matrices. By F. A. Ficken. Prentice-Hall, Englewood Cliffs, N. J., 1967. xii+398 pp. $10.95.

This book gives an excellent presentation of the basic theory of finite-dimensional real and complex linear spaces. Although the prerequisites are no more than standard high school mathematics, the level of sophistication of the treatment would probably make this book more suitable as a text at the Junior-Senior level than for the Mathematics 3 course proposed by the CUPM in "A General Curriculum in Mathematics for Colleges." Also it would probably take an extremely able class to do justice to this book in less than a standard two semester course.

The first four chapters give a brief survey of background material on such items as sets, the system of real numbers, vectors in three-dimensional Euclidean space, groups, and fields. This is a fairly rapid development, and many results are stated without proofs. The chapter on vectors in three-dimensional space is excellent and should be considered indispensable for providing motivation for definitions and theorems studied later.

In Chapter 5 linear spaces are formally introduced, and the remainder of the book is concerned with linear transformations, linear functionals, duality, basic matrix theory, systems of linear algebraic equations, bilinear and quadratic functionals and forms, determinants, similar operators, and unitary and Euclidean spaces. The book contains a more thorough treatment of duality than do many conventional texts. Since the author is assuming the reader has a limited background in algebra, some topics, such as the rational canonical form for a matrix, receive but brief mention.

There is an ample supply of problems of all shades of difficulty and theoretical importance, with solutions or hints for solution of many of these given.

P. W. CARRUTH, Middlebury College

Mathematical Methods in the Physical Sciences. By Mary L Boas. Wiley, New York, 1966. xix+778 pp. $11.95.

During the last twenty years there have been many books published under the general title, "Advanced mathematics for engineers," or similar titles. The purpose of these books is to give the student of engineering, physics, or chemistry a basic competence in the many areas of mathematics that he will use in his chosen field. Of the various books of this type that have been published in the last ten years, this is one of the best.

This book provides coverage that includes nearly every area of mathematical physics needed in the junior, senior, and first year of graduate work. The book is divided into

fifteen chapters. The subjects covered include: infinite series, complex numbers, determinants and matrices, partial differentiation and multiple integrals, vector analysis, Fourier series, ordinary differential equations, calculus of variations, special functions, coordinate transformations and tensor analysis, functions of a complex variable, orthogonal polynomials and functions, the Laplace and the Fourier transforms, partial differential equations, and probability.

The author has presented the elements of the various topics enumerated above with unusual clarity. Each chapter is concluded with a large number of problems extremely well chosen to illustrate the various topics discussed. Answers are given to most of the problems proposed in the text. The author has used original approaches to the presentation of many topics; one of the best is the development of Stirling's approximate formula for the factorial by the use of the gamma function.

A suggestion that may make the book more valuable for the students for whom it is intended, would be that in the next edition the author might add a more comprehensive discussion of linear differential equations expressed in matrix notation and also add a chapter on nonlinear differential equations.

L. A. PIPES, University of California, Los Angeles

The Role of Mathematics in the Rise of Science. By Salomon Bochner. Princeton University Press, Princeton, N. J., 1966. x+386 pp. $9.00.

It is not often that a distinguished mathematician takes the trouble of digging deeply into the history of his subject and for this reason alone this book should be welcomed. The history of science has gradually become an accepted academic subject, and many young and analytic minds have devoted themselves to research in this field. In the process history of science has grown in depth, together with its philosophy. This again is a challenge to those who themselves have contributed to the advance of modern science. Bochner offers his expositions not only with great mastery of the literature, but also, in the tradition of Heiberg and Dyksterhuis, with the erudition of a classical scholar.

The book is composed of a series of eight essays in which several aspects of mathematics are investigated in regard to their origin and often in relation to the natural sciences. The book, states the author, is about the uniqueness of mathematics as a force of our intellectuality and about the mystique of its creativity, about the growing efficacy of mathematics, its widening importance, and its continuing spread. The author allows himself to wander widely in these green pastures, admiring, probing and criticizing while explaining some of the achievements of the present and of yesteryear. A recurrent theme is the comparison of Greek mathematics with that of the seventeenth century and later. One essay is a highly specialized study of some of the fundamental concepts in Aristotle's physics seen in the light of today, in which somewhat of a limited rehabilitation of the Stagirite is—and I think successfully—attempted. Another essay concerns the role of mathematics in the rise of physics, another tries to establish the nature of mathematics. Seventy three pages are devoted to showing how history of science differs from other history. The last two essays are entitled "The essence of mathematics" and "The essence of analysis." The book ends with a great number of sometimes critical biographical sketches, "in lieu of an Epilogue."

In the course of these three hundred pages the author deals with so many subjects and expresses so many opinions that it is difficult to do him justice in a short review. The reader must find out for himself, but he better read up first a little on the history of mathematics in a less sophisticated book. He then can try to match his wits with those of the author, and be struck by an interesting or novel fact or idea, be perhaps a little amazed on occasion, and may find himself agreeing here and disagreeing there.

Let me give some examples from my own reading of the book. I followed with considerable profit Bochner's analysis of some concepts in Aristotle's physics, and especially was struck by his analysis of the concept of space (topos) and by his contention that

Aristotle's opposites such as cold and hot, dry and moist, are more appropriate to nineteenth century thermodynamics than to seventeenth century mechanics (p. 166). There is also some Aristotelianism in modern neurology. On the other hand, I believe that Bochner underestimates the possibility of using sociological concepts to explain trends in mathematics such as the growth and decay of Greek mathematics, or the concentration of the eighteenth century mathematicians on mechanics (p. 7). I can agree with his opinion that, with all respect for the usefulness of medieval studies, there exists a peculiar danger of overromanticizing the Middle Ages in so far as the history of science is concerned (p. 60). But when he writes (p. 109) that Stevin "appears on the scientific stage rather abruptly, as if from nowhere," then I fail to follow him, since the engineer-mathematician-accountant is rather a typical figure of the Renaissance. I also profit by the analysis of the meaning of the three dimensionality of space in the mathematical laws of physics (p. 248)—it reminds me of some ideas that intrigued Ehrenfest—but when Bochner says (p. 87) that it is questionable whether it is meaningful to force Heraclitus' sayings into any kind of "theory," then I must remark that Hegel did exactly that, and I, for one, consider Hegel's "theory" very meaningful indeed. I could go on like this, but these examples will show the richness of the material presented in this book. The reader is in for a treat, and we hope to see more essays from Bochner on the history of science.

D. J. STRUIK, Massachusetts Institute of Technology

Generalized Functions, Volume I. Properties and Operations. By I. M. Gel'fand and G. E. Shilov; translated from Russian by Eugene Saletan. Academic Press, New York 1964. xviii+423 pp. $13.50.

Since several reviews of this book have appeared since its first Russian edition in 1958, this review will essentially be a summary of the contents of the book. Chapter I: basic introduction to generalized functions, their convergence, differentiation and integration, convolutions of generalized functions, applications to solutions of differential equations; Chapter II: Fourier transforms of generalized functions of one or more variables and applications to differential equations. Chapter III: special types of generalized functions. Four appendices one of which gives additional details on generalized functions of one or more complex variables.

In the opinion of the reviewer, this book, contrasted with L. Schwartz's "Théorie des distributions" does not invoke the duality theory in locally convex spaces and is tremendously analytical in approach. A major feature of the book is that it has great details on several special types of generalized functions and their applications.

M. S. RAMANUJAN, University of Michigan

Computing Methods, Vols. I and II. By I. S. Berezin and N. P. Zhidkov. Addison Wesley, Reading, Mass., 1965. Vol. I, xxxiv+464 pp. $15.00; Vol. II, xv+678 pp. $15.00.

These two volumes constitute a laboriously detailed, stingily referenced, minimally exercise-endowed, exhaustive (and exhausting), largely theoretical account of numerical solution methods for the following general problem areas: interpolation; integration; polynomial and (generalized) Fourier series approximations; (systems of) linear, high degree algebraic, and transcendental equations; eigen-values and -vectors; and ordinary and partial differential equations. For each problem area, many direct, iterative, and variational solution techniques are presented; each technique is accompanied by its functional analytic basis and relevant discussions of error, convergence, and stability.

Nearly all of the method presentations of the book are in the classic static-factual mode rather than the dynamic-procedural mode more appropriate to modern computing practice. The task of converting factually-described methods into working algorithms is left entirely to the reader, as is the task of relating discussions of error, convergence, and stability of methods to the algorithmic expression of those methods.

J. B. JOHNSTON, General Electric Research and Development Center

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 15 (junior level)—18 (second graduate year).

Analysis

Exercises in Mathematics. By J. Bass. Translated by Scripta Technica. Academic Press, New York, 1966. xii+459 pp. $14.75. Solved problems in integration, series, Fourier series and integrals, analytic functions, ordinary and partial differential equations. S, P.

Introduction to Approximation Theory. By E. W. Cheney. McGraw-Hill, New York, 1966. xii+259 pp. $10.95. In order to "bring a large segment of approximation theory within the comfortable grasp of undergraduate and beginning graduate mathematics students," this book pre-supposes knowledge of such topics as sequences, vector spaces, series, uniform convergences, continuity, and the mean-value theorem, but not complex functions, modern algebra, Lebesgue integration or measure theory. Topics include Tchebycheff solution of inconsistent linear equations, Tchebycheff approximation, least-squares approximation, rational approximation, the Stone approximation theorem, the Müntz theorem, the converses of the Jackson theorems, polygonal approximation and bases, the Kharshiladze-Lozinski theorems and approximation in the mean. There are extensive historical notes and a substantial bibliography intended "to restore some of the historical prospective which is inevitably lost when the subject is presented in the dry manner of a *fait accompli*," T (16–17), S, P, L.

Problems and Solutions in Mathematical Physics. By Y. Choquet-Bruhat. Translated by C. Peltzer. Translation editor J. J. Brandstatter. Holden-Day, San Francisco, 1967. 314 pp. $9.00. In spite of the title, this is actually a collection of problems in linear algebra, linear analysis, function spaces, and differential and integral operators—and hence of use in courses in these fields. The title reflects the author's desire to make mathematics useful to practitioners rather than an especially physical content. S (14–16), P.

Advanced Calculus. By Jane Cronin-Scanlon. Heath, Boston, 1967. ix+320 pp. $8.50. Real numbers and euclidean n-space, continuous and differentiable functions, integration theory (Riemann and Lebesgue), infinite series of functions (including Fourier series), curves, surfaces and integrals on them. "The main emphasis is on the development of understanding of mathematical theory rather than a description of methods for solving particular problems." T (15–16).

Introduction to the Theory of Alegbraic Numbers and Functions. By Martin Eichler. Translated by George Striker (Göttingen). Academic Press, New York, 1966. xiv+323 pp. $14.50. A unified treatment of classic fields to which Kronecker, Dedekind, and Emmy Noether made essential contributions and which played a very significant role in the development of mathematics during the last 100 years. P, L.

Linear Systems of Ordinary Differential Equations with Periodic and Quasi-Periodic Coefficients. By Nikolay P. Erugin. Translated by Scripta Technica, (with revisions by the author for the English edition). Translation editor Richard Bellman. Academic Press, New York, 1966. xxi+271 pp. $12.00. Existence proofs and methods of constructing solutions. P.

Generalized Functions. Vol. 5. Integral Geometry and Representation Theory. By I. M.

Gelfand, M. I. Graev and N. Ya. Vilenkin. Translated by Eugene Saletan. Academic Press, New York, 1966. xvii+449 pp. $17.00. This volume in the series "uses the theory of generalized functions to discuss the Radon and other integral transforms on real and complex spaces." P.

Fundamentals of Abstract Analysis. By Andrew M. Gleason. Addison-Wesley, Reading, Mass. 1966. xi+404 pp. $13.75. Based on a nonaxiomatic formulation, this fine book explains the relation of set theory to mathematics. "It is a very abstract and highly formalistic book, but in several strategic places I have tried to point out how formalism is related to the elusive 'real mathematic' which exists only in our intuition." Sets, logic, configurations, equivalence, order induction, fields, real numbers, complex numbers, counting and size of sets, limits, sum and products, topology of metric spaces, analytic function. Very favorably reviewed by M. Hestenes in *Science*, 3 March, 1967. T (15–17).

An Introduction to Sequences, Series, and Improper Integrals. By O. E. Stanaitis. Holden-Day, San Francisco, 1967. vii+210 pp. $7.95. Based on experience teaching college sophomores and high school teachers, this introduction includes uniform convergence, trigonometric series, and improper integrals. T (14), S.

Applied Differential Equations, 2nd ed. By Murray R. Spiegel. Prentice-Hall, Englewood Cliffs, N. J., 1967. xv+412 pp. $8.50. This well-known text has been revised by adding sections on the Laplace transform, the Runge-Kutta method of numerical solution, Fourier-Bessel series and orthogonal functions, a discussion of linear dependence, and some additional exercises. T (14–15).

Applications

Stability of Motion. By A. M. Liapunov. Translated by Flavian Abramovici and Michael Shimshoni. Academic Press, New York, 1966. xi+203 pp. $9.75. S. Lefschetz ends his preface as follows: ". . . this volume offers a welcome addition to the growing literature centering around Liapunov's classical stability theory and available to the non-Soviet world." Three papers by Liapunov, a preface to one of them, a paper by V. A. Pliss, all previously unavailable and virtually unknown in English. P, L.

Applications of Undergraduate Mathematics in Engineering. Written and edited by Ben Noble. Mathematical Association of America, Macmillan, New York, 1967. xvii+364 pp. $9.00. Based on a selection from 200 contributions by engineers in industry and education to the Commission of Engineering Education and the Committee on the Undergraduate Program in Mathematics. An important contribution to collegiate education. S (13–16), P, L.

Geometric Programming—Theory and Application. By Richard L. Duffin, Elmor L. Peterson, and Clarence Zener. Wiley, New York, 1967. xi+278 pp. $12.50. A "new mathematical discipline" for dealing with nonlinear optimization problems by using concepts from linear geometry. P.

Vehicular Traffic Science. Proceedings of the Third International Symposium on the Theory of Traffic Flow, New York, June 1965, held under the Auspices of the Transportation Science Section of the Operations Research Society of America. Edited by Leslie C. Edie (The Port of New York Authority) and Robert Herman and Richard Rothery (Dept. of Theoretical Physics Research Laboratories, General Motors Corporation). American Elsevier, New York, 1967. x+373 pp. $16.00. Forty-five papers

devoted to "statistical and mathematical understanding," largely via stochastic models and optimum problems. P.

Shift Register Sequences. By Solomon W. Golomb, with portions co-authored by Lloyd R. Welch, Richard M. Goldstein, and Alfred W. Hales. Holden-Day, San Francisco, California, 1967. xiv+224 pp. $7.95. First in a series on information systems edited by Norman Abramson, this volume is intended to make available all information required for instruction and applications by those interested in coding theory, switching theory, and finite automata. T (17), S, P.

Digital Simulation in Operational Research. A conference under the aegis of the Scientific Affairs Division of NATO. Edited by S. H. Hollingdale. American Elsevier, New York, 1967. 392 pp. $14.50. Papers on theory and applications to linguistics, industry, warfare, S, L.

Algebraic Linguistics; Analytical Models. By Solomon Marcus. Academic Press, New York, 1967. xi+254 pp. $12.00. Analysis of language as a collection of sentences, axiomatic theory, linguistic origins, applications to natural languages. P, L.

Theory of Games. Techniques and applications. The proceedings of a conference under the aegis of the NATO Scientific Affairs Committee at Toulon, 29th June-3rd July, 1964. Edited by A. Mensch. American Elsevier, New York, 1966. 490 pp. $15.00. "The main problem was this—could the Theory of Games, which had been introduced as a means of simplifying human problems, be applied to military problems, both tactical and strategic? . . . For reasons of security, certain papers . . . have not been included." P.

Mathematics of Physics and Modern Engineering. By I. S. Sokolnikoff and R. M. Redheffer. 2nd ed. McGraw-Hill, New York, 1966. x+752 pp. $10.95. New material, some major revisions, and more exercises. Chapters (designed to be independent): Infinite series, Nonlinear differential equations, Linear differential equations and Laplace transform, Algebra and geometry of vectors; matrices, Functions of several variables, Field theory, Partial differential equations, Complex variable, Probability, Numerical analysis. The authors feel that they have included what "should be the minimum mathematical equipment of a scientific engineer" in sufficient detail to require "six consecutive semester courses meeting three hours a week" following "a sound first course in calculus." T(14–16).

Geometry

Resolution of Singularities of Embedded Algebraic Surfaces. By Shreeram Shankar Abhyankar. Academic Press, New York, 1966. ix+291 pp. $13.50. The primary aim is a self-contained exposition of "the marvelous memoir of Zariski entitled: Reduction of singularities of algebraic three-dimensional varieties" (*Annals of Mathematics* (2) 40 (1939) 639–689). A secondary aim is to partially extend some of the results to nonzero characteristics. P.

Differential Geometry. By Louis Auslander. Harper and Row, New York, 1967. xii+271 pp. $10.95. This treatment of differential geometry of surfaces "combines both the modern and classical approaches." Prerequisites are "about one year of modern algebra and two years of calculus." After algebraic preliminaries on vector spaces, topics are differentiable structures, matrix Lie groups and frame bundles, differential invariants, local theory of surfaces, global study of surfaces, integration of forms and the Gauss-Bonnet theorem. T (15–16).

The Method of Coordinates. By I. M. Gelfand, E. G. Glagoleva, and A. A. Kirillov. Translated and adapted from the Russian by Leslie Cohn and David Sookne. Prepared under the auspices of the Survey of Recent East European Mathematical Literature, a project conducted by Alfred L. Putnam and Izaak Wirszup, University of Chicago, under a grant from the National Science Foundation. Library of School Mathematics. Vol. I. The M. I. T. Press, Cambridge, Mass., 1967. viii+69 pp. $1.50 (paper). "Elementary" analytic geometry by masters. Part one deals with one, two, and three dimensions, part two with four-dimensional space. T(13), S, P, L.

Plane Geometry and its Groups. By Heinrich W. Guggenheimer. Holden-Day, San Francisco, 1967. xii+288 pp. $8.50. ". . . an attempt to revitalize the study of elementary geometry within the framework of modern mathematics." Generation of motions by reflections, geometry and the meta-mathematics of geometry. T (15–16), TT, S, P, L.

Geometric Transformations. Vols. I and II. By P. S. Modenov and A. S. Parkhomenko. Translated from the Russian by Michael B. P. Slater. Academic Press, New York, 1965. Vol. I: x+160 pp. Vol. II: xi+136 pp. (paper) $2.45. Orthogonal (59 pp), similarity (20 pp), affine (55 pp), and projective transformations (110 pp), inversions (14 pp). Not recommended as a text or supplement for students (because of a largely antiquated point of view and often careless exposition) but useful as a quarry for experts. P.

Lectures on the Foundations of Geometry. By A. V. Pogorelov. Translated from the second Russian edition by Leo F. Boron with the collaboration of Ward D. Bouwsma. Noordhoff, Groningen, Netherlands, 1966. 137 pp. $6.90. Historical sketch, axioms of Euclidean geometry, Lobachevskian geometry, projective geometry. T (14–15), TT, S, P. L.

Geometric Exercises in Paper Folding. By T. Sundara Row. Edited and revised by Wooster Woodruff Beman and David Eugene Smith. Dover, New York, 1966. xii+148 pp. $1.00 (paper). A reprint of the classic 2nd ed. of 1905 which saw the light of day due to the interest of Felix Klein, W. W. Beman, and David Eugene Smith. It has long been difficult to obtain and will now bring much pleasure to people, young and old, at many levels of sophistication. S, P, L.

Projective Geometry, Vols. I and II. By Oswald Veblen and John Wesley Young. Blaisdell, Waltham, Mass., Vol I (1938): x+345 pp. $2.25 (paper). Vol. II (1946): x+511 pp. $3.00 (paper). Reprint of a great classic and reference work. S, P, L.

Probability and Statistics

An Introduction to Stochastic Processes With Special Reference to Methods and Applications. By M. S. Bartlett. 2nd ed., Cambridge University Press, 1967. xvi+362 pp. $9.50. This compact, comprehensive treatise is designed for the applied mathematician, statistician, engineer, biologist, economist, etc., rather than for the "pure-minded mathematical critic" who may find it unsatisfactory for purposes not intended by the author. The second edition differs by the addition of sections on new developments. S, P.

Stationary and Related Stochastic Processes. Sample Function Properties and their Applications. By Harald Cramer and M. R. Leadbetter. Wiley, New York, 1967. xii+348 pp. $12.50. An outcome of collaboration sponsored by NASA from 1962. Assumes modern probability theory. P, L.

Queuing Theory, Recent Developments and Applications. A conference, under the aegis of the NATO Science Committee, September 27th-October 1965. Edited by R. Cruon. American Elsevier, New York, 1967. 224 pp. $13.50. P.

Fundamentals of Applied Probability Theory. By Alvin W. Drake. McGraw-Hill, New York, 1967. xi+283 pp. $9.75. ". . . from a relatively physical, applied point of view. . .", assuming a course in calculus. T (14–15).

Discrete Probability. By R. A. Gangolli and Donald Ylvisaker. Harcourt, Brace & World, New York, 1967. xii+223 pp. $4.50 (paper). This book is intended for an elementary one semester course. It is one of the Harbrace College Mathematics Series of single-topic textbooks to provide compact units and supplementary material. The reasonable price commends it for serious consideration to anyone planning a course of this kind. T (14–15).

Denumerable Markov Chains. By John G. Kemeny, J. Laurie Snell, and Anthony W. Knapp. Van Nostrand, Princeton, N. J. 1966. xi+439 pp. $14.75. A systematic monograph based on the recent research literature and dealing with the foundations as well as some topics in potential and boundary theory. "The book's theme is a discussion of relations among what might be called the descriptive quantities associated with Markov chains—probabilities of events and means of random variables that give insight into the behavior of the chains." T (17–18), S, P.

Probability, A Survey of the Mathematical Theory. By John Lamperti. Benjamin, New York, 1966. x+150 pp. $8.00 (cloth), $3.95 (paper). Based on a one-semester graduate course designed to present the main lines of probability theory up to 1950. Sums of random variables play a central role in dealing with laws of large numbers, limiting distributions, the central limit problem, and stochastic processes. One of a series of "brief topical booklets" edited by R. Gunning and H. Rossi in the hope of meeting the pedagogical problems that arise from expansion and increasingly complicated internal structure of mathematics. T (17), S, P, L.

Simultaneous Statistical Inference. By. Rupert G. Miller, Jr. McGraw-Hill, New York, 1966. xv+272 pp. $11.50. This is intended as a reference source or a textbook at the graduate level covering "simultaneous confidence intervals and multiple significance tests in all phases of statistics." T (after graduate training in statistics). S, P.

Elements of Mathematical Statistics, 2nd ed. By J. F. Ractliffe. Oxford, New York, 1967. x+224 pp. $3.00 (paper). A cook book with rather outdated content.

Exercises in Probability and Statistics for Mathematics Undergraduates. With answers and hints on solutions. By N. A. Rahman. Hafner, New York, 1967. xi+307 pp. $10.50. Over 400 problems in 140 pages under the headings of discrete probability (including simple difference equations and generating functions), continuous random variable (including joint and simple derived distributions), estimation, tests, inference (including bivariate correlation and regression), characteristic functions (including use in deriving sampling distributions). The hints on solutions (165 pages) include references to sources—a commendable feature that will help turn students toward the literature. T (15–16), S, P, L.

Stochastic Processes. Problems and Solutions. By Lajos Takacs, Translated by P. Zador. Methuen, London, 1966. xi+135 pp. $4.45 (cloth) $1.95 (paper). Brief exposition and problems with partial solutions relating to Markov chains, Markov processes, non-markovian processes. List of references. T, S, P.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor H. L. Alder, University of California, Davis, represented the Association at the inauguration of President John Summerskill of San Francisco State College on May 2, 1967.

Professor E. G. H. Comfort, Ripon College, represented the Association at the inauguration of President B. S. Adams of Ripon College on May 6, 1967.

Professor H. M. Gehman, SUNY at Buffalo, represented the Association at the inauguration of President Martin Meyerson of SUNY at Buffalo on May 29, 1967.

Professor V. C. Harris, San Diego State College, represented the Association at the inauguration of President J. A. Greenlee of California State College on May 8, 1967.

Professor R. J. Kohlmeyer, Albright College, represented the Association at the inauguration of President M. J. Mays of Elizabethtown College on April 15, 1967.

Professor A. W. McGaughey, Bradley University, represented the Association at the inauguration of President G. L. McConagha of Blackburn College on April 22, 1967.

Professor H. M. Nahikian, North Carolina State University, represented the Association at the inauguration of President E. B. Heilman of Meredith College on April 15, 1967.

Professor C. J. Pipes, Southern Methodist University, represented the Association at the inauguration of President D. W. Halladay of East Texas State University on April 17, 1967.

Professor Haim Reingold, Illinois Institute of Technology, represented the Association at the inauguration of President R. A. Weil of Roosevelt University on April 16, 1967.

Professor Norman Schaumberger, Bronx Community College, represented the Association at the inauguration of President J. A. Colston of Bronx Community College on April 23, 1967.

Professor W. J. Thomsen, Moorhead State College, represented the Association at the inauguration of President J. F. Nickerson of Mankato State College on May 11, 1967.

Amherst College: Assistant Professor D. E. Crabtree, University of Massachusetts, has been appointed Assistant Professor; Dr. Norton Starr, Massachusetts Institute of Technology, has been appointed Assistant Professor.

Florida Presbyterian College: Dr. B. H. Maddox, Presbyterian College, and Associate Professor R. G. Van Meter, Geneva College, have been appointed Associate Professors.

SUNY at Albany: Professor Violet H. Larney is Acting Chairman of the Mathematics Department; Assistant Professor G. E. Martin, University of Rhode Island, has been appointed Assistant Professor.

Mr. J. T. Anderson, Duke University, has been appointed Instructor at Hamilton College.

Mr. J. P. Becker, Stanford University, has been appointed Assistant Professor at Rutgers— The State University.

Professor R. H. Bing, University of Wisconsin, has been appointed Chairman of the Division of Mathematical Sciences, National Research Council.

Dr. I. E. Block has been promoted to Manager of AUERBACH Corporation's Scientific Information Division.

Mr. R. G. Bowers, University of Delaware, has been appointed Assistant Professor at Williamsport Area Community College.

Dr. L. E. Bragg, University of Minnesota, has been appointed Assistant Professor at the University of Kentucky.

Mr. D. A. Breault, Sylvania Applied Research Laboratory, has been appointed Senior Analyst at the Harvard Computing Center.

Dr. C. E. Burgess has been appointed Chairman of the Department of Mathematics at the University of Utah.

Mr. Marvin H. J. Gruber, SUNY at Buffalo, has been appointed Instructor at the Rochester Institute of Technology.

Assistant Professor S. L. Gulden has been promoted to Associate Professor at Lehigh University.

Dr. A. W. Landers, Brooklyn College, has been appointed Director of the Office of Grants and Research.

Mr. E. E. Mayer, Blue Mountain Community College, has been appointed Dean of Liberal Arts and Sciences.

Mr. O. B. McCowan, Kilgore College, has been appointed Assistant Professor at Belmont College.

Dr. G. P. Patil, Pennsylvania State University, has been elected an ordinary member of the International Statistical Institute.

Professor W. M. Perel has been appointed Head of the Mathematics Department at Wichita State University.

Dr. L. V. Quintas, CUNY, has been appointed Professor at Pace College.

Professor R. A. Raimi has been appointed Associate Dean for Graduate Studies at the University of Rochester.

Mr. J. W. Ridge has been appointed Director of Research in the Office of Institutional Research, Wisconsin State University.

Assistant Professor H. C. Wiser has been promoted to Associate Professor at Washington State University.

Mr. J. N. Herrmann, Young Radiator Co., died on March 6, 1967. He was a member of the Association for ten years.

Professor Emeritus H. W. Kuhn, Ohio State University, died in November 1965. He was a Charter member of the Association.

Mr. Jacob Schwartz, Goldfinger & Schwartz, died on February 24, 1967. He was a member of the Association for eight years.

Professor C. C. Torrance, U. S. Naval Postgraduate school, died on May 2, 1967. He was a member of the Association for forty years.

UNITED STATES-INDIA EXCHANGE OF SCIENTISTS AND ENGINEERS

The Governments of India and the United States signed an agreement in February, 1967 for an exchange of scientists and engineers to promote increased communication between the scientific communities of the two countries on problems of mutual interest. The national agencies which have been assigned responsibility for administering the program are the National Science Foundation (NSF) and the Indian Council for Scientific and Industrial Research (CSIR). Visits by scientists and engineers from one country to the other shall be for periods of two weeks to several months, and the total man-days per year are limited to 800 for each country. The NSF will pay transportation costs to India for American participants and the CSIR will pay subsistence and travel costs within India. Similarly, the NSF will be host to the Indian visitors while in the United States. For further information write to: United States - India Exchange Office of International Science Activities, National Science Foundation, Washington, D. C. 20550

BERNARD FRIEDMAN MEMORIAL PRIZE

The first Bernard Friedman Memorial Prize has been awarded to Ralph Ta-Shun Cheng, who has just received the Ph.D. degree from the Department of Mechanical Engineering, Aeronautical Sciences. Dr. Cheng was nominated by Professor S.A. Berger, and was selected from among all nominees by a committee under the chairmanship of Professor T. Kato. Cheng was cited for the mathematical quality of his dissertation, "An Investigation of the Laminar Flow Around the Trailing Edge of a Flat Plate."

The prize consists of a cash award bestowed annually, and is supported by a fund donated by friends of Professor Friedman. Additional contributions are welcome; they should be made out to "The Bernard Friedman Memorial Fund" and addressed to the Chairman, Department of Mathematics, University of California, Berkeley.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MARCH MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the MAA was held at the University of Arizona, Tucson, on March 31-April 1, 1967. There were fifty-one persons in attendance, including twenty-nine members of the Association. Professor D. E. Myers, Chairman of the section, presided.

The banquet was held on the evening of March 31. Dr. G. S. Rogers, New Mexico State University, was the guest speaker. The title of his talk was "How to pick a campsite."

The following officers were elected: Chairman, R. J. Wisner, New Mexico State University; Vice-Chairman, A. B. Gray, Jr., Northern Arizona University.

The following papers were presented:

1. *Reducibility and extended reducibility of linear differential operators*, by M. S. Klamkin, Ford Science Laboratory.

The reducibility of a class of nth order linear differential equations is first established. This is applied to solving certain nth order differential equations. The notion of reducibility is then extended and applied to 'solving' other nth order differential equations.

2. *Synthetic methods in differential geometry*, by J. H. Butchart, Northern Arizona University.

The neatness of applying elementary geometrical considerations to certain problems of differential geometry is stressed. For instance, the definition of curvature can be applied directly at times instead of appealing to the standard formulas in rectangular and polar coordinates.

3. *Matrix number theory and Fibonacci numbers*, by D. W. Hardy, New Mexico State University.

A matrix identity is used to motivate and prove some familiar formulas involving Fibonacci numbers. It is shown that more complicated formulas can also be discovered using this matrix identity. For example, a formula is given for $F_{a+b+c+\cdots+d}$ in terms of the Fibonacci numbers F_a, $F_b, \cdots, F_d$ and F_{a-1}, $F_{b-1}, \cdots, F_{d-1}$.

4. *At least 10 primes in arithmetical progression*, by Edgar Karst, University of Arizona.

Let a be the first prime in A.P., z the last, n the number of primes in A.P., and d the common difference of the form $d=p_1^r \cdot p_2^s \cdots p_k^w$, with $p_1, p_2, p_3, \cdots$ the sequence of consecutive primes,

2, 3, 5, $\cdots$, and $r, s, t, \cdots$ positive integers >0, not necessarily distinct. Then the sequence of primes in A.P. is called complete, if $n=p_{k+1}-1$ and $|a|>|p_{k+1}|$. In all other cases the sequence is called incomplete. The only complete sequence known since 1910 is the Escott sequence 199, 409, $\cdots$, 2089 with $d=210$ and $n=10$. There are now two more of this type: 52879, 53299, $\cdots$, 56659 with $d=420$ and $n=10$, and 34913, 37013, $\cdots$, 53813 with $d=2100$ and $n=10$. The only incomplete sequences with $n>10$ known since 1963 and 1964, respectively, are the Golubev sequence 23143, 53173, $\cdots$, 353473 with $d=30030$ and $n=12$, and the Seredinskij sequence -55117, 4943, $\cdots$, 725663 with $d=60060$ and $n=14$. There are now three more with $d=30030$ and $n=11$, starting at -24023, 75307, and 820793. The new results were accomplished by a program written for the IBM 1130 by the author.

5. *A noncommutative quotient ring*, by Merry Lomanitz, New Mexico Institute of Mining Technology.

6. *Mathematical control theory applied to management science*, by G. L. Thompson, Carnegie Institute of Technology.

7. *Game theory and the von Neumann model of an expanding economy*, G. L. Thompson, Carnegie Institute of Technology.

8. *Problem solving via transforms*, by M. S. Klamkin, Ford Scientific Laboratory.

Usually transform theory is thought of in terms of certain definite integrals. Here we take a much broader viewpoint. We illustrate (with slides, some in color) the philosophy and applications of transform theory by a series of problems starting off with some simple ones in arithmetic and geometry. We then consider some problems in probability, number theory, combinatorics, differential equations and physics.

The meeting was concluded with a panel discussion of the Conference Board of the Mathematical Sciences publication *The Role of Axiomatics and Problem Solving in Mathematics*. The outcome of the discussion was the following: 1. Mathematics without applications becomes sterile. 2. Axiomatics are necessary to bring order to the study of mathematics.

S. T. Kao. *Secretary-Treasurer*

APRIL MEETING OF THE IOWA SECTION

The fifty-fourth regular meeting of the Iowa Section of the MAA was held at Drake University, Des Moines, on April 21, 1967. Chairman W. L. Waltmann presided. Total attendance was ninety-six, fifty-eight of whom were members of the Association. Routine business was considered at the afternoon session.

The following officers were elected: Chairman, C. M. Lindsay, Coe College, Cedar Rapids; Vice-Chairman, Rev. J. C. Friedell, Loras College, Dubuque; Secretary-Treasurer, B. E. Gillam, Drake University, Des Moines.

The following motion was made and duly passed: "Prof. Earle Canfield was given a standing vote of thanks for his eleven years service to the Iowa Section of MAA as its Secretary-Treasurer."

The following papers completed the program:

1. *Karl Menger's contribution to ethics*, by E. S. Allen, Iowa State University.

Karl Menger's *Moral, Wille und Weltgestaltung* (1934) is a noteworthy application of the mathematical way of thought to ethics. Refuting the possibility of a universally accepted morality, Menger concentrates on the classification of men according to their desires and the norms of conduct which they choose. The graphical and tabular analysis even of a simple three-way choice is quite complex. The will-group (Willensgruppe) consists of men sufficiently like-minded to avoid collision within the group. The author envisages, and would hope for, will-groups whose codes, though different, would not prevent mutual toleration.

2. *The reversed upper central series in the polarization process in groups*, by Donald Pilgrim, Luther College.

Let G be the group and $\{Z_n\}$ be the upper central series of G. If $Z_n=G$ for some positive integer n and if c is the least such integer, we define a chain $\{K_i\}$ of subgroups of G by: (a) $K_i=Z_{c-i+1}$ if $1\leqq i\leqq c$ and (b) $K_i=\{1\}$ if $i\geqq c+1$. Call the chain $\{K_i\}$ the reversed upper central series of G. Now suppose we are considering a class, S, of groups defined by one or more identical relations involving commutators; for example, by the fourth Engel condition $(y, x, x, x, x)=1$. An illustration is given on how the reversed upper central series may be used to determine a bound for the nilpotency class of those groups in S which are nilpotent and have given number (finite) of generators and have no elements of certain prime orders. Finally, this chain is used to obtain polarized identical relations involving commutators in groups satisfying the fourth Engel condition.

3. *Functions "close" to continuous functions*, by A. Irudayanathan, Iowa State University, introduced by the Chairman.

Definitions: (1) $f\in Y^X$ is (weakly) nearly continuous iff for every open cover α of Y, there is a continuous $g\in Y^X$ such that for every $x\in X, f(x), g(x)\in U$ for some $U\in\alpha$ (i.e., g is α-near to f). (2) $A\subset Y$ is nearly connected (nearly compact) iff for every open cover α of Y, there is a connected (compact) set $B\subset Y$ such that $A\subset\alpha^*(B)$ and $B\subset\alpha^*(A)$. $[\alpha^*(A)=\bigcup\{U\in\alpha: U\cap A\neq\phi\}]$.

RESULTS. (1) Continuity implies near continuity; converse is true if range is regular (satisfies T_3), (2) composition of continuous and nearly continuous functions is nearly continuous; composition of two nearly continuous functions is weakly nearly continuous; (3) nearly continuous images of connected (compact) sets are nearly connected (nearly compact); (4) nearly connectedness and nearly compactness are both continuous invariants.

4. *Variance estimates in nested designs*, by F. C. Leone, State University of Iowa (by invitation).

Nested designs are widely publicized and used to isolate and estimate variances with multistage processes. Beyond the two-stage design, there is little information on the distributions of the estimates of variance components. "Staggered" designs and "inverted" designs are presently employed to decrease the variance of the estimates of the true variances. These are successful at the upper stages of the design at the expense of increasing the variability at the lower stages. Some results are presented, mostly empirical, for the designs of size forty and eight combinations of variances. The frequency of occurrence of negative estimates of variance components, as well as the very strongly biased results in estimation (in the cases where $H_0:\sigma_i^2=0$ is rejected) are emphasized.

5. Panel Discussion: *Problems in the first two years of college mathematics teaching.* J. C. Friedell (Chairman), Loras College; D. Q. Porter, Muscatine Community College; T. H. Price, State University of Iowa.

The main problem at the Chairman's college is the varied background and abilities that potential mathematics and science majors bring from high school. All register for the same course as freshmen, and several weeks of pre-calculus material are given, including tests; those to begin calculus immediately are selected, based on all information available. The rest take one or two semesters of preparatory material before beginning calculus. Mr. Porter asked that larger institutions help the Junior College departments while allowing them to maintain their independence and identity. It is very helpful if the larger universities send back the grades earned by transfer students. Prof. Price agreed that counselling the transfer student about the proper courses to take is of utmost importance. Articulation concerning calculus courses is needed. Several of those attending offered suggestions about these problems in the discussion that ensued.

6. *The role of axiomatics and problem solving in mathematics*, by F. W. Lott, Jr., State College of Iowa.

The Role of Axiomatics and Problem Solving in Mathematics, published by the Conference Board of the Mathematical Sciences, was discussed together with its implications for the college mathematics program.

7. *Convergent solutions of ordinary linear homogeneous difference equations in the neighborhood of an irregular singular point*, by W. J. A. Culmer, Mid-Western College, Denison, Iowa.

This paper deals with the solution of the linear homogeneous difference equation $X(s+1) = s^h A(s)X(s)$, where s is complex, $A(s)$ and $X(s)$ are 2 by 2 matrices and $A(s)$ is representable as a power series in s^{-1} with known coefficients convergent in some neighborhood of $s = \infty$. Formal power series solutions in powers of either s^{-1} or $s^{-1/2}$ are obtained. With one exception these are shown to be asymptotic representations of true solutions in certain regions of the s-plane. The asymptotic series solutions in general diverge, nevertheless are Borel summable and replaceable by convergent factorial series.

8. *Eigenvectors by norm reduction*, by Robert Lambert and Richard Sincovec, Iowa State University, Ames.

An iterative procedure for finding the eigenvectors of a symmetric matrix is developed based on a new theorem concerning the commutativity of a simple product matrix with the given symmetric matrix. The Euclidean norm of a residual commutator matrix is iteratively reduced by a modified gradient process which changes the simple product matrix successively until the commutator is forced to zero. This will yield both eigenvectors and eigenvalues to high precision. Several examples are given which give more precise results than some other known methods.

9. *Riemann integration in ordered fields*, by J. M. H. Olmsted, Southern Illinois University, Carbondale (by invitation).

Riemann integrability in an ordered field F is defined by upper and lower step-functions. The collection I of integrable functions is a vector space; also an algebra and lattice. An integrable function may or may not possess an integral. The set D of functions possessing integrals is a vector subspace of I, but may fail to be an l algebra or lattice. If F is Archimedean, then completeness of F is equivalent to each: $D = I$; D is an algebra; D is a lattice. However, there exist non-Archimedean (hence incomplete) fields in which all three are satisfied.

B. E. GILLAM, *Secretary-Treasurer*

APRIL MEETING OF THE KANSAS SECTION

The fifty-second annual meeting of the Kansas Section of the MAA was held at Fort Hays Kansas State College, Hays, Kansas, on April 22, 1967, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. There were 194 persons registered including 75 members of the Section. Chairman Jimmy Rice presided at the morning and afternoon sessions.

The following officers were elected: Chairman, Sister Mary Paul Buser, Marymount College, Salina; Vice-chairman, Jeneva J. Brewer, Wichita University; Secretary-Treasurer, Lyle Dixon, Kansas State University.

The following papers were presented:

1. *The structure of mathematics in grades K-12*, by S. J. Bezuszka, Boston College.

Contemporary mathematics includes both new content and revised methods for teaching mathematics at all levels. The structure of mathematics is a unifying factor which should play an important role in the mathematics curricula of the present. In the lower grades one should stress student creativity, intuition, and recognition of patterns. In the junior high school the formal development of a mathematical system may be introduced, including: a set of elements, a set of operations, a set of definitions, a set of unproved statements (assumptions), a set of proved statements (theorems), and a system of logic. The mathematical systems of the natural, rational, and real numbers are explored in this framework. Senior high school courses may also be approached from this viewpoint.

2. *Committee evaluation report on CUPM general college curriculum in mathematics*, by Lyle Dixon, Kansas State University.

The committee appointed by the executive council of the Kansas Section of MAA reported on its evaluation of CUPM's recommendations for a general curriculum in mathematics for colleges. The Section approved the report and approved a resolution expressing its thanks to CUPM for its continued leadership.

3. *Eight numbers*, by Ellen Veed, Fort Hays Kansas State College.

If A is a positive integer, let $F(A)$ be the sum of the squares of the digits of A. Define $F^k(A) = F(F^{k-1}(A))$. Either there is a k_1 such that $F^k(A)=1$ for all $k \geqq k_1$ or there is a k_2 such that $F^k(A) \in S$ for all $k \geqq k_2$, where $S=\{4, 16, 37, 58, 89, 145, 42, 20\}$. That is, under iteration of $F(A)$ all positive integers are divided into two classes, one with the 1-cycle $1\rightarrow 1$, and the other with the 8-cycle $4\rightarrow 16\rightarrow 37\rightarrow 58\rightarrow 89\rightarrow 145\rightarrow 42\rightarrow 20\rightarrow 4$.

4. *Topics from integral geometry in elliptic spaces*, by A. H. Simonson, Kansas State University.

The "measures" of linear subspaces for elliptic 2-space and 3-space were derived using information available from the hemisphere model of the single pole elliptic plane.

5. *Singular perturbations of the Dirichlet problem with homogeneous boundary data*, by W. M. Greenlee, University of Kansas.

A singular perturbation problem is a problem consisting of a differential equation involving a small parameter ϵ in which the order of the equation is smaller for $\epsilon=0$ than for $\epsilon\neq 0$. One wants to analyze the behavior of solutions, eigenvalues, etc. as $\epsilon\rightarrow 0$. The paper gives a rate of convergence result for solutions of the (generalized) Dirichlet problem with homogeneous boundary conditions. The result refines some work of Huet, Ann. Inst. Fourier Grenoble (1960).

HELEN KRIEGSMAN, *Secretary-Treasurer*

APRIL MEETING OF THE KENTUCKY SECTION

The fiftieth annual meeting of the Kentucky Section of the MAA was held at Murray State University, Murray, on April 1, 1967. A total of eighty-two persons attended the meeting, of whom fifty-five were members of the Association.

At the business meeting, the following members were elected: Chairman: W. C. Royster, University of Kentucky; Secretary-Treasurer: Henry Spragens, University of Louisville; High School Testing Program: Stephen Puckette, University of Kentucky.

Various items were discussed and acted upon and L. L. Scott reported on the MAA 1967 high school mathematics contest in Kentucky.

The following papers were presented:

1. *Some remarks on interpolation in the Hardy H^p spaces*, by J. H. Wells, University of Kentucky.

2. *Nonembeddable topological semigroups*, by J. W. Stepp, Eastern Kentucky University.

DEFINITION. A topological semigroup S is said to have property P if (1) S contains an open dense topological group G and (2) for every net $\{g_i\}_{i\in A} g \in G$, such that $\{g_i\}_{i\in A}$ does not have a cluster point in X, then $\{g_i^{-1}\}_{i\in A}$ has a cluster point in S. Several examples of topological semigroups with property P are given and the following theorem is discussed: *If S is a topological semigroup with property P then there does not exist a topological semigroup with S as a proper dense subsemigroup.*

3. *Peano spaces*, by Musa Shubbak, Murray State University.

4. *Remarks on a problem in musical composition*, by W. H. Spragens, University of Louisville.

5. *The "new math" and computerized arithmetic*, by A. C. R. Newbery, University of Kentucky.

Virtually all of the arithmetic that is performed today is executed on digital computers. In computerized arithmetic the distributive, associative and cancellation laws do not hold. The

speaker discusses some of the anomalies which arise from this, and he questions whether the "new math" provides an adequate basis for the understanding of computerized arithmetic.

6. *Numerical evaluation of improper integrals*, by R. H. Geeslin, University of Louisville.

The method of repeated interval halving is adapted to obtain the direct evaluation of integrals over infinite intervals and integrals of unbounded functions over finite intervals. The method is quite general and can be applied to any convergent integral. Several parameters provide for adjusting the accuracy and time of the computation. Without any appreciable increase in computing time, the Newton-Cotes formula for five points gave better accuracy than the trapezoid rule, and Simpson's Rule was least accurate and slowest for the functions integrated. Fortran programs are available on request.

7. *An ordered sample*, by J. C. Eaves, University of Kentucky.

Probability density functions are obtained for special character samples selected from a sequentially independent random sample of order 3.

8. *Vector lattices*, by J. M. H. Olmsted, Southern Illinois University, (invited address).

The concepts of vector space, lattice, and vector lattice, in both concrete and abstract form, were presented, with several illustrative examples and counterexamples chosen from familiar vector spaces of functions and polynomials. These examples serve to illustrate an important structural distinction between an algebraic polynomial and a polynomial function, as well as to clarify the idea of a sublattice. Finally, a single specific pair of mathematical objects was exhibited and examined from several points of view, depending on the vector lattice in which they are embedded.

9. *Problems of choice*, by Stephen Puckette, University of Kentucky.

Defects of election methods in several states are discussed as instances of Arrow's theorem and the possibility of weakening the assumption of independence from irrelevant alternatives.

J. C. EAVES, *Secretary-Treasurer*

MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The forty-first meeting of the Allegheny Mountain Section was held on May 6, 1967, at West Virginia University, Morgantown, with 166 people present including 114 members.

The Chairman of the Section, F. E. Justis, Geneva College, introduced Dr. H. B. Heflin, Acting President of the University, for welcoming remarks and Professor H. W. Gould, who reviewed the first century of mathematics and astronomy at West Virginia University in interesting detail.

Professor Mario Benedicty, University of Pittsburgh, gave an invited address titled *Singularities of algebraic varieties*. Professor Paulo Ribenboim, Queens College, Kingston, Ontario, gave an excellent presentation of *p-adic numbers*.

A panel of members discussed the 1965 CUPM curriculum proposals and suggested the following resolution which was voted on and passed by the Section in the business meeting: The Allegheny Mountain Section of MAA expresses its approval of the scope and general content of the 1965 CUPM report, *A General Curriculum in Mathematics for Colleges*. With certain exceptions the courses outlined therein appear to be both highly desirable and within the ultimate capabilities of most of the colleges represented here. Therefore it is resolved that the individual members of the Section should take whatever steps are feasible to bring their programs in line with the general content of this report. The Curriculum panelists, ably moderated by Craig Comstock, Penn State University, were: I. D. Peters and Franz Hiergeist, West Virginia University, Ward Bouwsma and Allan Krall, Penn State University, Frank Perry, Washington and Jefferson College, George Novak, California State College, Anthony Pagano, Slippery Rock State College, and C. F. Sebesta, Duquesne University.

In the business meeting, reports of the annual High School Mathematics Contest were given by I. D. Peters for West Virginia and Frank Perry for Western Pennsylvania.

Warm commendations for their work were voiced; Frank Perry's resignation was accepted with regret.

The winning team for the Section in the William Lowell Putnam Competition was coached by C. V. Coffman, Carnegie Institute of Technology, who received gift memberships in MAA for the three team members: Timothy Eaton, Robert Moll and David Wise.

I. D. Peters proposed the following resolution: Resolved that the Allegheny Mountain Section through the secretary extend in writing to the members of Cabell County, W. Va. an invitation to join the Allegheny Mountain Section and that a copy of the invitation be sent to the MAA Executive Director, Professor H. M. Gehman. The resolution was passed.

James Bartoo, Penn State University, chairman of the nominating committee including I. D. Peters and P. N. Carpenter, presented a slate of officers who were unanimously elected for two years: Chairman, H. L. Krall, Penn State University; Secretary-Treasurer, W. A. Hallam, West Virginia University; Member of Executive Committee, F. E. Justis, Geneva College.

The following ten-minute papers were given:

1. *Mathematics and astronomy at West Virginia University—the first century*, by H. W. Gould, West Virginia University.

The first significant mathematician at West Virginia University after its founding in 1867 was the Norwegian geometer John Arndt Eiesland (1867–1950), who headed the Department from 1907 till 1938. He published extensively and directed the first doctorates granted at the University. His personal library laid the cornerstone for a strong research library, and his name became a legend in Morgantown. The next most important mathematician at West Virginia was Clarence Newton Reynolds, Jr. (1890–1954), whose work on the four color conjecture brought him international fame in 1926–27 when he showed that an irreducible map must have at least 28 countries, thereby improving on a result of Franklin.

The work of Robert Maurice Mathews, who was an active problem solver in various journals prior to his death in 1929, was also discussed.

Astronomy at West Virginia traces back to 1872 when a telescope was procured; parts of this are still in existence. An observatory of some distinction existed in Morgantown from 1900 until about 1930. Strengths and weaknesses of the first century offer suggestions for future progress.

2. *The Doctor of Arts in mathematics at Carnegie*, by R. A. Moore, Carnegie Institute of Technology.

Beginning as early as June 1967, Carnegie Institute of Technology, (by that time, Carnegie-Mellon University) will grant the degree of Doctor of Arts in Mathematics. In its essentials, the degree program consists of three years of full-time study in mathematics and a fourth year as an intern in the teaching of mathematics. The course work at the graduate level in mathematics will amount to about 15 semester courses which is substantially equal to that of an average Ph.D. candidate. The dissertation, unlike the Ph.D. dissertation, will be expository in nature but, hopefully, of some scholarly merit. As a part of his training the candidate will not only teach in the Department, as all present Ph.D. candidates do, but he will also participate in the curriculum and course planning. In a way, the ideal candidate for this degree is a bright and energetic student who struggles against nature and instinct for six years, or so, to finally produce an acceptable but undistinguished Ph.D. thesis and then goes his way to a productive career in college teaching. Our objective is to systemize and shorten the path to a career of college teaching for the student who does not, and will never, make a significant contribution to mathematical research.

3. *Invariant definitions for vector calculus*, by Oswald Wyler, Carnegie Institute of Technology.

Coordinate-free definitions of gradient, divergence, and rotation are given, at the level of abstraction of an advanced calculus course.

4. *On a certain approximation theorem*, by S. Mrowka, Pennsylvania State University.

Let F be a real-valued function on a set X, and let a be a real number. A set having one of the following forms: $L_a(f)=\{p\in X: f(p)\leqq a\}$ or $L^a(f)=\{p\in X: f(p)\geqq a\}$ will be called a Lebesgue set of f. The following theorem holds: *Let F be a class a real-valued functions on X such that* (a) F *contains all constant functions on* X; (b) $f, g\in F$ *implies* $f+g\in F$ *and* $f\cdot g\in F$; (c) $f\in F$ *and* $f(p)\neq 0$ *for every* $p\in X$ *implies* $f\in F$; (d) *if* $f_n\in F$ *for* $n=1, 2, \cdots$ and $f_n\to f$ *uniformly on* X, *then* $f\in F$. *Let f_0 be an arbitrary function on X. The following conditions are equivalent:*

1. $f_0\in F$; 2. *every Lebesgue set of* f_0 *is a Lebesgue set of some functions in* F; 3. *for every* a, b, $a<b$, *there are* $g_1, g_2\in F$ *such that* $g_1[L_a(f_0)]\cap g_1[L^b(f_0)]=g_2[L_a(f_0)]\cap g_2[L^b(f_0)]=\varnothing$. The above theorem appears, under somewhat different formulation, in S. Mrowka, Notices AMS, October 1964, p. 666. It was given there for topological purposes. It turns out that this theorem can be used to simplify proofs of various classical theorems in the theory of functions of real variable.

5. *Orthogonal polynomials in two variables*, by H. L. Krall and I. M. Sheffer, Pennsylvania State University, read by Mr. Krall.

The classical polynomials of Jacobi, Laguerre and Hermite satisfy a differential equation of the type $(ax^2+bx+c)y_m''+(dx+e)y_m'=\lambda_m y_m$. In this paper we are interested in orthogonal polynomial solutions of a partial differential equation of the type $(a_1x^2+b_1xy+c_1y^2+d_1x+e_1y+f_1)w_{xx}+\cdots+(\cdots+f_3)w_{yy}+(g_1x+h_1y+k_1)w_x+\cdots=\lambda_m w$ where λ_m depends on the degree of the polynomial solution.

There exist (1) the solutions $\{L_k(x)L_{m-k}(y)\}$, $\{L_k(x)H_{m-k}(y)\}$, $\{H_k(x)H_{m-k}(y)\}$, where $L_m(x)$, $H_m(x)$ represent the Laguerre and Hermite polynomials; (2) the circle polynomials, with the weight function $\rho(x, y)=(1-x^2-y^2)^\alpha$ on $x^2+y^2\leqq 1$; (3) the triangle polynomials corresponding to $\rho(x, y)=x^\alpha y^\beta(1-x-y)^\gamma$ inside the triangle bounded by $x=0$, $y=0$, $x+y=1$; (4) four other classes of polynomials. Classes (2) and (3) have been studied by Hermite, Didou, Appel, Kampé de Fériet but the classes (4) are new. This listing contains all the orthogonal polynomials satisfying a differential equation of the above type, to within linear transformations of the independent variables.

6. *On classes of maximum numbers of symmetric equations in n reciprocals*, by J. L. Klemm, Indiana University of Pennsylvania.

This paper extends the work of H. A. Simmons on maximum numbers associated with symmetric equations in n reciprocals having positive coefficients to include a term $L_{s+1}\sum_{k=1}^{m}(x_1\cdots x_n)^{-k}$, including the limiting case, $m=+\infty$; consolidates the two equations considered by Simmons into a single treatment; and gives necessary and sufficient conditions, in terms of the coefficients, for the solution giving the maximum numbers to be obtained by the Kellogg process.

7. *Finding equivalent number pairs geometrically under addition, subtraction, multiplication and division*, by E. W. Bailey, Indiana University of Pennsylvania.

8. *On invariants of projective linear groups*, by J. H. Biggs, Pennsylvania State University.

In order to obtain the automorphisms of the projective linear groups over a skew field K, the connection between certain projective transformations and their eigenspaces is usually exploited. The linear transformations which represent two commuting projective transformations may anti-commute, inducing mappings which make standard invariants useless. A method is introduced which removes this difficulty. The new invariants have the same numerical connection as the standard invariants with the number of eigenspaces in a direct sum decomposition of the underlying vector space. One application is in obtaining the automorphisms of $PU_n^+K, f)$.

9. *The elimination method of solving a matrix sequencing problem*, by G. R. Hagan, I.B.M.

The traveling salesman problem is that of finding a permutation $P=(1i_2i_3\cdots i_n)$ of the integers 1 through n that minimizes the quantity $A_{1i_2}+A_{i_2i_3}+\cdots A_{i_n1}$, where A_{mn} are a given set of real numbers. The problem derives its name from the fact that a salesman wishing to travel by the shortest total distance from his home to each of $n-1$ cities, and then return home, could calculate every permutation if he were given the distances between each pair of cities on his tour.

This is closely related to the personnel-assignment problem, differing only in that P may be non-cyclic, that is, it is not necessary to return to the starting point. Most generally, this is the problem of assigning N men optimally to N different jobs. The personnel-assignment problem is mathematically equivalent to the transportation problem and both of these are special applications of the distribution problem which in turn is a special case of the linear-programming problem, i.e., the finding of a non-negative solution for a system of linear algebraic equations.

This paper gives a method of solving the traveling salesman problem—that is, a method for finding a non-cyclic solution to the personnel-assignment problem and a near-minimum permutation for non-symmetric matrices. Under these conditions (a near minimum non-cyclic solution is acceptable) this method is most useful, and in fact, appears to be stronger with larger matrices. This is a distinct advantage over the linear-programming approach.

10. *Some remarks on the Jordan canonical form*, by C. G. Cullen, University of Pittsburgh.

Let $J_m(\lambda)$ be the m by m simple Jordan matrix whose characteristic and minimum polynomials are $(x-\lambda)^m$. Let A be any n by n complex matrix, J its Jordan canonical form, and λ one of its characteristic values. Simple formulas are developed which provide the number, and the sizes, of the simple Jordan blocks of the form $J_m(\lambda)$, $m=1, 2, \cdots$, which appear in J.

BERTHA W. MATHER, *Secretary*

MAY MEETING OF THE ILLINOIS SECTION

The forty-sixth annual meeting of the Illinois Section of the MAA was held at the University of Illinois, Urbana, on May 12 and 13, 1967. Wayne McGaughey, Section Chairman, presided at the Friday afternoon session, business meeting, dinner meeting Friday evening, and the Saturday morning session. The highlight of the meeting was an invited address by Edwin Moise of Harvard University. An estimated 200 persons attended this lecture; a number of those attending this lecture, however, were students at the University of Illinois and not members of the Section. An estimated 135 members of the Association attended one or more of the various sessions.

The following officers were elected: Chairman, R.D. Boswell, Monmouth College; Vice-Chairman, Arnold Wendt, Western Illinois University; Secretary-Treasurer, Howard Saar, Western Illinois University.

The Section Governor, J. M. H. Olmsted of Southern Illinois University, reported on some of the highlights of the Meeting of the Board of Governors of the MAA at the winter meeting in Houston on January 25, 1967. The waiting period between submission of a paper and its publication in the MONTHLY is now 18 months.

Douglas Bey, chairman of the Secondary School Lecturer Committee, reported that of the 200 lecturers subsidized by the State Academy of Science under its NSF grant 30 were in mathematics. This compares to 17 out of 100 for last year. In any event, the program has been terminated by NSF; and if it is to be continued, it must be completely a section activity. The section took no action regarding continuation of the program either on a subsidized or nonsubsidized basis.

James Beach, chairman of the Membership Committee, reported that every junior college in the state had been sent a special invitation to the Section meeting.

Walter McCurdy, chairman of the Contest Committee, reported another highly successful year for this activity. Over 22,000 students from 318 schools participated. After including as an expense a $200 donation to the National MAA Contest Committee, the contest still showed a profit of about $424.

Franz Hohn, chairman of the Undergraduate Participation Committee, reported on the results of the Putnam Competition so far as Illinois students are concerned. The section will award cash prizes and/or MAA memberships to the top 15 Illinois contestants and voted to appropriate $200 to continue this activity next year.

Arnold Wendt, chairman of the Committee on the Strengthening of the Teaching of

Mathematics reported attending three meetings during the past year dealing with the pre-service and in-service training in mathematics of elementary teachers. An apparent lack of consensus at these meetings as to just what CUPM recommendations concerning this type of training meant lead CSTM to sponsor a resolution formalizing the creation of an ad hoc Section committee called the Commission on the Mathematical Preparation of Teachers of Elementary School Mathematics (CMPTESM), which is directed to consider in depth and in detail the question, "What mathematical background is necessary for the fully qualified teacher of mathematics in the elementary school?" The Section approved the resolution and appropriated $200 to support activities of the Commission, which will be chaired by Irwin Feinstein of the Circle Campus of the University of Illinois.

R. D. Boswell, chairman of the section's Study Committee for CUPM Recommendations on the Undergraduate Program, reported that the committee recommends the Illinois Section sponsor a two-day conference during the fall of 1967 devoted to a consideration of the CUPM recommendations on "Preparation for Graduate Study in Mathematics", "A General Curriculum in Mathematics for Colleges," and "Pregraduate Preparation of Research Mathematicians." The recommendation was approved by the Section.

The banquet address was given by Prof. Joseph Landin, University of Illinois Chicago Circle Campus, who spoke on *Current Developments of Higher Education in Illinois: Their Effects Upon Collegiate Mathematics.*

After a cordial welcome from Dean R. W. Rogers of the College of Liberal Arts, the following program was presented at the Friday afternoon session, dinner meeting, and the Saturday morning session.

1. *An inequality for the derivative of nonnegative polynomials*, by Michael Skalsky, Southern Illinois University.

Using a representation of nonnegative polynomials in terms of orthonormal polynomials, the following result has been obtained. If for $x \geqq 0$, $p_n(x) \geqq 0$, then $-[n/2]\int_0^\infty P_n(x)e^{-x}dx \leqq \int_0^\infty P_n'(x)e^{-x}dx \leqq \int_0^\infty P_n(x)e^{-x}dx$, where n is the degree of $P_n(x)$. In addition, the necessary and sufficient conditions are given under which the equality on the left or on the right can be obtained.

2. *Hyplanes of order* (n, m), by Subhash Saxena, Northern Illinois University.

A hyplane of order (n, m) (n and m being positive integers) is defined in terms of the following postulates. P.1: Two distinct points determine a unique line. P.2: There are exactly n points on a line ($n>1$). P.3: There are exactly $(n+m)$ lines through a point ($m>1$). P.4: There exists at least one point. It can be easily shown that a necessary condition for the existence of a hyplane is that $m(m-1)$ is a multiple of n. It so happens that this condition is also sufficient for $n=2$, and for $n=3$. For $n>3$ it may not be sufficient. It is proved that if $(n-1)=p^k$ (p being prime) and $m=(n-1)^2$, then there exists a hyplane of order (n, m); and so is true for $n=p^k$ and $m=n^2+1$.

3. *Functional analysis proofs of two theorems on analytic functions*, by L. A. Rubel, University of Illinois.

Methods of functional analysis are used to prove two basis theorems about functions of a complex variable. The first is Runge's theorem on approximation by rational functions. The second is that there exists an analytic function on an arbitrary region that interpolates any sequence of values on any discrete sequence of points. The proofs use the Hahn-Banach theorem and the Banach-Dieudonné theorem, and depend on a known characterization of the dual space of the space of analytic functions on a region, in the topology of uniform convergence on compact subsets.

4. *How to tell that a simple overhead knot is really knotted*, by E. E. Moise, Harvard University (by invitation).

It is shown that the trefoil knot does not have the knot-type of the circle. The main steps are as follows: (1) The fundamental group of the complement of a trivial knot is infinite cyclic (and

hence commutative). (2) Equivalent knots have homeomorphic complements, and therefore the fundamental groups of their complements are isomorphic. (3) Let G be the fundamental group of the complement of the trefoil. Then there is a homomorphism of G onto the symmetric group on three symbols. Therefore, (4), G is not commutative, and (5), the trefoil is not equivalent to a trivial knot.

The proof of (3) is by methods due to Artin and Fox. The rest of the methods and ideas are standard.

5. *The solution of Boolean equations*, by Helen Skala, Illinois Institute of Technology.

The ring properties of a Boolean algebra may be exploited with great advantage in the theory of Boolean equations. For, in absence of cancellation laws for union and intersection, the cancellation law for addition (symmetric difference) becomes highly significant in simplifying and solving Boolean equations. The general solution to an arbitrary Boolean equation in n unknowns may thus be obtained in a very easy manner.

6. *Recent developments in the theory of numbers*, by P. T. Bateman, University of Illinois (by invitation).

7. Panel Discussion: The Present and Future Freshman-Sophomore Program in Mathematics in the Junior and Senior Colleges.

Panel: W. A. Ferguson, University of Illinois (Chairman of Panel); J. M. H. Olmsted, Southern Illinois University; Franz Hohn, University of Illinois; Rupert Boswell, Monmouth College; Howard Saar, Western Illinois University.

ARNOLD WENDT, *Secretary-Treasurer*

MAY MEETING OF THE INDIANA SECTION

Eighty persons attended the May meeting of the Indiana Section of the MAA, held at Wabash College, Crawfordsville, on May 13.

Dr. Paul W. Cook, Jr., President of Wabash College, welcomed the group on behalf of his institution, and Professor Paul Mielke presided.

New officers were elected at the business meeting as follows: Kenneth Sidebottom, Indiana Central College, Chairman; Billy Rhoades, Indiana University, Vice-Chairman; Maynard Mansfield, Purdue University, Fort Wayne, Secretary-Treasurer.

The program consisted of the following three hour addresses:

1. *CUPM and the future*, by R. H. McDowell, Washington University, St. Louis.

2. *Some method in the four color problem*, by Oystein Ore, Yale University.

3. *Scales of differentiability*, by C. J. Neugebauer, Purdue University.

GEORGE PEDRICK, *Secretary-Treasurer*

MAY MEETING OF THE NEBRASKA SECTION

The forty-third annual meeting of the Nebraska Section of the MAA was held on May 6, 1967, at the University of South Dakota, Vermillion. Professor C. H. Frick, Chairman of the Section, presided. There were 72 persons in attendance of whom 45 were members of the Association. Professor E. Moise, President of the Association, gave a lecture during the morning session.

The following officers were elected for 1967–1968: Chairman, A. W. Zechmann, University of Nebraska; Vice-Chairman, W. E. Ekman, University of South Dakota; Secretary-Treasurer, H. M. Cox, University of Nebraska. Professor J. M. Earl of the University of Omaha was continued as Chairman of the Mathematics Contest Committee. Representatives of the Nebraska Section of the Mathematical Association of America, of the Nebraska Section of the National Council of Teachers of Mathematics, of the Ne-

braska Actuaries Club, and of the Nebraska Academy of sciences are on the Contest Committee.

The following papers were presented:

1. *The Nebraska-South Dakota Mathematics Contest*, by J. M. Earl, University of Omaha, and H. M. Cox, University of Nebraska.

Some 5752 contest papers from 179 Nebraska and 23 South Dakota high schools were scored and reports printed by use of IBM 1230, IBM 534, and IBM 360 machines. The median team score was 48 as compared with 59 in the 1966 Contest. Copies of the report are available upon request.

2. *Periodic solutions of second order nonlinear differential equations*, by Klaus Schmitt, University of Nebraska and Nebraska Wesleyan University.

We consider the second order ordinary differential equation (A) $x''=f(x, x', t)$, where f is assumed to be continuous on its domain of definition. If f satisfies a certain growth condition (Nagumo condition) in x', we deduce the existence of a periodic solution of (A) from the existence of "approximately" periodic solutions of both types of differential inequalities. This result is generalized and examples are given to illustrate the application of our results.

3. *One theorem on mappings of irreducibly connected spaces*, by G. L. Pfeifer, University of Nebraska.

A connected topological space X is said to be irreducibly connected about a subset A of X if there does not exist a proper subspace of X which is connected and contains A. THEOREM. If X is irreducibly connected about A and f on X onto Y is a continuous, open, closed function, and $f^{-1}f(a)$ is connected for one $a\in A$, then Y is irreducibly connected about $f(A)$.

4. *Differentiation with respect to a function*, by H. L. Hunzeker, University of Omaha.

Define the derivative of a function f with respect to a differentiator function g by $D_g f(x) = \lim_{t\to 0} 1/t[f[x+tg(x)]-f(x)]$ whenever the limit exists. The usual derivative and differential of f are obtained as special cases when 1 and Δx are the respective differentiator functions; generalizations to vector-valued functions of vector arguments follow.

5. *How to tell that a simple over-hand knot is really knotted*, by E. E. Moise, Harvard University (invited lecture).

6. *On rings with few zero divisors*, by M. D. Larsen, University of Nebraska.

Let R be a commutative ring with unity element. An ideal A of R is said to be regular if A contains at least one nonzero divisor. An ideal is said to be a 0-ideal if it consists entirely of zero divisors. A maximal 0-ideal is a maximal element in the set of 0-ideals. R is said to have few zero divisors if R has finitely-many maximal 0-ideals. [E. D. Davis, Trans. Amer. Math. Soc., 110 (1964) 196–212.] Let R be a ring with few zero divisors. Then (1) every regular ideal of R is generated by nonzero divisors and (2) if U, V are overrings of R such that the nonzero divisors of U are contained in V, then $U\subset V$. R is a quasi-valuation ring if for each pair of nonzero divisors of R, one divides the other. R is a P-ring if every overring of R is integrally closed. R is a D-ring if R is a Noetherian P-ring. Quasi-valuation ring, P-ring, and D-ring are natural generalizations of valuation ring, Prüfer domain, and Dedekind domain, respectively, to rings which allow zero divisors.

7. *Mathematics in a Korean university*, by M. M. Hasse, University of South Dakota.

A report on a year spent in Seoul National University as a Fulbright Professor.

8. *The integral of p from $-T$ to T and the differential equation $x''+px=0$*, by S. B. Eliason, University of Nebraska.

DEFINITION: $p(t)$ is said to be admissible on a closed finite interval provided there exists an associate solution $x(t)$ of $x''+p(t)=0$ with $x=0$ at the end points and positive throughout the open interval.

DEFINITION. $p(t)$ is said to be left balanced on $[-T, T]$ provided $p(t)\leqq p(-t)$ for $0\leqq t\leqq T$ with strict inequality for $0<t<d$ for some d, $0<d<T$.

From these definitions a theorem is discussed concerning the integrals of p_1 and p_2 from $-T$ to T, where p_1 and p_2 are admissible and left balanced.

9. *A terminal value problems*, by W. E. Shreve, University of Nebraska.

The Terminal Value Problem (λ) is defined as follows: (1) $y''=f(x, y, \lambda)$, $\lim_{x\to\infty} y(x)=\lambda$. y is said to be a solution to TVP (λ) if it satisfies (1) and if $y\in C^2[a, \infty)$. It is shown that if z is a solution to TVP (ν) and $\lambda\leq\nu$ then under certain conditions there exists a solution to TVP (λ).

10. *Adams-Cowell integration of oscillatory systems*, by J. F. Fox and D. D. Gibbs, University of South Dakota; introduced by W. E. Ekman.

High backward differences are computed from a few data points so as to enable the Adams and Cowell integrations to take advantage of the natural frequency of the function.

11. *Axiomatics and problem solving*, by Richard La Rue, University of South Dakota.

H. M. Cox, *Secretary-Treasurer*

MAY MEETING OF THE NEW JERSEY SECTION

The second joint meeting of the New Jersey Section of the Mathematical Association of America and the Association of Mathematics Teachers of New Jersey was held at Ocean County College on May 6, 1967. Dean A. E. Meder, Jr., Chairman of the Section, presided at the morning session and F. A. Brooks, Jr., Program Chairman, presided at the afternoon session. One hundred and thirty persons attended the meeting, including forty-five members of MAA.

There was no business meeting.

The morning session was as follows:

1. *Computer assisted instruction in modern mathematics*, by C. A. Riedesel, Penn. State University (introduced by A. E. Meder, Jr.).

2. *Advanced placement program in mathematics—present and future*, by Marion G. Epstein, ETS, Princeton, N. J. (by invitation).

The following paper was presented at the afternoon session:

The Programmed High School—A look at the future, by J. M. Honeycutt, American Cyanamid, Wayne, N. J. (introduced by F. A. Brooks, Jr.).

F. A. VARRICHIO, *Secretary*

MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the MAA was held at the State University College at Plattsburgh, New York, on Saturday, May 20, 1967. R. D. Larsson, Chairman of the section, presided at the morning session, and D. W. Hall, Vice-Chairman, presided at the afternoon session. There were 103 persons in attendance, including 88 members of the Association.

At the business meeting the following officers were elected: Chairman, D. W. Hall SUNY at Binghamton; Vice-Chairman, F. R. Olson, SUNY College at Fredonia; Secretary-Treasurer, Mary Williams, Skidmore College. Nura Turner reported on the 1967 MAA High School Mathematics Contest, and brought to the Section an invitation from Mr. Maurice Goldsmith, Chairman, Advisory Committee, Guiness Awards Ceremony, to bring a team of secondary school students from the N. Y. Upstate Contest Section to London in 1968 to compete in mathematics against a similar British team. The Section authorized the Contest Committee to accept the invitation, subject to the successful working out of the many problems such a meet would involve. R. D. Larsson reported that response to the Undergraduate Paper Contest was disappointing. Nevertheless, the

Section voted to continue the contest for another year. J. M. Perry, Chairman of the Committee to study the CUPM report, *General Curriculum in Mathematics for Colleges*, presented the findings of his committee, which, in general, were quite complimentary to the GCMC report. The Section voted to accept his report for transmittal to CUPM.

The program was as follows:

1. *Panel discussion: a general curriculum in mathematics for colleges—a critique of the CUPM recommendations*, by J. M. Perry, Wells College (moderator), and J. S. Klein, Hobart College; Violet H. Larney, SUNY at Albany; F. D. Parker, St. Lawrence University; F. S. Hawthorne, N. Y. State Education Department; P. W. Gilbert, Syracuse University (panelists).

2. *On the solution to Cantor's Problem*, by Andrey Feuerverger, McGill University, Undergraduate Paper Contest winner.

Sufficient background was given to make possible a rigorous statement of Cantor's Continuum Hypothesis. After a brief statement of the results of Gödel (1940) and Cohen (1963) regarding this problem, the basic structure of the simplified proof by Scott (1966) was outlined.

3. *How to tell that a simple overhand knot is really knotted*, by E. E. Moise, Harvard University, President of MAA.

4. *Dimension Theory*, by D. W. Henderson, Cornell University.

Two questions were presented from the dimension theory of infinite dimensional spaces. The first question asked for which spaces are the large (Ind) and the small (ind) transfinite inductive dimensions equivalent. The second question asked which infinite dimensional spaces have subsets of each finite dimension.

5. *Naming the irreducible representations of the symmetric group*, by A. J. Coleman, Queen's University.

Young diagrams, standard Young tableaux, Young's orthogonal irreducible representations, and a simple method of obtaining the dimension of a representation of S_N, as $N!$ divided by the content of the hook-graph, were explained by means of examples. (A more extensive account of this material will appear in Volume 4 of Advances in Quantum Chemistry, Academic Press, under the title "The Symmetric Group Made Easy.")

6. *Duo rings and a theorem of Jacobson*, by H. E. Bell, Union College.

A well-known proof of the Jacobson "$x^{n(x)}=x$ theorem" avoids structure theory arguments by establishing that the rings involved are duo rings—rings in which every one-sided ideal is two-sided. The same device is used to provide an elementary proof of Herstein's "$(xy-yx)^n=(xy-yx)$ theorem"; and an investigation of completely prime ideals in duo rings yields the result that a regular duo ring is commutative if it possesses a property which implies commutativity in division rings, and is preserved by homomorphisms.

7. *Recent CUPM proposals in mathematics for engineers*, by R. J. Walker, Cornell University.

Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, published by CUPM in 1962, has been revised. The changes are few, the most striking are being a greatly increased emphasis on automatic computing.

A five year program in applied mathematics, called "Mathematical Engineering," is presented for the consideration of Departments of Mathematics and Engineering. It combines a heavy central core of mathematics and physical science with any of a number of engineering specialties. The graduate of such a program should be able to make powerful applications of mathematics to many different fields.

N. G. GUNDERSON, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-first Annual Meeting, San Francisco Hilton, California, January 25–27, 1968.

Forty-ninth Summer Meeting, University of Wisconsin, Madison, Wisconsin, August 26–28, 1968.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Indiana University of Pennsylvania, Indiana, April 27, 1968.

FLORIDA

ILLINOIS, Southern Illinois University, Edwardsville Campus, May 10–11, 1968.

INDIANA, Marian College, Indianapolis, November 11, 1967.

IOWA, Wartburg College, Waverly, April 19, 1968.

KANSAS, Marymount College, Salina, Spring 1968.

KENTUCKY, University of Kentucky, Lexington, Spring 1968.

LOUISIANA-MISSISSIPPI, Broadwater Beach Hotel, Biloxi, Mississippi, February 16–17, 1968.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Morgan State College, Baltimore, December 9, 1967.

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, South Dakota State University, Brookings, November 11, 1967.

MISSOURI, Lindenwood College, St. Charles, April 27, 1968.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 26–27, 1968.

NEW JERSEY, St. Peter's College, Jersey City, November 4, 1967.

NORTHEASTERN, Phillips Academy, Andover, Mass., November 25, 1967.

NORTHERN CALIFORNIA

OHIO

OKLAHOMA-ARKANSAS, Federal Aviation Agency, Oklahoma City, March 29–30, 1968.

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 14–15, 1968.

PHILADELPHIA, University of Delaware, Newark, November 18, 1967.

ROCKY MOUNTAIN, University of Denver, Colorado, May 10–11, 1968.

SOUTHEASTERN, East Carolina College, Greenville, North Carolina, March 29–30, 1968.

SOUTHERN CALIFORNIA, Loyola University of Los Angeles, Los Angeles, March 9, 1968.

SOUTHWESTERN, New Mexico State University, University Park, Spring 1968.

TEXAS, Texas A and M University, College Station, April 19–20, 1968.

UPPER NEW YORK STATE, State University College at Buffalo, November 4, 1967.

WISCONSIN, Wisconsin State University, La Crosse, May 4, 1968.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, New York, New York, December 26–31, 1967.

AMERICAN MATHEMATICAL SOCIETY, San Francisco Hilton, January 23–26, 1968.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of California, Los Angeles, June 17–20, 1968.

ASSOCIATION FOR COMPUTING MACHINERY, New York Hilton, November 10, 1967.

ASSOCIATION FOR SYMBOLIC LOGIC, Sheraton-Boston Hotel, December 27–28, 1967.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 23–25, 1967.

INSTITUTE OF MATHEMATICAL STATISTICS

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Convention Hall, Philadelphia, Pennsylvania, April 17–20, 1968.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Sheraton-Chicago Hotel, November 1–3, 1967.

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SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, University of California at Santa Barbara, November 29–30, December 1–2, 1967. (Symposium on combinatorial mathematics.)



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NOVEMBER 1967

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$6.00 (\$8.00 in 1968). For nonmembers the subscription price is \$10.00 (\$15.00 in 1968).

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

HIGHER CURVATURES OF CURVES IN EUCLIDEAN SPACE, II

HERMAN GLUCK, Harvard University and the University of Pennsylvania

1. Introduction. As presented in elementary calculus, the curvature of a plane curve is a quantity which measures the rate of turning of the tangent line to the curve. If $\vartheta(s)$ denotes the angle between a fixed line and the tangent line, then the curvature is defined to be the derivative $\vartheta'(s)$ of this angle with respect to arc length. In sharp contrast to this, the curvatures of a curve in n-space are defined in an altogether different manner via Frenet theory [1]. My object here is to show that even in this more general setting, the curvatures can equally well be defined as the rates of turning of the appropriate osculating planes.

2. Angles between p-planes in R^n. The early portions of this section are likely to be familiar to the reader, and are in any case straightforward, so the discussion is purposely sketchy. More details may be found in Chapter 4 of [2].

The angle between a pair of lines, P^1 and Q^1, in Euclidean n-space R^n is the smaller of the two possible angles between any vector parallel to P^1 and any vector parallel to Q^1. The angle between a line P^1 and a p-plane Q^p in R^n is the smallest angle between P^1 and any line in Q^p. This is the same as the angle between P^1 and its orthogonal projection in Q^p, or $\pi/2$ in case this orthogonal projection degenerates to a point.

Consider now a pair of p-planes, P^p and Q^p, in R^n. Suppose that among all pairs of lines, one from P^p and one from Q^p, the lines S^1 and T^1 make the smallest possible angle, ϑ_1, with each other. Let P^{p-1} and Q^{p-1} be "orthogonal complements" of S^1 and T^1 in P^p and Q^p, respectively. Then it is easily seen that S^1 is orthogonal not only to P^{p-1} but also to Q^{p-1}, and similarly T^1 is orthogonal not only to Q^{p-1} but also to P^{p-1}.

If we iterate this procedure with P^{p-1} and Q^{p-1} in the roles of P^p and Q^p, we get another angle $\vartheta_2 \geqq \vartheta_1$. Doing this p times in all, we get p angles

$$0 \leqq \vartheta_1 \leqq \vartheta_2 \leqq \cdots \leqq \vartheta_p \leqq \pi/2.$$

These angles depend only on P^p and Q^p, and not on the various choices possible during the above procedure, and are called the *principal angles* between P^p and Q^p.

The procedure also yields orthonormal bases $\{u_1, u_2, \cdots, u_p\}$ and $\{v_1, v_2, \cdots, v_p\}$ for the subspaces P_0^p and Q_0^p parallel to P^p and Q^p, such that

$$\prec u_i, v_i \succ = \cos\vartheta_i \quad \text{for } 1 \leqq i \leqq p$$
$$\prec u_i, v_j \succ = 0 \qquad \text{for } i \neq j.$$

These bases are said to be in *normal form*. Unlike the principal angles, they are not uniquely determined by the procedure. For example, u_i and v_i may simultaneously (or individually if $\vartheta_i = \pi/2$) be replaced by their negatives for assorted values of i. This is the only ambiguity possible if all the principal angles are distinct, but more variations can occur if some of the successive principal angles

coincide. Note, for future use, that the orthogonal projection of u_i into Q_0^p is $(\cos \vartheta_i)v_i$, and similarly the orthogonal projection of v_i into P_0^p is $(\cos \vartheta_i)u_i$.

If the positive integers $p \leqq n$ and the p angles

$$0 \leqq \vartheta_1 \leqq \vartheta_2 \leqq \cdots \leqq \vartheta_p \leqq \pi/2$$

such that

$$\vartheta_i = 0 \quad \text{for } 1 \leqq i \leqq 2p - n$$

are given in advance, then there exists a pair of p-planes P^p and Q^p in R^n for which these angles are the principal angles between P^p and Q^p. Furthermore, these angles determine the relative position of P^p and Q^p in the following sense.

THEOREM 2.1. *Let P^p, Q^p, $\overline{P}^p$ and $\overline{Q}^p$ be p-planes in R^n. Then there exists a linear isometry F: $R^n \to R^n$ such that $F(P^p)$ is parallel to $\overline{P}^p$ and $F(Q^p)$ is parallel to $\overline{Q}^p$ if and only if the principal angles between P^p and Q^p coincide with those between $\overline{P}^p$ and $\overline{Q}^p$.*

The necessity of the condition follows from the fact that the construction of principal angles depends only on angles in R^n, and not on actual position. The sufficiency is proved by finding a linear isometry F which takes a pair of bases in normal form for P_0^p and Q_0^p onto a corresponding pair for $\overline{P}_0^p$ and $\overline{Q}_0^p$, and this is just an exercise in linear algebra.

THEOREM 2.2. *Let P^p and Q^p be p-planes in R^n, and let $(P^p)^\perp$ and $(Q^p)^\perp$ be $(n-p)$-planes orthogonal to P^p and Q^p, respectively. Then the nonzero principal angles between $(P^p)^\perp$ and $(Q^p)^\perp$ coincide with the nonzero principal angles between P^p and Q^p.*

The proof is left to the reader.

Let P^p and Q^p be p-planes in R^n. Then there are p principal angles between P^p and Q^p, which determine the relative position of these p-planes in the sense of Theorem 2.1. Suppose that it is desired to find a *single* angle which might reasonably be called *the* angle between P^p and Q^p. If one were constrained to choose from among the principal angles, one would have to select the largest principal angle ϑ_p for such a role, in order to insure that P^p and Q^p are parallel if and only if the angle between them is zero. Such a constraint, however, is artificial.

In order to arrive at the right definition, consider carefully the familiar case when $p=1$. Then there is just one principal angle ϑ_1 between the lines P^1 and Q^1, and it coincides with the ordinary angle ϑ between these lines. This angle ϑ, lying between 0 and $\pi/2$, has the following important property: if U is any measurable subset of P^1 with one-dimensional (Euclidean) measure $\mu(U)$, then the orthogonal projection of U into Q^1 is also measurable and has one-dimensional measure $(\cos \vartheta)\mu(U)$ in Q^1. Similarly, if V is a measurable subset of Q^1 with measure $\mu(V)$, then the orthogonal projection of V into P^1 has measure $(\cos \vartheta)\mu(V)$ in P^1. Thus the angle ϑ between P^1 and Q^1 might be characterized as

that angle between 0 and $\pi/2$ whose cosine is the reduction factor for one-dimensional measure under orthogonal projection of P^1 into Q^1, or vice versa of Q^1 into P^1. If u_1 is a unit vector in P_0^1 and v_1 a unit vector in Q_0^1, then the (1×1) matrix of the orthogonal projection of P_0^1 into Q_0^1 has a determinant whose absolute value is $\cos\vartheta$, and similarly in the reverse direction.

These observations lead directly to

DEFINITION 2.3. *Let P^p and Q^p be p-planes in R^n. Let the number α, $0\leqq\alpha\leqq 1$, be the reduction factor for p-dimensional measure under orthogonal projection of P^p into Q^p. Then the unique angle ϑ, $0\leqq\vartheta\leqq\pi/2$, such that $\cos\vartheta=\alpha$, will be called the* angle between P^p and Q^p.

Two immediate consequences of this definition are

(2.4) *If P^p is parallel to $\overline{P}^p$ and Q^p is parallel to $\overline{Q}^p$, then the angle between P^p and Q^p is the same as the angle between $\overline{P}^p$ and $\overline{Q}^p$.*

(2.5) *If $\{u_1, u_2, \cdots, u_p\}$ is any unimodular basis for P_0^p and $\{v_1, v_2, \cdots, v_p\}$ is any unimodular basis for Q_0^p, then the determinant of the matrix of orthogonal projection of P_0^p into Q_0^p has absolute value equal to $\cos\vartheta$, where ϑ is the angle between P^p and Q^p, or equivalently between P_0^p and Q_0^p.*

Information about the angle between p-planes is obtained by relating this angle to the principal angles.

THEOREM 2.6. *Let P^p and Q^p be k-planes in R^n, and let $\vartheta_1\leqq\vartheta_2\leqq\cdots\leqq\vartheta_p$ be the principal angles between them. Then the angle ϑ between P^p and Q^p is given by $\cos\vartheta=(\cos\vartheta_1)(\cos\vartheta_2)\cdots(\cos\vartheta_p)$.*

Let $\{u_1, u_2, \cdots, u_p\}$ and $\{v_1, v_2, \cdots, v_p\}$ be bases for P_0^p and Q_0^p which are in normal form. Then

$$\begin{aligned} &\prec u_i, u_j\succ = \prec v_i, v_j\succ = \delta_{ij} \\ &\prec u_i, v_i\succ = \cos\vartheta_i \qquad \text{for } 1\leqq i\leqq p \\ &\prec u_i, v_j\succ = 0 \qquad \text{for } i\neq j. \end{aligned}$$

Since the orthogonal projection of u_i into Q_0^p is $(\cos\vartheta_i)v_i$, the matrix of this projection with respect to the above bases is simply a diagonal matrix, with diagonal entries

$$\cos\vartheta_1, \cos\vartheta_2, \cdots, \cos\vartheta_p.$$

Since orthonormal bases are certainly unimodular, the theorem now follows from (2.5) and (2.4).

Alternatively, we may observe that the cube in P_0^p with leading edges u_1, $u_2, \cdots, u_p$ and p-dimensional measure 1 goes under orthogonal projection into the rectangular parallelepiped in Q_0^p with leading edges $(\cos\vartheta_1)v_1$, $(\cos\vartheta_2)v_2, \cdots,$ $(\cos\vartheta_p)v_p$ and p-dimensional measure

$$(\cos\vartheta_1)(\cos\vartheta_2)\cdots(\cos\vartheta_p).$$

And again, the theorem follows.

Some immediate consequences of Theorem 2.6 are

(2.7) *The angle between P^p and Q^p is the same as the angle between Q^p and P^p.*

(2.8) *If ϑ_p is the largest principal angle between P^p and Q^p, then the angle ϑ between P^p and Q^p is trapped by the inequality $\vartheta_p \leqq \vartheta \leqq \pi/2$.*

(2.9) *The angle between P^p and Q^p is zero if and only if P^p and Q^p are parallel.*

(2.10) *If $(P^p)^\perp$ and $(Q^p)^\perp$ are $(n-p)$-planes orthogonal to P^p and Q^p, respectively, then the angle between $(P^p)^\perp$ and $(Q^p)^\perp$ is the same as the angle between P^p and Q^p.*

This last statement follows from Theorems 2.6 and 2.2.

Suppose now that P^p and Q^p are p-planes in R^n. Corresponding to these p-planes are one-dimensional subspaces L^1 and M^1 of the exterior product $\wedge^p R^n$, which consist of the decomposable p-vectors representing P_0^p and Q_0^p, respectively. The inner product on R^n induces an inner product on $\wedge^p R^n$ in a natural way, relative to which we shall compute the angle φ between L^1 and M^1.

To do this, we need only find a unit p-vector $x \in L^1$ and a unit p-vector $y \in M^1$ such that $\prec x, y \succ \geqq 0$. Then φ will be the unique angle between 0 and $\pi/2$ such that $\cos\varphi = \prec x, y \succ$.

Let $\{u_1, u_2, \cdots, u_p\}$ and $\{v_1, v_2, \cdots, v_p\}$ be bases for P_0^p and Q_0^p which are in normal form. Consider the p-vector $x = u_1 \wedge u_2 \wedge \cdots \wedge u_p$ in $\wedge^p R^n$. Using the inner product in $\wedge^p R^n$, we have

$$\prec x, x \succ = \det(\prec u_i, u_j \succ).$$

Since $\{u_1, u_2, \cdots, u_p\}$ is an orthonormal basis for P_0^p, the matrix $(\prec u_i, u_j \succ)$ is the $p \times p$ identity matrix, and therefore has determinant 1. Thus $\prec x, x \succ = 1$, so x is a unit p-vector in L^1. Similarly,

$$y = v_1 \wedge v_2 \wedge \cdots \wedge v_p$$

is a unit p-vector in M^1.

Now $\prec x, y \succ = \det(\prec u_i, v_j \succ)$. Since the bases for P_0^p and Q_0^p are in normal form, the matrix $(\prec u_i, v_j \succ)$ is a diagonal matrix with diagonal entries $\cos\vartheta_1$, $\cos\vartheta_2, \cdots, \cos\vartheta_p$. Thus

$$\prec x, y \succ = (\cos\vartheta_1)(\cos\vartheta_2) \cdots (\cos\vartheta_p),$$

and since this product is $\geqq 0$, the angle φ between L^1 and M^1 is the unique angle between 0 and $\pi/2$ satisfying

$$\cos\varphi = (\cos\vartheta_1)(\cos\vartheta_2) \cdots (\cos\vartheta_p).$$

Comparing this formula with Theorem 2.6, we have

THEOREM 2.11. *Let P^p and Q^p be p-planes in R^n, and let L^1 and M^1 be the corresponding one-dimensional subspaces of $\wedge^p R^n$. Then the angle between P^p and Q^p in R^n coincides with the angle between L^1 and M^1 in $\wedge^p R^n$.*

As a corollary, we have

(2.12) *If P_1^p, P_2^p and P_3^p are three p-planes in R^n, then the angle between P_1^p and P_3^p is less than or equal to the sum of the angles between P_1^p and P_2^p and between P_2^p and P_3^p.*

For Theorem 2.11 reduces this to the case $p=1$, where the statement is evident.

The above considerations also yield a practical technique for computing the angle between two p-planes in R^n.

THEOREM 2.13. *Let P^p and Q^p be p-planes in R^n, and let $\{u_1, u_2, \cdots, u_p\}$ and $\{v_1, v_2, \cdots, v_p\}$ be arbitrary bases for P_0^p and Q_0^p, respectively. Then the angle ϑ between P^p and Q^p is given by the formula*

$$\cos\vartheta = \frac{|\det(\prec u_i, v_j\succ)|}{\sqrt{\det(\prec u_i, u_j\succ)}\sqrt{\det(\prec v_i, v_j\succ)}}.$$

Consider the p-vectors

$$x = u_1 \wedge u_2 \wedge \cdots \wedge u_p \quad \text{and} \quad y = v_1 \wedge v_2 \wedge \cdots \wedge v_p,$$

which lie in the one-dimensional subspaces L^1 and M^1 of $\wedge^p R^n$ corresponding to P^p and Q^p. The angle φ between L^1 and M^1 is given by

$$\cos\varphi = \frac{|\prec x, y\succ|}{\sqrt{\prec x, x\succ}\sqrt{\prec y, y\succ}},$$

and the desired result then follows upon application of Theorem 2.11 and the explicit formulas for the inner product in $\wedge^p R^n$.

3. Curvature of curves via Frenet theory. Recall the situation of Frenet theory (see [1] for details). We have an interval $I \subset R^1$ and a map $F: I \to R^n$ which is a C^k parametrization by arc length. We assume that at each point $s \in I$, the derived vectors

$$F'(s), F''(s), \cdots, F^{[r]}(s) \qquad r < k$$

are linearly independent. Applying the Gram-Schmidt orthonormalization process to these vectors, we get an orthonormal r-frame

$$V_1(s), V_2(s), \cdots, V_r(s),$$

called the *Frenet r-frame*. The derivatives of the frame vectors satisfy the following *Frenet equations*.

$$(3.1) \qquad V_1'(s) = k_1(s)V_2(s)$$

$$(3.2) \qquad V_i'(s) = -k_{i-1}(s)V_{i-1}(s) + k_i(s)V_{i+1}(s), \qquad 2 \leqq i \leqq r-1.$$

$V_r'(s)$ exists because we have assumed that $r<k$, but there is a bit of trouble with a formula for it, because $V_{r+1}(s)$ may not exist. Given a point $s_0 \in I$, if

$F^{[r+1]}(s_0)$ happens to be linearly independent of $F'(s_0)$, $F''(s_0)$, $\cdots$, $F^{[r]}(s_0)$, then this will also be true for all s in some neighborhood of s_0. In that neighborhood, $V_{r+1}(s)$ can be defined by carrying the Gram-Schmidt process one step further, and we can then write

$$V_r'(s) = -k_{r-1}(s)V_{r-1}(s) + k_r(s)V_{r+1}(s). \tag{3.3}$$

Alternatively, if $F^{[r+1]}(s_0)$ happens to be linearly dependent on $F'(s_0)$, $F''(s_0)$, $\cdots$, $F^{[r]}(s_0)$, then $V_r'(s_0)$ will also be linearly dependent on $V_1(s_0)$, $V_2(s_0)$, $\cdots$, $V_r(s_0)$, and we will have

$$V_r'(s_0) = -k_{r-1}(s_0)V_{r-1}(s_0). \tag{3.4}$$

In this case, $k_r(s_0)$ is defined to be zero.

The *curvatures* associated with the given curve are defined to be the coefficients $k_1(s)$, $k_2(s)$, $\cdots$, $k_r(s)$ which appear in the Frenet equations. We have $k_i(s)>0$ for $1\leqq i\leqq r-1$, while $k_r(s)\geqq 0$.

4. Curvature as the rate of turning of the osculating planes. Let $1\leqq p\leqq r$. The subspace $P_0^p(s)$ of R^n spanned by the vectors $F'(s)$, $F''(s)$, $\cdots$, $F^{[p]}(s)$, or equally well by $V_1(s)$, $V_2(s)$, $\cdots$, $V_p(s)$, is called the *p-th osculating subspace* associated with the curve at the given point. The p-plane $P^p(s)$ in R^n which passes through $F(s)$ and is parallel to $P_0^p(s)$ is called the *osculating p-plane* to the curve at $F(s)$. At any given point, the curve has exactly pth order contact with its osculating p-plane for $p<r$ and at least pth order contact with its osculating p-plane for $p=r$. In this latter case, the order of contact will be greater than $p=r$ if and only if $k_r(s)=0$ at the point. In all cases, however, the curve has less than pth order contact with every other p-plane in R^n at $F(s)$. Thus the osculating planes to the curve at a given point are those planes with which the curve has closest contact at the point.

Fix the integer p, $1\leqq p\leqq r$, and fix a point s_0 in the domain of the curve. Consider the osculating p-plane $P^p(s)$, and define the angle $\vartheta(s)$ to be the angle between $P^p(s_0)$ and $P^p(s)$. We are now in a situation which generalizes that of elementary calculus (where $p=1$ and $n=2$), save for the fact that the angle $\vartheta(s)$ is unsigned. This is unavoidable, as can already be seen when $p=1$ and $n=3$, but is easily compensated for by taking right-hand instead of ordinary derivatives.

The following theorem shows that the curvatures measure the rates of turning of the osculating planes, and therefore may equally well be defined in a manner which directly generalizes that of elementary calculus.

THEOREM 4.1. *The angle $\vartheta(s)$ between the osculating p-planes $P^p(s_0)$ and $P^p(s)$ has a right-hand derivative at s_0, and $\vartheta'(s_0)=k_p(s_0)$.*

We first isolate a short computation.

LEMMA 4.2. *Let G: $[s_0, s_1]\to R^m$ be a function such that $|G(s)|=1$ for all $s\in[s_0, s_1]$. Let $\vartheta(s)$ denote the angle between the vectors $G(s_0)$ and $G(s)$. If G has a (necessarily right-hand) derivative at s_0 then so does ϑ, and $\vartheta'(s_0)=|G'(s_0)|$.*

We simply capitalize on the geometric equality

$$| G(s) - G(s_0) | = 2 \sin \frac{\vartheta(s)}{2} \cdot$$

Now

$$\lim_{s \downarrow s_0} \frac{\vartheta(s) - \vartheta(s_0)}{s - s_0} = \lim_{s \downarrow s_0} \frac{\vartheta(s)}{s - s_0} = \lim_{s \downarrow s_0} \frac{2 \sin(\vartheta(s)/2)}{s - s_0}$$

$$= \lim_{s \downarrow s_0} \frac{| G(s) - G(s_0) |}{s - s_0} = \lim_{s \downarrow s_0} \left| \frac{G(s) - G(s_0)}{s - s_0} \right|$$

$$= \left| \lim_{s \downarrow s_0} \frac{G(s) - G(s_0)}{s - s_0} \right| = | G'(s_0) | ,$$

and the lemma follows. Note that the validity of the step

$$\lim_{s \downarrow s_0} \frac{| G(s) - G(s_0) |}{s - s_0} = \lim_{s \downarrow s_0} \left| \frac{G(s) - G(s_0)}{s - s_0} \right|$$

depends on the fact that $s - s_0$ is positive, and this is where the one-sidedness of the derivatives makes its formal appearance.

Proof of Theorem 4.1. Consider the p-vector

$$x(s) = V_1(s) \wedge V_2(s) \wedge \cdots \wedge V_p(s). \tag{4.3}$$

Since $\{ V_1(s), V_2(s), \cdots, V_p(s) \}$ is an orthonormal basis for the pth osculating subspace $P_0^p(s)$, the p-vector $x(s)$ representing $P^p(s)$ has unit length in $\wedge^p R^n$. It follows from Theorem 2.11 that, for s close to s_0, the angle $\vartheta(s)$ between $P^p(s_0)$ and $P^p(s)$ is the same as the angle between $x(s_0)$ and $x(s)$. We will show that

$$| x'(s_0) | = k_p(s_0), \tag{4.4}$$

and then Theorem 4.1 will follow immediately from Lemma 4.2.

Since $p \leqq r < k$, each $V_i(s)$, $1 \leqq i \leqq p$, is differentiable. Now differentiate both sides of equation (4.3). The exterior product $\wedge$ in the exterior algebra $\wedge R^n$ is bilinear, hence the ordinary rule for differentiating a product is valid. Thus

$$x'(s) = \sum_{i=1}^{p} V_1(s) \wedge \cdots \wedge V_{i-1}(s) \wedge V_i'(s) \wedge V_{i+1}(s) \wedge \cdots \wedge V_p(s).$$

The Frenet formulas, together with the alternating character of multiplication in $\wedge R^n$ ($V \wedge V = 0$ for all $V \in R^n$), imply that all but the last term in the above summation are 0. Thus

$$x'(s) = V_1(s) \wedge \cdots \wedge V_{p-1}(s) \wedge V_p'(s).$$

If $p < r$, or if $p = r$ and $k_r(s) > 0$, we use Frenet formula (3.2) or (3.3) to obtain

$$x'(s) = V_1(s) \wedge \cdots \wedge V_{p-1}(s) \wedge [-k_{p-1}(s)V_{p-1}(s) + k_p(s)V_{p+1}(s)]$$
$$= k_p(s)[V_1(s) \wedge \cdots \wedge V_{p-1}(s) \wedge V_{p+1}(s)].$$

If $p=r$ and $k_r(s)=0$, we use Frenet formula (3.4) to write

$$x'(s) = V_1(s) \wedge \cdots \wedge V_{p-1}(s) \wedge [-k_{p-1}(s)V_{p-1}(s)] = 0.$$

In either case,

$$|x'(s)| = k_p(s), \tag{4.5}$$

since $V_1(s) \wedge \cdots \wedge V_{p-1}(s) \wedge V_{p+1}(s)$ is a unit p-vector in $\wedge^p R^n$. Letting $s=s_0$ we get equation (4.4), and the theorem now follows from Lemma 4.2.

Note that equation (4.5) reduces, for $p=1$, to the familiar formula

$$|V_1'(s)| = k_1(s),$$

which says that the first curvature of a curve is the absolute value of the derivative of the unit tangent vector with respect to arc length. Also observe that while $|x'(s)| = k_p(s)$ for all $s \in I$, $\vartheta'(s)$ will generally equal $k_p(s)$ only at $s=s_0$. An appropriate example is easily constructible for $p=1$ and $n=3$. The angle between two lines in the plane is known once the signed angle that each makes with a fixed line is known, but nothing like this is true for p-planes in n-space.

Research partially supported by NSF grant GP-4433.

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ON GENERATING FUNCTIONS AND A FORMULA OF CHAUDHURI

DAVID ZEITLIN, Minneapolis, Minnesota

1. Introduction. Recently, Chaudhuri [1] obtained the generalized formula

$$F_n(x) = \left(\frac{1-x^2}{1-y^2}\right)^{n/2} \sum_{k=0}^{n} \binom{n}{k} \left[\frac{x(1-y^2)^{1/2} - y(1-x^2)^{1/2}}{(1-x^2)^{1/2}}\right]^{n-k} \cdot F_k(y), \tag{1.1}$$

where, if

$$E(u) = \sum_{n=0}^{\infty} u^n/n!, \qquad G(u) = \sum_{n=0}^{\infty} g_n u^n, \qquad g_n \neq 0, \tag{1.2}$$

then $F_n(x)$ is defined by the generating function

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ON SEMIGROUPS OF QUOTIENTS

B. M. PUTTASWAMAIAH, Carleton University, Ottawa

It was shown by Ore [1] that any right reversible cancellative semigroup can be embedded in a group. For an abelian semigroup this result can be generalized. The purpose of this note is to extend the later result for non-abelian case. This extension also applies for rings.

THEOREM 1. *Let G be a partially ordered semigroup and N be a subset in the centre of G such that $ax \leqq bx$ implies $a=b$, for any x in N. Then G can be embedded in a partially ordered semigroup with unity such that every element of N is positive.*

Proof. The semigroup S generated by N is contained in the centre of G and $ax \leqq bx$, $x \in S$ implies $a=b$. If for (a, s), (b, t) in $G \times S$, we define equality by $(a, s)=(b, t)$ if and only if $a=b$, $s=t$, and a relation by $(a, s) \sim (b, t)$ if and only if $at=bs$, then $\sim$ is an equivalence relation on $G \times S$. Let a/s be the equivalence class determined by (a, s). Let $G_S = \{a/s \mid a \in G, s \in S\}$. For a/s, b/t in G_S define $a/s \cdot b/t = ab/st$. This multiplication is well defined. For if $(a_1, s_1) \in a/s$ and $(b_1, t_1) \in b/t$, then $as_1 = a_1 s$ and $bt_1 = b_1 t$. These imply $a_1 b_1 st = ab s_1 t_1$, so that $(a_1 b_1, s_1 t_1) \sim (ab, st)$. Also multiplication is associative.

The element $s/s = \{(s, s) \mid s \in S\}$ is independent of s and

$$s/s \cdot b/t = sb/st = b/t = bs/ts = b/t \cdot s/s$$

for any b/t in G_S. Hence s/s is the unity of G_S.

The mapping $\phi: G \to G_S$ defined by $a\phi = sa/s$ is a homomorphism, since $ab\phi = sab/s = sa/s \cdot tb/t = a\phi \cdot b\phi$. If $a\phi = b\phi$, then $sa/s = tb/t$ implies $a = b$. Hence ϕ is a monomorphism. By identifying G with $G\phi$, we can regard G as being actually contained in G_S. An element of S (after identification) has the form st/s. But st/s, s/st are both members of G_S and satisfy $st/s \cdot s/st = s/s$. Thus G_S is a semigroup containing G such that every element of $N(\subset S)$ has an inverse in G_S.

For a/s, b/t in G_S, define $a/s \leqq b/t$ if and only if $at \leqq bs$. Again the relation $\leqq$ is well defined. For if $(a_1, s_1) \in a/s$, $(b_1, t_1) \in b/t$, then $a_1 s = a s_1$ and $b_1 t = b t_1$. Thus $a_1 st = a s_1 t \leqq b s s_1$; that is $a_1 t \leqq b s_1$ and $a_1 t t_1 \leqq b t_1 s_1 \leqq b_1 t s_1$. Hence $a_1 t_1 \leqq b_1 s_1$. Thus $\leqq$ is easily seen to be a partial order relation on G_S. If $a \leqq b$ in G, then $as \leqq bs$, so that $a/s \leqq b/s$ and conversely $as \leqq bs$ implies $a \leqq b$. Thus the partial order relation in G_S is an extension of the partial order relation in G.

Every element of S has an inverse. If P is the set of all elements of G_S which are greater than unity, then it is clear that every element of N is in P [2].

COROLLARY 1. *Every partially ordered abelian semigroup G with cancellation law can be embedded in a partially ordered abelian group such that every element of G is in the positive cone.*

COROLLARY 2. *If M is in the centre of a semigroup A and N is in the centre of semigroup B and if they satisfy the hypothesis of Theorem* 1, *then*

$$A_S \times B_T = (A \times B)_{S \times T}$$

where S is the semigroup generated by M, and T is the semigroup generated by N.

THEOREM 2. *Let N be a set of elements in the centre of a (multiplicative) semigroup G, such that $ax = bx$ implies $a = b$ for any x in N. Assume that G is also an additive abelian semigroup in which the right distributive law holds. Then G can be embedded in a (multiplicative) semigroup G_S with unity which is at the same time an additive abelian semigroup satisfying the right distributive law. If the cancellation law for addition holds in G then it holds in G_S. If the left distributive law holds in G, then it also holds in G_S.*

Proof. Let S be the semigroup generated by N. Then by Theorem 1, G can be embedded in a semigroup G_S with unity such that every element of N has an inverse. For a/s, b/t in G_S, define

$$a/s + b/t = at + bs/st.$$

This definition of addition is independent of the representatives of the equivalence classes. For if $(a_1, s_1) \in a/s$, $(b_1, t_1) \in b/t$, then $a_1 s = a s_1$ and $b_1 t = b t_1$ together imply $(a_1 t_1 + b_1 s_1)st = (at + bs)s_1 t_1$.

Next $(a/s + b/t)c/u = at + bs/st \cdot c/u = (at + bs)c/stu = act + bcs/stu = ac/su + bc/tu = a/s \cdot c/u + b/t \cdot c/u$. The other properties are similarly verified.

COROLLARY. *If G is an abelian semigroup with cancellation law in Theorem* 2,

it can be embedded in a semigroup with cancellation law which is also an additive abelian semigroup with cancellation law.

Example 1. Let G be the semigroup $\{r, 2r, 3r, \cdots\}$ where r is a natural number. If $N=\{r\}$, then $S=\{r^n \mid n=1, 2, 3, \cdots\}$ is a subsemigroup of G. Then G_S consists of all rational numbers of the form rm/r^n where $m, n=1, 2, 3, \cdots$.

THEOREM 3. *Let N be a set of elements in the centre of a (multiplicative) semigroup G, such that $ax=bx$ implies $a=b$ for any x in N. If G is an additive abelian semigroup with cancellation law satisfying the distributive laws, then G can be embedded in a ring $\overline{G}_S$ with unity such that every element of N has an inverse in the ring.*

Proof. Let S be the semigroup generated by N. Then S is in the centre of G and cancellation by elements of S holds. Let G_S be the semigroup constructed in Theorem 2. Every element of S has an inverse in G_S and G_S is an additive abelian semigroup with cancellation property. By Theorem 1, G_S can be embedded in an additive abelian group $\overline{G}_S$ as follows. For $(a/s, b/t)$, $(a'/s', b'/t')$ in $G_S \times G_S$, define $(a/s, b/t) \sim (a'/s', b'/t')$ if and only if $a/s+b'/t'=a'/s'+b/t$. Then $\sim$ is an equivalence relation on $G_S \times G_S$. Let $\overline{(a/s, b/t)}$ be the equivalence class determined by $(a/s, b/t)$ and $\overline{G}_S=\{\overline{(a/s, b/t)} \mid a, b \in G \text{ and } s, t \in S\}$. In $\overline{G}_S$ define

$$\overline{(a/s, b/t)} = \overline{(c/u, d/v)} = \overline{(a/s + c/u, b/t + d/v)}$$

and

$$\overline{(a/s, b/t)} \cdot \overline{(c/u, d/v)} = \overline{(ac/su + bd/tv, ad/sv + bc/tv)}.$$

The definitions of addition and multiplication are independent of the representatives of the equivalence classes. For if $(a_1/s_1, b_1/t_1) \in \overline{(a/s, b/t)}$ and $(c_1/u_1, d_1/v_1) \in \overline{(c/u, d/v)}$, then

(i) $$a_1/s_1 + b/t = a/s + b_1/t_1,$$

(ii) $$c_1/u_1 + d/v = c/u + d_1/v_1.$$

Adding (i) and (ii) one obtains

$$(a_1/s_1 + c_1/u_1) + (b/t + d/v) = (a/s + c/u) + (b_1/t_1 + d_1/v_1)$$

which implies that the addition is well defined. Again multiplying (i) by c_1/u_1 and d_1/v_1, and (ii) by a/s and b/t, then adding them, we get

$$(a_1c_1/s_1u_1 + b_1d_1/t_1v_1) + (ad/sv + bc/tu) = (ac/su + bd/tv) + (a_1d_1/s_1v_1 + b_1c_1/t_1u_1)$$

in view of the cancellation law. This proves that the multiplication does not depend on the representatives of the equivalence classes. It is easily verified that $\overline{G}_S$ is a ring, thus proving the theorem.

COROLLARY. *Let G be an abelian multiplicative semigroup with cancellation law, which is also an abelian additive semigroup with cancellation law, satisfying the distributive laws. Then G can be embedded in a field.*

Example 2. Let G be the semigroup of Example 1. Then it was seen that G_S consists of all rational numbers of the form rm/r^n where $m, n = 1, 2, 3 \cdots$. Then the ring $\overline{G}_S$ consists of all rational numbers of the form

$$rm/r^n$$

where $m = 0, \pm 1, \pm 2, \pm 3, \cdots$ and $n = 1, 2, 3, \cdots$.

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ON THE FUNCTIONAL INEQUALITY $f(x+y) \geqq f(x)f(y)$

J. E. WETZEL, University of Illinois

G. Pólya [3, p. 103] has given an elegant little proof* of the arithmetic-geometric mean inequality based on the inequality $e^x \geqq 1+x$, as follows: Let $a_1, a_2, \cdots, a_n$ be nonnegative real numbers,

$$A = \frac{1}{n} \sum_{k=1}^{n} a_k$$

their arithmetic mean, and $G = (\prod_{k=1}^{n} a_k)^{1/n}$ their geometric mean. The case $A = 0$ being trivial, we suppose that $A > 0$. Then for each $k = 1, 2, \cdots, n$,

$$\exp\left\{\frac{a_k}{A} - 1\right\} \geqq \frac{a_k}{A} \geqq 0.$$

Noting that the exponents on the left sum to 0, we multiply these n inequalities together and get

$$1 = e^0 = \exp\left\{\sum_{k=1}^{n}\left(\frac{a_k}{A} - 1\right)\right\} = \prod_{k=1}^{n} \exp\left\{\frac{a_k}{A} - 1\right\} \geqq \prod_{k=1}^{n} \frac{a_k}{A} = \frac{G^n}{A^n}$$

and $A \geqq G$. Since the equality $e^x = 1+x$ holds if and only if $x = 0$, the equality $A = G$ holds if and only if $a_k = A$ for each k.

The role of the exponential function in this argument could be played as well by any function f with the properties

* In a letter to the author dated January 31, 1967, Pólya wrote, "I had, especially in younger years, quite often mathematical dreams, but, usually, on awakening, the mirage dissolved and nothing tangible remained. The proof for the theorem of the means . . . is perhaps the only proof which I saw in a dream (when I was an undergraduate, perhaps 20 years old) and which on awakening turned out to be solid stuff." He has referred to it in conversations as "the best mathematics I ever dreamed."

$$f(x+y) \geqq f(x)f(y), \tag{1}$$

$$f(x) \geqq 1 + x. \tag{2}$$

The purpose of this note is to show that the *only* function f defined on an interval about 0 that satisfies both (1) and (2) on that interval is the function $f(x)=e^x$. An extension of this result and some related examples concerning the functional inequality (1) are also included.

THEOREM 1. *Let f be an arbitrary real function defined on an open interval I about 0 that satisfies* (1) *and* (2) *whenever x, y, and $x+y$ belong to I. Then $f(x)=e^x$ for each x in I.*

Proof. By (1), $f(x)=f((x/2)+(x/2))\geqq[f(x/2)]^2\geqq 0$ for each x in I. Thus if $f(x_0)=0$, then $f(x_0/2)=0$, and by induction, $f(2^{-n}x_0)=0$ for each positive integer n. But (2) demands that $f(x)>0$ near 0. It follows that $f(x)>0$ for every x in I.

Next we show that f must be differentiable at each point x of I and that $f'(x)=f(x)$. Using (1) and (2) we see that for small h,

$$f(x+h) - f(x) \geqq f(x)[f(h) - 1] \geqq hf(x).$$

Since $f(x)=f(x+h-h)\geqq f(x+h)f(-h)\geqq f(x+h)(1-h)$, we have similarly for small h,

$$f(x+h) - f(x) \leqq f(x)\left[\frac{1}{1-h} - 1\right] = h\,\frac{f(x)}{1-h}\,.$$

Thus for small, positive h,

$$f(x) \leqq \frac{f(x+h) - f(x)}{h} \leqq \frac{f(x)}{1-h}$$

and the reverse inequalities hold for small, negative h. Hence

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists and equals $f(x)$ for each x in I.

It follows that

$$\frac{d}{dx}\, e^{-x}f(x) = e^{-x}[f'(x) - f(x)] = 0$$

for every x in I, and so $f(x)=Ce^x$ for some constant C. Since (1) implies $f(0)\leqq 1$ and (2) implies $f(0)\geqq 1$, we have $C=f(0)=1$, proving the theorem.

We can generalize this result somewhat by relaxing condition (2).

THEOREM 2. *Let f be a function defined on an open interval I about* 0 *such that* (1) *holds whenever x, y, and $x+y$ belong to I. If f is bounded below by a function*

g that is differentiable at 0 *and satisfies* $g(0)=1$, *then* $f(x)=e^{kx}$ *for all* x *in* I, *where* $k=g'(0)$.

Proof. For convenience, we put $g(x)=1+\phi(x)$, so that $\phi(0)=0$, $\phi'(0)=g'(0)=k$, and $f(x)\geqq 1+\phi(x)$ for all x in I. Then precisely the same sequence of steps used in the proof of Theorem 1 yields the inequalities

$$\frac{\phi(h)}{h}f(x) \leqq \frac{f(x+h)-f(x)}{h} \leqq \frac{f(x)}{1+\phi(-h)}\,\frac{\phi(-h)}{-h}$$

for small, positive h and the reverse inequalities for small, negative h. It follows that f is differentiable, and $f'(x)=kf(x)$ for each x in I. Then $e^{-kx}f(x)$ has derivative zero on I and is therefore a constant C, and $f(x)=Ce^{kx}$. Since $C=f(0)=1$, the theorem is proved.

We noted above that $0\leqq f(0)\leqq 1$ is an immediate consequence of (1). Included in the theorem is the interesting assertion that the only solutions of (1) that are differentiable at 0 and take the extreme value $f(0)=1$ are exponentials.

COROLLARY 1. *If* f *satisfies* (1) *on* I, *if* $f(0)=1$, *and if* f *is differentiable at* 0, *then* $f(x)=e^{kx}$ *for all* x *in* I, *where* $k=f'(0)$.

The assumption in the hypotheses of this corollary that f be differentiable at 0 cannot be weakened to continuity, for the continuous function $f(x)=(1+|x|)^{-1}$ satisfies (1) for all x and y and fails to be differentiable only at $x=0$.

The situation should be compared with the state of affairs for the functional equation $f(x+y)=f(x)f(y)$. If f satisfies this equality for all real x and y, then the function $g=\log f$ satisfies the linear functional equation $g(x+y)=g(x)+g(y)$; and if f is assumed to be continuous (even at just one point), or if f is assumed to be bounded on some interval, or even if the graph of f is assumed not to be everywhere dense in the upper half-plane, then $g(x)=kx$ and $f(x)=e^{kx}$ for every x (see [1], p. 108ff. and the literature cited there for additional details and proofs).

No result similar to that of Corollary 1 can be expected in the other extreme case $f(0)=0$; *any* function f that vanishes for all nonpositive x, is monotonically increasing, and is bounded above by 1, satisfies (1). A more interesting example can be constructed as follows. Let h be any differentiable function on [5, 7] that satisfies $0\leqq h(x)\leqq 0.9$ and $h(5)=h'(5)=h(7)=h'(7)=0$. Define

$$f(x)=\begin{cases} 0 & \text{for } x<5 \\ h(x) & \text{for } 5\leqq x\leqq 7 \\ 0 & \text{for } 7<x<8 \\ 1-\exp\{-(8-x)^2\} & \text{for } x\geqq 8 \end{cases}$$

Then f is a differentiable function with $f(0)=0$ that satisfies (1) for all x and y.

Finally, it is worth recasting Theorem 2 as a result about the functional inequality

$$F(x+y) \leqq F(x) + F(y) \tag{3}$$

whose solutions were studied by Cooper [2, p. 428f.].

COROLLARY 2. *Let F be a function defined on an open interval I about 0 that satisfies (3) whenever x, y, and $x+y$ belong to I. If F is bounded above by a function G that is differentiable at 0 and satisfies $G(0)=0$ (and, in particular, if F is differentiable at 0 and satisfies $F(0)=0$), then $F(x)=kx$ for all x in I, where k is a constant.*

Proof. Apply the theorem to $f(x)=\exp\{-F(x)\}$ with $g(x)=\exp\{-G(x)\}$.

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GENERAL SOLUTION OF "ISOMOMENT" FUNCTIONAL EQUATIONS

J. ACZEL, University of Waterloo, Canada

1. In [7] S. Kotz has found that all *continuous* solutions of the "isomoment" functional equation:

$$f\left(\sum_{i=1}^{n} x_i^m/n\right) = \sum_{i=1}^{n} f(x_i)^m/n, \tag{1}$$

satisfied for all $x_i \geqq 0$ $(i=1, 2, \cdots, n)$ and for fixed integers $m>1$, $n>1$, are

$$\begin{aligned} &f(x) = 0,\ f(x) = 1,\ f(x) = x; \\ &\text{and if } m \text{ is odd also } f(x) = -1,\ f(x) = -x. \end{aligned} \tag{2}$$

In the present note we show that—even without supposing continuity or any other regularity-property—the functions (2) *are the most general real solutions of* (1) (for all $x_i \geqq 0$ and for fixed integers $m>1$, $n>1$), or that *all real solutions of* (1) *are continuous* (which is *not* the case if $m=1$).

2. First we observe, that (1) can be written with

$$x_i^m/n = y_i\ (i = 1, 2, \cdots, n), \qquad f((ny)^{1/m})^m/n = g(y) \tag{3}$$

Now, by (10), (13) and (14) for all nonnegative x and for all nonnegative rational r, s

$$(15)\quad \begin{aligned} a((rx+s)^m) &= (a(rx+s)+b)^m - b = (ra(x)+sa(1)+b)^m - b \\ &= r^m a(x)^m + mr^{m-1}sa(x)^{m-1}a(1) + mr^{m-1}a(x)^{m-1}b + \cdots + (sa(1)+b)^m - b \end{aligned}$$

and also

$$(16)\quad \begin{aligned} a((rx+s)^m) &= a(r^m x^m + mr^{m-1}sx^{m-1} + \cdots + s^m) \\ &= r^m((a(x)+b)^m - b) + mr^{m-1}sa(x^{m-1}) + \cdots + s^m a(1). \end{aligned}$$

Since r, s are *arbitrary* nonnegative rationals, the coefficients of $r^{m-1}s$ and similarly those of r^{m-1} have to be equal in both expressions (15) and (16):

$$(17)\qquad ma(x^{m-1}) = ma(x)^{m-1}a(1)$$

and

$$(18)\qquad ma(x)^{m-1}b = 0.$$

If $m>1$, equation (18) shows that *either* $a(x)=0$ *or* $b=0$ while (17) shows that a *is bounded* (from below or above or 0 depending upon whether $a(1)$ is positive, negative or 0, respectively) if m is *odd* (because $a(x)^{m-1}\geqq 0$). Equation (13) shows also that a *is bounded* from below if m is even. But *every bounded solution of* (10) *is continuous* and

$$(19)\qquad a(x) = ax$$

with constant a (see, e.g., [2]).

So the following solutions remain possible:

(i) $a(x)=0$ and (by (13)) $b^m=b$, i.e., $b=0$ or $b=1$ or, if m is odd, $b=-1$ and thus by (11)

$$(20)\qquad f(x) = 0, \qquad (21)\qquad f(x) = 1$$

and in case of odd $m>1$

$$(22)\qquad f(x) = -1.$$

(ii) $b=0, f(x)=a(x)=ax$ (by (11) and (19)) and by (13) $ay^m=a^m y^m$ so $a^m=a$, i.e., either $a=0$ (which was taken care of already in case (i)) or $a=1$ or, if m is odd, also $a=-1$, thus

$$(23)\qquad f(x) = x$$

and in case of odd $m>1$ also

$$(24)\qquad f(x) = -x$$

are solutions of (1) and only these.—We summarize our results:

THEOREM. *All real solutions of the functional equation* (1)— *supposed valid for all nonnegative* $x_i (i=1, 2, \cdots, n)$ *and for fixed integers* $m>1$, $n>1$—*are con-*

tinuous. The functions (20), (21), (23), *and in case of odd m, also* (22) *and* (24) *satisfy* (1) *and only these.*

The result in case (ii) can also be formulated so:

COROLLARY. *All real solutions of the system (with fixed integer* $m>1$)

$$f(y_1+y_2)=f(y_1)+f(y_2) \quad \text{and} \quad f(x^m)=f(x)^m \tag{25}$$

of functional equations—where $x>0$ *while* y_1, y_2 *may run through all real or all nonnegative numbers—are continuous; they are* (20), (23) *and in case of odd m also* (24), *and only these.*

3. REMARK 1. *Neither the theorem nor the corollary is true if* $m=1$; because then (12) and (13) are trivially satisfied and so every function of the form (11) with $a(y)$ satisfying only (10) is a solution of (1) and every solution of (10) with $f(y)=a(y)$ satisfies the system (25) too. (10) has, however, also noncontinuous solutions for all real or for all nonnegative y_1, y_2 (see, e.g., [2], [3]).

REMARK 2. *The Lemma, Theorem, Corollary and Remark* 1 *remain true, if in their statements "nonnegative" is replaced by "positive."* In fact, 0 was substituted into (4) only in the proof of the lemma, but the lemma remains true also if 0 is not in the set of validity of (4), (see, e.g., [1], [2], [4]).

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INEQUALITIES FOR ELLIPTIC FUNCTIONS

G. D. ANDERSON, Michigan State University

In recent papers in this MONTHLY P. L. Duren [4] and S. Fempl [6] have called attention to the apparent paucity of known inequalities involving elliptic functions. In this note, using well-known infinite product expansions, we compare the values of elliptic functions with those of closely associated trigonometric functions. We employ a similar method to present an inequality involving elliptic integrals. Most of these relations were encountered in the study of quasiconformal mappings in 3-space [1].

We show that the following inequalities hold for $0 \leqq k < 1$ and α real:

$$(1)\quad \frac{1}{2}\left(1 + \frac{2K}{\pi}\right) \leqq \coth\frac{\pi K'}{2K} \leqq \frac{2K}{\pi}, \qquad (3)\quad 1 \leqq \frac{sn((2K/\pi)\alpha, k)}{\sin\alpha} \leqq \frac{2K}{\pi},$$

$$(2)\quad \frac{2k'K}{\pi} \leqq \frac{cn((2K/\pi)\alpha, k)}{\cos\alpha} \leqq 1, \qquad (4)\quad \frac{2K}{\pi} \leqq \frac{tn((2K/\pi)\alpha, k)}{\tan\alpha} \leqq \frac{\pi}{2k'K},$$

$$(5)\quad 1 - k \leqq \frac{\sin\alpha}{sn((2K/\pi)\alpha, k)}\left[1 - k\, sn^2\left(\frac{2K}{\pi}\alpha, k\right)\right] \leqq \frac{\pi}{2K},$$

$$(6)\quad \frac{dn^2((2K/\pi)\alpha, k)}{cn^2((2K/\pi)\alpha, k)} \frac{\cos^2\alpha \cosh^2(\pi K'/2K)}{\cos^2\alpha + \sinh^2(\pi K'/2K)} \leqq 1,$$

where k is the modulus of the given elliptic functions and

$$K = \int_0^1 [(1 - t^2)(1 - k^2t^2)]^{-1/2}dt,$$

$$K' = K(k'), \qquad k' = (1 - k^2)^{1/2}.$$

By q we shall always mean Jacobi's Nome, $q = \exp(-\pi K'/K)$. The Nome increases with k, and $0 \leqq q < 1$ when $0 \leqq k < 1$.

To prove (1) we begin with the obvious series expansion

$$(7)\quad \coth\frac{\pi K'}{2K} = \frac{1+q}{1-q} = 1 + 2\sum_{n=1}^{\infty} q^n,$$

for $0 \leqq k < 1$. Next, it follows from the Fourier series for $dn\, u$ ([3], #908.03, [7], p. 286) that

$$(8)\quad \frac{2K}{\pi} = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}$$

for $0 \leqq k < 1$. Thus by (7) and (8) we see that

$$\frac{1}{2}\left(1 + \frac{2K}{\pi}\right) = 1 + 2\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \leqq 1 + 2\sum_{n=1}^{\infty} q^n = \coth\frac{\pi K'}{2K}$$

From this we easily deduce that

$$\frac{1}{sn^2((2K/\pi)\alpha, k)} \frac{\sin^2 \alpha \cosh^2 (\pi K'/2K)}{\sin^2 \alpha + \sinh^2 (\pi K'/2K)} \leqq 1.$$

Replacing α by $\alpha+\pi/2$ in this expression and using the identity $sn(u+K) = (cn\ u)/(dn\ u)$ ([2], p. 13, [3], #122.03) yields (6). There is equality when $\alpha=\pi/2$.

This work was supported in part by the National Science Foundation, Grant 18913. The material is included in a Ph.D. thesis written under the direction of Professor F. W. Gehring at the University of Michigan.

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THE GENERAL SOLUTION OF AN ARBITRARY BOOLEAN EQUATION

HELEN SKALA, Illinois Institute of Technology

If B is a Boolean algebra with operations union $\cup$, intersection $\cap$ and complementation $'$, then it is well known (cf., e.g., [6]) that B is a ring with regard to addition and multiplication defined as follows:

$$a + b = (a \cap b') \cup (a' \cap b) \quad \text{and} \quad ab = a \cap b.$$

Commutativity and associativity of addition (usually called symmetric difference) and distributivity of multiplication over addition are easily shown. The zero element of B considered as a ring is the zero element of B corresponding to union, i.e., the element 0 such that $0 \cup a = a$ for all a in B. For $a+0 = (a \cap 0') \cup (a' \cap 0) = a$. The additive inverse of any element a is a, and the complement of a is $1+a$, where 1 is the identity of B corresponding to intersection, i.e., the element such that $1 \cap a = a$ for all a in B. Union may be expressed in terms of addition and multiplication since

$$(1) \qquad a \cup b = (a' \cap b')' = 1 + (1 + a)(1 + b) = a + b + ab.$$

From this we easily deduce that

$$\frac{1}{sn^2((2K/\pi)\alpha, k)} \frac{\sin^2 \alpha \cosh^2 (\pi K'/2K)}{\sin^2 \alpha + \sinh^2 (\pi K'/2K)} \leqq 1.$$

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$$(1) \qquad a \cup b = (a' \cap b')' = 1 + (1 + a)(1 + b) = a + b + ab.$$

The ring properties of a Boolean algebra may be exploited with great advantage in the theory of Boolean equations. For in absence of cancellation laws for union and intersection, the cancellation law for addition becomes highly significant in simplifying and solving Boolean equations. Conditions for unique solvability of Boolean equations, for instance, have been obtained by Parker and Bernstein (cf. [4]) in terms of the ring operations of a Boolean algebra.

Frequently, however, one is interested in determining *all* solutions to some Boolean equation. One description of the solutions of an equation in n unknowns has been in parametric form, that is, by means of a family of n Boolean functions $\{f_i; i=1, \cdots, n\}$ satisfying the conditions:

(1) For every $t_1^*, \cdots, t_n^*$ in B, the elements $x_i^*=f_i(t_1^*, \cdots, t_n^*)$ is a solution of the given equation.

(2) If $(x_1^*, \cdots, x_n^*)$ is a solution of the given equation then there exist $t_1^*, \cdots, t_n^*$ in B such that $x_i^*=f_i(t_1^*, \cdots, t_n^*)$.

For example, the equation $x \cup a=1$ has the general solution $x=a' \cup k$ where k is an arbitrary element. Numerous examples of various types of Boolean equations with their general solutions may be found in [1]. More generally, Motinori Gotō in [2] has enumerated all solutions of an arbitrary equation in the two-element Boolean algebras and in [3] extended his results to the case of finite Boolean algebras. Sergei Rudeanu (cf. [5]) then proved that the results obtained in [3] are still valid in any Boolean algebra. Rudeanu's method of solution, essentially due to Gotō, was to first express the equation in complete disjunctive form and then to determine the family of functions mentioned above in terms of the coefficients appearing in the disjunctive form.

Similar results may be obtained in a very easy manner by making use of the ring properties of a Boolean algebra. By (1), each Boolean equation may be written using only the operations addition and multiplication. By use of the distributive and cancellation laws, any Boolean equation in one unknown x may be written as

$$ax = b. \tag{2}$$

For example, the equation $cx \cup dx'=ex$ which in view of $cx \cup dx'=cx+dx'$ $cdxx'=cx+d(1+x)=(c+d)x+d$ is equivalent to $(c+d+e)x=d$. So in order to solve the general Boolean equation in one unknown one need only find all solutions of (2). In order that (2) have a solution, it is necessary and sufficient that

$$ab = b. \tag{3}$$

If (3) is satisfied, all solutions of (2) will be given by

$$x = a'k + b, \tag{4}$$

where k is an arbitrary element of B. That (4) is indeed a solution is readily verified since $a(a'k+b)=ab=b$. Now suppose x^* is a solution of (2). Then letting $k=x^*+b$, we have

$$a'(x^* + b) + b = a'x^* + a'b + b = (a + 1)x^* + (a + 1)b + b = ax^* + x^* + ab = x^*$$

$$x(a + aby) = c + by. \tag{7}$$

Solving first for x one obtains

$$x = (a + aby)'k_1 + c + by. \tag{8}$$

The necessary condition (3) becomes $y(abc+b) = ac+c$ whose solutions are given by

$$y = (abc + b)'k_2 + ac + c \tag{9}$$

provided

$$(abc + b)(ac + c) = ac + c. \tag{10}$$

Substituting in (8) the expression (9) for y, one obtains with the help of (10)

$$x = (a' + abck_2)k_1 + abck_2 + ac. \tag{11}$$

So whenever (10) is satisfied (7) will have solutions, namely, x and y given by (9) and (11), respectively.

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MATHEMATICAL NOTES

MATRIX EXPRESSIONS IN RIGID BODY MECHANICS

M. N. Brearley, RAAF Academy, Victoria

In a recent Mathematical Education Note in this Monthly [1], R. K. Jarvis drew attention to the shortage of realistic applications of matrix theory at a level comprehensible to high school students, and appealed for examples of such applications. This note describes applications in mechanics which can be appreciated by a student whose knowledge of vector algebra extends only as far as the definitions of vector product and of the moment of a localized vector about a point. It also provides a method of bypassing the conventional dyadic or tensor treatment of the "inertia tensor" encountered when considering the rotation of a rigid body about a fixed point. The kinetic energy and angular

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where $I=-\Sigma mS_r^2$. In this notation the appropriate name for I is the inertia matrix of the body at O. It is easily verified that I has the expected component form

$$I=\begin{bmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{bmatrix}$$

in terms of the usual moments and products of inertia of the body with respect to the axes at O. In more conventional notations I is called the inertia dyadic or inertia tensor at O [2], and in dyadic notation (3) would be written

$$T=\tfrac{1}{2}\omega\cdot I\cdot\omega.$$

The transformation of axes by rotation to principal axes of the body at O may now be discussed in terms of the diagonalization of the matrix I by orthogonal transformation.

The angular momentum about O of the rotating body. This is denoted and defined by the moment sum

$$\boldsymbol{h}=\Sigma\boldsymbol{r}\times m\boldsymbol{v}=-\Sigma m\boldsymbol{r}\times(\boldsymbol{r}\times\omega).$$

Using the representation (1) this becomes

$$\begin{aligned}\boldsymbol{h} &= -\Sigma mS_r(S_r\omega),\\ &= -(\Sigma mS_r^2)\omega,\\ &= I\omega,\end{aligned}$$

where I is the inertia matrix at O. The corresponding dyadic notation is $\boldsymbol{h}=I\cdot\omega$.

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A NOTE ON DIRECT PRODUCTS OF FREE MODULES

J. S. HAINES, New Mexico State University

An elegant characterization of those rings R such that the direct product of any family of projective R-modules is projective is given by S. Chase in [1]. (All rings have identities and all modules are left unitary.) However, Chase noted [1, page 467] that it was not known whether such a ring R had the property that the direct product of any family of free R-modules was free. This question turns out to be largely set-theoretical, and is settled in the negative in this note.

Let R and S be rings, $R\oplus S=T$ be the ring direct sum, and $\Pi_\alpha T$ ($\Sigma_\alpha T$) the direct product (sum) of α copies of T, with everything considered as a T-module.

If $\Pi_\alpha T$ is free, then $\Pi_\alpha T \approx \Sigma_\beta T$ for some cardinal β. Letting $|X|$ denote the cardinal of the set X, and α, R, and S be infinite, then

$$|R(\Pi_\alpha T)| = |\Pi_\alpha R| = |R|^\alpha = |R\Sigma_\beta T| = |\Sigma_\beta R| = \beta|R|, \text{ and } |S|^\alpha = \beta|S|.$$

Thus to answer the question in the negative, it suffices to find infinite rings R and S, and an infinite cardinal α such that products of projective T-modules are projective, but such that the equations $|R|^\alpha = \beta|R|$ and $|S|^\alpha = \beta|S|$ have no simultaneous solution for β. This can be accomplished by letting R be a field with $|R| = \chi_\omega$ (such a field clearly exists), S be a field with $|S| = \chi_0$, and $\alpha = \chi_0$. For then, $|R|^{\chi_0} > |R|$, β must be $|R|^{\chi_0}$, but $|R|^{\chi_0}|S| > |S|^{\chi_0}$. Since products of projective $R \oplus S$ modules are projective [1, Theorem 3.4], $B \oplus S$ is the desired example.

This work was supported by NSF GP-377.

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1. S. U. Chase, Direct products of modules, Trans. Amer. Math. Soc., 3 (1960) 457–473.

A CURIOSITY IN MEASURE THEORY

MAURICE MACHOVER, St. John's University

Below is presented an interesting curiosity in the theory of Lebesgue Measure in E^n. The result may be formulated as follows: If the outer measure of two sets is additive, then so is their inner measure; but the converse is not true. In addition to the usual definitions, the proof is based on the following properties of outer and inner measure, all of which may be found in [1] (all sets are assumed to have finite outer measure):

(1) If $m^*(E_1 \cap E_2) = 0$, then

$$m^*(E_1) + m^*(E_2) \geqq m^*(E_1 \cup E_2) \geqq \begin{matrix} m_*(E_1) + m^*(E_2) \\ m^*(E_1) + m_*(E_2) \end{matrix} \geqq m_*(E_1 \cup E_2)$$
$$\geqq m_*(E_1) + m_*(E_2).$$

(2) If E is any set, there are measurable sets F and G such that

$$F \subseteq E \subseteq G \quad \text{and} \quad m(F) = m_*(E),\ m(G) = m^*(E).$$

(3) If E_1 and E_2 are measurable, $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$. It is immediately clear from (1) that addition or subtraction of a set of measure zero does not alter the outer or inner measure of a set.

LEMMA. *Let E_1 and E_2 be given sets. Then $m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2)$ if and only if there exist measurable sets M_1 and M_2 such that $E_1 \subseteq M_1$, $E_2 \subseteq M_2$, and $m(M_1 \cap M_2) = 0$.*

If $\Pi_\alpha T$ is free, then $\Pi_\alpha T \approx \Sigma_\beta T$ for some cardinal β. Letting $|X|$ denote the cardinal of the set X, and α, R, and S be infinite, then

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Proof. Suppose such sets M_1 and M_2 exist. Given $\epsilon>0$, choose O, open, $\ni (E_1\cup E_2)\subseteq O$ and $m(O)<m^*(E_1\cup E_2)+\epsilon$. Now

$$E_1 \subseteq M_1 \cap O,\ E_2 \subseteq M_2 \cap O, \quad \text{and} \quad m[(M_1 \cap O) \cap (M_2 \cap O)] = 0.$$

Thus

$$\begin{aligned} m^*(E_1) + m^*(E_2) &\leqq m(M_1 \cap O) + m(M_2 \cap O) = m[(M_1 \cap O) \cup (M_2 \cap O)] \\ &= m[(M_1 \cup M_2) \cap O] \leqq m(O) < m^*(E_1 \cup E_2) + \epsilon \\ &\leqq m^*(E_1) + m^*(E_2) + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, $m^*(E_1)+m^*(E_2)=m^*(E_1\cup E_2)$.

Conversely, suppose $m^*(E_1)+m^*(E_2)=m^*(E_1\cup E_2)$. Let M_1 and M_2 be measurable $\ni E_1\subseteq M_1$, $E_2\subseteq M_2$, and $m(M_1)=m^*(E_1)$, $m(M_2)=m^*(E_2)$. Suppose $m(M_1\cap M_2)>0$. Then

$$\begin{aligned} m^*(E_1 \cup E_2) &= m^*(E_1) + m^*(E_2) = m(M_1) + m(M_2) \\ &= m(M_1 \cup M_2) + m(M_1 \cap M_2) > m(M_1 \cup M_2) \geqq m^*(E_1 \cup E_2), \end{aligned}$$

which is contradictory.

The main result of the paper can now be established.

THEOREM. *If* $m^*(E_1\cup E_2)=m^*(E_1)+m^*(E_2)$, *then* $m_*(E_1\cup E_2)=m_*(E_1)+m_*(E_2)$.

Proof. By the lemma $\exists M_1$, M_2, measurable, $\ni E_1\subseteq M_1$, $E_2\subseteq M_2$, and $m(M_1\cap M_2)=0$. Given $\epsilon>0$, choose Q, closed, $\ni Q\subseteq(E_1\cup E_2)$ and $m(Q)>m_*(E_1\cup E_2)-\epsilon$. Now $Q\cap M_1\subseteq E_1\cup(M_1\cap M_2)$, $Q\cap M_2\subseteq E_2\cup(M_1\cap M_2)$. Since $m[(Q\cap M_1)\cap(Q\cap M_2)]=0$, it follows that

$$\begin{aligned} m_*(E_1) + m_*(E_2) - \epsilon &\leqq m_*(E_1 \cup E_2) - \epsilon < m(Q) = m[Q \cap (M_1 \cup M_2)] \\ &= m[(Q \cap M_1) \cup (Q \cap M_2)] = m(Q \cap M_1) + m(Q \cap M_2) \\ &\leqq m_*[E_1 \cup (M_1 \cap M_2)] + m_*[E_2 \cup (M_1 \cap M_2)] \\ &= m_*(E_1) + m_*(E_2). \end{aligned}$$

Since ϵ is arbitrary, $m_*(E_1)+m_*(E_2)=m_*(E_1\cup E_2)$.

The converse of this theorem is not true, as the following counter-example will show. Let $I=[1, 2]$, $J=[5, 6]$. Let N be a nonmeasurable set of inner measure zero, and contained in $[3, 4]$. Then the sets $E_1=I\cup N$, $E_2=J\cup N$ constitute a counter-example.

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DERIVATIVES OF THE JACOBI ELLIPTIC FUNCTIONS

D. A. PELLETT, U. S. Naval Postgraduate School, Monterey, California

Although it is not at all difficult to obtain the derivative of any of the twelve (Glaisher's notation) Jacobi elliptic functions, the following quickly memorized mnemonic device provides the result instantly. It is based on the following antisymmetric array

	c'	d'	n'	s'
c	0	$-m_1$	-1	-1
d	m_1	0	$-m$	-1
n	1	m	0	-1
s	1	1	1	0

in which the row and column indices appear in alphabetic order. Here m is the parameter, $m=k^2$, and $m_1=1-m$ is the complementary parameter. Prime denotes differentiation with respect to argument. Argument and parameter (or modulus) need not be displayed, but we will do so in the first example so as to eliminate any confusion in the mind of the reader. Let α, β, γ, δ stand for the letters c, d, n, s in any order. Then

$$\alpha\beta' = A_{\alpha\beta'}\cdot\gamma\beta\cdot\delta\beta,$$

where $A_{\alpha\beta'}$ is the element in the α-row β'-column of the array.

Example 1. $(d/dx)ds(x|m)=ds'=-1\cdot cs\cdot ns=-cs(x|m)ns(x|m)$

Example 2. $dc'=m_1\cdot nc\cdot sc$.

Memorizing the array is aided by noting that the subdiagonal is "em-one, em, one."

A CHARACTERIZATION OF ORTHOGONAL MATRICES

K. A. BUSH, Washington State University

The following elementary theorem has been of occasional use to the author in trying to decide whether a matrix about which some information is available but where the precise matrix is not known could possibly be orthogonal, and it is apparently not in the literature.

THEOREM. *An n-square real matrix $A=(a_{ij})$ is orthogonal if and only if, for any subsets L and M of $(1, \cdots, n)$,*

$$\sum_{j=1}^{n}\Big(\sum_{i\in L} a_{ij}\Big)\Big(\sum_{i\in M} a_{ij}\Big) = p, \tag{1}$$

where p is the number of elements in $L\cap M$.

Proof. If (1) holds, then choosing L and M as singletons we have that the

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where $A_{\alpha\beta'}$ is the element in the α-row β'-column of the array.

Example 1. $(d/dx)ds(x|m)=ds'=-1\cdot cs\cdot ns=-cs(x|m)ns(x|m)$

Example 2. $dc'=m_1\cdot nc\cdot sc$.

Memorizing the array is aided by noting that the subdiagonal is "em-one, em, one."

A CHARACTERIZATION OF ORTHOGONAL MATRICES

K. A. BUSH, Washington State University

The following elementary theorem has been of occasional use to the author in trying to decide whether a matrix about which some information is available but where the precise matrix is not known could possibly be orthogonal, and it is apparently not in the literature.

THEOREM. *An n-square real matrix $A=(a_{ij})$ is orthogonal if and only if, for any subsets L and M of* $(1, \cdots, n)$,

$$\sum_{j=1}^{n}\Big(\sum_{i\in L} a_{ij}\Big)\Big(\sum_{i\in M} a_{ij}\Big) = p, \tag{1}$$

where p is the number of elements in $L\cap M$.

Proof. If (1) holds, then choosing L and M as singletons we have that the

inner product is zero unless the two choices coincide while it is one if the choices agree. Now assuming that A is orthogonal we reverse the order of summation in (1). We secure:

$$\sum_{k\in M}\sum_{i\in L}\sum_{j} a_{ij}a_{kj}.$$

The summation on j yields the element in the (i, k) position in AA^T where A^T is the transpose of A. The summations over the index sets therefore yield the sum of the elements in that submatrix of AA^T whose rows are the rows in the index set L and whose columns are the columns in the index set M. But A orthogonal implies that this is a submatrix of the unit matrix, and therefore the only non-zero elements are on the main diagonal or precisely the number of common elements in the two index sets. We note that a similar theorem holds if the roles of rows and columns are interchanged by applying (1) to A^T.

A SET OF LINEARLY INDEPENDENT PERMUTATION MATRICES

G. R. BLAKLEY, SUNY at Buffalo, W. E. COPPAGE, Wright State University,
R. D. DIXON, Wright State University

The set Ω_n of doubly stochastic n by n matrices is a convex polyhedron whose vertices are the n by n permutation matrices. The maximal dimension of any simplex in Ω_n is $(n-1)^2$ since Ω_n is, in the n^2 dimensional real vector space of all n by n matrices, a subset of a translate of an $(n-1)^2$ dimensional subspace D_n^*, the set of matrices with row and column sums equal to zero. M. Marcus and H. Minc [1] give an example of such an $(n-1)^2$ dimensional simplex. It seems natural to seek a maximal dimensional simplex whose vertices are also vertices of the polyhedron Ω_n. In this note we exhibit $(n-1)^2+1$ permutation matrices which are linearly independent, an even stronger property than that of being the vertices of an $(n-1)^2$-simplex.

We number the positions in a matrix of order m consecutively from 1 to m^2, referring to the position in row i and column j as the $[i+(j-i)m]$-th (mod m^2) position. This procedure assigns distinct numbers mod m^2 to distinct positions since if

$$i_1 + (j_1 - i_1)m \equiv i_2 + (j_2 - i_2)m \mod m^2$$

then $i_1 \equiv i_2 \mod m$, whence $i_1 = i_2$. Hence $j_1 m \equiv j_2 m \mod m^2$, implying $j_1 \equiv j_2 \mod m$. Consequently $j_1 = j_2$. If $m = 4$ the positions are numbered by

1	5	9	13
14	2	6	10
11	15	3	7
8	12	16	4

MORE HIGHLY COMPOSITE CONSECUTIVE INTEGERS

D. P. GIESY, University of Southern California

In [1] Subramanian and Becker show that for every n there are infinitely many sequences of n consecutive integers in which each integer contains a preassigned number of pairwise relatively prime factors, each factor to a preassigned power. I improve this result by noting that the (highly composite) factors of the consecutive integers can themselves be preassigned subject to certain limitations. In fact, the result is an immediate consequence of a well known Chinese remainder type

THEOREM. *The system* $x \equiv c_i \pmod{m_i}$ *for* $i = 1, \cdots, n$ *is solvable if and only if*

$$(m_i, m_j) \mid (c_i - c_j)$$

for $i, j = 1, \cdots, n$. *The solution, if it exists, is unique modulo the least common multiple of* $m_1, \cdots, m_n$.

(See [2] p. 34, Theorem 3–12.) The result follows by letting $c_i = -i$ in the theorem. Appropriate choices of the m_i's (for example, pairwise relatively prime) then furnish the required highly composite factors.

References

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AN INFINITE CLASS OF HYPOHAMILTONIAN GRAPHS

W. F. LINDGREN, South Dakota School of Mines and Technology

We consider only finite, undirected, connected graphs. A circuit is called *simple* provided it has at most two edges incident with any given vertex. A simple circuit is a *hamiltonian circuit* (abbreviated *hc*) if it is incident with every vertex of the graph under consideration. A graph is *hypohamiltonian* (abbreviated *HH*) provided it has no hamiltonian circuit, but every subgraph obtained by deleting any one vertex and all edges incident with that vertex has such a circuit [1], [2]. We denote an edge between p_1 and p_2 by (p_1, p_2).

It has been shown that an *HH* graph of minimum order has 10 vertices, and further, that the solution is unique. We shall give a method for constructing an *HH* graph having an arbitrarily larger number of vertices.

Consider a regular polygon P with $6K+3$ vertices (K positive) numbered in counterclockwise order. Next add a point in the center of P and label it $c = 6K+4$. Now add all edges of the form $(c, 1+3j)$, where $j = 0, 1, \cdots, 2K$; these edges we shall call *spokes*. Next draw all edges of the form $(2+3j, 6+3j)$, $(j = 0, 1, 2, \cdots, 2K-1)$, and add $(c-2, 3)$. These edges we shall call *chords*.

There are the following types of edges: 1. $2K+1$ spokes, 2. $2(2K+1)$ edges

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There are the following types of edges: 1. $2K+1$ spokes, 2. $2(2K+1)$ edges

It remains to find an *hc* with each point deleted.

1. An *hc* with c deleted is P.

2. For $K>1$, an *hc* with 1 deleted is composed of the two spokes $(c, 7)$, $(c, 10)$; all chords except $(c-2, 3)$; and all edges of P except $(1, c-1)$, $(1, 2)$, $(5, 6)$, $(7, 8)$, $(9, 10)$, and the edges $(11+3j, 12+3j)$, $(j=0, 1, 2, \cdots, 2K-4)$. For $K=2$, an *hc* with 1 deleted meets the following vertices in order: 2, 3, 4, 5, 9, 8, 12, 13, 14, 15, 11, 10, 16, 7, 6, 2. For sufficiently large K, this circuit meets, in order: 2, 3, 4, 5, 9, 8, 12, 13, 14, 18, 19, 20, 24, 25, 26, 30, 31, 32, $\cdots$, $c-4$, $c-3$, $c-2$, $c-1$, $c-5$, $c-6$, $c-7$, $c-11$, $c-12$, $c-13$, $\cdots$, 11, 10, c, 7, 6, 2. For $K=1$, see [2].

3. An *hc* with 2 deleted is composed of the two spokes $(c, 1)$, $(c, c-3)$; the chord $(c-2, 3)$; and all edges of P except $(1, 2)$, $(2, 3)$, and $(c-2, c-3)$. We may obtain the other cases from symmetry.

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ON PROJECTION OF MEASURE MATRICES

A. R. Amir-Moéz, Texas Technological College

In a three-dimensional Euclidean space, let $\mathfrak{A}$ be the area of a portion of a plane and $\mathfrak{B}$ be the area of the projection of that portion on another plane. Then $\mathfrak{B}=\mathfrak{A}|\cos\theta|$, where θ is the angle between the two planes. In this article we generalize the idea to the projection of an m-cell onto a subspace of R_n, a real unitary space.

1. Definitions and notations. The inner product of two vectors will be denoted by (ξ, η). The subspace spanned by vectors $\xi_1, \cdots, \xi_k$ will be denoted by $[\xi_1, \cdots, \xi_k]$. Let $\{\xi_1, \cdots, \xi_m\}$ be a linearly independent set of vectors in R_n. Suppose $\xi_i=x_{i1}\alpha_1+\cdots+x_{in}\alpha_n$, where $\{\alpha_1, \cdots, \alpha_n\}$ is an orthonormal basis in R_n. Then the matrix A, whose ith row is $(x_{i1}, \cdots, x_{in})$, $i=1, \cdots, m$, is called a measure matrix of the m-cell defined by $\xi_1, \cdots, \xi_m$. It is well known that the measure of this m-cell is

$$\mathfrak{A} = (1/m!)[\det AA']^{1/2},$$

where A' is the transpose of A and $\det AA'$ means the determinant of AA' [1], [2].

It is clear that the measure $\mathfrak{A}$ is independent of the choice of $\{\alpha_1, \cdots, \alpha_n\}$.

In what follows all projections are perpendicular or Hermitian.

2. Projection of an m-cell. Let $\{\alpha_1, \cdots, \alpha_n\}$ be an orthonormal basis for R_n. Choose an orthonormal set $\{\beta_2, \cdots, \beta_m\}$ such that $\beta_i \in [\alpha_i, \alpha_{i+2}]$,

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Here, cases in which $b_i = 0$ for some i have been omitted.

Therefore the measure of this ellipsoid is

$$\frac{(\sqrt{\pi})^m}{\Gamma((m/2)-1)}\, a^m \prod_{i=2}^{m} \frac{b_i}{a} = \frac{(\sqrt{\pi})^m}{\Gamma((m/2)-1)}\, a \prod_{i=2}^{m} b_i.$$

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FINITELY GENERATED DOMAINS OVER JACOBSON SEMI-SIMPLE RINGS ARE JACOBSON SEMI-SIMPLE

J. A. EAGON, University of Illinois

The above statement, which does not seem to be in the literature, is a sort of "local Hilbert Nullstellensatz." Indeed, it follows that if P is a prime ideal in a commutative ring with unit R such that P is the intersection of maximal ideals, then any prime in $R[x]$ which contracts to P in R is the intersection of maximal ideals in $R[x]$. From this follows Theorem 3 in Goldman [1]. On the other hand the result here is more general since there are individual primes in quite arbitrary rings which are intersections of maximal ideals. Goldman proves the result for Hilbert rings, i.e., those rings where *every* prime is the intersection of maximal ideals. In this paper we also obtain one part of Theorem 5 in [1]. These are Goldman's main results, and other forms of the Nullstellensatz follow a la Goldman or directly. Proposition 1 is derived from Zariski ([2], Lemma 1) but the proof is short and included here.

PROPOSITION 1. *Let D be an integral domain and $D[t]$ a simply generated domain over it. Then, if t is algebraic with minimal polynomial g, leading coefficient a and M is a maximal ideal of D not containing a, or if t is transcendental and M is any maximal ideal of D, there exists a maximal ideal N of $D[t]$ such that $N \cap D = M$.*

Proof. We will show that $MD[t]$ is a proper ideal of $D[t]$ and then embed this ideal in a maximal ideal N which clearly has the required property. If $MD[t]$ is not proper then $1 = f(t)$ where $f(t)$ is a polynomial with coefficients in M. In the transcendental case this would imply that 1 is in M so we are done. In the algebraic case we divide g into f getting

$$a^n f(x) = g(x)q(x) + r(x), \qquad \deg r < \deg g, \text{ or } r = 0,$$

where the coefficients of q and r may be taken in the ideal generated by those of f, according to the well-known division algorithm. Thus we get

Here, cases in which $b_i = 0$ for some i have been omitted.

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where the coefficients of q and r may be taken in the ideal generated by those of f, according to the well-known division algorithm. Thus we get

$$a^n = a^n f(t) = r(t).$$

Since $a^n - r(x)$ is a polynomial of degree less than g such that $a^n - r(t) = 0$ we have that $r(x) = a^n$ by the minimality of the degree of g. Thus a^n and hence a is in M, contradicting our hypothesis.

PROPOSITION 2. *If D is Jacobson semisimple then so is any simply generated domain over D.*

Proof. We observe that the intersection of all maximal ideals of D not containing a given nonzero element a is still zero. For, if x is in the intersection of such ideals then ax is in the intersection of all maximal ideals. Hence $ax = 0$ which implies $x = 0$.

Suppose $D[t]$ is not Jacobson semi-simple. Let $p(t)$ be a nonzero element in the Jacobson radical of $D[t]$. Then $1 + p(t)$ is a unit. In the transcendental case this implies the degree of $1 + p(t)$, and hence the degree of $p(t)$, is 0. In the algebraic case $p(t)$ is also algebraic over D, and the constant term of a minimal equation for $p(t)$ will be a nonzero element in the ideal generated by $p(t)$. Either way we have a nonzero element of D in the Jacobson radical of $D[t]$. But this element will be in every maximal ideal of D which is the contraction of a maximal ideal of $D[t]$. In the transcendental case this includes all maximal ideals, in the algebraic case all maximal ideals not containing a given element, by Proposition 1. Either way the intersection of these ideals is 0. This is a contradiction.

The remarks in the first paragraph give us

COROLLARY 1. *If R is a Hilbert ring so is $R[x]$.* ([1] Theorem 3.)

COROLLARY 2. *If D is Jacobson semi-simple and $D[t_1, \cdots, t_n]$ is a domain then the latter is also Jacobson semi-simple.*

Proof. Apply Proposition 2 inductively.

PROPOSITION 3. *If D is Jacobson semi-simple and if $D[t]$ is a field, then D is a field.*

Proof. Clearly, t must be algebraic, say with minimal polynomial g, leading coefficient a. By Proposition 1, every maximal ideal not containing a would be (0). If D were not a field then every maximal ideal would contain a, contradicting the hypothesis on D.

COROLLARY 2. *If R is a Hilbert ring then every maximal ideal in $R[x]$ contracts to a maximal ideal in R,* ([1], Theorem 5).

Proof. Apply Proposition 3.

The author wishes to thank Ed Davis for a helpful conversation.

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A CONCISE PROOF OF BOREL'S THEOREM ON COEFFICIENT BOUNDS

R. A. WHITEMAN, IIT Research Institute

A classical theorem of Borel ([1], [2]) concerning the coefficient bounds of analytic functions with a positive real part defined on $|z|<1$ may be stated as:

Let $f(z)$ be an analytic function of a complex variable defined on $|z|<1$. If the real part of $f(z)$ is positive and the function is normalized to have the power series representation

$$f(z) = 1 + \sum c_n z^n, \tag{1}$$

then $|c_n| \leqq 2$ for $n=1, 2, \cdots$.

A concise generalized proof of this theorem may be achieved through F. Riesz's and Herglotz's Stieltjes integral representation of $f(z)$ [3]. The integral representation is

$$f(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \cdot d\phi(t) + iq, \tag{2}$$

where $\phi(t)$ is a bounded nondecreasing function of a real variable t such that $\phi(0)=0$ and $\phi(2\pi)>0$ and q is a real constant.

By expressing the integrand of (2) as a power series for $|z|<1$, and rearranging terms, the following power series for $f(z)$ is obtained;

$$f(z) = \int_0^{2\pi} d\phi(t) + \sum \left[2 \int_0^{2\pi} e^{-int} d\phi(t) \right] z^n + iq. \tag{3}$$

By comparing the power-series coefficients of (1) with those of (3), the following relations are obtained:

$$c_n = 2 \int_0^{2\pi} e^{-int} d\phi(t), \qquad \int_0^{2\pi} d\phi(t) = 1 \text{ and } q = 0.$$

Since $\phi(t)$ is a nondecreasing function, $|c_n| \leqq 2\int_0^{2\pi} |d\phi(t)| = 2\int_0^{2\pi} d\phi(t)$ and therefore, $|c_n| \leqq 2$, for $n=1, 2, \cdots$.

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FUNCTIONS AND INVERSES OF ASYMPTOTIC FUNCTIONS

R. C. ENTRINGER, University of New Mexico

In this note we consider the following two problems: (1) *Given* $f(x)\to\infty$ *and* $f(x)\sim g(x)$ *as* $x\to\infty$, *for what functions* $h(x)$ *is* $h(f(x))\sim h(g(x))$? (2) *For what functions* $f(x)$ *does* $f(x)\sim g(x)$ *imply* $f^{-1}(x)\sim g^{-1}(x)$ *as* $x\to\infty$? (Here and throughout this note the -1 superscript will denote the inverse function.)

In problem (1) we note that if $h(x)=x^r$ where r is any real number then $h(f(x))\sim h(g(x))$, but if $h(x)=e^x$, $f(x)=x$, $g(x)=x+1$, then $h(f(x))\nsim h(g(x))$. For a more interesting example the reader may consult [1] where it is shown that $\prod_{p\leq x} p\sim e^x$ even though $\sum_{p\leq x}\log p\sim x$ (p prime).

In problem (2) we see that if $f(x)=\log x$, $g(x)=\log 2x$ then $f(x)\sim g(x)$, but $f^{-1}(x)=e^x\nsim\frac{1}{2}e^x=g^{-1}(x)$. On the other hand (employing the usual number-theoretic notation) $\pi(n)\sim n/\log n$ and

$$(\pi(n))^{-1}=p_n\sim n\log n\sim(n/\log n)^{-1}.$$

In Theorems 1 and 2 we obtain restrictions on the size of $h(x)$ and $f(x)$ so that the conclusions of the problems obtain. We also show that these restrictions cannot be relaxed to any great extent.

THEOREM 1. (i) *If* $f(x)\to\infty$ *and* $f(x)\sim g(x)$ *as* $x\to\infty$, *and if* $h(x)$ *is monotonic and* $h'(x)/h(x)=O(1/x)$ *for all sufficiently large* x, *then* $h(f(x))\sim h(g(x))$ *as* $x\to\infty$.

(ii) *If* $h'(x)$ *is strictly monotonic for all sufficiently large* x *and* $h(x)/h'(x)=o(x)$ *as* $x\to\infty$ *then there are functions* $f(x)$ *and* $g(x)$ *such that* $f(x)\to\infty$, $f(x)\sim g(x)$ *and* $h(f(x))\nsim h(g(x))$ *as* $x\to\infty$.

Proof. (i) Let $M(x)=\max(f(x), g(x))$ and $m(x)=\min(f(x), g(x))$. Then, by the mean value theorem there is a c, $m(x)\leqq c\leqq M(x)$ for which

$$|h(f(x))-h(g(x))| = |h'(c)\{f(x)-g(x)\}| = |h(c)O(1/c)o(M(x))|$$
$$\leqq |\max\{h(M(x)), h(m(x))\}O(1/m(x))o(M(x))| = o(\max\{h(M(x)), h(m(x))\})$$

as $x\to\infty$. This completes the proof of (i) since $\max\{h(M(x)), h(m(x))\}$ is $h(f(x))$ or $h(g(x))$.

(ii) We first consider the case where $h'(x)$ is strictly increasing. For all sufficiently large x, $h'(x)\neq 0$ and if we set

$$f(x)=x,\quad g(x)=x+\frac{|h(x)|}{h'(x)}$$

we have $f(x)\to\infty$ and $f(x)\sim g(x)$ as $x\to\infty$. Now for some c between x and $x+(|h(x)|/h'(x)$, we have

$$|h(f(x))-h(g(x))| = \left|h'(c)\frac{h(x)}{h'(x)}\right| \geqq |h'(x)|\left|\frac{h(x)}{h'(x)}\right| \neq o(h(x))$$

and hence $h(f(x))\nsim h(g(x))$. If $h'(x)$ is strictly decreasing we set

$$g(x) \sim \sqrt{2\pi}\,\frac{x^{x+1/2}}{e^x} \quad \text{and} \quad h(x) \sim \frac{\log x}{\log\log x},$$

then $h(\sqrt{2\pi}(x^{x+1/2}/e^x)) \sim x$ as $x \to \infty$. Hence $g^{-1}(x) \sim \log x/\log\log x$.

Research supported by the National Science Foundation, Grant GP-4532.

Reference

1. S. Chowla and F. B. Correia, The falsity of a certain asymptotic relation, this MONTHLY, 67 (1960) 884.

SEMIGROUPS WHOSE REGULAR REPRESENTATION IS A RIGHT GROUP

J. L. CHRISLOCK, University of California, Santa Cruz

1. Introduction. As in [3], if S is a semigroup we denote by ρ_x the mapping $a\,\rho_x = ax$, but, for convenience, we will call the semigroup $\{\rho_x : x \in S\}$ the regular representation of S.

Given a transformation semigroup T on a set S it is natural to ask whether or not there exists a binary operation on S making S a semigroup with T as its regular representation. The general problem seems too difficult, but if we restrict T to the class of right groups we will see in Theorem 2 that (C) is necessary and sufficient for such an operation to exist.

To prove this fact we determine in Theorem 1 all semigroups whose regular representation is a right group, and it turns out that these are exactly the M-inversive semigroups introduced by Yamada [5].

2. M-inversive semigroups. A right group is a semigroup with a left identity and with right solvability. The following lemma is proved in [3, p. 38; 2; 4].

LEMMA 1. *A semigroup is a right group if and only if it is the direct product $G \times R$ of a group G and a right zero semigroup R.*

Yamada [5] defines an M-inversive semigroup to be one in which for all $a \in S$ there exist $x, y \in S$ such that

$$(1) \qquad caxd = cd, \qquad (2) \qquad cyad = cd,$$

for all $c, d \in S$. We now give a necessary definition [3] and then Yamada's characterization, Lemma 2, of these semigroups.

DEFINITION. *Let S be a semigroup. For each $x \in S$, let A_x be a set containing x such that the sets A_x, $x \in S$, are pairwise disjoint. If $S' = \bigcup\{A_x : x \in S\}$ and if $a \cdot b = xy$, where $a \in A_x$ and $b \in A_y$, then $(S', \cdot)$ is called an inflation of S.*

LEMMA 2. *S is M-inversive if and only if it is an inflation of a direct product $G \times L \times R$ of a group G, a left zero semigroup L, and a right zero semigroup R.*

If on a semigroup we define a mapping λ_a by $x\lambda_a = ax$, then we will call the

$$g(x) \sim \sqrt{2\pi}\,\frac{x^{x+1/2}}{e^x} \quad \text{and} \quad h(x) \sim \frac{\log x}{\log\log x},$$

then $h(\sqrt{2\pi}(x^{x+1/2}/e^x)) \sim x$ as $x \to \infty$. Hence $g^{-1}(x) \sim \log x/\log\log x$.

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LEMMA 2. *S is M-inversive if and only if it is an inflation of a direct product $G \times L \times R$ of a group G, a left zero semigroup L, and a right zero semigroup R.*

If on a semigroup we define a mapping λ_a by $x\lambda_a = ax$, then we will call the

semigroup of all such mappings the regular anti-representation of S. In the following we will give only proofs for the regular representation of S, but the proofs for the regular antirepresentation are identical.

Using Lemma 2, we can now characterize the M-inversive semigroups by their regular [anti-] representation.

THEOREM 1. *S is M-inversive if and only if its regular [anti-] representation is a right group.*

Proof. Assume that S is M-inversive. The definition of M-inversive is equivalent to the following: for all $a \in S$ there exist x, $y \in S$ such that

$$(1') \qquad \rho_a \rho_x \rho_d = \rho_d, \qquad (2') \qquad \rho_y \rho_a \rho_d = \rho_d,$$

for all $d \in S$. From (1') and (2') it follows trivially that the regular representation T is M-inversive with a left identity. The fact that T has a left identity implies that $T^2 = T$, which in turn implies from the definition that the inflation in Lemma 2 is trivial. Hence, $T \cong G \times L \times R$ for some G, L, and R. Let $(e, a, b) \in G \times L \times R$ be a left identity, where e is the identity of G. For arbitrary $(x, c, d) \in G \times L \times R$,

$$(x, c, d) = (e, a, b)(x, c, d) = (x, a, d).$$

We conclude that $a = c$ and that $|L| = 1$, which implies that $T \cong G \times R$ or that T is a right group.

COROLLARY. *The regular [anti-] representation T of a semigroup S is a group if and only if S is an inflation of a left [right] group $G \times L$ [$G \times R$].*

Proof. Assume that T is a group. Theorem 1 tells us that S is an inflation of a direct product $G \times L \times R$. Also, since T is a group, there exists an element $e \in S$ such that

$$(3) \qquad \rho_x \rho_e = \rho_x, \quad \text{for all } x \in S.$$

We can assume that $e \in G \times L \times R$. Let $e = (f, x, y)$ and let (g, a, b) and (h, c, d) be arbitrary members of $G \times L \times R$. It follows from (3) that f must be the identity of G and that

$$(gh, a, y) = (g, a, b)(h, c, d)(f, x, y) = (g, a, b)(h, c, d) = (gh, a, d).$$

This proves that $y = d$, and since (h, c, d) was arbitrary, $|R| = 1$. Hence, S is an inflation of a left group.

Conversely, if S is an inflation of a left group, we can easily show (3), which along with Theorem 1 proves that the regular representation of S is a group.

3. The main theorem.

THEOREM 2. *If T is a right group of transformations on a set S, then there exists a binary operation on S such that S is a semigroup with T as its regular [anti-] representation if and only if, for all $x \in S$ and all $\alpha, \beta \in T$,*

(C) *$x\alpha = x\beta$ implies $\alpha = \beta$.*

Proof. Let E be the set of idempotents of S indexed by the set I. We will call any congruence relation η on S which satisfies $e_\alpha x \eta x$, for all $x \in S$ and for all $\alpha \in I$, an i-congruence on S. Now define the congruence σ by $a\sigma b$ if $\rho_a = \rho_b$. If $a\sigma b$ then $e_\alpha \rho_a = e_\alpha \rho_b$ or $e_\alpha a = e_\alpha b$, for all $\alpha \in I$. Let η be any i-congruence on S. Then $a\eta e_\alpha a = e_\alpha b \eta b$. Hence $a\eta b$, and $\sigma \subseteq \eta$. Now any group-congruence on S is also an i-congruence on S. It follows that σ is the smallest group-congruence on S. Further, $S/\sigma \cong T$.

The author would like to thank Professor Takayuki Tamura for directing this research and the referee who offered many helpful remarks, and suggested the use of the antiregular representation.

Weaker results were published under a different title as a brief note in [1] and were presented to the Society, August 25, 1964. Although the method is changed, the earlier results appear here.

References

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2. A. H. Clifford, A system arising from a weakened set of group postulates, Ann. of Math., 34 (1933) 865–871.

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ON MÖBIUS' FUNCTION AND RELATED MATTERS

A. A. MULLIN, University of California, Livermore (Now at Aberdeen Proving Ground, Maryland)

1. Introduction. This note relates Möbius' number-theoretic function [1] to its analogue relative to the author's "mosaic model" of elementary unique factorization [2]. By "mosaic model" of elementary unique factorization is meant the 1-1 mapping of the natural numbers to "configurations" of prime numbers alone obtained by applying recursively the canonic prime-power factorization of natural numbers to its own exponents. For a given natural number the "configuration" of primes is called a *mosaic*. For example, (1) the mosaic of 192 is $2^{2\cdot 3}\cdot 3$; and (2) the mosaic of a square-free number coincides with its canonic prime decomposition, although this can be the case with nonsquare-free numbers such as 666. This self-recursion technique is quite old, as one may use it to obtain the Euclidean algorithm from the division algorithm. Further, conjectures are given in [4] bearing on the prime number theorem [3] and the ordinary Riemann hypothesis for the ordinary zeta-function.

2. Definitions and propositions. Define μ^* (*Modified Möbius Function*) as follows: $\mu^*(1) = 1$; $\mu^*(n) = 0$, if the mosaic of $n > 1$ has any prime number repeated; and $\mu^*(n) = (-1)^m$ if the mosaic of $n > 1$ has no prime repeated, where m is the number of (distinct) primes in the mosaic of n.

By analogy with the multiplicativity of μ one has, clearly,

Proof. Let E be the set of idempotents of S indexed by the set I. We will call any congruence relation η on S which satisfies $e_\alpha x \eta x$, for all $x \in S$ and for all $\alpha \in I$, an i-congruence on S. Now define the congruence σ by $a\sigma b$ if $\rho_a = \rho_b$. If $a\sigma b$ then $e_\alpha \rho_a = e_\alpha \rho_b$ or $e_\alpha a = e_\alpha b$, for all $\alpha \in I$. Let η be any i-congruence on S. Then $a\eta e_\alpha a = e_\alpha b \eta b$. Hence $a\eta b$, and $\sigma \subseteq \eta$. Now any group-congruence on S is also an i-congruence on S. It follows that σ is the smallest group-congruence on S. Further, $S/\sigma \cong T$.

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By analogy with the multiplicativity of μ one has, clearly,

C1. $\sum_{m \leq x} \mu^*(m) = o(x)$ entails the prime number theorem and

C2. $\sum_{m \leq x} \mu^*(m) = O(x^{1/2})$ entails the ordinary Riemann hypothesis for the ordinary zeta-function.

If C1 or C2 is, indeed, valid, how can either be weakened so that full equivalence with the prime number theorem or the Riemann hypothesis can be obtained? Finally it might be interesting to study the author's *Modified Liouville Function* [6] of exponential number-theory from the point of view of its possible relations to the Prime Number Theorem and the Riemann Hypothesis, and its known relations with the standard Liouville and Möbius functions [9].

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ON THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL WITH REAL ROOTS

GIDEON PEYSER, Newark College of Engineering

In this note we investigate the location of the roots of the derivative of a polynomial with all real roots. The location of these roots is of interest in analysis, in particular in investigations of hyperbolic partial differential equations, whose main property is that the characteristic polynomial has only real roots.

From elementary considerations it follows that if a polynomial has only real roots, then between two adjacent (not necessarily simple) distinct roots lies a unique simple root of the derivative. This note is concerned with the location of these roots relative to the roots of the original polynomial.

All polynomials will be written in increasing order of their roots. In particular, throughout the paper we denote

$$P(x) = (x - a_1) \cdots (x - a_n) \qquad a_i \text{ real}, \quad a_i \leqq a_{i+1} \tag{1}$$

$$P'(x) = (x - c_1) \cdots (x - c_{n-1}) \qquad a_i \leqq c_i \leqq a_{i+1}. \tag{2}$$

The main result is the following:

C1. $\sum_{m \leq x} \mu^*(m) = o(x)$ entails the prime number theorem and

C2. $\sum_{m \leq x} \mu^*(m) = O(x^{1/2})$ entails the ordinary Riemann hypothesis for the ordinary zeta-function.

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The main result is the following:

LEMMA 4. *Let* $T(x)=(x-a_1)\cdots(x-a_{n-1})(x-a_n+\epsilon)$, *where* $\epsilon\geqq 0$ *is such that* $a_n-\epsilon\geqq a_2$ *and let* $T'(x)=(n-1)(x-g_1)\cdots(x-g_{n-2})$ *then* $g_1\leqq c_1$.

We now can prove the main theorem. We consider the polynomial $W(x)=(x-a_k)^k(x-a_{k+1})$. $W'(x)$ has a root

$$c=\frac{ka_{k+1}+a_k}{k+1}=a_{k+1}-\frac{a_{k+1}-a_k}{k+1}.$$

From repeated application of Lemmas 1 and 3 it follows that

$$c_k\leqq c=a_{k+1}-\frac{a_{k+1}-a_k}{k+1}.$$

Similarly it follows from the consideration of the polynomial $(x-a_k)(x-a_{k+1})^{n-k}$ and Lemmas 2 and 4 that

$$c_k\geqq a_k+\frac{a_{k+1}-a_k}{n+1-k}.$$

This completes the proof.

From (3) we have the following

COROLLARY. *The root* c_k *of* $P'(x)$, $1\leqq k\leqq n-1$, *satisfies the inequality*

$$a_k+\frac{a_{k+1}-a_k}{n}\leqq c_k\leqq a_{k+1}-\frac{a_{k+1}-a_k}{n}. \tag{7}$$

This means that the root c_k cannot be closer to either a_k or a_{k+1} than $1/n$ of the distance between a_k and a_{k+1}. This estimate is independent of the index k and depends only on the degree n of $P(x)$.

As another application of (3) we note that c_1 is always in the first half of the first interval, i.e., $c_1\leqq(a_1+a_2)/2$ and c_{n-1} is always in the second half of the last interval, $c_{n-1}\geqq(a_n+a_{n-1})/2$.

ON BURCHNALL'S GENERATING RELATION FOR BESSEL POLYNOMIALS

J. W. BROWN, Oberlin College

1. Since 1949 there has been considerable interest in what Krall and Frink [4] have called the Bessel polynomials:

$$y_n(x,a,b)={}_2F_0(-n,a-1+n;-x/b). \tag{1}$$

While Krall and Frink were the first to study these polynomials for their own sake, various subsequent authors have concentrated on direct connections between these and certain of the more familiar classical polynomials. For instance, Agarwal [1] noted that the Bessel polynomials are special limiting cases of the

LEMMA 4. *Let* $T(x)=(x-a_1)\cdots(x-a_{n-1})(x-a_n+\epsilon)$, *where* $\epsilon\geqq 0$ *is such that* $a_n-\epsilon\geqq a_2$ *and let* $T'(x)=(n-1)(x-g_1)\cdots(x-g_{n-2})$ *then* $g_1\leqq c_1$.

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$$a_k+\frac{a_{k+1}-a_k}{n}\leqq c_k\leqq a_{k+1}-\frac{a_{k+1}-a_k}{n}. \tag{7}$$

This means that the root c_k cannot be closer to either a_k or a_{k+1} than $1/n$ of the distance between a_k and a_{k+1}. This estimate is independent of the index k and depends only on the degree n of $P(x)$.

As another application of (3) we note that c_1 is always in the first half of the first interval, i.e., $c_1\leqq(a_1+a_2)/2$ and c_{n-1} is always in the second half of the last interval, $c_{n-1}\geqq(a_n+a_{n-1})/2$.

ON BURCHNALL'S GENERATING RELATION FOR BESSEL POLYNOMIALS

J. W. BROWN, Oberlin College

1. Since 1949 there has been considerable interest in what Krall and Frink [4] have called the Bessel polynomials:

$$y_n(x,a,b)={}_2F_0(-n,a-1+n;-x/b). \tag{1}$$

While Krall and Frink were the first to study these polynomials for their own sake, various subsequent authors have concentrated on direct connections between these and certain of the more familiar classical polynomials. For instance, Agarwal [1] noted that the Bessel polynomials are special limiting cases of the

Jacobi polynomials and showed how various properties of the former are inherited from the latter. Indeed, he obtained from a well-known generating relation for the Jacobi polynomials Burchnall's formula [3]

$$(2)\quad \sum_{n=0}^{\infty} y_n(x, a, 1)t^n/n! = \frac{1}{\sqrt{(1-4xt)}}\left(\frac{2}{1+\sqrt{(1-4xt)}}\right)^{a-2}\exp\left(\frac{2t}{1+\sqrt{(1-4xt)}}\right)$$

which is the earliest known generating relation for Bessel polynomials with the parameter a arbitrary. As Burchnall pointed out, the parameter b in (1) is merely a scaling factor and may be taken as unity without loss of generality.

More recently, Al-Salam [2] observed that the Bessel polynomials are essentially Laguerre polynomials with the parameter of the latter varying with the index. That is,

$$(3)\quad y_n^*(x, a, 1)/n! = (-1)^n L_n^{1-a-2n}(x),$$

where it is understood that $p_n^*(x)$ denotes the *reverse* $x^n p_n(1/x)$ of a given polynomial $p_n(x)$. Al-Salam pointed out that with (3) it is a simple matter to rewrite various known properties of the Laguerre polynomials in terms of Bessel polynomials. While his viewpoint thus paralleled Agarwal's, he did not include a corresponding derivation of (2). Our purpose here is to furnish such a derivation.

2. From the familiar generating relation [5]

$$(4)\quad \sum_{n=0}^{\infty} L_n^a(x)t^n = (1-t)^{-a-1}\exp\left(\frac{xt}{t-1}\right)$$

we derive

$$(5)\quad \sum_{n=0}^{\infty} (-1)^n L_n^{1-a-2n}(x)t^n = \frac{1}{\sqrt{(1-4t)}}\left(\frac{2}{1+\sqrt{(1-4t)}}\right)^{a-2}\exp\left(\frac{2xt}{1+\sqrt{(1-4t)}}\right)$$

which, by (3), is the equivalent form of (2) in the language of Laguerre polynomials. Concentrating on the way in which the parameter enters into the right-hand side of (4), we find it convenient to present our derivation in the context of more general polynomials $f_n^a(x)$ generated by a relation of the form

$$(6)\quad \sum_{n=0}^{\infty} f_n^a(x)t^n = (1-t)^{-a-1}F(x, t),$$

where $F(x, t)$ is independent of the parameter a. Keeping in mind that $f_n^a(x) = L_n^a(x)$ when $F(x, t) = \exp(xt/(t-1))$, we find that this more general setting furnishes a deeper insight into the nature of the connection between (4) and (5).

THEOREM. *If there is a generating relation of the form* (6) *for given polynomials* $f_n^a(x)$, *then*

$$(7)\quad \sum_{n=0}^{\infty} (-1)^n f_n^{1-a-2n}(x)t^n = \frac{1}{\sqrt{(1-4t)}}\left(\frac{2}{1+\sqrt{(1-4t)}}\right)^{a-2}F\left(x, \frac{-4t}{[1+\sqrt{(1-4t)}]^2}\right).$$

The author is indebted to the late Professor E. D. Rainville, of the University of Michigan, for his help and encouragement.

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ON A REMARK OF UTZ

FRED GROSS, Bellcomm, Inc., Washington

W. R. Utz in introducing the problem "The equation $f'(x)=af(g(x))$" (Bull. Amer. Math. Soc. Vol. 71, No. 694, Jan. 1965) commented that the corresponding problem for certain special cases is fairly simple for analytic functions.

I think it is worth mentioning that for entire functions the problem can be completely solved. We shall use the following theorem of Polya [1]:

For two entire functions f and g such that $g(0)=0$ one has $M_{f\circ g}(r)\geqq M_f(cM_g(r/2))$, where $M_h(r)=\max_{|z|=r}|h(z)|$ and c is an appropriate positive constant.

We now prove:

THEOREM 1. *If $f(z)$ is entire and not identically zero, then the only entire solutions $g(z)$ of the equation $f'(z)=af(g(z))$ are linear, where a is any complex number.*

Proof. Let us first assume that $g(0)=0$. In this case

$$M_{f'}(r) = |a| M_{f\circ g}(r) \geqq |a| M_f(cM_g(r/2)) \geqq |a| M_{f'}(c'M_g(r/2))(c-c')M_g(r/2)$$
$$\geqq M_{f'}(c'M_g(r/2))$$

for sufficiently large r provided that g is not a constant, where $0<c'<c$. The second inequality relating f to f' follows from the inequality

$$M_{f'}(r) \leqq \frac{M_f(R)}{R-r}, \quad \text{where} \quad R>r.$$

See Titchmarsh [2].

Since $M_{f'}$ is monotonic increasing [3] it follows that $c'M_g(r/2)\leqq r$ and consequently that g must be linear.

When $g(0)\neq 0$ let $\bar{g}(z)=g(z)-g(0)$ and $\bar{f}(u)=f(u+g(0))$, then $f(g)=\bar{f}(\bar{g})$. Using an argument similar to the one above we get

$$M_{f'}(r) \geqq M_{\bar{f}'}(c'M_{\bar{g}}(r/2)) \geqq M_{f'}(c'M_{\bar{g}}(r/2) - |g(0)|),$$

which as before leads to the conclusion that $\bar{g}$ and hence g must be linear.

Utz gave the example $f(x)=A(\sin ax+\cos ax)$. This function is entire and satisfies $f'(x)=af(-x)$. One can easily verify that for any $|A|\leqq 1$, $A\neq 1$ and any constant B there exists an entire $f(z)$ such that $f'(z)=f(Az+B)$.

For example the function

$$f(z)=\sum_{n=0}^{\infty}\frac{A^{n(n-1)/2}z^n}{n!}$$

is entire for $|A|\leqq 1$ and one can easily verify that $f'(z)=f(Az)$.

Thus, for $D=-B/(A-1)$ and $f_1(z+D)=f(z)$, $f_1(A(z+D)+B)=f_1'(z+D)$ or letting $z+D=z'$, $f_1(Az'+B)=f_1'(z')$.

Note that similar solutions can be found for $f'(z)=af(Az+B)$, where a is a constant and $|A|\leqq 1$.

The identical argument can also be used to show more generally that, for any $f^{(n)}(z)n=0, 1, 2, \cdots$, where $f^{(0)}=f$, we have

THEOREM 2. *If $f(z)$ is entire, then the only entire solutions $g(z)$ of the equation $f^{(n)}(z)=af(g(z))$ are linear.*

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A CONJECTURE ON THE EXTENSION OF BILINEAR FUNCTIONALS

T. L. HAYDEN, University of Kentucky

In analogy with the Hahn-Banach Theorem the following conjecture seems plausible:

CONJECTURE. *Suppose each of U and V is a linear space and that p and q are subadditive, positively homogeneous functions on U and V respectively. Then if S and T are subspaces of U and V respectively and $f(s, t)$ is a bilinear functional defined on $S\times T$ such that $f(s, t)\leqq p(s)q(t)$, there is a bilinear functional F on $U\times V$ such that $F(s, t)=f(s, t)$ on $S\times T$ and $F(u, v)\leqq p(u)q(v)$ on $U\times V$.*

We show that the conjecture is true for Hilbert spaces.

THEOREM. *Suppose $S_1, S_2, \cdots, S_n$ are subspaces respectively of the complex Hilbert spaces $X_1, X_2, \cdots, X_n$ and $f(s_1, s_2, \cdots, s_n)$ is a multilinear functional defined on $\times_{k=1}^{n}S_k$ with norm K. Then there exists a multilinear functional F on $\times_{k=1}^{n}X_k$ with norm K such that F is f on $\times_{k=1}^{n}S_k$.*

Proof. First we show that f can be extended to a function defined on

$\times_{k=1}^{n}(S_k \cup \{x_k\})$ for $x_k \notin S_k$, $k=1, 2, \cdots, n$. In fact, take $x_k \perp S_k$, and define $h(a_1x_1+s_1, \cdots, a_nx_n+s_n) = K\prod_{k=1}^{n} a_k\|x_k\| + f(s_1, \cdots, s_n)$ where $a_1, a_2, \cdots, a_n$ are scalars and K is the norm of f. It is immediate that the extension is multilinear, and we now show that the norm is the same.

$$|h(a_1x_1 + s_1, \cdots, a_nx_n + s_n)| \leqq K\left(\prod_{k=1}^{n}\|a_kx_k\| + \prod_{k=1}^{n}\|s_k\|\right)$$

$$\leqq K\prod_{k=1}^{n}\left(\|a_kx_k\|^{u_k} + \|s_k\|^{u_k}\right)^{1/u_k} \tag{1}$$

$$\leqq K\prod_{k=1}^{n}\left(\|a_kx_k\|^2 + \|s_k\|^2\right)^{1/2} = K\prod_{k=1}^{n}\|a_kx_k + s_k\|, \tag{2}$$

where $u_k \geqq 2$ and $1 \geqq \sum_{k=1}^{n} u_k^{-1}$. In (1) we have used a generalization of Hölder's inequality [1, 20], while (2) follows from an application of Jensen's inequality [1, 18].

The remainder of the proof follows from a straightforward application of Zorn's Lemma similar to that in the Hahn-Banach Theorem. (The author thanks the referee for correcting an error in the above proof.)

The generalizations of the usual corollaries of the Hahn-Banach Theorem follow easily and we mention only the following:

COROLLARY. *If $x \neq 0$ and $y \neq 0$ are in the Hilbert spaces X and Y respectively, then there exists a bilinear functional f such that $f(x, y) = \|x\|\,\|y\|$ and $\|f\| = 1$.*

Added in Proof. The conjecture is false [2]. Does the bilinear extension property characterize Hilbert spaces?

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TOPOLOGICAL ISOMORPHISM AND CONJUGATE SPACES

JAMES DUEMMEL, Western Washington State College

1. Let X be a Banach space with conjugate X' (also called its normed conjugate or dual [2, p. 185]). Suppose that a second norm is introduced on X' so that the two normed linear spaces on X' are topologically isomorphic or linearly homeomorphic [2, p. 85]. Can X' with this new norm be obtained as the conjugate of X with a suitable norm? The example of this paper shows that this cannot always be done.

2. Let (X, N) be a Banach space with conjugate (X', N'). Suppose M' is a

$\times_{k=1}^{n}(S_k \cup \{x_k\})$ for $x_k \notin S_k$, $k=1, 2, \cdots, n$. In fact, take $x_k \perp S_k$, and define $h(a_1x_1+s_1, \cdots, a_nx_n+s_n) = K\prod_{k=1}^{n} a_k\|x_k\| + f(s_1, \cdots, s_n)$ where $a_1, a_2, \cdots, a_n$ are scalars and K is the norm of f. It is immediate that the extension is multilinear, and we now show that the norm is the same.

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By the remarks at the close of section 2, we can conclude that (Y'', M'') is *not* the conjugate of Y' with any norm. This gives the example mentioned in the opening paragraph.

4. Another interpretation. In the notation of section 2, let B' denote the closed unit sphere in (X', M'). From (1), the closed unit sphere B in $(X, P_{M'})$ is just the polar in X of B'. Let B^* be the polar in X' of B. From standard results [1, p. 52; 3, p. 244], B^* is the weak* closure of B', and B^* is the unit sphere of $P'_{M'}$. The norms $P'_{M'}$ and M' are the same iff $B^* = B'$, i.e., iff B' is weak* closed.

Thus, we can interpret our example as the construction in Y'' of a set U'' which is balanced, absorbing, bounded, convex, norm closed (also weak closed) but not weak* closed.

5. The identity map on Y'' is a topological isomorphism of (Y'', N'') onto (Y'', M''). In this sense, these spaces are the "same" although (Y'', M'') has lost the very property used to define (Y'', N''). For Banach spaces, it might be argued that congruence or isometric isomorphism [2, p. 84] is a better criterion of "sameness." The author has not been able to determine whether or not there exists a norm P' on Y' such that (Y'', P'') is congruent to (Y'', M'') as defined in the above example.

Acknowledgment. I am indebted to the referee for many helpful suggestions.

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NEAR-RINGS WITH TRIVIAL MULTIPLICATIONS

J. J. Malone, Jr., University of Houston and Texas A and M University

In [1] the following question is raised: "Every group may be taken as the additive group of a near-ring. What characterizes the class of near-rings with the property that, if the group operation is taken as the addition of the near-ring, the only possible multiplications are $ab = 0$ or b?" This note answers that question.

A type of multiplication which may be defined on any group $(G, +)$ so that $(G, +, \cdot)$ is a near-ring will be called a trivial multiplication. Thus, a trivial multiplication does not depend upon the structure of $(G, +)$. Obviously, the two multiplications given in the question are trivial. The question is settled by showing that, if the order of G is greater than one, there exist additional trivial multiplications.

Let H be a subset of G such that $0 \notin H$. For $h \in H$ define $h \cdot x = x$, for every $x \in G$. For $g \in G - H$ define $g \cdot x = 0$, for every $x \in G$. Consider the product $x \cdot y \cdot z$ of three elements of G. If x or y is in $G - H$ or $z = 0$, the product is 0 regardless of

the manner of association. If x and y are in H, the product is z regardless of the manner of association. That the left-distributive law holds may be seen by comparing $x\cdot(y+z)$ and $x\cdot y+x\cdot z$ first for $x\in G-H$ and then for $x\in H$. In the first case both expressions reduce to 0 and in the second case both reduce to $y+z$. Thus $(G, +, \cdot)$ is a near-ring.

For each of the multiplications just defined $(G, +, \cdot)$ is a C-ring. These multiplications include the case in which $ab=0$, i.e., H is empty. If we remove the restriction $0\in H$ and take $H=G$, then $(G, +, \cdot)$ is a Z-ring, i.e., $ab=b$.

If G is of finite order n, we see that a minimum of $2^{n-1}+1$ near-rings may be defined on $(G, +)$. For the case in which G is cyclic, Clay [2] has shown, by also considering some nontrivial multiplications, that there are at least $2^{n-1}+n$ near-rings, $n>2$. For the case in which G is cyclic and has prime order, Jacobson [3] has shown that the exact number of near-rings is

$$2+\sum_{k}\left\{\sum_{j=1}^{k} C(k, j)\left(\frac{p-1}{k}\right)^{j}\right\}, \qquad k \mid p-1.$$

Since the group of order two admits only three multiplications (all of which are trivial), we conclude that all trivial multiplications have been described.

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2. J. R. Clay, The near-rings on a finite cyclic group, this MONTHLY, 71 (1964) 47–50.
3. R. A. Jacobson, The structure of near-rings on a group of prime order, this MONTHLY, 73 (1966) 59–61.

CLASSROOM NOTES

Material for this department should be sent to George Raney, Department of Mathematics, University of Connecticut, Storrs, CT06268.

A MATHEMATICAL MICROSCOPE

G. T. CARGO, Syracuse University

1. Introduction. Pedagogically speaking, the purpose of this paper is to show that there are still frontiers of science where meaningful problems can be successfully attacked by persons possessing only a modest knowledge of undergraduate mathematics.

2. Recalcitrant microorganisms. The exponential function arises naturally in connection with many processes of change such as cooling of a hot body, damping on a telephone line, continuous compounding of interest, decay of radioactive material, and organic growth. Therefore, it came as no surprise to microbiologists (see [2]) when, time after time, experimental data indicated

the manner of association. If x and y are in H, the product is z regardless of the manner of association. That the left-distributive law holds may be seen by comparing $x\cdot(y+z)$ and $x\cdot y+x\cdot z$ first for $x\in G-H$ and then for $x\in H$. In the first case both expressions reduce to 0 and in the second case both reduce to $y+z$. Thus $(G, +, \cdot)$ is a near-ring.

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that the thermal death of microorganisms obeys first-order kinetics; in other words, if a population of P microorganisms is subjected to heat at a constant temperature, there exists a positive constant κ such that the number of microorganisms still alive at time t is $Pe^{-\kappa t}$. It is easy to understand the interest of applied microbiologists in this matter, for example, in connection with food processing and sterilization.

But Nature sometimes yields her secrets reluctantly. During the last decade, several investigators (see [1]) have reported instances of nonexponential thermal death for certain strains of bacterial spores, e.g., *Bacillus stearothermophilus*, where the deviations were not attributable to experimental errors. In this paper we shall devise a mathematical model which "explains" this phenomenon.

3. The model. A spore may be in one of three mutually exclusive states: activated, nonactivated, or dead. An *activated spore* is one which is capable of colony formation when properly nourished; in other words, it is a germinated spore. A *nonactivated spore* is a dormant spore which is capable of passing to the activated state upon being subjected to appropriate heat conditions.

It is difficult to determine which state a spore is in by means of a microscope. It is possible, however, to count the number of activated spores in a population by simply giving the population suitable nourishment and then counting the colonies which spring up—one colony for each activated spore. Instead of nourishing the whole population, it is often desirable to use an appropriate sampling procedure. Since the spore populations used in experiments are usually of the order of magnitude of 100,000, the use of some sampling technique is almost imperative. Moreover, the use of sampling permits one to continue an experiment, so that if a population is subjected to heat, one can obtain accurate estimates of the activated spore population at various times throughout the experiment.

When a population is heated at a constant temperature, it is customary to speak of the plot of the number of activated spores with respect to time as the *thermal death curve.*

If the initial population consists entirely of activated spores, then the thermal death curve is an exponential die-away curve whose equation is of the form $P(t)=Pe^{-\kappa t}$. If the initial population is a mixture of activated and nonactivated spores, then the thermal death curve usually exhibits an initial lag which is followed by an exponential die-away portion after a certain time delay (evidently, after almost no spores are left in the nonactivated state).

If the nonactivated spores and the activated spores were of equal heat resistance with respect to death and if the rate of activation were quite large, then the exponential die-away portion of the thermal death curve would extrapolate to the initial number of living spores. For certain strains of spores, however, the exponential portion of the death curve extrapolates to a point in excess of the initial number of living spores. There are several alternative hy-

potheses which could conceivably explain this anomaly. We shall incorporate the most reasonable hypothesis in our mathematical model.

In order to construct a mathematical model, we assume that: (a) the spores in a given population are all of a specific strain; (b) when heat is applied to a population, the temperature is held fixed; (c) a nonactivated spore must pass to the activated state before it can die; (d) an activated spore never enters the nonactivated state; (e) given a population of A activated spores, there is a positive number κ, the death-rate constant (which depends on the temperature and on the strain of spore but not on the number A), such that the number of spores which are still alive t units of time after the heating commences is $Ae^{-\kappa t}$; and (f) given a population of N nonactivated spores, there is a positive number α, the activation-rate constant, such that $Ne^{-\alpha t}$ spores are still in the nonactivated state t units of time after heating commences.

Now suppose that heat is applied to a hypothetical population consisting of $N(0)$ nonactivated and $A(0)$ activated spores. Since the population is not being nourished, the total number of spores in all three states remains constant. According to assumption (f), the number of nonactivated spores at time t is

$$N(t) = N(0)e^{-\alpha t} \qquad (0 \leqq t < \infty). \tag{1}$$

We seek an analytic expression for the number $A(t)$ of activated spores at time t. If we denote the number of living spores at time t by $L(t)$, then

$$L(t) = N(t) + A(t) \qquad (0 \leqq t < \infty). \tag{2}$$

In order to discover a differential equation involving $A(t)$, we shall examine the behavior of $A(t)$ over a small interval of time. To this end, let t be a fixed positive number, let h be a positive increment of time, and consider $A(t+h)$. Of the $A(t)$ spores in the activated state at time t, $A(t)e^{-\kappa h}$ are still in the activated state at time $t+h$ according to assumption (e). Of the $N(t)$ nonactivated spores at time t, $N(t)(1-e^{-\alpha h})$ become activated and possibly die by time $t+h$ according to assumption (f). Thus

$$A(t+h) = A(t)e^{-\kappa h} + N(t)(1 - e^{-\alpha h}) - R(t, t+h), \tag{3}$$

where $R(t, t+h)$ denotes the number of spores which are in the nonactivated state at time t but become activated and die in the time interval from t to $t+h$. Even if the $N(t)(1-e^{-\alpha h})$ spores which leave the nonactivated state in the time interval from t to $t+h$ were activated instantaneously at time t, only $N(t)(1-e^{-\alpha h})(1-e^{-\kappa h})$ of them would die by time $t+h$. Consequently,

$$0 \leqq R(t, t+h) \leqq N(t)(1 - e^{-\alpha h})(1 - e^{-\kappa h}). \tag{4}$$

From (4) we see at once that

$$\lim_{h \to 0+} \frac{R(t, t+h)}{h} = 0,$$

and, upon applying this last result to (3), we get

$$\lim_{h\to 0+} \frac{A(t+h)-A(t)}{h} = -\kappa A(t) + \alpha N(t). \tag{5}$$

If we assume that $A(t)$ is differentiable (or merely continuous) on the interval $0 \leqq t < \infty$, then (1) and (5) yield

$$A'(t) + \kappa A(t) = \alpha N(0) e^{-\alpha t}. \tag{6}$$

Depending upon whether $\alpha \neq \kappa$ or $\alpha = \kappa$, the solution of (6) is

$$A(t) = A(0)e^{-\kappa t} + \alpha N(0)\{e^{-\alpha t} - e^{-\kappa t}\}/(\kappa - \alpha) \qquad (\alpha \neq \kappa) \tag{7}$$

or

$$A(t) = \{A(0) + \kappa N(0)t\}e^{-\kappa t} \qquad (\alpha = \kappa). \tag{8}$$

If $N(0)=0$, the expressions for $A(t)$ given by (7) and (8) both reduce to $A(t)=A(0)e^{-\kappa t}$ as, indeed, they should according to assumption (e).

An examination, the details of which are left to the reader, of (7) and (8) shows that $A(t)$ attains a maximum for a positive value of t if and only if $\alpha N(0) > \kappa A(0)$; this accounts for the mysterious "peaks" in the thermal death curves observed by certain investigators.

4. The initial conditions. If, in a specific situation, the constants α, κ, $A(0)$, and $N(0)$ are known, then $A(t)$ is given by (7) or (8). The constant $A(0)$ is readily obtained by the process described in Section 3, but α, κ, and $N(0)$ are more evasive. Since values of $A(t)$ are relatively easy to obtain experimentally, one might substitute several of these values in (7) or (8) and attempt to solve the resulting system of simultaneous equations. We shall use a more sophisticated approach.

From (6) and (7) we see that if $\alpha \neq \kappa$, then

$$(d/dt)\{\log A(t)\} = -\kappa + \alpha N(0)\{A(0)e^{(\alpha-\kappa)t} + \alpha N(0)(1 - e^{(\alpha-\kappa)t})/(\kappa - \alpha)\}^{-1};$$

and, consequently, if $\alpha < \kappa$ and $N(0) \neq 0$, then

$$\lim_{t\to\infty} (d/dt)\{\log A(t)\} = -\alpha \qquad (\alpha < \kappa,\ N(0) \neq 0). \tag{9}$$

A similar argument shows that if $\alpha \geqq \kappa$, then

$$\lim_{t\to\infty} (d/dt)\{\log A(t)\} = -\kappa \qquad (\alpha \geqq \kappa). \tag{10}$$

If $\alpha \geqq \kappa$, then (10) shows that $-\kappa$ is the "limiting slope" as $t \to \infty$ of the graph of $\log A(t)$; in this case, κ can be estimated easily from the experimental data.

In order to obtain $N(0)$, it will suffice to determine $L(0)$ since $N(0)=L(0)-A(0)$ by (2).

From (1), (2), and (6) one easily deduces that $L'(t) = -\kappa A(t)$. Thus, for

complish indirectly with the naked eye things that one cannot easily do with a microscope, e.g., estimate the number of living spores at a given time t. Analogous situations of a qualitative nature are, of course, well known—it is very difficult to differentiate, by the use of a microscope, between the streptococcus of blood poisoning and the *Streptococcus cremoris* which gives the special flavor to sour cream butter, but they are easily distinguished by their biochemical activities.

In conclusion, we note that a mathematician cannot always be indifferent to the source of the differential equations which he undertakes to solve. In our model, the determination of the initial conditions involves as much mathematics as the derivation of the model itself (and we assumed that we knew *a priori* whether or not $\alpha<\kappa$). Some individuals would even go so far as to say that this phenomenon is characteristic of mathematical microbiology.

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1. J. J. Shull, G. T. Cargo and R. R. Ernst, Kinetics of heat activation and of thermal death of bacterial spores, Applied Microbiology, 11 (1963) 485–487.

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TWO DEFINITIONS OF SIMPLE CONNECTIVITY

RICHARD JOHNSONBAUGH, University of Oregon

Chevalley [1, p. 44] defines a simply connected topological space X as a space whose only covering spaces are those of the form (X^*, f) where f is a homeomorphism from X^* to X. Hereafter such spaces are called covering-simply-connected. This paper gives two examples contrasting this definition with the more usual definition given in terms of loops. (See, for example, [3] p. 234.) When we say "simply connected" we will be referring to this loop definition.

The following definitions are from [1, pp. 40–44]:

DEFINITION. *Let f be a continuous mapping of a space X^* into a space X. A subset E of X is evenly covered by X^* (with respect to f) if $f^{-1}(E)$ is not empty and every component of $f^{-1}(E)$ is mapped homeomorphically onto E by f.*

DEFINITION. *Let X be a topological space. A covering space (X^*, f) of X is a pair formed by a connected and locally connected space X^* and a continuous mapping f of X^* onto X such that each point of X has a neighborhood which is evenly covered by X^* (with respect to f).*

DEFINITION. *A space X is said to be covering-simply-connected* (c.s.c.) *if it is connected and locally connected and if, given any covering space (X^*, f) of X, f is a homeomorphism.*

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DEFINITION. *A space X is said to be covering-simply-connected* (c.s.c.) *if it is connected and locally connected and if, given any covering space (X^*, f) of X, f is a homeomorphism.*

Example of a space which is simply connected but not covering-simply-connected. Let L' denote the set of ordinals less than the first uncountable ordinal. Let $L''=L'\times[0, 1[([0, 1[=\{x|\,0\leqq x<1\})$ with lexicographic order. Adjoin an endpoint to L'' and give the resulting space L the order topology. Identify the first and last points of L and give the resulting space C the quotient topology. C is a generalized circle. Notice that C is path connected. Since any loop f: $[0, 1]\to C$ has a homeomorph of $[0, 1]$ as its range and $[0, 1]$ is simply connected, it follows that C is simply connected.

Let I denote the integers and let $C^*=I\times L''$ with lexicographic order. Define f: $C^*\to C$ by $f(n, x)=x$. Then (C^*, f) is a covering space of C, f is not a homeomorphism, and hence C is not c.s.c.

It can be shown that C^* is c.s.c. and if the fundamental group, π, is defined as the set of homeomorphisms ϕ of C^*, the c.s.c. covering space, such that $f\circ\phi=f$ (see [1] p. 52), then π is isomorphic to the integers under $+$. Of course, the fundamental group given in terms of loops is trivial.

For the second example we require the following results:

LEMMA 1. [1, p. 42]. *Let (X^*, f) be a covering space of X and let Y be a connected and locally connected subspace of X. Let Y^* be any component of $f^{-1}(Y)$. If g is the restriction of f to Y^*, then (Y^*, g) is a covering space of Y.*

A special case of Lemma 1 is:

LEMMA 2. *If (X^*, f) is a covering space of X and Y is a* c.s.c. *subspace of X, then Y is evenly covered by X^*.*

LEMMA 3 [1, p. 57]. *Let (X^*, f) be a covering space of X. Let A and B be two closed, connected, and locally connected subsets of X which are both evenly covered by X^*. If $A\cap B$ is connected and not empty, then $A\cup B$ is evenly covered by X^*.*

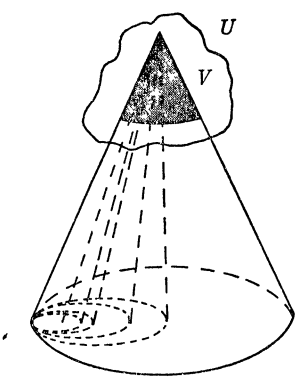


FIG. 1

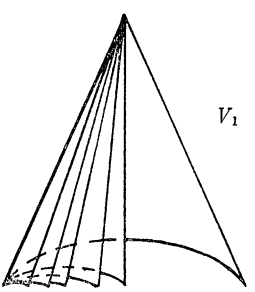


FIG. 2

Example of a space which is covering-simply-connected but not simply connected. Let B be the subset of the plane consisting of all circles with center at $(1/n, 0)$

fundamental group of $X_1 \vee X_2$ is not the free product of the groups of X_1 and X_2, these groups being trivial.

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A THIRD DEFINITION OF LOCAL COMPACTNESS

J. L. Gross, Dartmouth College

1. Introduction. Two definitions of local compactness were recently discussed in "Classroom Notes," [1]. We rephrase these slightly:

(1.1) Definition. *A space X is locally compact if and only if each point of X has a compact neighborhood.*

(1.2) Definition. *A space X is locally compact if and only if each point of X has a closed compact neighborhood.*

In this paper we consider a third definition for local compactness. In general, if P is a topological property, we define a space X to be a locally-P space if and only if each point of X has a neighborhood basis consisting of P-subsets of X (e.g., local connectedness [2] p. 61). We now make the following definition:

(1.3) Definition. *A space X is locally compact if and only if each point of X has a neighborhood basis consisting of compact subsets of X.*

If we restrict our attention to Hausdorff or regular spaces, we know that (1.1) and (1.2) are equivalent [1]. It is not hard to see that, in such spaces, (1.1) implies (1.3) [2, p. 146]. Since (1.3) implies (1.1) in any space, we observe that in Hausdorff or regular spaces, all three local compactness properties are equivalent. (The terms "compact," "normal," and "regular" in this paper are those of Kelley [2].) This note provides counterexamples to show that, in arbitrary spaces, (1.3) does not imply (1.2), and that neither (1.2) nor compactness implies (1.3).

2. Counterexamples. (2.1) Example of a locally compact: (1.3) space which is not locally compact: (1.2): Let X be the set of real numbers. The open sets will be X, the empty set, and intervals $(-n, n)$ such that n is a positive integer. The set X is evidently the only closed set containing 0. But the open cover $\{(-n, n): n = 1, 2, \cdots\}$ of X has no finite subcover, and so X is not compact. Hence 0 does not have a closed compact neighborhood. Thus X is not locally

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nor in $B(1-1/M, M-1)$. Hence $\mathcal{C}$ does not contain a finite subcover of U and so U is not compact. Thus Y is not locally compact: (1.3).

(2.3) Remark: Notice that the space Y of Example (2.1) is normal but not regular. It is normal because all nonvoid closed subsets of Y contain the point (1, 1). It is not regular because (0, 0) is in every open subset of Y.

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A STURMIAN THEOREM FOR POSITIVE MATRICES

KURT KREITH, University of California, Davis

The similarity between many of the concepts of matrix theory and linear differential equations should be a source of interest to the undergraduate student who is exposed to both of these subjects. While a full explanation of the relationship between matrix theory and other linear problems must wait for a course on functional analysis, it does seem worthwhile to point out appropriate similarities even before full explanations can be given.

One subject which is often treated in elementary differential equations but for which matrix analogues are apparently neglected is that of Sturmian theorems. In order to motivate such an analogue it is convenient to state a slightly weakened version of Sturm's theorems as follows. (The positiveness of $a(t)$ and $b(t)$ is not essential except in this formulation.)

Let A and B be formal differential operators defined by

$$\begin{aligned} Ax(t) &= -\frac{1}{a(t)}\frac{d^2}{dt^2}x(t), \\ Bx(t) &= -\frac{1}{b(t)}\frac{d^2}{dt^2}x(t), \end{aligned} \tag{1}$$

where $a(t)$ and $b(t)$ are continuous positive functions. Let $t_1<t_2$ be two arbitrary but fixed real numbers and define the boundary conditions

$$\begin{aligned} \theta x(t_1) - x'(t_1) &= 0, \\ \phi x(t_2) + x'(t_2) &= 0, \end{aligned} \tag{2}$$

where θ and ϕ are real numbers. The notation $\theta=\infty$ $(\phi=\infty)$ will be used to denote the boundary condition $x(t_1)=0$ $(x(t_2)=0)$.

We use the formal operators (1) and the boundary conditions (2) to define differential operators $A_{\theta\phi}(B_{\theta\phi})$ whose domain consists of functions which are twice continuously differentiable and satisfy (2), and which operate on their domains according to the rule (1).

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ON THE GEOMETRY OF NUMBERS IN ELEMENTARY NUMBER THEORY

G. E. ANDREWS, Pennsylvania State University

In most elementary texts on number theory, the geometry of numbers is rarely mentioned. This is probably due partly to the feeling that the introduction of convexity, n-dimensional geometry, etc., in an elementary course unnecessarily burdens students; also most of the theorems in the geometry of numbers seem beyond the field of interest of a first course in number theory. The following proof of the theorem on the linear diophantine equation gives the flavor of the geometry of numbers but does not call for any knowledge beyond a little analytic geometry.

We remark that the geometric interpretation of the linear Diophantine equation used here was originally given by H. J. S. Smith [4, p. 147], but Smith's proof is entirely algebraic, depending on the theory of continued fractions. A totally different geometric proof depending on the construction of regular polygons was given by P. Johnson [2].

In the following, $P=(p_1, p_2)$, $Q=(q_1, q_2)$, $\cdots$ will denote points of the plane; $O=(0, 0)$. A lattice point is a point with integral coordinates.

LEMMA. *If there exists no integer $b>1$ such that $b\mid a_1$ and $b\mid a_2$ (where a_1 and a_2 are integers with $a_1>0$, $a_2>0$), then all the lattice points on the line $L_0=\{(x, y); y=a_2x/a_1\}$ are given by (ta_1, ta_2) with $t=0, \pm1, \pm2, \cdots$.*

Proof. Let R be the lattice point with positive coordinates nearest the origin on L_0. Clearly R exists since there are at most a_1a_2 lattice points (x^*, y^*) with $1\leqq x^*\leqq a_1$ and $1\leqq y^*\leqq a_2$. Assume that the n lattice points with positive coordinates on L_0 nearest O are (r_1, r_2), $(2r_1, 2r_2)$, $\cdots$, (nr_1, nr_2). Suppose there is a lattice point P such that $nr_1<p_1<(n+1)r_1$ and $nr_2<p_2<(n+1)r_2$. Then $0<p_1-nr_1<r_1$ and $0<p_2-nr_2<r_2$; hence, (p_1-nr_1, p_2-nr_2) is a lattice point on L_0 nearér to O than R. Hence by mathematical induction (and by the fact that L_0 is symmetric with respect to O), all lattice points on L_0 are given by (tr_1, tr_2) with $t=0, \pm1, \pm2, \cdots$. Thus $(a_1, a_2)=(t_0r_1, t_0r_2)$ for some positive integer t_0. Thus by the hypothesis of the lemma, we must have $t_0=1$. Hence $A=R$.

For further use note that the distance between (ta_1, ta_2) and $((t+1)a_1, (t+1)a_2)$ is given by $(a_1^2+a_2^2)^{1/2}$.

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THEOREM. *If there exists no integer $b>1$ such that $b \mid a_1$ and $b \mid a_2$ (where a_1 and a_2 are integers with $a_1>0$, $a_2>0$), then there exist integers k and m such that $ka_1+ma_2=1$.*

Proof. Consider the set Σ of all lines parallel to $L_0=\{(x, y); y=a_2x/a_1\}$ and containing lattice points. If $L\in\Sigma$, P is a lattice point on L, and $d(L)$ is the distance between L and L_0, then the area of the triangle $\triangle(OAP)$ is $\left|\frac{1}{2}(a_1p_2-a_2p_1)\right| = \frac{1}{2}(a_1^2+a_2^2)^{1/2}d(L)$. Hence $d(L)=n\cdot(a_1^2+a_2^2)^{-1/2}$ where n is an integer. It is easy to verify that the adjacent lines of Σ are all the same distance d apart, for translating a lattice point on one line to a lattice point on another line transforms Σ into itself. Consider the square S of side ρ centered on O with two sides parallel to L_0. There are $\rho^2+O(\rho)$ lattice points inside or on S (a fact easily established by the technique of Gauss which is well described in [1, pp. 29–30]). But there are at most $\rho/d+1$ lines in S containing lattice points and parallel to L_0, and on each such line there are at most $\rho/(a_1^2+a_2^2)^{1/2}+1$ lattice points (by the note following the lemma and by the remarks concerning the translation properties of Σ). Hence

$$\rho^2 + O(\rho) \leqq \rho^2 d^{-1}(a_1^2 + a_2^2)^{-1/2} + \rho d^{-1} + \rho\cdot(a_1^2 + a_2^2)^{-1/2} + 1$$

which is true for all positive ρ only if $d\leqq(a_1^2+a_2^2)^{-1/2}$. Since we have already established that $d\geqq(a_1^2+a_2^2)^{-1/2}$, we obtain $d=(a_1^2+a_2^2)^{-1/2}$. Thus there is a lattice point P on a line a perpendicular distance $(a_1^2+a_2^2)^{-1/2}$ from L_0. Thus the area of the triangle $\triangle(OAP)$ is

$$\tfrac{1}{2}(a_1^2 + a_2^2)^{1/2}(a_1^2 + a_2^2)^{-1/2} = \tfrac{1}{2} = \left|\tfrac{1}{2}(a_1p_2 - a_2p_1)\right| .$$

Hence one of the pairs $(k, m)=(p_2, -p_1)$ or $(-p_2, p_1)$ must fulfill the conclusion of the theorem.

Thus a standard technique from the geometry of numbers yields the essential result needed to develop such properties as unique factorization, etc., [3, Chapter 2]. In conclusion, I should like to express my appreciation to a very helpful referee; in particular, he brought references [1] and [4] to my attention.

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MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D.C. 20005.

INSTITUTIONAL INFLUENCES IN THE GRADUATE TRAINING OF PRODUCTIVE MATHEMATICIANS

B. R. SIEBRING, University of Wisconsin, Milwaukee

Recently the American Council on Education published an evaluation of graduate education [1]. Essentially this evaluation was a compilation of the opinions of experts in the various academic disciplines on the quality of graduate faculties and the effectiveness of graduate programs. This paper is the result of an attempt at another type of evaluation of graduate instruction in mathematics. The criterion of this evaluation is the productivity of the doctoral graduates of the various mathematics departments. Essentially this study compares graduate departments in mathematics in terms of (a) number of productive Ph.D. mathematicians graduated, (b) number of papers published by the Ph.D. graduates, (c) the proportion of Ph.D. graduates who become productive mathematicians, and finally (d) the number of papers published per Ph.D. graduate in mathematics. For the purposes of this investigation a "productive mathematician" is defined as one who has published at least one paper which was reviewed in the *Mathematical Reviews* through the year 1964.

The investigation is limited to mathematicians who received their doctorates in the 1950–59 decade, who were listed in the *Combined Membership List (1963–1964) of the American Mathematical Society, Mathematical Association of America, and Society for Industrial and Applied Mathematics* [2] and whose biographies were found in the 9th or 10th editions of the *American Men of Science* [3].

The top ten institutions in terms of the number of productive mathematicians produced are listed in Table I. Upon comparison of this listing to rankings of the American Council on Education, one notices a high degree of similarity. Eight of the top ten institutions according to the A.C.E. rating on quality of graduate faculty are among the top ten institutions in Table I. Also seven of the top ten institutions according to the A.C.E. rating on effectiveness of the graduate programs are among the top ten institutions in Table I. One institution which stands out more favorably in Table I than in the A.C.E. study is New York University. It should be noted that the top ten institutions in Table I provided the graduate education for over forty percent of the productive mathematicians of this period.

Table II lists the top ten institutions according to the number of papers published by their doctoral graduates in mathematics. Again one notices the high degree of similarity to the A.C.E. rankings. Over 47 percent of the papers published by mathematicians who received their doctorate degrees in the 1950–

TABLE I. Institutions ranked by the number of Ph.D.'s granted to productive mathematicians during the 1950–59 decade.

Rank	*Institution*	*A.C.E. Program*	*A.C.E. Faculty*	*Number*	*Cumulative Percent*
1	New York University	12	8	54	5.2
2	Harvard University	2	1	53	10.3
3	Princeton University	1	3	51	15.2
4	The University of Chicago	6	4	50	20.0
5	University of California (Berkeley)	3	2	43	24.1
6	Massachusetts Institute of Technology	5	5	38	27.8
7	The University of Michigan	10	11	36	31.2
8 & 9	Columbia University	15	9	35	34.6
	University of Wisconsin (Madison)	8	10	35	38.0
10	University of Illinois	13	12	33	41.1
	Institutions ranked 11 through 21 (Includes no foreign institutions)			234	63.6
	Institutions ranked 22 through 30 (Includes no foreign institutions)			116	74.7
	Total of 104 institutions (Includes 29 foreign institutions)			1041	100.0

TABLE II. Institutions ranked by the number of papers published by doctorate graduates who received their Ph.D. degrees in the 1950–59 decade

Rank	*Institution*	*A.C.E. Program*	*A.C.E. Faculty*	*Number*	*Cumulative Percent*
1	The University of Chicago	6	4	501	7.1
2	Princeton University	1	3	465	13.6
3	Columbia University	15	9	406	19.3
4	Harvard University	2	1	386	24.8
5	New York University	12	8	349	29.7
6	University of California (Berkeley)	3	2	327	34.3
7	The University of Michigan	10	11	286	38.3
8	University of Wisconsin (Madison)	8	10	218	41.4
9	Massachusetts Institute of Technology	5	5	211	44.4
10	University of California (Los Angeles)	17	16	174	46.8
	Institutions ranked 11 through 20 (Includes 1 foreign institution)			1421	66.9
	Institutions ranked 21 through 30 (Includes 2 foreign institutions)			921	79.8
	Total of 104 institutions (Includes 29 foreign institutions)			7096	100.0

1959 decade were written by graduates of the top ten institutions.

One might conclude that the institutions which ranked high in Tables I and II were there because of the large output of Ph.D.'s in mathematics. That is,

TABLE III. Institutions ranked by percent of total output of Ph.D. mathematicians in the 1950–59 decade who became productive mathematicians

Rank	*Institution*	*A.C.E. Program*	*A.C.E. Faculty*	*Productive Mathematicians*	*Total No. of Doctorates*	*Percent*
1	Johns Hopkins University	A.P.[1]	20	6	7	85.7
2	Harvard University	2	1	53	71	74.6
3	Case Institute of Technology	—	—	5	7	71.4
4	Vanderbilt University	—	—	4	6	66.7
5	Tulane University of Louisiana	A.P.	G.[2]	15	23	65.2
6	University of Pennsylvania	A.P.	22	21	34	61.8
7	University of Wisconsin (Madison)	8	10	35	57	61.4
8	University of Oregon	A.P.	A.P.[3]	14	23	60.9
9, 10, 11	Lehigh University	—	—	3	5	60.0
	Rice University	A.P.	G.	9	15	60.0
	Rutgers University	—	—	3	5	60.0
12	Cornell University	9	13	15	26	57.7
13	University of Washington (Seattle)	13	17	19	33	57.6
14	University of Rochester	A.P.	G.	4	7	57.1
15	University of Texas	A.P.	A.P.	24	43	55.8
16	University of Washington (St. Louis)	A.P.	A.P.	5	9	55.6
17	Massachusetts Institute of Technology	5	5	38	71	53.5
18, 19, 20	University of Maryland	A.P.	G.	11	22	50.0
	Syracuse University	A.P.	A.P.	9	18	50.0
	University of Tennessee	—	—	8	16	50.0
	All American Institutions			907	2326	39.0

[1] The letters A.P. in this column refer to institutions not ranked by the A.C.E. but classed as "Acceptable plus."

[2] The letter G. in this column refers to institutions not ranked by the A.C.E. but classed as "Good."

[3] The letters A.P. in this column refer to institutions not ranked by the A.C.E. but classed as "Adequate plus."

institutions which produced large numbers of Ph.D.'s in mathematics would naturally be expected to account for a large number of productive mathematicians.

A better evaluation of an institution's productivity might therefore be based upon an institution's total output of Ph.D.'s in mathematics. Tables III and IV present this type of evaluation. Table III ranks institutions according to the percent of the total output of Ph.D. mathematicians in the 1950–59 decade who became productive mathematicians. The total output of Ph.D.'s in mathematics by institution was obtained from the U.S. Office of Education document, *Degrees in the Biological and Physical Sciences, Mathematics, and Engineering* [4]. Table IV ranks the institutions according to the number of publications per

TABLE IV. Institutions ranked by the number of publications per Ph.D. mathematician graduated in the 1950–59 decade

Rank	*Institution*	*A.C.E. Program*	*A.C.E. Faculty*	*Papers Published*	*Total No. of Doctorates*	*Ratio*
1	Tulane University of Louisiana	A.P.[1]	G.[2]	145	23	6.30
2	Harvard University	2	1	386	71	5.44
3	The University of Chicago	6	4	501	101	4.96
4	Columbia University	15	9	406	82	4.95
5	Indiana University	A.P.	—	130	29	4.48
6	University of Washington (Seattle)	13	17	138	33	4.18
7	Cornell University	9	13	108	26	4.15
8	Princeton University	1	3	465	114	4.08
9	University of Missouri	—	—	48	12	4.00
10	University of Wisconsin (Madison)	8	10	218	57	3.82
11	University of Notre Dame	A.P.	A.P.[3]	61	16	3.81
12 & 13	Wayne State University	—	A.P.	81	22	3.68
	University of Maryland	A.P.	G.	81	22	3.68
14	Syracuse University	A.P.	A.P.	66	18	3.67
15	University of Pennsylvania	A.P.	22	123	34	3.62
16	Rice University	A.P.	G.	53	15	3.53
17	University of Texas	A.P.	A.P.	128	43	2.98
18	University of California (all campuses)	—*	—*	501	170	2.95
19	New York University	12	8	349	120	2.91
	All American Institutions			6191	2326	2.66

[1] The letters A.P. in this column refer to institutions not ranked by the A.C.E. but classed as "Acceptable plus."

[2] The letter G. in this column refers to institutions not ranked by the A.C.E. but classed as "Good."

[3] The letters A.P. in this column refer to institutions not ranked by the A.C.E. but classed as "Adequate plus."

* The A.C.E. study ranked University of California campuses individually.

Ph.D. mathematician graduated in the 1950–59 decade. Upon comparing these rankings with those in the A.C.E. report, one notices an appreciable difference. Only two of the top ten institutions in each of the A.C.E. rankings appear among the top ten in Table III. The top ten institutions in Table IV include five from the top ten institutions in each of the A.C.E. rankings. Tables III and IV show quite clearly that small departments unable to support a program or faculty of national renown are still capable of providing quality education for small numbers of Ph.D. candidates. It is interesting to note that only one institution, Harvard University, maintains a position among the top ten institutions in all four rankings in this study, as well as in the two rankings of the A.C.E. report.

In order that institutions might be compared by geographical location, composite data for each of the major geographical divisions were tabulated. These

TABLE V. Composite data for the major geographical divisions

Geographical Division	*Productive Mathematicians*		*Total No. of Doctorates* (3)	(1) —×100 (3) (4)	*Publications*		(5) — (3) (7)
	Number (1)	*Percent* (2)			*Number* (5)	*Percent* (6)	
Northeastern	304	29.2	791	38.4	2552	35.8	3.23
North Central	284	27.3	730	38.9	1889	26.5	2.59
West	157	15.1	357	44.0	908	12.7	2.54
South	162	15.6	448	36.2	842	11.8	1.88
Foreign	134	12.9	—	—	947	13.3	—
Total	1041	100.1	2326*	39.0*	7138	100.1	2.66*

* These data exclude foreign institutions.

data are presented in Table V.

Although this study presents evaluations which are less subjective than those in the A.C.E. report, it may be criticized on the following grounds:

(1) A study of this nature can never be current. The A.C.E. report has a time lag of only several years, but this study has a time lag of from five to fifteen years. That is, this study evaluates the performance of the departments as they were in the fifties.

(2) Admittedly publication counting is a very narrow measure of research productivity, as the research value of all papers is not uniform. One article may represent as much work as several others.

(3) Whether a productive mathematician is included in this study or not depends on whether his biography appeared in the *American Men of Science*. The reliability of this study depends on the assumption that the proportion of productive mathematicians who submit their biographies to the *American Men of Science* is independent of doctoral origin. Yet it is possible that graduates of some institutions are more likely to submit their biographies than are graduates of other institutions.

In spite of these deficiencies, the writer feels that evaluations of this type should be considered in contemplating the relative merits of departments offering graduate instruction in mathematics.

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incorporate such a seemingly mundane feature as specific training for teaching elementary school mathematics? If so, is there a substantial number of prospective elementary teachers who would persevere in the rigorous training necessary to excel both in humanitarian and cultural endeavors as well as in the teaching of mathematics?

Unless such challenges are honestly met, we will continue to develop mathematical neuroses in elementary teachers and their pupils. The details of a teacher preparation program such as that prescribed by CUPM will have to be resolved by each institution according to its own peculiar structure, but its ultimate impact must rely upon a core of college teachers who possess the rare talent of communicating mathematical knowledge and technique to students who are prospective teachers. Professors who serve in such a capacity must be willing to, and be free to, consider teaching their primary function. Will you fight the necessary curriculum battles and the publish or perish deterrents for the privilege of being one of these professors?

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

COLLABORATING EDITORS: J. BARLAZ, Rutgers—The State University; L. CARLITZ, Duke University; HASKELL COHEN, University of Massachusetts; H. EVES, University of Maine; M. S. KLAMKIN, Ford Scientific Laboratory; R. C. LYNDON, University of Michigan; and A. WILANSKY, Lehigh University.

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P.O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before March 31, 1968. ***Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.***

E 2000 [1967,719] **Correction.** *Proposed by Howard Kleiman, Queensborough Community College, New York*

Let q be any prime number >2. Show that

$$\sum_{n=1}^{q} x^{q-n} \left| \left\{ \sum_{m,n=1}^{q-1} x^{mn} + q - 1 \right\} . \right.$$

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$$\sum_{n=1}^{q} x^{q-n} \left| \left\{ \sum_{m,n=1}^{q-1} x^{mn} + q - 1 \right\} . \right.$$

E 2031. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

a is any real number. Prove the inequality

$$\left|\sqrt[3]{-\frac{a}{2}+\sqrt{\frac{a^2}{4}+\frac{1}{27}}}+\sqrt[3]{-\frac{a}{2}-\sqrt{\frac{a^2}{4}+\frac{1}{27}}}\right|<\sqrt[3]{|a|}.$$

E 2032. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Prove the following inequality

$$\min[(b-c)^2,(c-a)^2,(a-b)^2]\leqq\tfrac{1}{2}(a^2+b^2+c^2),$$

with a, b, c, real numbers.

Study the analogous problem for $\min[(a_k-a_i)^2]$, $k<i$, k, $i=1, 2, \cdots, n$.

E 2033. *Proposed by D. R. Rao, Secunderabad, India*

Show that $p=1093$ simultaneously satisfies

$$2^{(p-1)/2}+1\equiv 0 \pmod{p^2};$$

$$2^j-2^{j-k}+2^{j-2k}-\cdots+1\equiv 0 \pmod{p^2},$$

where $j=\frac{1}{2}[p-(2k+1)]$, $k=2+4s_n$ $(n=0, 1, 2, 3, 4)$, and s_n is the sum of the first n natural numbers.

E 2034. *Proposed by Merrill Barneby, Wisconsin State University*

Show that there exist solutions in positive integers of the system:

$$a+b+c=x+y, \qquad a^3+b^3+c^3=x^3+y^3.$$

In particular, show that there are infinitely many in which a, b, c, form an arithmetic progression.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Polynomial Identity

E 1829 [1965, 1021; 1967, 323]. *Proposed by M. J. Pascual and S. D. Beck, Watervliet Arsenal, N.Y.*

If m is a positive integer and x is arbitrary, establish the following identity:

$$\sum_{k=m}^{2m-1}\binom{2m-1}{k}x^k(1-x)^{2m-1-k}=\sum_{k=m}^{2m-1}\binom{k-1}{m-1}x^m(1-x)^{k-m}.$$

II. *Additional Comments by W. C. Guenther, University of Wyoming.*

(1) The identity is more general than claimed. It should be

$$(*)\qquad \sum_{k=m}^{r}\binom{k-1}{m-1}x^m(1-x)^{k-m}=\sum_{k=m}^{r}\binom{r}{k}x^k(1-x)^{r-k},$$

where m and r are positive integers, $r \geqq m$, which also holds for all x.

(2) An important use of (*) is in evaluating cumulative negative binomial sums in terms of ordinary binomial sums. By using both the Harvard table [2] and the Ordnance Corps Table [7] one can obtain values to at least 5 significant figures for the right hand side of (*) for

$$x = .01(.01)\ .99,\ r = 1(1)\ 150(10)\ 200(20)\ 500(50)\ 1000.$$

This useful fact is apparently not well known although it appears in several texts, e.g. [1].

(3) The literature has many proofs for $0 < x < 1$. See [4] and [5] for bibliography. The earliest paper mentioned is [3], 1933, but the result was probably known earlier. The result is also implied by a paper by Rider [6] appearing in this MONTHLY.

(4) If one is acquainted with the binomial and negative binomial, then the proof is easy by a probability argument. Suppose we set as a goal the obtaining of m successes in a binomial type experiment, with probability of success on a single trial being x, but agree to repeat the experiment no more than r times. Our goal can be achieved in two ways. First, if the mth success is obtained on either the mth, $(m+1)$th, $\cdots$, rth repetition, the goal is accomplished. The probability that this series of events happens is given by the left hand side of (*). The second way to achieve the goal in r repetitions of the experiment is to plan to repeat the experiment r times and obtain m, $m+1$, $\cdots$, or r successes. The probability that the latter series of events happens is given by the right hand side of the identity. Hence for $0 < x < 1$ the identity is established. A well-known theorem of algebra states that if two polynomials of degree r are equal for more than r values, they are equal for all values.

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Minimal Subseries—A Class of Convergent Series

E 1836 [1965, 1128]. *Proposed by D. E. Daykin, University of Malaya, Kuala Lumpur*

Given a convergent series $\sum_{i=0}^{\infty} u_i$ of positive terms u_i, find the greatest

lower bound g of $\sum_{i=0}^{\infty} \epsilon_i u_i$ over all sequences $\{\epsilon_i\}$, where ϵ_i is 0 or 1 and $\epsilon_{i-1}+\epsilon_i+\epsilon_{i+1}>0$ for $i \geqq 1$.

Solution by the proposer. If a sequence $\{\epsilon_i\}$ has $\epsilon_{k-1}=\epsilon_k=\epsilon_{k+1}=1$ for some $k \geqq 1$, then $\sum_{i=0}^{\infty} \epsilon_i u_i > \sum_{i=0}^{\infty} \epsilon_i' u_i$ where $\epsilon_i' = \epsilon_i$ for $i \neq k$ but $\epsilon_k' = 0$. Hence we may also assume that $\epsilon_{i-1}+\epsilon_i+\epsilon_{i+1}<3$ for $i \geqq 1$. For $n \geqq 1$ consider all possible nth partial sums $\sum_{i=0}^{n} \epsilon_i u_i$. Let p_n, q_n, r_n and s_n denote the smallest of these sums which end with $(\epsilon_{n-1}, \epsilon_n)$ equal to (0, 0), (1, 0), (0, 1) and (1, 1) respectively. Then the numbers p_n, q_n, r_n, s_n satisfy the recurrence relations

$$p_1 = 0, \qquad q_1 = u_0, \qquad r_1 = u_1, \qquad s_1 = u_0 + u_1,$$

and for $n \geqq 1$,

$$p_{n+1} = q_n, \quad q_{n+1} = \min(r_n, s_n), \quad r_{n+1} = u_{n+1} + \min(p_n, q_n), \quad s_{n+1} = u_{n+1} + r_n.$$

For $n \geqq 1$ let $m_n = \text{minimum } (p_n, q_n, r_n, s_n)$. Then the sequence $\{m_n\}$ is non-decreasing bounded above by $\sum_{i=0}^{\infty} u_i < \infty$, and so has a finite limit h. We wish to show that $h=g$ for then we will have a method for determining g.

Now $h \leqq g$ because $m_n \leqq \sum_{i=0}^{n} \epsilon_i u_i < \sum_{i=0}^{\infty} \epsilon_i u_i$ for all sequences $\{\epsilon_i\}$. Hence it only remains to show that, given $\delta>0$, there is some sequence $\{\epsilon_i\}$ with $\sum_{i=0}^{\infty} \epsilon_i u_i < h+\delta$. For this δ there is an integer G such that $\sum_{i=G+1}^{\infty} u_i < \delta$, and for this G there is a sequence $\{\epsilon_i\}$ such that $\sum_{i=0}^{G} \epsilon_i u_i = m_G < h$, so for this sequence $\sum_{i=0}^{\infty} \epsilon_i u_i = m_G + \sum_{i=G+1}^{\infty} \epsilon_i u_i < h+\delta$ and we are through.

Equivalence Classes of Poker Hands

E 1865 [1966, 309]. *Proposed by E. O. Thorp, New Mexico State University*

(a) How many distinct hands (i.e. of different values) are there in a 5-card draw poker? More precisely, the hands can be grouped into equivalence classes such that the hands in any equivalence class are equal (tie), and such that, given any pair of equivalence classes, all the hands in one equivalence class beat all the hands in the other. The problem is to find how many equivalence classes there are.

(b) Which equivalence class (or classes) determines the median hand?

Solution by D. C. B. Marsh, Colorado School of Mines. By the very nature of five-card draw, this is an ill-defined problem; on the matter of equivalence classes one might be tempted to include "Skip straights" et al.; also, the category full house Kings over Queens would technically beat full house Kings over Jacks, but in practice the winner would be the one with the quicker draw . . . and so it goes.

However, one may compute 7,462 equivalence classes with the median falling between "High card, AKQJ7" and "High card, AKQJ6."

TABULATED:

Name of class	*No. of classes*	*No. of hands per class*	*Total no. of hands*
Straight flush	10	4	40
Four of a kind	156	4	624
Full house	156	24	3,744
Flush	1,277	4	5,108
Straight	10	1,020	10,200
Three of a kind	858	64	54,912
Two pair	858	144	123,552
One pair	2,860	384	1,098,240
High card	1,277	1,020	1,302,540

Also solved by Walter Bluger, John Berslem, Allan Gottleib, Rudolf Ondrejka, Robert Prener, David Tranberg, and J. E. Wilkins, Jr.

Solvers noted that the definition of "equivalence classes" should be more explicit. Do hands below a pair go into one class or should "ace high" and "king high" be in separate classes? Should "four of a kind" be a single class or broken up according to the fifth card? How can "aces full on kings" and "aces full on queens" be compared since they cannot compete in a showdown? Different interpretations lead to different solutions. (Several solvers failed to observe that the median hand does not occur in the median class.

Determinants Related to Pascal's Theorem

E 1870 [1966, 310]. *Proposed by Arun Sanyal, Indian Institute of Technology, Kharagpur*

(1) Prove that the third order determinant (a_{ij}) vanishes identically, where

$$a_{11} = (A + B)DE - (D + E)AB, \quad a_{12} = AB - DE, \quad a_{13} = A + B - D - E,$$
$$a_{21} = (B + C)EF - (E + F)BC, \quad a_{22} = BC - EF, \quad a_{23} = B + C - E - F,$$
$$a_{31} = (C + D)FA - (F + A)CD, \quad a_{32} = CD - FA, \quad a_{33} = C + D - F - A.$$

(2) The same, where

$$a_{11} = \sin(A + B)\cos(D - E) - \sin(D + E)\cos(A - B),$$
$$a_{12} = \cos(A + B)\cos(D - E) - \cos(D + E)\cos(A - B),$$
$$a_{13} = \cos(A + B + D + E), \quad a_{23} = \cos(B + C + E + F),$$
$$a_{33} = \cos(C + D + F + A),^*$$
$$a_{21} = \sin(B + C)\cos(E - F) - \sin(E + F)\cos(B - C),$$
$$a_{22} = \cos(B + C)\cos(E - F) - \cos(E + F)\cos(B - C),$$
$$a_{31} = \sin(C + D)\cos(F - A) - \sin(F + A)\cos(C - D),$$
$$a_{32} = \cos(C + D)\cos(F - A) - \cos(F + A)\cos(C - D).$$

Solution suggested by the proposer. Consider six points P_i chosen on the parabola $y^2 = x$ as follows: $P_1(A^2, A)$, $P_2(B^2, B)$, $P_3(C^2, C)$, $P_4(D^2, D)$, $P_5(E^2, E)$,

$P_6(F^2, F)$. The point of intersection of lines P_1P_2 and P_4P_5 is determined routinely to be

$$\left(\frac{(A+B)DE-(D+E)AB}{D+E-A-B}, \frac{DE-AB}{D+E-A-B}\right).$$

Finding, similarly, the intersections of P_2P_3 with P_5P_6 and of P_3P_4 with P_6P_1, and recalling Pascal's theorem, the determinant which indicates the collinearity of the three points is immediately reduced to (1).

We choose $(\cos 2A, \sin 2A)$, $(\cos 2B, \sin 2B)$, etc., as 6 points on the circle $x^2+y^2=1$, and follow the same procedure with some additional trigonometric reductions. This gives a determinant which agrees with (2) except that the third column (*) should be $a_{13}=\sin(A+B-D-E)$, $a_{23}=\sin(B+C-E-F)$, $a_{33}=\sin(C+D-F-A)$.

Also solved, (1) only, by Michael Goldberg, D. C. B. Marsh, and P. V. Murty (India).

An Increasing Function

E 1880 [1966, 411]. *Proposed by Robert Schlesinger, Washington University, St. Louis, Mo.*

Show that $f(t)=(t-\sin t)(\pi-t-\sin t)$ is an increasing function of t for $0<t<\frac{1}{2}\pi$.

I. *Solution by R. G. Kayel, Grumman Aircraft, Bethpage, N.Y.* Assume that $f'(t)\leqq 0$ for some t, $0<t<\frac{1}{2}\pi$. Then

$$f'(t) = (1-\cos t)(\pi - t - \sin t) - (t - \sin t)(1+\cos t) \leqq 0$$

implies

$$\sin t - t \leqq \frac{1-\cos t}{1+\cos t}\cdot(\sin t + t - \pi) < \sin t + t - \pi,$$

which gives $\pi<2t$ for some t, $0<t<\frac{1}{2}\pi$. This contradiction shows $f'(t)>0$ for all t, $0<t<\frac{1}{2}\pi$.

II. *Solution by L. A. Steen, Saint Olaf College.* We need only show that $f'(t)=\pi-\pi\cos t-2t+\sin 2t$ is positive for $0<t<\frac{1}{2}\pi$. Since $f''(t)=\sin t(\pi-4\sin t)$, f' is strictly increasing on $(0, \sin^{-1}\pi/4)$ and strictly decreasing on $(\sin^{-1}\pi/4, \frac{1}{2}\pi)$. But $f'(0)=f'(\frac{1}{2}\pi)=0$. Therefore $f'>0$ on $(0, \frac{1}{2}\pi)$.

Also solved by Anders Bager (Denmark), Merrill Barnebey R. A. Bell, J. P. Celenza, Yu Chang & Sidney Spital, R. J. Cormier, P. M. Ellis, M. A. Ettrick, H. J. Fletcher, A. S. Galbraith, Michael Goldberg, J. H. Graham, H. A. Greenbaum, Louise S. Grinstein, Emil Grosswald, Erwin Just, Roman Kaluzniacki, M. S. Kaplan, M. S. Klamkin, E. S. Langford, Arthur Lieberman, Dan Marcus, D. C. B. Marsh, Renate McLaughlin, Lieselotte Miller, R. E. Moyer, P. V. Murty (India), C. B. A. Peck, Stanton Philipp, R. S. Ray, Simeon Reich (Israel), C. H. Sampson, P. A. Scheinok, Allan Wachs, Mitchell Wand, John Wessner, and the proposer.

A Locus Problem

E 1881 [1966, 411]. *Proposed by Omar Khayyam, Jr., University of California, Berkeley*

Describe the locus of a point P in the Euclidean plane for which there is a function f with the property that the graph of a function g defined on the whole real line is symmetric with respect to P if and only if g commutes with f (i.e., $f(g(x))=g(f(x))$ for all real x).

Solution by Daniel Marcus, Student, Adelphi University. For the point (a, b), $a\neq b$, it is necessary that f (assuming it exists) commute with $g_m(x)=m(x-a)+b$ for all real m, so that $f(g_m(x))=g_m(f(x))$, or

$$f(m(x-a)+b)=m(f(x)-a)+b \tag{1}$$

for all real x and all real m.

First suppose that $x=\alpha\neq a$ and take $m=(\alpha-b)/(\alpha-a)$. Then (1) reduces to $f(\alpha)=\alpha$, $\alpha\neq a$. Now take $x=a+1$ and $m=a-b$, and (1) reduces to $f(a)=a$. This means that f must be the identity function, which commutes with every function, even those without symmetric graphs. Hence if $a\neq b$, there is no f with the desired property.

Now suppose that the graph of $g(x)$ is symmetric about the point (a, a), so that $g(x)=2a-g(2a-x)$ for all real x. Then take $f(x)=2a-x$, which gives $f(g(x))=g(2a-x)=g(f(x))$.

Conversely, suppose that $f(g(x))=g(f(x))$, so that, if $f(x)=2a-x$, we have $2a-g(x)=g(2a-x)$, which implies symmetry. Thus $f(x)=2a-x$ has the desired property for the point (a, a).

The locus of points for which such a function exists is the line $y=x$.

Square Roots of Unit Matrix

E 1882 [1966, 411]. *Proposed by W. C. Beckham, Kirtland Air Force Base*

Let I represent the $n\times n$ unit matrix. A is an $n\times n$ matrix which has only one nonzero entry in each row and column, these nonzero entries being either 1 or -1. If $A^2=I$, determine the number of ways there are of writing A.

I. *Solution by D. P. Roselle, University of Maryland.* Call the matrix $A=(a_{ij})$ *admissible* if it satisfies the conditions stated. It is clear from the row by column rule of multiplication that an admissible matrix is symmetric. Thus, if $\phi(n)$ denotes the number of admissible $n\times n$ matrices, then

$$\phi(n+1)=2\phi(n)+2n\phi(n-1), \qquad n\geq 1,$$

as follows if we first set $a_{11}=\pm 1$ and then $a_{1k}=\pm 1$ $(2\leq k\leq n+1)$.

Using the initial values $\phi(0)=1$, $\phi(1)=2$, we find that

$$\phi(n)=n!\sum 2^k/k!r!$$

where the summation extends over those nonnegative integers k and r which satisfy $k+2r=n$.

II. *Solution by Daniel Marcus, student, Adelphi University.* From the given conditions it is clear that A must be symmetric. Aside from the possibility of either of two nonzero entries, this problem is identical to E 1759 [1966, 413], the solution of which states that there are

$$\sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{2k} \frac{(2k)!}{2^k k!}$$

ways of placing n rooks on an $n \times n$ chessboard so that no rook can take any other, and so that they are placed symmetrically about a given diagonal.

The necessary trivial modification is to multiply each term in the summation by 2^{n-k}, $n-k$ being the number of choices for positive or negative one, when there are $2k$ nonzero entries off the main diagonal. Hence the number of ways of writing A is

$$\sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{2k} \frac{(2k)!2^{n-2k}}{k!}.$$

Also solved by J. C. Abad, J. A. Carpenter & J. D. Fulton, R. B. Eggleton (Australia), Scott Harrod, Donald Jeffords, Erwin Just, E. S. Langford, Norman Locksley, D. C. B. Marsh, Norman Miller, Robert Patenaude (Germany), C. B. A. Peck, Stanton Philipp, B. V. Rao & M. N. Sastry (India), Bernard Rosner, Steven Russ, Sidney Spital, E. T. Wong and the proposer.

Solvers wrote the final formula in a number of different, equivalent ways. Marsh gives it in the form

$$\phi(n) = \frac{d^n}{dx^n}(e^{x^2+2x})\Big]_{x=0}.$$

A Generalization of the μ-function

E 1883 [1966, 411]. *Proposed by A. M. Vaidya, Texas Technological College*

The Möbius function $\mu(n)$ provides a criterion for deciding whether or not a given integer is squarefree. Find a function (in terms of μ) which has different constant values for primes, for composite squarefree integers, and for other integers.

I. *Solution by H. S. Hahn, Pennsylvania State University.* Such a function is given by

$$\phi(n) = \prod_{d|n} \mu(d), \qquad n > 1,$$

which is -1 if n is prime, 0 if n has a squared factor, and $+1$ if n is a squarefree composite. We explain the last part.

Suppose n is a product of $k(>1)$ distinct primes. Then the number of divisors

of n with j primes is $\binom{k}{j}$ and, for each such divisor d, $\mu(d)=(-1)^j$. Thus $\phi(n)=(-1)^{\alpha(k)}=+1$, where

$$\alpha(k)=\sum_{j=1}^{k}\binom{k}{j}j=k\sum_{j=1}^{k}\binom{k-1}{j-1}=k\cdot 2^{k-1},\ \text{which is even.}$$

II. *Solution by R. B. Eggleton, Avondale College, Cooranbong, N.S.W., Australia*

The function defined below meets the requirements. Let $p_1, p_2, \cdots, p_r$ be the distinct primes which do not exceed $\sqrt{n}$. Then

$$\mu(n)=\begin{cases}(-1)^s & \text{if } n \text{ is squarefree and has } s \text{ distinct prime factors}\\ 0 & \text{if } n \text{ is not squarefree.}\end{cases}$$

$$\mu(p_1p_2\cdots p_rn)=\begin{cases}(-1)^{r+1} & \text{if } n \text{ is prime}\\ 0 & \text{if } n \text{ is composite.}\end{cases}$$

Thus $\mu^*(n)\equiv|\mu(n)|+|\mu(p_1p_2\cdots p_rn)|$ is 2 if n is prime, 1 if n is composite and squarefree, and 0 otherwise.

Also solved by Gabriel Bastien, M. G. Greening (Australia), F. T. Howard, Donald Jeffords, Daniel Marcus, D. C. B. Marsh, Lieselotte Miller, and Charles Wells.

Convergence of a series

E 1884 [1966, 411]. *Proposed by A. F. Beardon, University of Maryland*

Prove that the series

$$\sum_{n_1,\cdots,n_k=1}^{\infty}\frac{1}{(n_1^2+\cdots+n_k^2)^p}$$

converges if and only if $p>k/2$.

I. *Solution by Yu Chang and Sidney Spital, California State Polytechnic College.* The sufficiency part of the proof follows almost directly from the arithmetic mean-geometric mean inequality, $n_1^2+\cdots+n_k^2\geqq k(n_1\cdots n_k)^{2/k}$. Hence, denoting the given series by $S(p)$,

$$S(p)<\frac{1}{k}\left(\sum_{n_1=1}^{\infty}n_1^{-2p/k}\right)\cdots\left(\sum_{n_k=1}^{\infty}n_k^{-2p/k}\right).$$

For $2p/k>1$, each factor on the right converges, therefore so too must $S(p)$.

It suffices to demonstrate the necessity part for $p=k/2$. This is begun with $n_1^2+\cdots+n_k^2<(n_1+\cdots+n_k)^2$ and followed by a diagonal rearrangement of the resulting series. Namely,

$$S(k/2)>\sum_{m=k}^{\infty}s(m),\qquad s(m)=\sum_{n_1+\cdots+n_k=m}1/m^k$$

and this is a finite sum. But the number of solutions of $n_1+\cdots+n_k=m$, in positive integers, is $\binom{m-1}{k-1}$. Hence

$$S\left(\frac{k}{2}\right) > \frac{1}{(k-1)!}\sum_{m=k}^{\infty}\left(\frac{m-1}{m}\right)\cdots\left(\frac{m-k+1}{m}\right)\frac{1}{m}.$$

Each of the first $k-1$ factors within the summation tends to one for m large; therefore divergence of $S(k/2)$ follows from a comparison with the harmonic series.

II. *Solution by M. S. Klamkin, Ford Scientific Laboratory.* By the integral test (for m-dimensions) the series will converge or diverge with the integral

$$\int_1^{\infty}\int_1^{\infty}\cdots\int_1^{\infty}\frac{dn_1dn_2\cdots dn_k}{(n_1^2+n_2^2+\cdots+n_k^2)^p}$$

or equivalently (using spherical coordinates) with the integral $\int_c^{\infty}(r^{k-1}dr)/r^{2p}$, which converges if and only if $2p-k+1>1$, or $p>k/2$.

We are assuming p is a constant, otherwise the result is not valid, i.e. $\sum_{n=1}^{\infty} n^{1+1/n}$ diverges.

Also solved by E. S. Langford, J. Lehner, D. E. Nixon, M. Shiba (Japan), and Al Somayajulu.

A Triangle Inequality

E 1887 [1966, 538]. *Proposed by Simeon Reich, Haifa, Israel*

Let I, O, H, M, respectively, denote the incenter, circumcenter, orthocenter, and centroid of an acute-angled triangle ABC. Show that

$$9\frac{AM^2+BM^2+CM^2}{AO^2+BO^2+CO^2}-\frac{AH\cdot BH\cdot CH}{AI\cdot BI\cdot CI}\geqq 8,$$

with equality if and only if the triangle is equilateral.

Solution by Leon Bankoff, Los Angeles, California. The given expression can be transformed into $\Pi\sin^2(\frac{1}{2}A)\leqq 1/64$ by making the following substitutions (see Roger A. Johnson, *Advanced Euclidean Geometry*, pp. 163, 174, 186, 191):

(1) $AO^2+BO^2+CO^2=3R^2$,

(2) $AM^2+BM^2+CM^2=3(R^2-OM^2)$,

(3) $AH\cdot BH\cdot CH=8R^3\Pi\cos A$,

(4) $AI\cdot BI\cdot CI=64R^3\Pi\sin^2(\frac{1}{2}A)$,

(5) $9OM^2=(3OM)^2=OH^2=R^2(1-8\Pi\cos A)$.

Since it is known that $\Pi\sin(\frac{1}{2}A)\leqq 1/8$, with equality only when $A=B=C$, the result follows.

Also solved by A. N. Aheart, M. G. Greening (Australia), J. M. Quoniam (France), K. R. Rajagopalan (India), R. Sivaramakrishnan (India), C. S. Venkataraman (India), and the proposer.

Sum of the Divisors of an Integer

E 1888 [1966, 538]. *Proposed by C. C. Lindner, Coker College, Hartsville, N. C.*

Prove that if $n>2$, then $\sigma(n)\leqq n\sqrt{n}$, where $\sigma(n)$ is the sum of divisors of n.

I *Solution by Anders Bager, Hjørring, Denmark.* Put $f(n)=\sigma(n)/n$. We shall prove that $f(n)<\sqrt{n}$ for all $n\geqq 3$. As the general case follows easily by multiplicativity, we need only consider the following three special cases:

1) $n=2^\alpha$, $\alpha\geqq 2$. $f(n)<2\leqq 2^{\alpha/2}=\sqrt{n}$.
2) $n=p^\alpha$, p odd prime. $f(n)<1/(1-1/n)\leqq 1/(1-1/3)=3/2<\sqrt{3}\leqq\sqrt{n}$.
3) $n=2p^\alpha$, p odd prime. $f(n)<\frac{3}{2}/(1-\frac{1}{3})=\frac{9}{4}<\sqrt{6}\leqq\sqrt{n}$.

II. *Solution by Steven Russ, Rice University.* To be a divisor of n, m can be only one of $n, n/2, n/3, \cdots, n/n$ (and certainly not all of these are integral for $n>2$). So $\sigma(n)\leqq n\sum_{k=1}^{n}1/k<n(1+\log n)$.

For $n>20$, $1+\log n<n^{1/2}$, which proves the problem for $n>20$. For the integers $2<n\leqq 20$, a numerical check is easily made.

By the same method it can be shown for every $\epsilon>0$, there exists a sufficiently large N such that if $n>N$, $\sigma(n)\leqq n^{1+\epsilon}$, namely N is such that $\log N\leqq N^\epsilon$.

Also solved by P. N. Bajaj, Peter Bundschuh (Germany), Jens Cartensen (Denmark), P. R. Chernoff, H. M. W. Edgar, R. B. Eggleton (Australia), Neal Felsinger, N. J. Fine, E. D. Gingerich, C. R. Glenn, Jerry Goodman, M. G. Greening (Australia), Donald Jeffords, Erwin Just, Kenneth Kramer, E. S. Langford, P. A. Lindstrom, D. C. B. Marsh, Alexander Morgan, Sam Newman, Robert Patenaude (Germany), Stanton Philipp, P. A. Scheinok, D. L. Silverman, Al Somayajulu, Sidney Spital, D. R. Stark, A. M. Vaidya (India), C. S. Venkataraman (India), and Allan Wachs.

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5530. *Proposed by S. S. Mitra and J. R. Porter, University of Oklahoma*

Let I be the unit interval, and let $f\colon I\to I$ be a continuous function such that $f(f(x))\equiv x$. Prove or disprove that if f is monotone decreasing, then $f(x)\equiv 1-x$.

5531. *Proposed by R. O. Davies, The University, Leicester, England*

Prove that if $\sum u_m$ is a convergent series of nonzero terms and u_m-u_{m+1} is decreasing, then $u_{m+1}^{-1}-u_m^{-1}\to\infty$ as $m\to\infty$.

5532. *Proposed by Ka Menehune, University of Hawaii.*

Let G be an abelian p-group, and f an endomorphism of G. If $f/G[p]=1$, where $G[p]=\{x\colon x\in G, px=0\}$, then f is an automorphism of G. Is f necessarily the identity of G?

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5533. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Let n be a natural number and a, b, c real numbers. Determine

$$\int_0^\infty \frac{1}{a^2 + 2b^2x^n + c^2x^{2n}}\,dx.$$

5534. *Proposed by Richard Sinkhorn, University of Houston*

Let A and B be $n \times n$ irreducible doubly stochastic matrices. Show that any matrix X satisfying $AX = XB$ has constant row and column sums.

5535. *Proposed by C. J. Mozzochi, University of Connecticut*

Suppose $0 < (\delta - \frac{1}{2}) < \epsilon < \frac{1}{2}$; m and n positive integers such that m is not a multiple of n; $G = \bigcup_{k=-\infty}^{\infty}(k,\ k+\delta)$; $H = \sim G$; $g(\nu) = m(x + \nu/n)$, $\nu = 0,\ 1,\ \cdots,\ (n-1)$; $mx \in H$. Then for arbitrarily small $\epsilon > 0$:

(A) if n is even, then (number of $g(\nu) \in G) \geqq (n-2)/2$;
(B) if n is odd, then (number of $g(\nu) \in G) \geqq (n-3)/2$.

5536. *Proposed by Herbert Robbins, Purdue University*

We confront an urn containing an unknown number N of similar balls. We are allowed to draw as many balls as we like, one at a time, marking each ball drawn (so we will recognize it if we ever draw it again) and replacing it before the next ball is drawn. We must stop eventually and guess the value of N. Is there a procedure for doing this with the property that the probability that our guess will be exactly correct is $\geqq .999$ *uniformly for all* $N \geqq 1$?

5537. *Proposed by Sidney Heller, Brookhaven National Laboratory*

Show that

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{A_n}{n} = C, \quad \text{Eulers constant,}$$

where A_n is the determinant $|a_{ij}|$ such that

$$a_{ij} = \begin{cases} (i - j + 2)^{-1} & \text{for } i - j + 2 > 0 \\ 0 & \text{for } i - j + 2 \leqq 0. \end{cases}$$

5538. *Proposed by L. F. Meyers, Ohio State University*

Is it possible to find a Hamel basis of the real numbers over the rational numbers which is closed under ordinary multiplication of real numbers?

5539. *Proposed by Robert Breusch, Amherst College*

Find all the continuous, nonnegative-valued, strictly increasing functions f, defined for $x \geqq 0$, such that for all $x \geqq 0$,

$$f(f(x)) = k^2x, \quad \text{with} \quad k > 1.$$

Also solved by J. Aczel, J. P. Celenza, Robert Cohen, G. Di Antonio, E. J. Eckert, Robert Goldstein (England), D. A. Hejhal, H. E. Lahmann (Germany), Dorembus Leonard (Israel), M. D. Mavinkurve (India), Charles McCracken, Jürg Rätz (Switzerland), and the proposers.

Goldstein generalizes by replacing (2) with the condition $f(g(x))=h(f(x))$, where h is bounded above (or below) and the range of g contains an interval of length greater than unity. He also refers to R. P. Boas, *A Primer of Real Functions*, p. 111 for other results of the same nature. For a further generalization, Aczel and Rätz refer to S. Kurepa, *Note on Additive Functions*, Glasnik Mat.-Fiz. Astr., 20 (1965) pp. 75–78. Cohen, with his solution, raises the question as to the possibility of replacing p in (2) by g, an entire transcendental function.

Roots of Transfinite Ordinals

5431 [1966, 1018]. *Proposed by G. F. Schumm, University of Chicago*

Given the ordinal numbers ξ and γ, we call ξ a root of γ if there exists an ordinal number $\alpha>1$ such that $\xi^\alpha=\gamma$. What is the kth smallest transfinite number for which there exists no root of degree >1?

I. *Solution by K. A. Bowen, Princeton University.* We first note the following simple facts: if $1\leqq m<\omega$ and $1<n<\omega$, then $n^m<\omega$ and $n^\omega=\omega$. Moreover, if α is a limit number so is n^α. Hence if $\alpha\geqq\omega$, then $n^\alpha=\omega$ or $n^\alpha=\omega\cdot 2$ according as $\alpha=\omega$ or $\alpha>\omega$.

Finally, if $0\leqq n$, $m<\omega$, $(\omega+m)(\omega+n)\geqq\omega\cdot\omega=\omega^2$. Hence we see that if $1\leqq s<\omega$, $\omega+s$ has no roots of degree >1, while if $1<n<\omega$, n is a root of ω. Thus the kth transfinite "prime" is $\omega+k$.

II. *Additional Comments by Leonard Gillman, University of Rochester.* It is easy to continue the enumeration up to ω^2; but shortly thereafter, it gets complicated. Between ω and ω^2, the only numbers with roots are limit numbers. In general, the transfinite ordinals having roots are precisely those of the form

$$(1)\qquad \omega^\beta\cdot k\ (\beta>0;\ 0<k<\omega)\qquad \text{or}\qquad (2)\qquad \mu^\nu(\mu+m)^n,$$

where

$$(2')\qquad (\mu+m)^n=\mu^n+\mu^{n-1}\cdot m+\cdots+\mu\cdot m+m$$

(μ a limit ordinal, ν a limit ordinal or 0, $n<\omega$, $\nu+n>1$, and $m<\omega$).

The numbers (1) are the powers of finite ordinals: since $k_1^\omega=\omega$ for $1<k_1<\omega$, we have

$$\omega^\beta\cdot k=(k_1^\omega)^\beta\cdot k=k_1^{\omega\beta+k_2}\qquad (k=k_1^{k_2},\ k_1>1,\ k_2\geqq 0).$$

The numbers (2) are the powers of transfinite ordinals: $(\mu+m)^{\nu+n}$ reduces to (2), since $(\mu+m)^\nu=\mu^\nu$; and (2′) is established by induction on n, with the help of the relation $\mu^{n-1}\cdot m+\mu^n=\mu^n$ $(n>0)$. For necessary and sufficient conditions of ξ and γ in order that ξ be a root of γ, see P. W. Carruth, *Roots and Factors of Ordinals*, Proc. Amer. Math. Soc., 1 (1950) 470–480; Math. Reviews, 12 (1951) 166.

Also solved by the proposer.

An Orthogonal Sequence

5433 [1966, 1019]. *Proposed by Benjamin Volk, Yeshiva University*

Let ϕ be a function, continuous on $(0, 1]$ of period 1 and satisfying

$$\sum_{j=1}^{n} \phi\left(x + \frac{j-1}{n}\right) = 0, \qquad n = 1, 2, \cdots.$$

Then the sequence $\{\phi(n!\,x)\}$ is orthogonal on $(0, 1)$. (Cf. 5143 [1964, 1050].)

Solution by L. Carlitz, Duke University. The hypothesis should read $\sum\phi(x+(j-1)/n)=0$, $(n=2, 3, \cdots)$. Let $m<n$ and put $k=n!/m!$. Then

$$\begin{aligned}
\int_0^1 \phi(m!x)\phi(n!x)\,dx &= \frac{1}{n!}\int_0^{n!} \phi\left(\frac{1}{k}\,x\right)\phi(x)\,dx \\
&= \frac{1}{n!}\int_0^1 \phi(x) \sum_{j=0}^{n!-1} \phi\left(\frac{x+j}{k}\right) dx \\
&= \frac{1}{n!}\int_0^1 \phi(x) \sum_{r=0}^{m!-1} \sum_{s=0}^{k-1} \phi\left(\frac{x+rk+s}{k}\right) dx \\
&= \frac{m!}{n!}\int_0^1 \phi(x) \sum_{s=0}^{k-1} \phi\left(\frac{x+s}{k}\right) dx = 0.
\end{aligned}$$

Also solved by M. D. Mavinkurve (India).

Orthogonal Projections from a Normal Matrix

5434 [1966, 1019]. *Proposed by H. Kestleman, University College, London, England*

Let N be a normal $n\times n$ matrix (i.e. N commutes with its Hermitian transpose), $N^k=I$ for some integer $k\geqq 2$, and $N\neq I$. If $M=(1/k)\cdot(I+N+N^2+\cdots+N^{k-1})$ and $M\neq 0$, show that M projects orthogonally on to a subspace of complex-euclidean n-space, and identify this subspace in relation to N. (Note that if $n=k=2$, then N is a reflection and $\frac{1}{2}(I+N)$ is the projection onto the mirror.)

Solution by G. M. Leibowitz, Northwestern University. By the spectral theorem, $N=\sum\lambda_i E_i$ where the λ_i are the distinct eigenvalues of N and the E_i are mutually orthogonal self-adjoint projections. So $p(N)=\sum p(\lambda_i)E_i$ for every complex polynomial p. In particular, $M=p_0(N)$ where

$$p_0(\lambda) = \frac{1}{k}\sum_{j=0}^{k-1}\lambda^j = \begin{cases} 1 & \text{if } \lambda = 1 \\ \dfrac{1}{k}\,\dfrac{1-\lambda^k}{1-\lambda} & \text{if } \lambda \neq 1, \end{cases}$$

so that $M=\sum p_0(\lambda_i)E_i$. Since $N^k=I$, $\lambda_i^k=1$ for each i. If all $\lambda_i\neq 1$, $p_0(\lambda_i)=0$ for all i, so $M=0$. Otherwise, some $\lambda_j=1$ and so $p_0(\lambda_i)=\delta_{ij}$; thus $M=E_j$. In

either case, M is a self-adjoint projection.

We note (i) The assumption of finite dimensionality is unnecessary. For $N=\int\lambda dE(\lambda)$ where the integral extends over the spectrum of N. If $N^k=I$ then $\{1\}=\sigma(I)=\sigma(N^k)=\sigma(N)^k$, so $\sigma(N)$ is contained in the set of kth roots of unity. Thus the measure $E(\cdot)$ is supported on that set, and so $N=\sum\lambda_iE_i$ with the λ_i all the kth roots of unity (some of the projections E_i may be 0).

(ii) The range of M is $\{x: Nx=x\}$, the set of invariant elements for N. For if $Nx=x$, then $N^jx=x$ for each j, so $Mx=(1/k)\sum N^jx=x$. Conversely, $NM=(1/k)\sum_{j=1}^{k-1}N^{j+1}=M$ since $N^k=I$, so if $Mx=x$, $Nx=NMx=Mx=x$.

(iii) We can generalize in the following way: Suppose $p(\lambda)=(\lambda-\lambda_0)q(\lambda)$ with p, q polynomials and $q(\lambda_0)=1$. If N is a normal operator and $p(N)=0$, then $q(N)$ is the projection onto the null space of $N-\lambda_0I$.

Also solved by D. W. Bailey, J. P. Celenza, John de Pillis, D. Ž. Djoković (Yugoslavia), W. G. Dotson, Jr., Harley Flanders, A. S. Householder, C. M. Joiner & S. J. Pierce, C. R. MacCluer, M. D. Mavinkurve (India), G. W. Soules, Art Steger, R. C. Thompson, and the proposer.

Norm of Complex Homomorphisms on a Commutative Banach Algebra

5435 [1966, 1019]. *Proposed by J. P. Williams, University of Michigan*

In the appendix to his book, *Fourier Analysis on Groups*, (Interscience, 1962), W. Rudin asserts that if A is a commutative Banach algebra then each nonzero complex homomorphism on A has norm 1. Is this true in general?

I. *Solution by Chi Song Wong, University of Illinois.* It is not true. Let D be the closed unit disc of the complex plane K and let $A(D)$ be the disc algebra. Consider the Banach subalgebra of $A(D)$:

$$B = \{f \in A(D): f(0) = 0\}.$$

Let t_1, t_2 be any elements in K such that $0<|t_1|<1$, $|t_2|=1$, and take multiplicative linear functionals h_1, h_2 on B such that for any $f\in B$

$$h_i(f) = f(t_i), \qquad i = 1, 2.$$

It is familiar that $\|h_i\|\leqq 1$, $i=1, 2$. For any nonzero f in B, f is not a constant, so by the maximum modulus principle, $|f(t_1)|\neq\|f\|$. Therefore $\|h_1\|<1$. On the other hand, let f be the function on D defined by

$$f(z) = \tfrac{1}{2}z(z + t_2), \qquad z \in D.$$

Then $f\in B$ and $\|f\|=1$. Thus $\|h_2\|\geqq|h_2(f)|=|f(t_2)|=1$ and therefore $\|h_2\|=1$.

II. *Solution by John Rainwater, University of Washington.* The assertion is not generally true. Let A be any normed algebra without a unit, and with norm $x\to\|x\|$. Let K be a real number greater than 1. The mapping $x\to K\|x\|$ is an algebra norm on A for which all multiplicative linear functionals have norm not exceeding $1/K$. This example also proves that a homomorphism of a commutative Banach algebra into a semisimple Banach algebra can have arbitrarily small norm. (See Theorem in D5 of Rudin, *loc. cit.*, p. 263.)

Also solved by G. M. Leibowitz, and the proposer.

A counterexample to Rudin's statement occurs in A. Wilansky's *Functional Analysis*, p. 278, Problems 2 and 3.

Some Trigonometric Integrals

5438 [1966, 1020]. *Proposed by M. L. Glasser, Battelle Memorial Institute*

Express the following integral in terms of Fresnel functions:

$$\int_0^\infty e^{x \sin y} \sin\left\{x \cos y \pm \tfrac{1}{2}(y + n\pi)\right\} dy.$$

Solution by Lowell Schoenfeld, Pennsylvania State University. The integrals as printed diverge. To show this, let λ be $+1$ or -1; then the integrals in question are given by

$$\int_0^\infty e^{x \sin y} \sin\left\{x \cos y + \tfrac{1}{2}\lambda(y + n\pi)\right\} dy. \tag{1}$$

Let $I_m(\alpha, \beta)$ be the integral which results when the integration limits 0 and ∞ are replaced by $2\pi(2m+\alpha)$ and $2\pi(2m+\beta)$ respectively. The divergence of (1) is a consequence of the following fact which we demonstrate: For fixed values of λ, x, n there correspond numbers α, β, η which are independent of m, $0 \leqq \alpha < \beta$ and $\eta > 0$, and are such that for all integers m we have $|I_m(\alpha, \beta)| \geqq \eta$.

On making the substitution $u = (y - 4\pi m)/2\pi$, we obtain, for integral m,

$$I_m(\alpha, \beta) = 2\pi \int_\alpha^\beta f(u) e^{x \sin 2\pi u} du,$$

where $f(u) = \sin(x \cos 2\pi u + \lambda\pi u + \frac{1}{2}\lambda n\pi)$. (Note that $f(u)$ and the last integral are independent of m.) If $x + \frac{1}{2}\lambda n\pi$ is not an integral multiple of π, we define $\alpha = 0$ and find that $f(\alpha) = \sin(x + \frac{1}{2}\lambda n\pi) \neq 0$. If $x + \frac{1}{2}\lambda n\pi = k\pi$, where k is an integer, then we define $\alpha = \frac{1}{2}$ and find that

$$f(\alpha) = f(\tfrac{1}{2}) = \sin\left\{(\lambda n - k)\pi + \tfrac{1}{2}\lambda\pi\right\} = \pm 1 \neq 0.$$

In either case, $f(\alpha) \neq 0$. By continuity, there exists $\beta > \alpha$ such that for all $u \in [\alpha, \beta]$, $f(u)$ has the same sign as $f(\alpha)$, and $|f(u)| \geqq \frac{1}{2}|f(\alpha)|$. Hence

$$\begin{aligned} |I_m(\alpha, \beta)| &= 2\pi \int_\alpha^\beta |f(u)| e^{x \sin 2\pi u} du \\ &\geqq \pi(\beta - \alpha) |f(\alpha)| e^{-|x|} \equiv \eta, \end{aligned}$$

as we set out to prove.

Editorial Note. Through an unfortunate corruption in the typescript, the integrals came through with an infinite limit whereas the proposer intended integrals with finite limits:

$$\int_0^\pi e^{x \sin y} \sin\left\{x \cos y \pm \tfrac{1}{2}(y + n\pi)\right\} dy \tag{2}$$

which are expressible in terms of the Fresnel integrals. Further contributions for the evaluation and representation of the integrals in (2) will be welcomed and are now solicited.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada, Correspondence about reviews is welcomed.

Non-Standard Analysis. By A. Robinson. Edited by L. E. J. Brouwer, A. Heyting. A. Robinson and P. Suppes. North Holland, Amsterdam, 1966. xi+293 pp. $10.00.

Ideas developed in the study of the foundations of mathematics, notably those of set theory, have been applied increasingly in most branches of modern mathematics. Professor Robinson has been a pioneer in a new field: the direct application of the central ideas of metamathematics, the study of proof theory, to algebra. The present work is an application of these ideas to analysis, and also to applied mathematics, and it is even more remarkable; firstly, in view of the lesser role that axiomatics plays in these subjects, in showing that metamathematics is so fruitful a tool in their study, and secondly, in that it combines these very modern methods with a rigorous revival of the use of the oldest method of analysis, that of infinitesimals.

The central idea of the book is a method by which any mathematical system can be imbedded in a larger system in such a way that all statements true in the original system remain true, provided that they are interpreted in a certain manner, in the extended system. When this construction is applied to the real numbers it produces a system containing, in addition to the ordinary real numbers, elements which we would recognize as infinitely large or infinitesimal, though from the point of view of the new system they obey all the laws to which the real numbers are subjected. Because the interpretation of these laws involves certain restrictions on the interpretation of terms such as "set," these systems are called nonstandard models of the real numbers.

The way in which such systems are constructed is described in the central second chapter of the book, which is, unfortunately, by far the most difficult. Some of the difficulty involved is inevitable in the very abstract subject matter, but it is increased by a sparseness of explanation which makes great demands on any reader not acquainted with metamathematical theory on the one hand and on the theory of ultrafilters on the other. These difficulties are further increased by a rather large number of misprints and similar mistakes. The reader qualified to follow this chapter or even to take its results for granted and press on (a feasible method of reading this book, and perhaps to be recommended for the non-logician on his first reading) will find the later chapters considerably easier going and very rewarding. They show in detail how infinitesimals and infinitely large numbers can be applied in the calculus, topology, the theory of groups and linear spaces and in certain parts of applied mathematics, such as boundary layer theory or St. Venant's Principle.

Many of these arguments are on lines which one might more or less expect once one accepts that the use of infinitesimals and infinitely large integers is legitimate, but there are others which indicate that a powerful new method has been added to analysis. In particular one may mention a problem on invariant subspaces of operators, which was

first solved by the methods of nonstandard analysis, and the ingenious applications to the theory of functions of a complex variable.

The last chapter contains observations on the history of the calculus. These have already been reviewed in this MONTHLY (May 1966, p. 608). One need say only that they show the greater flexibility in interpretation of the sometimes elliptical statements of the great mathematicians of the past which is provided by the existence of new mathematical models in terms of which their words can be understood.

This is certainly a most important and pathbreaking work. It is to be hoped that the author will find time to rewrite the second chapter so as to make it accessible to a wider circle of readers. In any case it should prove of value to very many mathematicians by showing them, as practically no other work that I know of can do, the significance to mathematics and the meaning of the concepts of metamathematics.

J. L. B. COOPER, University of Toronto and Chelsea College of Science and Technology, University of London.

Introduction to Algebra. By Sam Perlis. Blaisdell, Waltham, Mass., 1966. xx+440 pp. $9.75.

(A review based on classroom use.)

In the past few years there have appeared a number of abstract algebra texts intended for a first course and containing advanced topics, e.g., the Sylow Theorems and Galois Theory, but only a little linear algebra. Of course, the trend today is toward the introduction of linear algebra into the calculus sequence, but until this is universally done, the effectiveness of these books as first course algebra texts is greatly reduced. Indeed, linear algebra would not be available to motivate and illustrate other algebraic structures and the students would miss the experience of learning to construct proofs in the more concrete situations of linear algebra. Perlis' book, on the other hand, goes in the opposite direction. Many of the advanced topics are omitted in order to leave room for an integrated treatment of linear and abstract algebra, presuming no previous work in algebra.

Perlis' book is intended for a year's course in algebra, but it is flexible enough to be suitable for a semester course in abstract algebra or a semester course in linear algebra. It covers linear algebra, first with n-tuple spaces and later with abstract spaces, and goes through the reduction to diagonal form of real symmetric matrices. More advanced topics, like the Jordan canonical form, are not included. Group theory does not include homomorphisms, except for a few exercises, but homomorphisms, ideals, and residue class rings are studied in ring theory. There is an excellent chapter on divisibility for integers and for polynomials, with principal ideals used as a unifying device. The discussion of polynomials is very detailed and there is a chapter on classical algebra, containing results on multiplicities of roots and irreducibility of polynomials over the rational, real, and complex fields. The chapter on field extensions contains a proof of the impossibility of the trisection of an angle.

In using this book for a year's course, I modified the order to take up group theory and all the linear algebra before covering the material on rings and fields. I found that there is more in the book than can be covered in a year, but this arrangement allowed the class to cover linear algebra thoroughly.The problems are excellent; there are many nonroutine but workable problems that can be assigned to the better students and also enough routine ones for the poorer students. The writing is clear and detailed; many sections are suitable for self-study. There are very few misprints or errors. The only serious source of confusion I found was with formula 14.43 on p. 411, which is true only for symmetric matrices.

In summary, I regard this as an outstanding textbook, well-planned and well-written.

I recommend it, in particular, for a first course in algebra where there has been no previous work in linear algebra.

R. B. REISEL, Loyola University, Chicago

Abstract Theory of Groups. By O. U. Schmidt. Translated from the Russian by Fred Holling and J. B. Roberts. Edited by J. B. Roberts. Freeman, San Francisco, 1966. vi+174 pp. $5.00.

In 1916 when this book was written it admirably fulfilled the purpose of supplying an introductory text in group theory in Russian. However, very few books in mathematics can hope to survive for half a century and the current translation is primarily of historical interest. I would not recommend this book to a beginner in the subject as many things in group theory have been clarified since 1916. The most striking example of this is the concept of a homomorphism, which as defined in the text need not even be a function. Also some of the terminology used here sounds strange to modern ears. For instance "special" is used in place of "nilpotent" and "group union of" in place of "group generated by." (The latter is due to the translators. According to a footnote, Schmidt had used "least common multiple of.") The main impression I received on reading the book is that mathematics progresses and many things that appear difficult to one generation may become elementary for the next.

WALTER FEIT, Yale University

Vector Spaces. By D. A. Raikov. Translated from the Russian by Leo F. Boron, with the assistance of H. R. Stevens and G. F. Votruba. Noordhoff, Groningen, 1965. xii+190 pp. $7.60.

This book is an algebraic introduction to the theory of topological linear spaces, and consists of three chapters: Preliminary information about partially ordered sets and about groups, General theory of vector spaces, L-spaces. The first two chapters take up about two-thirds of the book, and give a masterly account of the general theory of vector spaces. The last section of the second chapter is devoted to a discussion, in algebraic form, of the Hahn-Banach theorem concerning the extension of a linear function. In the final chapter the L-spaces which were first introduced by the author, are considered. These spaces are essentially the algebraic analogues of linear topological spaces. There is a detailed study of the algebraic analogues of many notions from the theory of topological linear spaces. The chapter ends with a study of L-spaces over the fields of real and complex numbers, and the algebraic analogue of the Krein-Milman theorem on extreme points of bicompact convex sets is proved. There is an index. There are no exercises, which is a pity.

DANIEL PEDOE, University of Minnesota

The Foundations of Intuitionistic Mathematics. By S. C. Kleene and R. E. Vesley. North-Holland Publishing Company, Amsterdam, 1965. 206 pp. $7.00.

This monograph on the foundations of intuitionistic mathematics has been long awaited. It reports on research that has been continuing for fifteen years; the research has been reported in various papers, particularly by the senior author, but this is the unified and definitive product of many years of work.

As might be expected, it is written from a formalistic and model-theoretic point of view. This is not as inappropriate as it sounds; even Heyting has formalized a fragment of intuitionist mathematics. As Kleene writes, "The formation rules and postulates help the outsider by separating out for him a relatively small number of concepts he must grasp, and of propositions and methods of inference to which he must assent, in order that in principle nothing else should remain problematical for him in the body of intuitionistic mathematics deducible within the system. (I need hardly add that the outsider often needs just this kind of help.) Furthermore, metamathematical investigations are made

possible, as well as investigations, which we may call model-theoretic or semantical, . . " (p. 6).

The book consists of four chapters. Chapter I, by Kleene (90 pp.), gives a formal system of intuitionistic analysis, and chapter III, by Vesley (40 pp.) extends this to include an intuitionistic theory of the continuum. Chapter II, by Kleene (43 pp.) provides several notions of *realizability*, helpful in developing relations between classical and intuitionistic analysis. (It should be observed that not only are there classical assertions that cannot be proved in intuitionistic systems, but that the converse holds as well: there are intuitionistic assertions that do not hold in classical mathematics.) This chapter is particularly interesting. Chapter IV, also by Kleene (23 pp.), deals with some problems of order in the continuum.

The style of the book is very much like that of Kleene's *Introduction to Metamathematics*, and the content is heavily dependent on that book. Most of the axioms of the formal system developed in the first chapter are to be found written out only in the earlier book. Having to look in a different book for the axioms of the system you are studying in one book would be very inconvenient, except that the ordinary reader will need to have Kleene's earlier book open on his desk the whole time he is reading the present book anyway. (Indeed, this monograph is, among other things, an excellent study work-book for *Introduction to Metamathematics*.) Nevertheless, the book constitutes a rich and accessible study of intuitionistic analysis.

H. E. KYBURG, JR., University of Rochester

Introduction to Linear Algebra. By Marvin Marcus and Henryk Minc. Macmillan, New York. Collier-Macmillan, London, 1965. x+261 pp. $7.95.

Here we have yet another text in linear algebra. Such a book must justify its publication by offering new, or otherwise unusual, material or pedagogical excellence. This text does indeed come up with some new ideas, although it is not entirely satisfactory.

The text is intended, according to the preface, for a one semester course for sophomores or juniors although the prerequisite is only the "successful completion" of a standard course in college algebra. The text belies this preparation in its level of sophistication.

There are three chapters entitled: Vector spaces and linear transformations; Linear equations and determinants; Characteristic roots. The text combines the matrix and linear transformation approach leaning somewhat more heavily on matrices. The material on determinants is fuller than one usually sees; the approach here is not through multilinear functions. The last chapter includes some interesting material on matrix inequalities. Here we find Gershgorin disks, the Courant-Fischer theorem and Cauchy's inequalities for characteristic roots of submatrices.

The style is somewhat thick and there are some confusing and difficult spots. For example: Definition 1.3 on pages 4 and 5, which includes some intercalary remarks, runs to 31 lines of type, almost a full page; bases are defined to be sets but are used as if they were ordered sets; fields are permitted to have 1 element; arbitrary rings sometimes mysteriously acquire identities.

The approach to the problems is the most unusual feature of the book. Each problem section includes true-false questions; solutions to all the problems, of which there are many, are to be found at the end of the text. Whether or not this is the best part of the book, or the worst, is up to the reader.

NEIL GRABOIS, Williams College

Handbook of Nonparametric Statistics, II. By John E. Walsh. Van Nostrand, Princeton, N. J., 1965. xxvi+686 pp. $17.50.

This is Volume II of a three volume set covering nonparametric statistics up to 1958.

Volume III has not yet appeared. The introductory chapter of Volume II states that "the first volume contains procedures for testing the random sample hypothesis, estimates and tests for expected values, estimates and tests for population percentiles, results for distribution-free tolerance regions, and procedures for investigating probability distributions."

Volume II contains seven chapters. Chapter I contains necessary information on how to use the handbook such as: notation, format, interpretation of "nonparametric" and some general remarks on nonparametric procedures. The format discussion needs careful study because in the remainder of the book the presentations of the various tests are written in a very abbreviated way requiring considerable decoding by the reader. Chapters II and III are concerned with the two sample problems for noncategorical and categorical data. Then follow two chapters on the several sample problem. The last two chapters include tests of symmetry and tests for extreme populations and observations. Each chapter and each section of each chapter begins with explanatory and introductory material. These introductions are followed by the highly coded "presentations." Some of the more lengthy "presentations" are so formidable in complexity that one would probably be better off going to the original literature. (A reference is included at the end of each "presentation.") The bibliography at the end of the book includes 339 titles, each one listing the pages on which it is cited in the Handbook. There is also both a very detailed subject index and an extensive table of contents (16 pages).

The handbook represents an enormous amount of work. Many important gaps found in the literature have been filled in by the author of the handbook. The handbook should prove valuable to both applied and mathematical statisticians, about to study a particular area, who wish a comprehensive review of the literature.

H. J. ARNOLD, Oakland University

Orthogonal Families of Analytic Functions. By Bernard Epstein. Macmillan, New York, 1965. viii+80 pp. $2.50 (paperback).

The study of Hilbert spaces whose elements are analytic functions has held the interest of many mathematicians, especially since the appearance of S. Bergman's works on kernel functions and conformal mapping. Some other topics of study have been zeros, interpolation, approximation, boundary behavior, and (more recently) relations between spaces of analytic functions and the general theory of linear operators.

In this monograph the author restricts his attention to the space $A_2(D)$ consisting of all analytic functions f on the bounded domain D in the plane for which $\int_D \int |f(z)|^2 dxdy < \infty$. Brief mention is also made of the Bergman class of antiderivatives of elements of $A_2(D)$ (i.e., all F such that $\int \int |F'(z)|^2 dxdy < \infty$). Omitted entirely are the Hardy class H^2 and the important classes of entire functions which are square integrable with respect to various weight functions. Kernel functions, conformal mapping, and interpolation problems are discussed.

The present work is a very brief introduction to the books of Bergman. Prerequisites are a knowledge of basic complex variables and linear algebra. No acquaintance with Hilbert space theory is required of the reader, and indeed Chapter 2 is a good concrete introduction to the rudiments of that theory. The presentation is pleasant and unhurried, and for the most part rigorous. Unfortunately there is a great deal of handwaving in the geometric parts of the chapter on conformal mapping (but perhaps this is an inherent weakness of the subject itself).

G. M. LEIBOWITZ, Northwestern University

Functional Analysis. By Kosaku Yoshida. Springer-Verlag, Berlin, 1965. xi+458 pp. $16.50.

Like many books with similar titles, the book under review contains a large cross

section of topics that go under the broad heading of functional analysis.

Topics include the basic facts on various topological vector spaces, generalized functions, uniform boundedness, closed graph theorem, interior mapping, Hormander's theorem on hypoelliptic partial differential operator, Hilbert spaces (including orthogonalization, the Riesz representation theorem and the Lax-Milgram theorem), the Hahn-Banach theorem, dual spaces, various topologies on the space of linear maps, strong and weak topologies, strong and weak measureability, vector valued analytic functions, the Gelfand-Mazur theorem, a wide class of results formally to be put under the title of 'distributions,' Fourier series of distributions, Mikusinksi's operational calculus, spectral theory, semigroups of operators (here one finds, perhaps for the first time in a book, an updated version of the theory of semigroups of operators presented by the author, who is a pioneer among students of this field), ergodic theory, integration of the equation of evolution, compact operators, normed rings and spectral representation, and vector lattices.

In the opinion of the reviewer the significant virtues of the book are (1) that a wide variety of topics is presented in a single book, and (2) the theory of semigroups of operators and their applications is given in a more general context than in earlier publications and covers the literature up to 1965. The author's style and presentation are very pleasing. The reader needs to be warned, however, that while the book certainly contains the basic material on spectral theory and also certain topics in topological vector spaces, it makes no claims of being devoted to these. A very exhaustive bibliography enhances the utility of the book. The book will long remain a useful reference and a text in a suitably planned graduate course.

M. S. RAMANUJAN, University of Michigan

Probability and Statistics. By H. Freudenthal. Elsevier, Amsterdam, London, New York, 1965. 139 pp. $5.00.

This book by Hans Freudenthal, Utrecht University, is a translation of the 1957 Dutch edition. It is quite different in format and flavor from the usual probability and statistics books.

The format is one of nine chapters broken up into curt, but precise, sections. The flavor, but not the accompanying wit, is possibly indicated by the following brief summary of Section 1.9.14; the section begins with the statement that it is possible for three stochastics (random variables) to be independent in pairs, but nevertheless interdependent. The example then given of this phenomenon is one of a hall lamp controlled, in a common manner, by a switch upstairs and a switch downstairs such that the lamp is off if both switches are "up" or "down" and on otherwise. The switches are used with equal frequency and the lamp burns half the day. Stochastics relating to the states of the switches and lamp turn out to be pairwise independent, but not fully independent.

The chapters of the book present in turn definitions and descriptions of the elements of probability, limit theorems, the classical distributions, Markov chains, estimation, significance and hypothesis testing, sequential analysis and game theory. There is also a discussion of the philosophical nature of probability. Frequently the chapters are highlighted by amusing illustrations and problems.

A disadvantage that this book may well prove to have with certain readers is that frequently nonstandard terminology is employed and at the same time few references to alternative sources are provided. This latter is especially sad in view of the fact that Freudenthal typically introduces a topic in an arousing manner, but does not develop it in any great detail.

The book would appear to be a good text for mathematics majors having to satisfy a statistics requirement or alternatively a good supplementary text, in a mathematical direction, for statistics students.

D. R. BRILLINGER, London School of Economics

The open mapping and closed graph theorems in topological vector spaces. By Taqdir Husain. Oxford Mathematical Monographs, 1965. x+106 pp. $4.80.

The two theorems named in the title of this monograph are related; they are amongst the most important and powerful results of functional analysis. Let E and F be two topological vector spaces (i.e., vector spaces provided with a topology under which the algebraic operations are continuous). A linear mapping f of E into F is called closed if its graph is closed in $E \times F$; the closed graph theorem asserts that, under certain conditions on E and F, every such closed linear mapping is continuous. One form of the open mapping theorem asserts that, again under suitable conditions on E and F, every closed linear mapping of E onto F is open, i.e., maps open sets onto open sets. First proved by Banach nearly thirty years ago under the hypothesis that E and F were complete metrizable spaces, these two theorems have received successive generalisations by various authors to wider classes of spaces.

The present book aims to collect together recent work on these two theorems and on related topics. In order to make it self-contained, the author has included two chapters of background material, containing a brief thirty-page summary of the theory of topological vector spaces. Then the book goes on to prove Banach's classical forms of the theorems, with some related results involving inductive limits. Chapters 4 and 5 deal with duality theory. The defining properties of several important classes of locally convex spaces can be given in terms of the so-called *ew** topology on the dual space (the finest coinciding with the weak* topology on equicontinuous sets). In particular, a space in whose dual every *ew**-closed subspace is weak*-closed is called B-complete, or fully complete. The important part played by these spaces in the study of the closed graph and open mapping theorems, first noticed by Ptâk, is expounded in these chapters. Finally, in the last two chapters, the author gives the contents of some of his own publications. These comprise the theory of a class of spaces called by the author S-spaces, in the duals of which the *ew** topology coincides with the topology of uniform convergence on precompact sets, and the theory of general classes of spaces which admit the closed graph theorem.

This is undoubtedly a specialist book, likely to be of interest to the reader who has some knowledge of the theory of topological vector spaces. In terminology and notation it follows fairly closely that of Bourbaki's tract. Its merit is that the contents of the author's papers, together with those of many other writers who have contributed to the subject, are available here, with a uniform style and notation, and compressed into about a hundred pages of print. The author offers some historical notes at the end, and there is also a bibliography.

A. P. ROBERTSON, University of Keele, England

Introduction to Algebra. By Donald J. Lewis. Harper and Row, New York, 1965. 318 pp. $8.50.

This is a book suitable for a two-semester course for able undergraduates. No previous acquaintance with abstract algebra is assumed (e.g., the book begins with a chapter on the integers) but the pace is brisk. A student of some ability who studies this book will gain much in mathematical sophistication as well as in knowledge of the basic concepts of algebra.

The number of topics covered is small, but each is treated in sufficient depth to make the concepts meaningful. Groups are studied through the structure of finite abelian groups. Rings, integral domains and fields are studied simultaneously, including some valuation theory and field extensions (but no Galois theory). The last chapter contains a fairly abstract treatment of vector spaces and linear transformations, with little mention of the geometric interpretations.

The author uses examples generously, for both illustration and motivation. For in-

stance, twenty pages are devoted to permutation groups before the study of group theory is begun. Exercises are numerous and varied in difficulty. On the whole, it appears to be a very teachable book.

In the preface, the author states that the original form of the book was a list of definitions, examples and theorems, used in a "prove-it-yourself" course. He states that it was thought that a book in that format would not meet with wide approval. and hence the present form. Perhaps such an "Aufgaben und Lehrsätze aus der Algebra" would not be widely used, but where used it would fill a real need. It is regrettable that this book could not have been made available in its original form as well as in the present form.

H. E. CHRESTENSON, Reed College

Quasilinearization and Nonlinear Boundary-Value Problems, vol 3. By R. E. Bellman and R. E. Kalaba. American Elsevier, New York, 1965. 224 pp. $8.50.

This book is a well written and well motivated introduction to quasilinearization and its applications to nonlinear boundary-value problems. A number of analytical concepts are introduced and illustrated through the use of simple nontrivial examples. The book begins with a discussion of the application of quasilinearization to the study of the Riccati equation. Here quasilinearization results in an explicit analytic representation for the solution in terms of quadratures and a maximum operation. This representation yields simple upper and lower bounds for the solution. In nonlinear two point boundary value problems the authors then show how quasilinearization may lead to a sequence of approximations with quadratic convergence. Chapter 3 examines conditions under which the sequence of approximations converges monotonically. Here the Green's function is encountered and its variation diminishing property emphasized. Chapter 5 deals with higher order equations and systems and develops techniques for using successive approximations without excessive reliance on rapid-access storage. Various methods for overcoming storage problems are discussed. Some nonlinear partial differential equations are dealt with in Chapter 5, and a number of examples are given in Chapter 6 which illustrate ways in which quasilinearization may be applied to treat various descriptive and variational processes. The final chapter illustrates how quasilinearization and dynamic programming may be effectively combined in the treatment of multi-dimensional variational problems, and identification problems. Computer programs for a number of problems are contained in the various appendices.

Each chapter begins with an introductory paragraph which motivates and describes the content of the chapter. The exposition is clear throughout, the references are numerous, and the misprints are few.

L. E. PAYNE, Cornell University

Begründung der euklidischen und nichteuklidischen Geometrien nach F. Klein, von László Rédei. Akadémiai Kiadó, Budapest, 1965. 363 pp. $10.00.

The first 227 pages of the book are devoted to a very careful construction of real projective space out of the Hilbert (nonprojective) axioms of incidence, order, and continuity. This construction uses some remarkable ideas such as construction of a further Desargues configuration from a given one, and a systematic use of central collineations to study the group of projectivities. After a representation of real projective space by homogeneous coordinates is obtained, the space is extended to complex projective geometry by one sentence. No geometric interpretation is offered. This is a pity in two respects. First, it contradicts the spirit of the book and makes questionable its synthetic approach. Second, a modern exposition of von Staudt's theory of conjugate complex elements over a real projective space would serve a great need, since it is carefully eliminated from all contemporary textbooks of projective geometry. Modulo the use of com-

plex numbers, and following Klein very strictly, the construction of the euclidean and noneuclidean groups proceeds in a very lucid way. The last chapter in which this is done can also serve as an illustration to chapter III of F. Bachmann's book (Aufbau der Geometrie aus dem Spiegelungsbegriff, Springer, Berlin, 1959). The formulas of all three geometries are quite complete, including Klein's formula for the euclidean distance as the square root of the product of two cross-ratios involving isotropic lines. As a development of geometry, the reviewer still prefers F. Schilling's book (Projektive und nichteuklidische Geometrie, 2 vols., Teubner, Leipzig und Berlin, 1931), from which he learned non-euclidean geometry.

H. GUGGENHEIMER, University of Minnesota

Principles of Hindu Reckoning. By Kūshyār Ibn Labbān. Translated by Martin Levey and Marvin Petruck. University of Wisconsin Press, Madison, 1965. xii+114 pp. $6.00.

This book will be of interest to persons with three types of concern for the history of mathematics. It will please the serious historical scholar as a careful translation and analysis, accompanied by a photostatic copy of the original manuscript, of a previously little known Arabic work. This may be the earliest work of this type (c. 1000 A.D.) since the only other copy of Kushyar's work is in Hebrew, and the still earlier works of Al Khowarizmi and others have come to us only in Latin translations. In this sense the book is a valuable contribution to the history of mathematics.

The person with a serious interest in the history of mathematics whose concern is more for the development and communication of ideas will also find the book interesting. He will probably feel, however, that it documents and reinforces much that he has known previously and adds little to our fund of knowledge and insights.

The teacher of the history of mathematics may find it a useful teaching device, as the reviewer has. The ideally extensive use which should be made of source materials is largely frustrated by language difficulties and lack of multiple copies of original documents. The reviewer cited this book to his class and asked them to decipher, explain, and interpret two problems copied from it for them. In doing this they "discovered" and finally understood why we write numerals "backward" today (Arabs still write from right to left), that sexagesimal fractions persisted long after the Babylonians, and that operations with them possess many of the advantages and follow the algorithms the students associate with decimals. This latter was especially brought out in the translators' association of Kushyar's rules for determining the place value of sexagesimally expressed products with modern rules for operations with exponents.

On the whole there were few errors, the introductory discussions and explanations were clear and the documentation will lead an interested reader a fine and merry chase.

Marshall Clagett correctly claims in his foreword to the book that "the reader will turn with greater perception to the medieval Latin arithmetics of the high and late Middle Ages."

P. S. JONES, University of Michigan

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 15 (junior level)—18 (second graduate year).

Algebra

Algebraic Groups and Discontinuous Subgroups. Proceedings of Symposia in Pure Mathematics, Volume IX. American Mathematical Society, 1966. vii+426 pp. $10.20.

plex numbers, and following Klein very strictly, the construction of the euclidean and noneuclidean groups proceeds in a very lucid way. The last chapter in which this is done can also serve as an illustration to chapter III of F. Bachmann's book (Aufbau der Geometrie aus dem Spiegelungsbegriff, Springer, Berlin, 1959). The formulas of all three geometries are quite complete, including Klein's formula for the euclidean distance as the square root of the product of two cross-ratios involving isotropic lines. As a development of geometry, the reviewer still prefers F. Schilling's book (Projektive und nichteuklidische Geometrie, 2 vols., Teubner, Leipzig und Berlin, 1931), from which he learned non-euclidean geometry.

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" . . . A comprehensive exposition of the arithmetic aspect of algebraic groups." P, L.

The Theory of Spinors. By Elie Cartan. Foreword by Raymond Streater. M.I.T. Press, Cambridge, Mass., 1967. 157 pp. $7.00. A translation, first published in Paris in 1938, of notes on lectures by this great French mathematician. Rotations, reversals, multivectors, tensors, and spinors from a geometrical point of view. S, P, L.

A First Course in Abstract Algebra. By John B. Fraleigh. Addison-Wesley, Reading, Massachusetts, 1967. xvi+447 pp. $9.75. Intended as "a text from which an *average* student of mathematics" can benefit the most in a year course. Groups, rings, and fields are dealt with in forty-nine sections, reaching to such topics as algebraic extensions and homology groups. A subset of starred sections is designed to bring the student to Galois theory in a single semester. T(15—16).

Abstract Algebra. By Andrew O. Lindstrum, Jr. Holden-Day, San Francisco, 1967. xii+211 pp. $9.00. Sets, mappings, laws of composition, natural numbers, semigroups, equivalence relations, rational integers, groups, systems with more than one law of composition, polynomials, factorization, ideals, fields extensions, fields, linear mappings and matrices. Bourbaki terminology and logical symbols are used where appropriate. The book is based on classroom experience with four editions. T(16–17, year course).

Algebra. By Saunders MacLane and Garrett Birkhoff. Macmillan, New York, 1967. xix+598 pp. $11.95. This book is directed to the same audience as the authors' *Survey of Modern Algebra* which revolutionized undergraduate algebra following its publication in 1941. In order to present algebra for undergraduates on the basis of new insights and concepts, such as module, tensor product, category, and morphism, the authors have decided to make a "wholly new start." As in the previous book, abstract notions are developed from particular cases. Chapter headings are: Sets, Functions and Universal Elements, Integers, Groups, Rings, Special Fields, Modules, Vector Spaces, Matrices, Determinants and Tensor Products, Similar Matrices and Finite Abelian Groups, Quadratic Forms, Affine and Projective Spaces, Structure of Groups, Lattices, Categories and Adjoint Functors, Multilinear Algebra. Every textbook becomes obsolete in time, and it is to the credit of these authors that they decided to write a book that may well eliminate the use of their previous one. They even made it easy for us to distinguish the two by reversing the order of their names on the title page. T(15–17), S, P, L.

Varieties of Groups. By Hanna Neumann. Springer-Verlag, New York, 1967. viii+192 pp. $11.50. A "report" on a field that goes back to the 1930's but has developed for the most part since the 50's. P.

Abstract Algebra. By Chih-Han Sah. Academic, New York, 1967. xii+342 pp. $9.75. Preliminaries on set theory, from Peano's axioms to the rationals, groups, rings, integral domains, modules and associated algebras over commutative rings, vector spaces, fields, Galois theory, real and complex numbers. Pitched at a high level of abstraction, e.g. the determinant is defined, without any reference to matrices, in terms of the exterior algebra of a module over a commutative ring. T(16–17).

Applications

Optimal Control Theory, A Course in Automatic Control Theory. By R. Pallu de la Barriere. Translated by Scripta Technica. Edited by B. R. Gelbaum. Saunders, Philadelphia, 1967. xii+412 pp. $8.00. After mathematical prerequisites, the book deals

with linear servo systems, filtering, prediction, anticipation, discrete systems, programming, Markov control systems, minimum-time, and Pontryagin's principle. P.

Magneto-Fluid and Plasma Dynamics. Proceedings of Symposia in Applied Mathematics. Volume XVIII. American Mathematical Society, 1967. xii+294 pp. $6.80.

Seismic Love Waves. Z. S. Andrianova, V. I. Keilis-Borok, A. L. Levshin, and M. G. Neigauz. Translated from Russian by F. M. C. Goodspeed. Consultants Bureau, New York, 1967. xiv+91 pp. $15.00. The title is self-explanatory, but it should be remarked that the first and last authors are women and work at the Institute of Chemical Physics, while the two authors between are men and work at the Institute of Physics of the Earth. (The price is more than 15¢ per page!) P.

Spectral Theory and Wave Processes. Edited by M. Sh. Birman. Vol. I of Topics in Mathematical Physics. Translated from Russian. Consultants Bureau, New York, 1967. vii+114 pp. $15.00. The series is intended for "mathematicians specializing in the fields of mathematical physics, functional analysis, and the theory of differential equations." (The price is more than 13¢ a page!) P.

Applied Group Theory. By G. G. Hall. American Elsevier, New York, 1967. vii+128 pp. $6.00. "Applied" refers to physical applications. The emphasis is on "the theory of algebras" rather than "matrix representations." Topics include group algebra, character of groups, group representations and symmetry-adapted vectors, translation groups, continuous groups, rotation groups, product representations, and an appendix on connectivity. The author misspells Gordan. P.

The Advanced Theory of Language. By Gustav Herdan. Springer-Verlag, New York, 1966. xiii+495 pp. $14.00. A systematic treatise which includes portions of the author's more elementary *Language as Choice and Chance* (1956), now out of print. The author divides the subject of quantitative linguistics into four main branches: statistical linguistics, stylostatistics, optimal systems of language structure, and linguistic duality. The mathematical derivations appear in a systematic way in a final chapter of 75 pages. There are historical comments and bibliographies. This appears to be a significant contribution to a rapidly growing and very interesting field. P, L.

An Introduction to Mathematics for Students of Economics. By J. Parry Lewis. St. Martin's Press, New York, 1966. x+394 pp. $3.95 (paper). This is a paperback reprint of a book first published in 1959. It is slightly shorter and starts at a slightly lower level than *Mathematical Analysis for Economists*, by R. G. D. Allen, also available in paperback from the same publisher for $3.50. Both should be considered as possible textbooks or supplements in courses covering classical analysis. T, S, L.

Essentials of Statistics for Scientists and Technologists. By C. Mack. Plenum Press, New York, 1967. vii+174 pp. $5.95. For a low level service course. T.

Mathematics for the Physical Sciences. By Laurent Schwartz. Addison-Wesley, Reading, Mass., 1966. 358 pp. $14.00. An expanded and revised translation of the French original. After preliminary results in calculus it has chapters on elementary theory of distributions, convolution, Fourier series, the Fourier transform, the Laplace transform, the wave and heat conduction equations, the gamma function, and Bessel functions. There are exercises at the end of each chapter. Prerequisites include some linear algebra and functions of a complex variable. T(16), S, P.

Linear and Convex Programming. By S. I. Zukhovitskiy and L. I. Avadeyeva. Translated by Scripta Technica, edited by B. R. Gelbaum. Saunders, Philadelphia, 1966. viii+286 pp. $8.00. The editor's description, "a happy combination of mathematical

rigor, patient exposition and a wide variety of illustrative material" appears to be justified. Topics include Jordan elimination, the simplex method, the transportation problem, Chebyshev approximation, and convex programming. Though the jacket describes it as suitable for those "interested in advanced programming," the authors do not appear to assume any previous acquaintance with linear programming. T (16–17, if supplemented by exercises, since there are none in this book), S, T, L.

History

Bullettino di bibliografia e di storia delle scienze matematiche e fisiche, Vol. 1–20 (all published), Rome 1868–1887. With general index in volume 20. Reprint by Johnson Reprint Corporation. Cloth bound set $400, paper bound set $375; per volume paper bound $20. This famous serial, also known as "Boncompagni," is indispensable for the history of mathematics and ought to be in every mathematics research library. L.

The Encyclopedia of Philosophy. Edited by Paul Edward. Macmillan, New York, 1967. 8 volumes, app. 4300 pp. This attractive and readable encyclopedia testifies to the close ties between mathematics and philosophy and contains much biographical, historical and expository material of interest to mathematicians and suitable for supplementary reading by students. Among many articles, we mention Theory of Types (Y. Bar-Hillel), Number (S. F. Barker), David Hilbert (P. Bernays), Probability (M. Black), Combinatory Logic (H. Curry), Recursive Function Theory (M. Davis), G. Frege (M. Dummett), Set Theory (A. Fraenkel), Formal Systems and Their Models (L. Henkin), Continuity (S. Korner), Foundations of Mathematics (C. Parsons), History of Logic (A. N. Prior and others), Continuum Problem (R. M. Smullyan), Decision Theory (P. Suppes), Logical Paradoxes (J. van Heijenoort), Gödel's Theorem (ditto), Computing Machines (G. M. Weinberg). The bibliographies are good, but the Glossary of Logical Terms is unreliable. There is an index of 157 pages. S, P, L.

Britannica Book of the Year 1967. Encyclopaedia Britannica, Chicago, 1967. 832 pp. +a statistical supplement of 64 pages. Of interest to mathematicians are a two column article by Irving Kaplansky, which briefly mentions the main achievements of the year 1966, two paragraphs in the article on economics, which describe the triumph of mathematical methods in economic research and in Soviet planning, a description of the increased importance of mathematics in the article on geology, and two paragraphs in the article on philosophy relating to logic and mathematics. S, L.

The Basic Laws of Arithmetic. By Gottlob Frege. Translated and edited with an introduction by Montgomery Furth. University of California Press, Berkeley, 1967. lxiii+143 pp. (paper) $1.95. This is the first paperbound edition of a translation of *Grundgesetze der Arithmetik* (1893). The editor's introduction is a substantial 54 pages. S, L.

Mathematics Dictionary, Spanish-English/English-Spanish. By Mariano Garcia. Hobbs, Dorman, New York, 1965. vii+78 pp. $2.50. This brief dictionary is, as far as we know, the only English-Spanish mathematical dictionary now available. It appears to be up to date and should be useful to both English-speaking and Spanish-speaking mathematicians. P, L.

The Mathematical Papers of Sir William Rowan Hamilton. Vol. III, Algebra. Edited for the Royal Irish Academy by H. Halberstam and R. E. Ingram. Cambridge University Press, New York, 1966. xx+647 pp. $37.50. Vols. I (Optics) and II (Dynamics) appeared in 1931 and 1940. The editors promise a fourth "and final" volume of papers on analysis and geometry "in a few years." There is a facsimile of Hamilton's first notes on quaternions and a nine page introduction. P, L.

Fom Frege to Gödel, a Source Book in Mathematical Logic, 1879–1931. Edited by Jean van Hiejenoort. Harvard, Cambridge, Mass., 1967. xii+660 pp. $18.50. Forty-six papers, originally written in seven languages, are here translated into English and presented in chronological order beginning with Frege's epoch making *Begriffsschrift.* They were selected to depict the main strand of development. The last paper is by Herbrand on the consistency of arithmetic. This is one of a series of source books "of classical papers that have shaped the structure of the various sciences." The series is intended to cover all major sciences from ancient times to the middle of the present century, and several mathematical volumes are in preparation. This volume complements the collection edited by Copi and Gould (See below under Logic.) S, T, L.

The Six-Cornered Snowflake. By Johannes Kepler. Edited and translated by Colin Hardie, with essays by L. L. Whyte and B. F. J. Mason. Oxford, London, 1966. xiv+74 pp. $3.40. The English translation and modernized Latin are printed on facing pages. The essays deal respectively with the modern theory of snow crystals and with Kepler's effort in its historical context. S, P, L.

Mathematics and the Sciences of the Heavens and the Earth. Volume 3 of Science and Civilization in China. By Joseph Needham. Cambridge University Press, 1959. xlvii+878 pp. $30.00. Pages 1–168 deal with Chinese mathematics. The most thorough treatment of its subject, this work has forced the rewriting of ancient and medieval mathematical history. It ought to be in mathematics libraries as well as in reference sections of general libraries. P, L.

Festband zum 70. Geburtstag von Rolf Nevanlinna. Edited by H. P. Kunzi and A. Pluger. Springer-Verlag, Berlin, New York, 1966. 149 pp. 24 DM. Thirteen papers in honor of Nevanlinna's seventieth birthday plus a five-page biographical sketch and a bibliography of his published works. P, L.

The Mathematical Papers of Isaac Newton. Vol. I. 1664–1666. Edited by D. T. Whiteside with the assistance in publication of M. A. Hoskin. Cambridge University Press, 1967. xlvii+590 pp. $40. This magnificent volume is the first of eight in which all the known mathematical papers (many previously unpublished) will appear in an annotated and critical edition in which Latin materials will be printed with opposing English translations. This volume covers Newton's final year at Trinity College and the two following spent at home to escape the plague, during which he laid the basis of the calculus and the theory of light. There is a twenty-one page general introduction to the series and one of equal length for this volume, both written by the editor, who has already devoted a decade at Cambridge to researches related to Newton. P, L (!).

The Mathematical Works of Newton. Volume 1. Assembled with an introduction by Derek T. Whiteside. The Sources of Science, No. 3. Johnson Reprint Corporation, New York, 1964. xix+160 pp. $14.50. This fine volume contains facsimile reproductions of early English translations of three important mathematical works: *Analysis by equations of an infinite number of terms* (1745), *A treatise on the method of fluxions and infinite series* (1737), and *Quadrature of curves* (1710). This and a second planned volume complement *The Mathematical Papers* under the same editor. P, L(!).

Regiomontanus on Triangles. De triangulis omnimodis. By Johann Müller, otherwise known as Regiomontanus, translated by Barnabas Hughes, O.F.M., with an Introduction and Notes. University of Wisconsin Press, Madison, 1967. 298 pp. $10.00. In 1464, Regiomontanus "gathered together the trigonometric knowledge of his predecessors and enriched it with his own improvements." This fine edition of the first treatise on trigonometry gives the original in facsimile with translation on facing

pages. The introduction includes a ten page biography. Congratulations to Father Hughes and to the University of Wisconsin Publications in Medieval Science, edited by Marshall Clagett! S, P, L. Reviewed by D. J. Struik in *Science*, 28 July 1967.

Mathematical and Physical Papers. By George Gabriel Stokes. A reprint of the second edition of 1880–1905 which included annotations, biographical articles by Lord Kelvin and Lord Rayleigh, and portions of the original papers omitted from the first edition. With a new preface by C. Truesdell. No. 33 of The Sources of Science series. Johnson Reprint Corporation, New York, 1966. Five volumes, 360, 366, 421, 385, 402 pp. Cloth $75.00. L.

A Concise History of Mathematics. By Dirk J. Struik. Third edition. Dover, New York, 1967. x+195 pp. (paper) $2.00. The additional material on China, based on the work of Needham and others, makes this new edition of interest even to those who are completely familiar with its predecessors, which have been translated into more languages and been more widely read than any other contemporary history of mathematics. The bibliographies have been extended. T (as an introduction in a history course), S (in any mathematics course!), P, TT, L.

Logic and Foundation

The Undecidability of the Domino Problem. By Robert Berger. Memoirs of the American Mathematical Society, Number 66. American Mathematical Society, 1966. 72 pp. $1.90 (paper). A domino set is a finite number of infinite sets of congruent squares. Whether the infinite plane can be covered by squares chosen from an arbitrary domino set is proved undecidable. P, L.

Set Theory and the Continuum Hypothesis. By Paul J. Cohen. Benjamin, New York, 1966. 154 pp. $8.00. These notes on a course given at Harvard in 1965 are intended to give a self-contained exposition of the author's proof of the independence of the continuum hypothesis, for which he received a Fields Medal in 1966. Cohen addresses himself to the nonspecialist, avoids a formalistic style, and gives background material on logic, set theory, the consistency of the continuum hypothesis, formal systems, undecidable statements and recursive functions—all of which may be omitted by "the reader who has had an introductory course in logic." The author and publisher are to be congratulated for publishing these notes rather than waiting for the more polished version that doubtless will be forthcoming in due time. By so doing they afford the mathematical public a direct view of an exciting and significant sector of contemporary research in the foundations. S (17), P, L.

Contemporary Readings in Logical Theory. By Irving M. Copi and James A. Gould Macmillan, New York, 1967. 342 pp. $4.50 (paper). This is a book of readings for supplementary use in courses in symbolic logic. The essays are grouped in nine parts history of logic (Lewis, Langford, Kneale), the formal approach (Wang, Myhill, Nagel, Newman), meaning and reference (Frege, Russell, Strawson), theory of types (Russell, Fitch), logic and ontology (Quine, Carnap, Church, Goodman), logic and ordinary language (Ryle, Strawson, Quine, Geach), modal logic (Lewis, Quine, Fitch, Marcus), deontic logic (Wright, Anderson, O. K. Moore), and many valued logics (Rosser, Turquette). S, P, L.

Juristische Logik. By Ulrich Klug. Third enlarged and revised edition. Springer-Verlag, New York, 1966. xv+189 pp. Topics include the concept of juridical logic, pure logic and logic applied to the law, special arguments of legal logic, computers and the law, axiomatization of the law. Of interest to both logicians and jurists, this book is a candidate for translation. S, P.

theory and a one semester undergraduate course in general topology, the book introduces "as painlessly as possible" 2-dimensional manifolds, the fundamental group, and covering spaces, plus the group theory needed in these topics. Homology theory is intentionally not included. This is part of a series edited by Salomon Bochner and W. G. Lister, who in their foreword advocate more short, single-topic books to permit greater flexibility in college teaching—an idea that seems to be taking hold and deserves encouragement. T (16–17), P. L.

The Four-Color Problem. By Oystein Ore. Academic Press, New York, 1967. xv +259 pp. $12.00. This comprehensive study is certainly *the* book on its subject. It is an analysis of the entire literature, including important related studies in graph theory and map coloration problems. It combines history, exposition, analysis, and contributions—the latter via revision of many proofs. Pointing out that "one often sees in the history of mathematics, a fresh and comprehensive review of a subject has been the starting point for strong progress by the younger generation of mathematicians," the author expresses the hope that his work will have "some influence of this kind." It should be equally useful as a summary of achievements or a starting point for new departures. S, P, L.

Lecture Notes on Elementary Topology and Geometry. By I. M. Singer and John A. Thorpe. Scott, Foresman, Glenview, Ill., 1967. 220 pp., $6.25 paper. These notes are motivated by a desire to break up the compartmentalization of undergraduate education "at least in topology-geometry." "What the student has learned in algebra and advanced calculus is used to prove some fairly deep results relating geometry, topology, and group theory." The idea for such a course was conceived during the first author's "tour of duty for the Committee on the Undergraduate Program in Mathematics." Topics include the fundamental group, covering spaces, simplicial complexes, manifolds, homology theory, the De Rham theory, intrinsic Riemannian geometry of surfaces, and imbedded manifolds in R^3. T(16–17), S, T, L. The price of less than 3¢ a page compares very favorably with some books selling for more than three times this rate.

Connectivity in Graphs. By W. T. Tutte. University of Toronto Press, Toronto, 1966. ix+145 pp. $6.00. For those interested in research in the field. P.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Charles Hayes, University of California, Davis, represented the Association at the Dedication of Sonoma State College on May 27, 1967.

University of Hawaii: Miss Dorothy I. Koehler and Mr. J. K. M. Siu have been promoted to Assistant Professors; Assistant Professor Z. Z. Yeh has been promoted to Associate Professor; Assistant Professor W. J. Leahey has been promoted to Associ-

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ate Professor and appointed Chairman of the Mathematics Department; Associate Professor E. H. Mookini has been promoted to Professor.

University of Illinois: Assistant Professors K. I. Appel, Hyman Gabai and P. M. Weichsel have been promoted to Associate Professors; Associate Professor R. W. Carroll has been promoted to Professor; Professor K. T. Chen, SUNY at Buffalo, and Dr. J. L. Selfridge, Pennsylvania State University, have been appointed Professors; Associate Professor H. S. Leonard, Jr., Carnegie Institute of Technology, has been appointed Visiting Associate Professor; Mr. H. E. Benzinger, Jr., Syracuse University, Mr. D. K. Friesen, Dartmouth College, and Mr. D. G. McRae, University of Washington, have been appointed Assistant Professors.

Montana State University: Dr. L. C. Barrett, Clarkson College of Technology, has been appointed Head of the Mathematics Department; Dr. B. L. McAllister, South Dakota School of Mines and Technology, has been appointed Associate Professor; Doctors N. H. Eggert, Colorado State College, Vinnie H. Miller, Brigham Young University, G. H. Ryder, Clarkson College of Technology, and K. J. Tiahrt, Oklahoma State University, have been appointed Assistant Professors.

Wesleyan University: Dr. J. F. Berglund, Tulane University, has been appointed Assistant Professor; Mr. E. M. Coven, Yale University, has been appointed Assistant Professor; Professor W. W. Comfort, University of Massachusetts, has been appointed Professor.

Dr. D. C. Bossard, Dartmouth College, has accepted a position with Daniel H. Wagner Associates.

Professor H. L. Dorwart, Trinity College, has been appointed Dean of the College.

Dr. E. J. Eckert has been appointed Chairman of the Department of Mathematics at California State College at Los Angeles.

Dr. J. B. Goebel, Pacific Northwest Laboratory, has been appointed Professor and Head of the Department of Mathematics at the Montana School of Mines.

Assistant Professor Dagmar Henney, The George Washington University, has been promoted to Associate Professor.

Mr. R. R. Holbert, Bob Jones University, has accepted a position with the IBM Corporation in the Systems Development Division.

Associate Professor Taqdir Husain, McMaster University, has been promoted to Professor and appointed Acting Chairman of the Mathematics Department.

Professor J. S. Mac Nerney, University of North Carolina at Chapel Hill, has been appointed Professor at the University of Houston.

Professor A. O. Boatman, Carthage College, died on January 19, 1967. He was a member of the Association for twenty-eight years.

Professor Emeritus P. P. Boyd, University of Kentucky, died on December 19, 1966. He was a Charter Member of the Association.

Professor Emeritus R. D. Carmichael, University of Illinois, died on May 2, 1967 at the age of 88. He was for many years dean of the graduate school of the University of Illinois. He was a Charter Member of the Association. He served as President in 1923, Vice President in 1921–22, Member of the Committee on the Official Journal 1916–18, and was an elected member of the Board of Governors in 1920, 1924–29, and 1939–41.

Professor Emeritus Mabel M. Heren, Knox College, died on February 19, 1967. She was a member of the Association for forty-seven years.

Instructor Emeritus W. W. Johnson, Cleveland State University, died on November 1, 1966. He was a Charter Member of the Association.

Professor Emeritus G. W. Spenceley, Miami University, died on March 6, 1967. He was a member of the Association for forty-six years.

CONFERENCE ON COMMUNICATION PROBLEMS IN THE MATHEMATICAL SCIENCES

The American Mathematical Society will hold a Conference on Communication Problems in the Mathematical Sciences on December 5, 6, and 7, 1967, in Providence, Rhode Island. The scientific sessions will take place at the Sheraton-Biltmore Hotel, and the program will include the official opening of the new AMS headquarters office building in Providence.

The purpose of the Conference is to consider improved means of retrieval, categorization, distribution, and collection of mathematical material. The Conference will encourage cooperation among organizations in the mathematical community in the development of new guidelines for an integrated, balanced information system. The participation of distinguished scientists in other disciplines and from other countries will insure an overall perspective of communication problems.

All interested mathematicians and scientists are invited to attend the Conference. Information concerning accommodations and a complete Conference program may be obtained from the American Mathematical Society, P. O. Box 6248, Providence, Rhode Island 02904.

THE UNIVERSITY OF TEXAS—SIXTH ANNUAL SYMPOSIUM ON BIOMATHEMATICS AND COMPUTER SCIENCE IN THE LIFE SCIENCES

The University of Texas Graduate School of Biomedical Sciences at Houston, Division of Continuing Education, is pleased to announce the Sixth Annual Symposium on Biomathematics and Computer Science in the Life Sciences, which will be held at the Shamrock Hilton Hotel in Houston, Texas, on March 14–16, 1968. The theme of the symposium is MAN, MACHINES, MATHEMATICS, AND MEDICINE. The topics for the sessions of the symposium are: 'Quantitative and Theoretical Biology,' 'Mathematical and Applied Statistical Problems in Biomedical Research,' 'Experimental Design in the Biomedical Sciences', 'Bioengineering,' 'Information Management: Storage and Retrieval,' 'Computer Applications in the Biomedical Sciences,' and 'The Man-Machine Interface.'

Papers of an interdisciplinary nature will be particularly welcomed by the Program Committee, as will appropriate papers on education. Those interested in submitting abstracts should contact the Office of the Dean, The University of Texas Graduate School of Biomedical Sciences at Houston, Division of Continuing Education, P. O. Box 20367, Houston, Texas 77025. *Abstracts will not be reviewed for inclusion in the program after December 15, 1967.*

FELLOWSHIP AND RESEARCH OPPORTUNITIES IN MATHEMATICS

The Division of Mathematical Sciences of the National Research Council calls attention to a number of fellowships and other support for research in the mathematical sciences at both the predoctoral and postdoctoral levels to be awarded during the year 1967–68. Copies of the complete announcement are available from: Division of Mathematical Sciences, National Research Council, 2101 Constitution Avenue, N. W., Washington, D. C. 20418.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-EIGHTH SUMMER MEETING OF THE ASSOCIATION

The Forty-eighth Summer Meeting of the Mathematical Association of America was held at the University of Toronto, Toronto, Ontario, Canada, from Monday, August 28, to Wednesday, August 30, 1967, in conjunction with meetings of the American Mathematical Society, the Canadian Mathematical Congress, the Society for Industrial and Applied Mathematics, the Econometric Society, the Pi Mu Epsilon Fraternity, and Mu Alpha Theta. The lectures by Professors Coxeter and Lehner on Monday and by Professor Samuel on Tuesday were joint sessions with the Canadian Mathematical Congress. The last lecture on Wednesday afternoon was a joint session with the Econometric Society. There were registered 2250 persons, including 850 members of the Association.

Sessions of the Association were held on Monday morning and afternoon, on Tuesday morning, and on Wednesday afternoon. All sessions were held in Convocation Hall of the University of Toronto. Presiding officers at the three Earle Raymond Hedrick Lectures were President E. E. Moise, First Vice-President G. S. Young, and Second Vice-President R. J. Walker; at the presidential address to the Canadian Mathematical Congress by Professor Coxeter, Professor A. W. Tucker; at the lecture by Professor Lehner, Professor Maurice L'Abbe; at the panel discussion on Monday afternoon, Professor V. L. Klee; at the lecture by Professor Samuel on Tuesday morning, Professor G. de B. Robinson; at the panel discussion on Wednesday afternoon, Professor P. D. Lax; and at the joint session with the Econometric Society, Professor Daniel McFadden. The sixteenth series of Earle Raymond Hedrick Lectures was delivered by Professor Gian-Carlo Rota of the Massachusetts Institute of Technology and the Rockefeller University. The Program Committee consisted of G. de B. Robinson, Chairman; T. E. Hull, P. D. Lax, A. H. Stone, and W. T. Tutte.

FIRST SESSION OF THE ASSOCIATION

Welcome on behalf of the University by Professor G. de B. Robinson, Vice-President, University of Toronto, and response by Professor E. E. Moise, President of the Association.

The Earle Raymond Hedrick Lectures: *Combinatorial Analysis as a Theory*, Lecture I, by Professor Gian-Carlo Rota, Massachusetts Institute of Technology and the Rockefeller University.

The Problem of Apollonius, Presidential Address to the Canadian Mathematical Congress, by Professor H. S. M. Coxeter, University of Toronto.

In the real inversive plane, three nonintersecting circles may be either Apollonian (so that exactly eight circles touch them all) or "nested" (so that no circle touches them all). Any two nonintersecting circles have a unique "mid-circle" which inverts each of them into the other. The three mid-circles of an Apollonian triad are coaxal. Any two nonintersecting circles have an "inversive distance," which is the logarithm of the ratio of the radii of two concentric circles into which the given circles can be inverted. Let a, b, c be the inversive distances between pairs of circles in an Apollonian triad. Among the eight circles that touch the three, two are nonintersecting if and only if each of the three numbers $\cosh \frac{1}{2}a$, $\cosh \frac{1}{2}b$, $\cosh \frac{1}{2}c$ is less than the sum of the other two

The Picard Theorems, by Professor Joseph Lehner, University of Maryland.

The theorems are: (1) a nonconstant entire function assumes every complex value with one possible exception; (2) in every neighborhood of an isolated essential singularity a holomorphic function has the same property. In graduate courses (1) is commonly proved by the elliptic modular function, but (2) is then established by the so-called "elementary" (Landau-Schottky) approach.

Here it is shown that once the modular function has been developed, (2) can be proved without much additional work. The modular function can also be used to prove Landau's and Schottky's theorems and to derive the exact values for the constants therein.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Rota.

Panel Discussion on Geometry in the University, by Professor Walter Prenowitz, Brooklyn College, Professor Seymour Schuster, University of Minnesota, and Professor F. A. Sherk, University of Toronto, with Professor V. L. Klee, University of Washington, as moderator.

Professor Schuster described the College Geometry Project at the University of Minnesota, including its aims and accomplishments. While this project is primarily concerned with the education of prospective secondary teachers, the work of the project has considerable bearing on, and non-trivial implications for, the geometry courses for mathematics majors. These implications were elucidated.

Professor Sherk outlined and discussed the undergraduate geometry courses given at the University of Toronto.

Professor Prenowitz's theme was that n-dimensional (linear) geometry can be successfully studied by employing intrinsic methods without assuming the existence of coordinates or vectorial representations. The principal illustrations given were the join theoretic methods that appear in the speaker's Slaught Memorial Paper (this MONTHLY, Vol. 68 No. 1, January 1961). Specific topics considered were the generation of convex sets and linear spaces, properties of half-spaces, separation theorems and the theory of linear independence and dimension. Much of the treatment described is not merely coordinate-free but dimension-free as well.

Professor Klee then listed and discussed six desiderata for undergraduate geometry courses and explained how the geometry of convex bodies can be presented so as to include all six desirable features. An article based on this presentation will appear in the MONTHLY.

General Discussion by the Panel and the Audience

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Rota.

Business Meeting of the Association; presentation of L. R. Ford Awards.

Unique Factorization, by Professor Pierre Samuel, University of Paris.

Arithmetic and geometric characterizations of unique factorization in commutative rings were discussed, in particular, the methods used for proving that a ring is factorial; relations of factorial rings with more general (e.g. Krull) rings and more special classes of rings (e.g. principal or Euclidean rings); the behaviour of factorial rings under various ring-theoretic operations (rings of quotients, polynomials, power-series, rings of invariants). Some deeper properties, related with Grothendieck's theory of schemes, were stated.

FOURTH SESSION OF THE ASSOCIATION

Panel Discussion on What to Do Till the Computer Scientist Comes, by Professor G. E. Forsythe, Stanford University, Professor T. E. Hull, University of Toronto, and Professor Marvin Minsky, Massachusetts Institute of Technology, with Professor P. D. Lax, New York University, as moderator.

Professor Forsythe defined computer science and discussed its many applications, its relation to mathematics, and the activities of the approximately 35 existing computer science departments. He gave the following answers to the question of the title: (1) get a little bit acquainted with computing by programming, and by reading some books from a recommended list; (2) study how computing intersects mathematics and keep this in mind as you teach; (3) help the computer scientist feel welcome when he comes but expect a big blast of computing activity; (4) (optional) tackle a mathematical research problem that would help computer science and your publication list (sample problem areas were given).

Professor Hull felt that much of what is now being taught about computing to freshmen and sophomores will soon be available in most high schools. About one in ten Ontario high schools now offers a two-year option in Computer Science, and the number is increasing rapidly. This development can have a very beneficial effect on existing courses in mathematics, by reinforcing the concepts of sets, functions, the nature of proof, etc., and by introducing the concept of an algorithm in a fundamental and meaningful way.

Professor Minsky suggested that one should first ask how to tell when Computer Science has arrived. This will be true when the graduate students know more than the undergraduates. The most serious problems center around the lack of realistic mathematical models for Computation, and young mathematicians should be encouraged to work on: theory of reciprocity between memory-size and number of computation-steps; reciprocity between local and global methods of pattern-recognition (to shed more light on the relation between serial vs. parallel algorithms); finding methods for establishing lower bounds for what is necessary for particular algorithms, as in numerical, sorting, and algebraic-manipulation problems. These are much more important than most workers realize, and good results that seem special and *ad hoc* at first may lead to large new theories.

The worldly demands of Computation Science will bring heavy pressure on Mathematics Departments to become more "Applied." The priceless integrity of Pure Mathematics must be protected by expansion rather than dilution, for Computation is, he believes, to become the largest and most powerful of our technological and intellectual disciplines.

JOINT SESSION WITH THE ECONOMETRIC SOCIETY

Numerical Representation of Preference Orderings, with an Application to Infinite Consumption Sequences, by Professor T. C. Koopmans, Cowles Foundation for Research in Economics, Yale University.

The speaker started with a set of axioms about a preference ordering on a space of infinite sequences of future consumption vectors. He constructed a numerical representation of such a preference ordering by a discounted sum of future utilities. A slight weakening of one axiom permits a wider class of representations in which the discount rate for constant consumption sequences varies with the utility of the consumption vector in question.

SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in Concert Hall of the Edward Johnson Building as follows:

Sunday, 7:00–7:47 P.M. MEASURES AND SET THEORY: A Lecture by Stanislaw Ulam (A CEM Individual Lectures film in b & w).

8:00–9:03 P.M. NIM AND OTHER ORIENTED GRAPH GAMES: A Lecture by Andrew M. Gleason (A CEM Individual Lectures film in b & w).

Films of the College Geometry Project of the University of Minnesota (in color).

Monday, 7:00–7:12 P.M. ORTHOGONAL PROJECTION by Daniel Pedoe; 7:13–7:23 P.M. CENTRAL SIMILARITIES by Daniel Pedoe; 7:24–7:36 P.M. DIHEDRAL KALEIDOSCOPES by H. S. M. Coxeter; 7:37–7:54 P.M. GEOMETRIC VECTORS by Wm. O. J. Moser and S. Schuster; 8:05–8:18 P.M. INVERSION by Daniel Pedoe; 8:19–8:35 P.M. CURVES OF CONSTANT WIDTH by J. D. E. Konhauser; 8:36–8:49 P.M. CENTRAL PERSPECTIVITIES by S. Schuster.

CEM Animated Calculus Films (in color).

9:00–9:15 P.M. THE DEFINITE INTEGRAL by Charles E. Rickart; 9:16–9:24 P.M. VOLUME BY SHELLS by George F. Leger; 9:25–9:39 P.M. FUNDAMENTAL THEOREM OF THE CALCULUS by Morris Schreiber; 9:40–9:50 P.M. THE THEOREM OF THE MEAN by Felix P. Welch;

Tuesday, 7:00–8:03 P.M. JOHN VON NEUMANN (A CEM Individual Lectures film in b & w).

CEM Animated Level I Films (in color).

8:15–8:29 P.M. WHAT IS A SET? 8:29–8:39 P.M. ONE-TO-ONE CORRESPONDENCE; 8:39—8:48 P.M. COUNTING, 8:48–8:56 P.M. UNION AND INTERSECTION; 8:56–9:04 P.M.

ADDITION AND SUBTRACTION.

9:20–10:20 P.M. WHO KILLED DETERMINANTS: A Lecture by Kenneth O. May (A CEM Individual Lectures film in b & w).

10:22–10:30 P.M. MULTIPLICATION AND DIVISION. (A CEM Animated Level I film in color).

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Sunday at 10:00 A.M. in Room 134 of the McLennan Laboratories at the University of Toronto, with thirty-eight members present. Among the items of business transacted were the following:

The Board elected Professor A. B. Willcox of Amherst College as Executive Director of the Association for a term beginning on or about August 1, 1968.

The Board elected Professor C. J. A. Halberg, Jr. of the University of California, Riverside, as Governor from the Southern California Section to fill the unexpired part of the term ending June 30, 1968, of Professor R. C. James, who has moved from the Southern California Section.

The Board voted to invite Professor Hyman Bass of Columbia University to deliver the seventeenth series of Earle Raymond Hedrick Lectures at the 1968 Summer Meeting.

In accordance with a petition submitted by 34 members of the Association residing in Florida and upon the recommendation of the Committee on Sections, the Board voted to authorize the establishment of a Florida Section as of September 1, 1967, with the exact territory to be covered by the Section to be determined at a later date. The Board then elected Professor D. B. Goodner of Florida State University as the first Governor from the Florida Section for a term ending June 30, 1968, with the possibility that the Section can re-elect him for a full three-year term if it so desires.

Upon the recommendation of the Finance Committee, the Board set the annual dues for academic members at $50, beginning with 1968.

The Executive Director reported the membership of the Association as 17,596 individual members, 3 corporate members, and 245 academic members.

Upon the recommendation of the Executive and Finance Committees, the Board approved a proposal, originally made in a slightly different form by Professor Grosswald, for additional awards to be made to high ranking participants in the Putnam Competition as follows:

(a) The Board encourages the Sections to offer free one-year memberships in the MAA to participants in the Putnam Mathematical Competition who have achieved high scores, to be financed, wherever possible, from funds available to the Sections.

(b) The national office shall offer (in addition to the present prizes) five free one-year memberships in the MAA to the five students who scored individually highest in the Putnam Mathematical Competition, regardless of the success of their respective teams.

(c) Sections whose resources might be severely strained by offering two free one-year memberships in the Association might wish to apply for funds from the annual appropriation of $500 available to the Committee on Sections for worthwhile projects of the Sections.

The Board passed a resolution of thanks to the Du Pont Company for its contribution of $5000 toward printing and distributing the brochure YOU'LL NEED MATH.

The Board of Governors authorized the Secretary to send a letter of appreciation and thanks to the National Science Foundation recognizing their valuable support in mathematical education and research; copies of this letter are to be sent to the appropriate congressional committees and to officers of the National Science Foundation.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Tuesday morning with President Moise presiding.

The third set of L. R. Ford Awards were presented by President Moise to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE in 1966. The Awards, in the amount of $100 each, were presented for five articles (for

further details on these Awards, see the August–September issue of this MONTHLY, page 908).

The Secretary then reported on some of the actions taken by the Board on Sunday. He announced that, in accordance with the recommendation of the Committee on the Future Administrative Structure of the Association, the Association has decided to move its central office to Washington, D.C. The reasons for the decision to move to Washington are best expressed in a resolution adopted by the Executive and Finance Committees at their meeting in Washington, D.C. on July 9. He then read the resolution which appears on page 1176 of this issue of the MONTHLY.

This resolution has been sent to the presidents and secretaries of all the major organizations in the mathematical sciences. Expressions of interest in cooperation have already been received from many of these organizations.

The Secretary reported the receipt of a grant of $716,495 from the National Science Foundation for the support of the Committee on the Undergraduate Program in Mathematics. He stressed the Association's gratitude to the Foundation for this grant and the confidence which it expresses in the Association's past work in improving undergraduate instruction in mathematics.

The Secretary also reported the receipt of $5000 from an anonymous donor to support certain activities of the Committee on Educational Media which were not provided for in the grant from the National Science Foundation.

The Secretary announced that copies of the second edition of the GUIDEBOOK TO DEPARTMENTS IN THE MATHEMATICAL SCIENCES IN THE UNITED STATES AND CANADA are available from the Buffalo Office of the Association for 50 cents per copy. This book provides in summary form information about the location, size, staff, library facilities, course offerings, and special features of departments in the mathematical sciences in 1200 four-year colleges and universities in the United States and Canada.

He also announced that the seventh edition of PROFESSIONAL OPPORTUNITIES IN MATHEMATICS has just been published and is available from the Buffalo office of the Association for 25 cents per copy. To be published early in 1968 are: MAA STUDIES No. 5: Studies in Modern Topology, edited by Peter Hilton and CARUS MONOGRAPH No. 15, Non-Commutative Rings, by I. N. Herstein.

He announced the appointment of a Committee on Assistance to Negro Colleges, which is to consider various possibilities in which the Association might be able to assist Negro colleges. This Committee consists of George Springer, Chairman; R. D. Anderson, Murray Braden, L. L. Clarkson, C. O. Oakley, B. E. Rhoades, M. B. Smith, Jr., and F. M. Stewart.

The Secretary expressed the gratitude of the Association to the University of Toronto for having made their excellent facilities available for this meeting and ascribed the success of this meeting to the careful and energetic planning of the local Committee on Arrangements under the chairmanship of Professor Chandler Davis. He expressed special appreciation also to Professor F. A. Sherk for his imaginative handling of the publicity for this meeting.

He announced the following schedule of future meetings: San Francisco, California, January 25–27, 1968; University of Wisconsin, Madison, August 26–28, 1968; New Orleans, Louisiana, January, 1969; University of Oregon, August 25–27, 1969; Miami, Florida, January, 1970.

A motion to amend Article V of the By-Laws of the Association, as it appears on pages 624–25 of this MONTHLY, was then approved.

MEETING OF SECTION OFFICERS

The meeting of representatives of the Sections was held on Monday evening in Room 102 of the McLennan Laboratories. Professor L. E. Mehlenbacher, Chairman of the

Committee on Sections, presided. Fifty-three persons were present, representing twenty-six of the twenty-seven Sections of the Association.

President Moise reported on the visits he has made to the Sections. Of all the activities connected with the presidency, he considered his visits to the sections the most pleasant. While we meet nationally—or internationally—twice a year, the work of the Association is carried out in the Sections; for most of the members most of the year, the Sections are the Association. The work of CUPM, for example, could not be carried out successfully without the assistance of the Sections. He wished luck and success to the Sections in their future work.

Sister Mary Paul Buser, Chairman of the Kansas Section, reported on the special awards for the Putnam Mathematical Competition made by the Kansas Section. In addition to a one-year membership in the MAA, each of the top five students in the State receives a monetary award, which is made at the luncheon of the spring meeting of the Section.

Professor C. T. Long, Chairman of the Pacific Northwest Section, gave a report on efforts on the Pacific Northwest Section to reach junior college personnel. Discussion with junior college teachers revealed that they felt a need for special programs at Section meetings for teachers at this level as well as possible representation on the Executive Committee of the Section. Such programs, stressing the teaching of mathematics at the appropriate level, were arranged for the 1966 and 1967 meetings. These sessions were well attended, and continued efforts along these lines are expected to receive support. At the 1967 Section meeting, the By-Laws of the Section were amended to create the position of Second Vice-Chairman, who will represent the two-year college personnel on the Executive Committee, arrange the program for the two-year college sessions at the annual Section meetings, and preside at these sessions. Professor Long noted that the Section has an MAA representative at almost all of the two-year colleges, and that the Section was making an effort to attract members from these colleges.

Professor Dale Maness presented the report of the Texas Section Committee on "A General Curriculum in Mathematics for Colleges" (GCMC). The report concludes that "the evaluation committee feels that GCMC is a valuable contribution from CUPM even though its outline may not be specifically followed in many schools. It points out areas of the curriculum that will be of increasing importance . . . "

In answer to a question as to what disposition will be made of these reports by Sections on the GCMC report, Professor Alder reported that CUPM is planning to publish a representative compilation or synthesis of such reports when additional reports have been received from the Sections who have not as yet sent in their evaluations.

Professor D. T. Finkbeiner reported that the Ohio Section plans a special meeting this fall on collegiate mathematics and that they have had excellent support from CUPM and the officers of the Association in providing good speakers; this program has generated a considerable amount of enthusiasm in the Section.

Dr. Richard Balomenos reported on trends in the National Science Foundation, specifically, (1) the decision to discontinue the Visiting Scientist Program for Secondary Schools, (2) the construction of mathematical problems for use with secondary students; and (3) the new Office of Computing Activities.

As regards (1), NSF was confronted with a relatively level budget for support of science and mathematics in the pre-college area. Faced with the problem of providing for the continuing support of programs designed to meet longstanding needs in science and mathematics education, the Foundation has recently failed to meet its responsibility for providing support for new ideas. As a result, a program entitled "Special Projects in Science Education" was established and additional support was given to the Cooperative College-School Science Program for teacher training activities in the context of the needs of school systems. To provide for part of the increased support of these programs, it was necessary to discontinue the Visiting Scientist Program. It is hoped that as a re-

sult of the encouragement from NSF such visits to the secondary schools can continue with support from local sources.

In reference to (2), NSF has received numerous requests for support of projects to construct sets of mathematical problems for use by groups involved in talent search or mathematical contest activities. NSF has supported some of these projects. However, funds and talent are not available to provide for all the local requests. Hence, the question by NSF, "Is there need for a national or regional project to construct problem sets and assist local groups?"

Concerning (3), NSF has established an Office of Computing Activities under Dr. Milton Rose. One of the major aims of the office is to develop the potential of computers in the educational process at all levels. The comments and views from individual mathematicians or groups, such as the MAA, as well as specific proposals, will aid the office in establishing priorities for its activities.

President Moise, in commenting on the Visiting Scientist Program for Secondary Schools, felt that one of its main values was the impact made on students' attitudes toward mathematics when they realize that there are such people as mathematicians. One of the problems is to present mathematics as a live field. A way of demonstrating this is to present an example of one of the human beings who is an active worker in this field. He felt that the program is so important that he would like to see it continued on a voluntary basis. The impact a mathematician makes when spending an afternoon visiting a school will be well worth the effort. Dr. Balomenos also emphasized that discontinuance of NSF support should not mean the end of the program. He reported that in some areas the program is being supported by industry and others.

In the absence of Professor H. M. MacNeille, Director of the Calculus Film Project, a letter was read from him announcing that the films produced by the Calculus Film Project are now available from Modern Learning Aids.

Professor C. T. Salkind, Chairman of the Committee on High School Contests, reported on some changes in the Contest. He requested that the Section officers be familiar with, and make known through appropriate means, the following changes in the Annual High School Mathematics Contest for 1968:

(a) The number of questions is being reduced from forty to thirty-five with the maximum score remaining at 150.

(b) Problem subdivision into four parts is replacing the present three: Part I, consisting of ten "very easy problems," 3 credits each, Part II, with ten "easy" problems, 4 credits each, Part III, with ten "medium" problems, 5 credits each, and Part IV, with five "hard" problems, 6 credits each.

This revision (or one very similar to it) will be tried for a period of two years. If it receives general acceptance, and, nevertheless, maintains discriminatory power on various levels, it will be retained.

He also announced certain revisions in the composition of the ten regions in which the United States and Canada have been divided for the purposes of the Contest. He announced that the 1968 Contest would be held on Tuesday, March 12, 1968, and not on a Thursday as has been previously the case.

Professor R. E. Gaskell, Chairman of the Committee on Visiting Lecturers, discussed the history of this program from its initiation in 1954. He remarked that the number of day-visits is receding from a high in 1964–65, and the Committee seeks to extend the program to schools which have not previously been visited. The Committee proposes to contact Section Chairmen, asking for the names of schools in this category within the various Sections.

Professor R. D. Boswell, Jr., a member of the Committee, reported on the efforts made by the Committee to find out which lectures have been successful. They are usually asked to continue. This year, there will be 65 lecturers, of which approximately 25 will be new lecturers in the region in which they will make their visits.

Professor J. H. McKay, Director of the Putnam Mathematical Competition, re-

There was a party with dancing held at Casa Loma Castle on Tuesday at 9:00 P.M. SIAM conducted a beer party on Wednesday at 9:30 P.M. in Hart House. The University offered the participants a free concert on Thursday at 8:30 P.M. in Concert Hall. The artists were Lorand Fenyves, violinist, and Bela Siki, pianist. All-day bus tours to Niagara left on Wednesday and Thursday from Sidney Smith Hall at 9:40 A.M. on both days and returned at approximately 7:45 P.M. In addition, there were a number of local tours to points of interest in the Toronto area.

HENRY L. ALDER, *Secretary*

ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendment adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting in Toronto, Canada, on August 27, 1967, elected to membership the twelfth set of applicants for academic membership (for election of the other eleven sets, see the April and November issues for 1962–67). Approval for election was given to the following 9 applicants for academic membership:

Brock University, St. Catharines, Ontario, Canada
College of Artesia, Artesia, New Mexico
College of St. Thomas, St. Paul, Minnesota
Holy Trinity High School, Hicksville, New York
Inter American University, Puerto Rico
Miami Dade Jr. College, Miami, Florida
Mt. Hood Community College, Gresham, Oregon
Northern Virginia Community College, Bailey's Crossroads, Virginia
Pacific Lutheran University, Tacoma, Washington

In addition, the Board elected the American Mathematical Society as a corporate member.

HENRY L. ALDER, *Secretary*

MOVE OF MAA CENTRAL OFFICE TO WASHINGTON

The following resolution was approved by the Executive and Finance Committees of the Association on July 9, 1967, in Washington, D.C.:

"The Mathematical Association of America has decided to move its headquarters office to Washington, D.C., late in 1968. The decision was made at a meeting of the Executive and Finance Committees of the Association which was held in Washington on July 8, 1967.

The Mathematical Association of America takes this opportunity to invite other organizations in the mathematical sciences to join it in locating their headquarters in Washington, so that we all can realize, to the fullest extent, the advantages of a common location and the special advantages of Washington.

We believe that the work of the learned societies, in the mathematical sciences and in general, will become easier and more effective if informed day-to-day communication among them is made convenient. For this reason, in the long run, the learned societies should have their headquarters in some one city. And the city on which the learned societies have tended to agree is Washington because all these societies need to communicate not only with one another but also with the federal government.

In attempting to facilitate communication within the scientific community, the Mathematical Association of America is concerned primarily with the problems of scientific and educational policy. We hope, however, that if other organizations in the mathematical sciences also estab-

lish headquarters in Washington, then cooperative arrangements can be made for more economical handling of such operations as the editing of journals, publication of books, maintenance of membership lists, collection of dues, and so on."

HENRY L. ALDER, *Secretary*

RETIRED MATHEMATICIANS

The next issue of the List of Retired Mathematicians Available for Employment will be published in January 1968 by the Mathematical Sciences Employment Register. Besides being distributed to academic and industrial employers, who request the list from the Register office, copies will be available when the Register is open at the annual meeting in San Francisco, California, January 24–26, 1968. Retired mathematicians who are interested in being included in the list may either request a form from the Register office or send the following information: name, date of birth, highest degree earned and where it was obtained, most recent employment, present address, date available, references, preference for academic or industrial employment, and geographic location preferred. The deadline for receipt of either the completed form or the above information is January 1, 1968.

SUMMER EMPLOYMENT OPPORTUNITIES

The List of Opportunities for Summer Employment for Mathematicians and College Mathematics Students is being compiled and will be issued in January 1968. The list will be available free of charge at the annual meeting in San Francisco, California, in January and upon request from the Register office. Institutions and industrial concerns that have summer openings and would welcome applications from mathematicians and students of mathematics may request forms for listing from the Register office. The deadline for receipt of the completed forms is January 1, 1968.

THE EMPLOYMENT REGISTER

During the annual meeting in San Francisco, California, the Mathematical Sciences Employment Register will be open for the scheduling of interviews between applicants and employers. The Employment Register which is sponsored by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be located in the Imperial Ballroom of the San Francisco Hilton Hotel, and will be open from Wednesday, January 24, through Friday, January 26, from 9:00 A.M. to 5:00 P.M. on each of the three days. Registration for the Employment Register is *separate and apart* from meeting registration, and it is, therefore, most important that both applicants and employers sign in at the Employment Register Desk as early as they can on Wednesday morning. A separate visual index will be maintained for Employment Register use only. Appointments may be requested only with applicants and employers who have actually signed in at the Register.

There is no charge for registration except when the late registration fee of $5.00 is applicable. Provision will be made for anonymity of applicants upon payment of $5.00 to defray the cost involved in handling such a listing.

Applicants and employers who wish to be listed should write to the Mathematical Sciences Employment Register, Post Office Box 6248, Providence, Rhode Island 02904, for applicant qualification forms or position description forms. These forms must be completed and returned to the Employment Register not later than December 15, 1967, in order to be included in the January lists. Those forms which arrive too late to be included in the printed lists are taken to the meeting where they may be seen by applicants and/or employers who are interested in them. The printed lists will be mailed to sub-

scribers during the first week in January. Lists can be ordered from the Employment Register office in Providence. They will also be available at the meeting.

A subscription to the lists, which includes three issues (January, May, and August) of both the applicants list and the positions list is available for $25.00 a year. The individual issues of both lists may be purchased in January, May, and August for $12.50. A subscription to the applicants list alone or single copies of that list are not available. Copies of the positions list only may be purchased for $3.00. Checks should be made payable to the American Mathematical Society and sent to the address given above.

MARCH MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The forty-fourth annual meeting of the Louisiana-Mississippi Section of the MAA was held in the Jung Hotel, New Orleans, Louisiana, on March 3, 1967, in conjunction with a place of site meeting of the NCTM. There were 144 persons registered at the MAA sectional meeting, including 112 members of the Association.

The following officers were elected: N. A. Childress, University of Mississippi, Chairman; Patrick Ford, McNeese State College, Louisiana, Vice Chairman; Z. T. Gallion, Mississippi State College for Women, Mississippi, Vice Chairman; L. Virginia Carlton, Centenary College, Secretary-Treasurer.

The following papers were presented:

1. *Partially ordered Abelian groups*, by Robert Pendleton, Louisiana State University.

2. *Uniqueness theorems for a class of second-order differential equations*, by J. B. Garner, Louisiana Polytechnic Institute.

THEOREM 1. Let $f(x, y, z) \in C$ in $[a, b]XR^2$ and let $\beta_1, \beta_2 \in R$ and $a \leqq \alpha_1 \leqq \alpha_2 \leqq b$. If $f(x, y, z)$ is nonincreasing in y and nondecreasing in z for each fixed $x \in [a, \alpha_1]$, nondecreasing in y and z for each fixed $x \in [\alpha_1, \alpha_2]$, and nonincreasing in y and z for each fixed $x \in [\alpha_2, b]$, then there exists at most one solution of (1) $y'' = f(x, y, y')$ satisfying either (2) $y(\alpha_1) = \beta_1$, $y'(\alpha_2) = \beta_2$, or (3) $y(\alpha_1) = \beta_1$, $y(\alpha_2) = \beta_2$, $\alpha_1 \neq \alpha_2$.

THEOREM 2: Suppose the conditions of Theorem 1 hold except that for each fixed $x \in [\alpha_1, \alpha_2]$, $f(x, y, z)$ is nondecreasing in y and nonincreasing in z. Then there exists at most one solution of (1), (4) $y'(\alpha_1) = \beta_1$, $y(\alpha_2) = \beta_2$.

3. *Concerning the maximal convex subsets of a non-convex set of functions*, by C. R. Smith, Northeast Louisiana State College.

The author has shown that in any linear space the convex kernel of a nonconvex set is the intersection of the maximal convex subsets of the nonconvex set (to appear, this MONTHLY). In the space of real functions defined on the unit interval I, the class of functions defined by $D = \{f \mid f(I) \subset I, f(0) = 0, f$ discontinuous at $x_i \in I, 0 < i < N\}$, is a nonconvex set and has as its convex kernel the set, $ckD = \{f \in D \mid f$ is discontinuous at $x = 0\}$ (this MONTHLY, 73, 1966, 927). Interesting situations which arise when one attempts to find some maximal convex subsets of D and a possible relationship between this example and the unsolved problems of the dimension of convex kernels were discussed.

4. *Concerning absolute continuity*, by W. L. Bynum, Louisiana State University.

5. *A combinatorial problem from a design viewpoint*, by Lonnie T. Bennett, Northeast Louisiana State College.

A method of solving the combinatorial problem of generating Partially Balanced Incomplete Block Designs was given. The construction method replaces each treatment of a given Balanced or Partially Balanced Incomplete Block Design with an entire Balanced or Partially Balanced Incomplete Block Design. A principal result is that such a composition of a design having g associate classes with one having f classes yields a Partially Balanced Incomplete Block Design having at most $g+f+gf$ classes.

6. *A discussion of Robinson's non-standard model of the reals*, by J. B. Deeds, Louisiana State University.

7. *A simple method for evaluating the determinant of a square matrix*, by J. O. Koehl, University of Southwestern Louisiana.

The method proposed for the evaluation of a square determinant is a simple means which eliminates the expansion into minor determinants and permits the user to quickly obtain the answer. The method and proof are elementary and can easily be done in an introduction to determinants.

8. *Congruence without measure*, by D. R. Hester, Nicholls State College.

Congruence of two line segments is taken as a primitive notion and is characterized by postulates. Continuity of a line is postulated and several basic theorems concerning betweenness of points are assumed. A two term order relation for line segments is defined such that if AB and CD are two line segments, and if there is a point Q between points C and D such that segment AB is congruent to segment CQ, then segment AB is said to be less than segment CD. It is then shown that: (1) a segment is never less than itself, (2) given a point B between points A and C, then segment AB is less than segment AC, (3) if a first segment is congruent to a second segment, and the second segment is less than a third segment, then the first segment is less than the third, and (4) if a point B lies between points A and C, and if segment AC is congruent to segment $A'C'$, then there is a point B' such that segment AB is congruent to segment $A'B'$, and point B' lies between points A' and C'.

9. *Report on International Congress of Mathematicians, Moscow, August, 1966*, by Margaret M. LaSalle, University of Southwestern Louisiana.

This report included a synopsis of the outstanding mathematical work done by the Award Winners for discovery and generalization of meta-mathematical and algebro-geometric theorems. The Congress and trips to Moscow and Leningrad were outstanding events with excellent balance provided in intellectual, social, and cultural pursuits.

L. VIRGINIA CARLTON, *Secretary-Treasurer*

MARCH MEETING OF THE OKLAHOMA-ARKANSAS SECTION

The annual spring meeting of the Oklahoma-Arkansas Section of the MAA was held on March 31 and April 1, 1967 at Northeastern State College, Tahlequah, Oklahoma. There were 114 persons registered, of whom 76 were members of the Association.

Dr. James Scroggs of the University of Arkansas presided at the business session on Saturday morning. Dr. James Nickel reported that approximately 3800 high school students from the Oklahoma-Arkansas section participated in the MAA High School Contest. The section went on record expressing its appreciation to Dr. Nickel for his work with the high school contest.

Dr. R. B. Deal, representative on the Board of Governors, gave a report on the activities of the Board. Dr. James Scroggs reported on the Sectional Officers meeting held in August at Rutgers University.

A resolution, offered by Dr. Gene Levy, commending the National Science Foundation for its institute programs and urging their continuance was passed without dissent.

The meeting in 1968 will be at the Federal Aviation Agency in Oklahoma City, and the 1969 meeting will be at Arkansas State University at Jonesboro.

Officers elected for 1967–68 were: Chairman, Herbert Monks of Northeastern State College; Vice-Chairman, John Reiger of Federal Aviation Agency; Secretary-Treasurer, Harold Huneke of The University of Oklahoma.

The Saturday session was devoted to the following invited papers:

1. *A computer simulation of atrial fibrillation*, by Thomas Cairns, University of Tulsa.

6. *A discussion of Robinson's non-standard model of the reals*, by J. B. Deeds, Louisiana State University.

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contributions to the theory of coloring maps. It treats several related areas such as the coloring of maps on surfaces of higher genus, Tait's conjecture and Petersen's theorem.

8. *On some definite integrals involving trigonometrical functions*, by Tetsundo Sekiguchi and Naoki Kimura, University of Arkansas, Fayetteville.

We shall deal with the integral $\int_0^\infty [(\sin^m x \cos^n x)/x^p]dx$ and generalizations to several directions by making use of only elementary methods.

9. *On ordered sets with some finitary conditions*, by Naoki Kimura, University of Arkansas, Fayetteville.

Relations between ordered sets and projective semilattice generated by them will be settled.

10. *Time-optimal controls of the equation of evolution in a separable and reflexive Banach space*, by J. K. Cole, University of Oklahoma, Norman.

11. *An oscillation theorem for a nonlinear second order differential equation*, by John Hooker, University of Oklahoma, Norman.

H. V. HUNEKE, *Secretary-Treasurer*

MAY MEETING OF THE MINNESOTA SECTION

The spring meeting of the Minnesota Section of the MAA was held at St. John's University in Collegeville, on May 6, 1967. There were 110 persons in attendance, including 92 members of the Association. Professor J. E. Lange of St. John's University presided over the morning session; Professor Dale Varberg of Hamline University presided over the afternoon session. The sessions were held in the Science Center.

At the business meeting, R. J. Dowling of the University of Manitoba was elected Chairman; Rev. Walbert Kalinowski of St. John's University was elected Secretary-Treasurer; Sylvan Burgstahler of the University of Minnesota, Duluth, and D. A. Storvick of the University of Minnesota, Minneapolis, were elected to the Executive Committee. A report was given on the Minnesota High School examination contest. This was followed by a report on the Putnam examination. Finally, the report of the Committee which had examined the CUPM recommendations for the college curriculum was given by the chairman Murray Braden. The report was accepted by the section.

The program of the sessions was as follows:

1. *Biorthogonal polynomials suggested by the Laguerre polynomial*, by Joseph Konhauser, University of Minnesota, Minneapolis.

2. *Some a priori estimates for the Riemann mapping function for a star shaped domain with a cone property*, by J. R. Cannon, Carleton College.

3. *A differential equation in life contingencies*, by F. C. Smith of George V. Stennes and Associates.

4. *Mnemonics for curvature and Christoffel symbols*, by L. E. Bragg, University of Minnesota, Minneapolis.

5. *Another solution of the Morley triangle problem*, by C. N. Mills, Sioux Falls College (Charter Member of MAA).

6. *Some new applications of generating series*, by Leo Moser, University of Alberta (by invitation).

7. *Generalization of the Cauchy integral test*, by Eugene Hobbs, St. Olaf College (introduced by O. E. Stanaitis).

8. *Remarks on theorems of Fort and Schuster*, by G. U. Brauer, University of Minnesota, Minneapolis.

9. *Reaction of the Minnesota Section to the CUPM Report*, by Murray Braden, Macalester College.

WALBERT KALINOWSKI, *Secretary-Treasurer*

MAY MEETING OF THE WISCONSIN SECTION

The annual meeting of the Wisconsin Section of the MAA was held at St. Norbert College in West DePere, on May 6, 1967. Chairman Sister Madeleine Sophie of Alverno College, Milwaukee, presided. Approximately 150 persons attended. The morning session was devoted to a short business meeting and the presentation of several short papers.

During the business meeting, Professor E. F. Wilde of Beloit College, Beloit, was elected Chairman, Dr. J. A. Raab of Wisconsin State University, Oshkosh, was elected Vice-Chairman, and Dr. R. W. Christensen of Wisconsin State University, La Crosse, was elected Secretary-Treasurer.

The remainder of the morning session was devoted to the presentation of the following papers:

1. *Variations on the Dance Problem*, by R. A. Brualdi, University of Wisconsin, Madison.

The following "dance problem" was formulated and solved many years ago: Suppose m boys and m girls attend a dance and that each boy knows exactly k girls and each girl knows exactly k boys. Without making further introductions, is it possible to pair the boys and girls so that each person dances with an acquaintance? Not only is the answer affirmative, but in precisely k dances it is possible for each person to have danced with everyone of his or her acquaintances. When no restrictions are placed on the number of acquaintances of each person, the maximal number of dances possible so that no person dances with an acquaintance more than once is known. (Assuming there are no wallflowers at any dance.) What if the dance hall is small and only r couples can dance at a time? This question is treated and results analogous to the "uncrowded" dance hall are obtained.

2. *Some properties of the algebra of functions vanishing at infinity*, by R. E. Mullins, Marquette University, Milwaukee.

Let $\overline{X}$ denote a σ-compact, locally compact Hausdorff space, $B(X)$ the algebra of bounded complex-valued continuous functions on $\overline{X}$ and $C_0(\overline{X})$ the algebra of functions in $B(\overline{X})$ vanishing at infinity. An algebra A is said to satisfy (P) if $\{x \in \overline{X} : f(x) = 1\}$ is compact for all f in A. It is shown that every closed algebra of real-valued functions satisfying (P) must be contained in $C_0(\overline{X})$. An example is given to show this is not true for complex-valued functions. A Zorn's lemma argument shows there can be many algebras other than $C_0(\overline{X})$ maximal with respect to (P). The distinguishing property of $C_0(\overline{X})$ is that it is the *unique* closed sub-algebra of $B(\overline{X})$ maximal with respect to (P).

3. *Some thoughts on prime numbers*, by Merrill Barnebey, Wisconsin State University, La Crosse.

A brief historical presentation of prime number theory, punctuated with a few suggestions of how to regard prime numbers, is given. The expectation of orderly arrangement of the primes may be due more to psychology than to mathematics. To look for much structure on the set of primes is a little like trying to study the laws of combustion by poking around in the ashes. Even so, some limited pattern is seen.

4. *On a property of the quaternion group*, by C. P. Seguin, Beloit College, Beloit.

In the quaternion group the cosets modulo the center Z, excluding the coset consisting of the center itself, are precisely the conjugate classes consisting of more than one element. The speaker proved the following:

Let G' denote the commutator subgroup of a group G, $C(X)$ the centralizer of the element x, $|C(x)|$ the order of the centralizer and $(G:Z)$ the index of the center. A group G has the property mentioned above if and only if $G' = Z$ and $|C(x)| = (G:Z)$ for each element $x \notin Z$; furthermore, any such group is a p-group.

5. *A convergence theorem for continued fractions*, by W. A. Sanders, Wisconsin State University, Platteville.

In this paper, a rough sketch of the parabola theorem by H. S. Wall was given. This theorem characterizes completely all convergence sets for the continued fraction $1/1+ a_2/1+ a_3/1+ \cdots$ which are symmetric with respect to the real axis. The theorem which was proved was the following: A set of points, S, in the complex plane, which is symmetrical with respect to the real axis, is a convergence set for the continued fraction $1/1+ a_2/1+ a_3/1+ \cdots$ if, and only if, S is a bounded set contained in the parabolic domain $|Z| - R|Z| \leq 1/2$.

6. *The Bayesian approach to statistics*, by D. R. Lund, Wisconsin State University, Eau Claire.

A few of the salient features of the Bayesian approach to statistical inference are pointed out. Under this approach, which is based on the well-known theorem of Thomas Bayes, parameters in a probability model are themselves allowed to have probability distributions. This makes it possible for an experimenter to inject his opinions about the parameters into an analysis through a prior distribution, and inferences about the parameters are then made from a posterior distribution. Two examples are utilized to illustrate the approach.

The afternoon session began with the viewing of the MAA films "Let's Teach Guessing," by George Polya and "Predicting at Random" by David Blackwell. It concluded with a panel discussion with Creighton Buck on the CUPM publication, "A General Curriculum in Mathematics for Colleges."

R. W. CHRISTENSEN, *Secretary-Treasurer*

INCREASES IN MAA CHARGES

At the Houston meeting in January of this year, the bylaws of the Association were amended to provide that dues of individual members be increased to $8 effective with 1968 dues. As was explained in the announcement of the proposed amendment, the need for the increase arose in part from the general economic inflation and in part from the increased cost of providing additional services required by the expansion in Association activities.

At the Toronto meeting and earlier, the Finance Committee and the Board of Governors took appropriate action to increase other charges so that the Association might derive additional income from sources other than members dues. Some of these increased charges are as follows:

Prices of volumes of Carus Monographs and MAA Studies were increased from $2 to $3 for members and from $4 to $6 for nonmembers on July 1, 1967.

Effective January 1, 1968, subscriptions to the AMERICAN MATHEMATICAL MONTHLY will be increased from $10 to $15, and subscriptions to MATHEMATICS MAGAZINE will be increased from $3 for one year to $4 for one year, $7 for two years and $10 for three years. At the same time, the special subscription rate to MATHEMATICS MAGAZINE for MAA members and members of Mu Alpha Theta will be increased from $5 to $6 for a two-year subscription.

Annual dues of academic members will be increased from $25 to $50 also effective with 1968 dues.

Effective July 1, 1968, the price of back numbers of the MONTHLY including Slaught Memorial Papers, the Dunkel Problem Book, and the Fiftieth Anniversary Issue will be increased from $1.50 to $2.00 per copy, with a special rate of $1.50 each for purchases of five or more Slaught Papers.

Increases in advertising rates for the MONTHLY and MATHEMATICS MAGAZINE will not be made this year since these rates were increased in the summer of 1966. It is expected that these rates will be increased again in the summer of 1968.

In brief, those responsible for the finances of the Association have taken actions to increase income so that the dues of individual members may be kept as low as possible.

HARRY M. GEHMAN, *Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-first Annual Meeting, San Francisco Hilton, California, January 25–27, 1968.

Forty-ninth Summer Meeting, University of Wisconsin, Madison, Wisconsin, August 26–28, 1968.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Indiana University of Pennsylvania, Indiana, April 27, 1968.

FLORIDA, Miami Dade Junior College, South Campus, Miami, March 22–23, 1968.

ILLINOIS, Southern Illinois University, Edwardsville Campus, May 10–11, 1968.

INDIANA

IOWA, Wartburg College, Waverly, April 19, 1968.

KANSAS, Marymount College, Salina, Spring 1968.

KENTUCKY, University of Kentucky, Lexington, Spring 1968.

LOUISIANA-MISSISSIPPI, Broadwater Beach Hotel, Biloxi, Mississippi, February 16–17, 1968.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Morgan State College, Baltimore, December 9, 1967.

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA

MISSOURI, Lindenwood College, St. Charles, April 27, 1968.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 26–27, 1968.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, San Francisco Hilton, January 27, 1968.

OHIO

OKLAHOMA-ARKANSAS, Federal Aviation Agency, Oklahoma City, March 29–30, 1968.

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 14–15, 1968.

PHILADELPHIA.

ROCKY MOUNTAIN, University of Denver, Colorado, May 10–11, 1968.

SOUTHEASTERN, East Carolina College, Greenville, North Carolina, March 29–30, 1968.

SOUTHERN CALIFORNIA, Loyola University of Los Angeles, March 9, 1968.

SOUTHWESTERN, New Mexico State University, University Park, Spring 1968.

TEXAS, Texas A and M University, College Station, April 19–20, 1968.

UPPER NEW YORK STATE

WISCONSIN, Wisconsin State University, La Crosse, May 4, 1968.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, New York, December 26–31, 1967.

AMERICAN MATHEMATICAL SOCIETY, San Francisco Hilton, January 23–26, 1968.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of California, Los Angeles, June 17–20, 1968.

ASSOCIATION FOR COMPUTING MACHINERY, Chicago, Illinois, August 20–22, 1968.

ASSOCIATION FOR SYMBOLIC LOGIC, Sheraton-Boston Hotel, December 27–28, 1967.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, St. Louis, November 28–30, 1968.

INSTITUTE OF MATHEMATICAL STATISTICS

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Convention Hall, Philadelphia, Pennsylvania, April 17–20, 1968.

OPERATIONS RESEARCH SOCIETY OF AMERICA, St. Francis Hotel, San Francisco, May 1–3, 1968.

PI MU EPSILON

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, University of California at Santa Barbara, November 29–30, December 1–2, 1967. (Symposium on combinatorial mathematics.)

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A SURVEY OF MATHEMATICS
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This new graduate text provides a thorough treatment of modern set theory, the basic definitions of abstract algebra theory, and the major theorems of universal algebra. Emphasizing applications of the theory, the author also presents some of the fundamental representation theorems of Boolean algebras, distributive lattices, and lattice-ordered groups and rings. (**Athena** Series)

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MATRICES AND LINEAR ALGEBRA

HANS SCHNEIDER and GEORGE P. BARKER, both of the University of Wisconsin, Madison

Intended for students of mathematics as well as of other disciplines, this book combines concrete aspects of the subject with more abstract facets to provide a balance not found in other texts. While matrices are emphasized, a chapter is devoted to linear transformations.

February 1968 352 pp. $7.00 (tent.)

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The January, 1967 issue of *The American Mathematical Monthly* reprinted Professor Nura D. Turner's paper entitled "National Aspects of the MAA-SA Contest in the Development of Talent," which was originally presented at the International Congress of Mathematicians in Moscow in 1966. One of the conclusions that Professor Turner drew after a study of the academic and professional progress of 117 outstanding mathematics students was: "There is a weakness in teaching mathematics at the calculus level, both elementary and advanced. Too many mathematics students are turning to other fields at this point. Students have been frank in saying this has been where they have lost interest."

Although numerous calculus texts appear on the market each year, they generally accept a more or less standard course content, and either rearrange the order of topics or offer some improvements on the way the topics are presented.

In the preface of his recently published text POLYNOMIALS, POWER SERIES, AND CALCULUS, Professor Howard Levi of Hunter College of The City University of New York has this to say about the problem: "The whole situation suggests that the course itself is at fault and that no text for it can be satisfactory. This text is designed for a *new* course in analysis, covering the calculus of functions of one variable, but having a mathematical base which differs from the traditional. Specifically, it builds the course around the notion of *best approximating polynomials, deemphasizes limits of functions and derivatives,* and *omits the Riemann integral entirely* (obtaining all its usual applications in the other ways)."

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By the late L. J. HEIDER, formerly of Marquette Univ., and J. E. SIMPSON of the Univ. of Kentucky. 379 pages. Illus. $8.50. *Published July, 1967*

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LINEAR ALGEBRA: AN INTRODUCTORY APPROACH, Second Edition; by Charles W. Curtis, University of Oregon.

INTRODUCTORY MATHEMATICAL ANALYSIS, Third Edition; by Edgar D. Eaves, University of Tennessee; and Robert L. Wilson, Ohio Wesleyan University.

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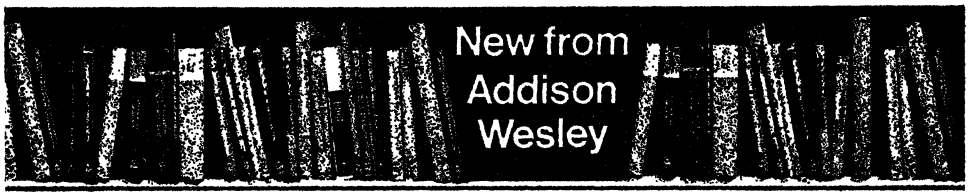
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THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

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DECEMBER 1967

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$6.00 (\$8.00 in 1968). For nonmembers the subscription price is \$10.00 (\$15.00 in 1968).

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

PROJECTIVE PLANES AND ISOTOPIC TERNARY RINGS

G. E. MARTIN, State University of New York at Albany

1. Introduction and summary. This article concerns itself with a certain coordinatization of a projective plane, the ternary systems induced by this coordinatization, and isotopism, an equivalence relation on such algebraic structures. The remainder of this section is a brief summary. All the necessary definitions are included in the text. The article is self-contained.

Two distinct points A and A' are picked in a projective plane. The points not on the special line $l=AA'$ are given coordinates in the standard fashion. Essentially, such a point corresponds to the ordered pair of lines through A and A' which pass through the point. Let l' be a line distinct from l which passes through A. Coordinates are now assigned to the lines not through the special point A in a fashion dual to that for the points. Essentially, such a line corresponds to the ordered pair of points on l and l' which lie on the line.

The incidence relation between the points and the lines of the projective plane determines a ternary system which is not, in general, a Hall ternary ring (HTR). We shall call such a ternary system an intermediate ternary ring (ITR). A planar ternary ring (PTR) is a ternary system which has the same axioms as a HTR without those concerning a zero or unity. Structurally, the ITR lie between the PTR and the HTR. Right and left zeros are defined in the expected way. An ITR has a right zero if and only if it has a left zero, although they may be distinct. Isotopism is defined for ternary systems in the general sense as a composite of four arbitrary permutations on the elements. It turns out that the ITR are exactly the isotopes, in this general sense, of the HTR.

For a summary of related material, *Contributions to Geometry*, Number 4 of the Slaught Memorial Papers of this MONTHLY, is an excellent reference. Other references on projective planes as well as ternary rings include Artzy [2], Blumenthal [3], and Chapter 20 of Hall [4]. Pickert [7] is the standard advanced reference.

2. A coordinatization of a projective plane. A *projective plane* consists of a set whose elements are called points and another set whose elements are called lines together with an incidence relation between the points and lines such that (using the familiar language of geometry to designate the incidence relation): (i) Two points lie on exactly one line. (ii) Two lines intersect in exactly one point. (iii) There exist four points such that no three are collinear. From these axioms it easily follows that there are four lines such that no three are concurrent. This is the dual of the third axiom. The second is the dual of the first. Thus we may assume that the axioms are symmetric with respect to "point" and "line."

From the axioms of a projective plane, it follows that the number of points on any line is equal to the number of lines through any point. If this number is $n+1$, then n is called the *order* of the plane.

We shall give a coordinatization to a projective plane of order n. Let S be any set of cardinality n. Let ∞ be any element which is not in S. We pick two distinct points A and A' in the plane. Let $l=AA'$, the line through A and A'. Let l' be any line through A different from l. Our coordinatization depends on these choices.

To the line l we assign coordinate $[\infty]$. To each line, different from l, through A we assign a coordinate $[x]$ where x is in S. To each line, different from l, through A' we assign a coordinate $[y]'$ where y is in S. In each case the correspondence between the lines and the elements of S is one-to-one and onto. Then to every point P off l we assign coordinates (x, y) if and only if PA is $[x]$ and PA' is $[y]'$.

We shall now dualize the above construction in the following sense. To the point A we assign the coordinate (∞). To each point, different from A, on l we assign a coordinate (m) where m is in S. To each point, different from A, on l' we assign a coordinate $(b)'$ where b is in S. In each case the correspondence between the points and the elements of S is one-to-one and onto. Then to every line t off A we assign coordinates $[m, b]$ if and only if t and l intersect in (m) and t intersects l' in $(b)'$.

Thus every point and line has been associated with at least one type of coordinate. Only the points different from A on l' and the lines different from l through A' have two types of coordinates. Note that the points off l on a line through A all have the same first coordinate, the points off l on a line through A' all have the same second coordinate, the lines off A concurrent on l all have the same first coordinate, and the lines off A concurrent on l' all have the same second coordinate.

3. Planar ternary rings. A *ternary system* $R=(S, \langle\ \rangle)$ is a nonempty set S together with a ternary operation defined on S, i.e., for every ordered triple $\langle abc\rangle$ of elements of S there corresponds a unique element $d=\langle abc\rangle$ of S. A *planar ternary ring* or PTR is a ternary system $(S, \langle\ \rangle)$ such that S has at least two distinct elements and such that for all a, b, c, d in S:

I. $\langle abx\rangle=c$ has a unique solution x in S.

II. $\langle xab\rangle=\langle xcd\rangle$, $a\neq c$, has a unique solution x.

III. $\langle axy\rangle=b$, $\langle cxy\rangle=d$, $a\neq c$, has a solution x and y.

THEOREM 1. *Let a projective plane be coordinatized as above by elements from a set S. Define a ternary operation by $y=\langle xmb\rangle$ if and only if (x, y) is on $[m, b]$. Then $(S, \langle\ \rangle)$ is a PTR.*

Proof. The proof is straightforward. One has only to check I, II, and III in turn.

THEOREM 2. *Let $(S, \langle\ \rangle)$ be a PTR. Then $(S, \langle\ \rangle)$ defines a projective plane as ollows. Points: (x, y), (m), (∞) for every x, y, m in S, ∞ not in S. Lines: $[m, b]$, $[c]$, $[\infty]$ for every m, b, c in S. Incidence: $[\infty]$ contains exactly (∞) and (m) for*

all m in S. $[c]$ *contains exactly* (∞) *and all points* (c, y) *for every* y *in* S. $[m, b]$ *contains exactly* (m) *and all points* (x, y) *for which* $y = \langle xmb \rangle$.

Proof. Again the proof is straightforward. One has only to check the three axioms for a projective plane. The plane in the theorem is called the *plane induced by* $(S, \langle\ \rangle)$.

THEOREM 3. *A solution in* III *is unique.*

Proof. Suppose $\langle axy \rangle = b = \langle auv \rangle$ and $\langle cxy \rangle = d = \langle cuv \rangle$ with $a \neq c$. If $x \neq u$, then $z = a$ and $z = c$ are both solutions of $\langle zxy \rangle = \langle zuv \rangle$, contradicting II. If $x = u$, then $y = v$ by I.

If $R = (S, \langle\ \rangle)$ is a finite PTR, meaning S is a finite set, the "unique" may be omitted from I since the mapping x into $\langle abx \rangle$ is one-to-one if it is onto. Also if R is finite, then III may be omitted from the list as it is implied by I and II. In fact:

THEOREM 4. (Hughes) *If R is a finite ternary system which satisfies* I, *then R satisfies* II *if and only if R satisfies* III *with unique solutions.*

Proof. See Hughes [6].

A *Hall ternary ring* or HTR is a planar ternary ring $(S, \langle\ \rangle)$ with elements 0 and e in S such that for all a, c in S

$$\langle a0c \rangle = \langle 0ac \rangle = c, \qquad \langle ae0 \rangle = \langle ea0 \rangle = a.$$

It must be pointed out that in the literature a HTR is frequently called a "planar ternary ring." The HTR were first studied by M. Hall [5]. D. R. Hughes seems to be the first to have studied the PTR. A proof of the following theorem may be found in Hughes [6].

THEOREM 5. (Hughes) *Let $(S, \langle\ \rangle)$ be a PTR. Let T be a finite subset of S containing at least two distinct elements such that $(T, \langle\ \rangle)$ is a ternary system. Then $(T, \langle\ \rangle)$ is a PTR. Further, if $(S, \langle\ \rangle)$ is a HTR, then so is $(T, \langle\ \rangle)$.*

4. Intermediate ternary rings. Given a PTR$(S, \langle\ \rangle)$ and the projective plane induced by $(S, \langle\ \rangle)$, we have a coordinatization given by Theorem 2 which, in general, may have undesirable properties. We would prefer that points with the same second coordinate be collinear. That is, (a, d) and (c, d) on $[m, b]$ with $a \neq c$ implies (x, d) on $[m, b]$ for all x in S or, equivalently,

IV. $\langle amb \rangle = \langle cmb \rangle = d$, $a \neq c$, implies $\langle xmb \rangle = d$ for all x in S.

We would also prefer that lines with the same second coordinate be concurrent or, equivalently,

V. $\langle mad \rangle = \langle mcd \rangle = b$, $a \neq c$, implies $\langle mxd \rangle = b$ for all x in S.

We shall define an *intermediate ternary ring* or ITR to be a PTR $(S, \langle\ \rangle)$ such that IV and V hold.

The following examples illustrate the independence of the axioms IV and V.

Example 1. $S=\{r, s, t\}$. The tables below give the ternary product $y=\langle xmb\rangle$ on S.

$x = r$

m \ b	r	s	t
r	r	t	s
s	r	t	s
t	r	s	t

$x = s$

m \ b	r	s	t
r	r	t	s
s	s	r	t
t	t	r	s

$x = t$

m \ b	r	s	t
r	r	s	t
s	s	t	r
t	t	s	r

$(S, \langle\ \rangle)$ is a PTR such that both IV and V fail since $\langle rrt\rangle=\langle srt\rangle=s$ but $\langle trt\rangle=t$ and $\langle rrt\rangle=\langle rst\rangle=s$ but $\langle rtt\rangle=t$.

Example 2. $S=\{r, s, t\}$. The entries in the tables above are replaced by:

r t s	r t s	r t s
r t s	s r t	t s r
r s t	t r s	s t r.

$(S, \langle\ \rangle)$ is a PTR such that IV holds. V does not hold since $\langle rrt\rangle=\langle rst\rangle=s$ but $\langle rtt\rangle=t$.

Example 3. $S=\{r, s, t\}$. The entries in the tables above are replaced by:

r t s	r t s	r s t
r t s	s r t	s t r
r t s	t s r	t r s.

$(S, \langle\ \rangle)$ is a PTR such that V holds. IV does not hold since $\langle rrt\rangle=\langle srt\rangle=s$ but $\langle trt\rangle=t$.

Example 4. Let v and w be two distinct elements of a field. Define a ternary operation on S, the set of elements of the field, in terms of the field operations by:

$$\langle xyz\rangle = (x - v)(y - w) + z. \qquad (S, \langle\ \rangle) \text{ is an ITR.}$$

We have seen that in order that the plane induced by a PTR have a coordinatization consistent with that of Section 2, it is necessary that the PTR be an ITR. The question of sufficiency is answered by the next theorem.

THEOREM 6. *Let a projective plane be coordinatized, as in Section 2, by the elements from a set S. Define a ternary operation by $y=\langle xmb\rangle$ if and only if (x, y) on $[m, b]$. Then $(S, \langle\ \rangle)$ is an ITR. Conversely, the plane induced by an ITR has such a coordinatization.*

Proof. At this point we need only prove the second statement. In particular we must recapture A' and l'. $A=(\infty)$ and $l=[\infty]$.

Let $(S, \langle\ \rangle)$ be an ITR, and let b be any element of S. By IV all points with second coordinate b are collinear, say on $[c, d]$. Suppose (s, t) is also on $[c, d]$.

Then $t=\langle scd\rangle$. But $b=\langle scd\rangle$ since, in particular, (s, b) is on $[c, d]$. Hence $t=b$, and we may now say $[c, d]$ is $[b]'$. Likewise for any $g\neq b$, let $[g]'$ be the line containing all points with second coordinate g. So $[b]'$ and $[g]'$ intersect on $[\infty]$ as otherwise they would intersect in a point (x, y) with $y=b=g$. This point of intersection determines $A'=(c)$ uniquely.

Again let b be any element of S. By V all lines with second coordinate b intersect in some point, say (f, d). If $[s, t]$ also passes through (f, d) then $d=\langle fst\rangle$. But $d=\langle fsb\rangle$ as, in particular, $[s, b]$ passes through (f, d). Hence $t=b$ by I. So it is exactly $[f]$ and those lines with second coordinate b which pass through (f, d). Hence (f, d) is $(b)'$. Likewise for any $g\neq b$, all lines with second coordinate g intersect in a point $(g)'$. Thus the line through $(b)'$ and $(g)'$ must pass through A, as otherwise it would have a set of coordinates $[x, y]$ with $y=b=g$. This line determines $l'=[f]$ uniquely.

5. The dual of a ternary system. Two planes π and π' are said to be *dual* if there exists a one-to-one mapping of the points of π onto the lines of π' and a one-to-one mapping of the lines of π onto the points of π' such that a point and line in π are incident if and only if their images are incident in π'.

As we remarked in Section 2, the axioms for a projective plane π are symmetric in "point" and "line." Thus, by interchanging these, we define a plane π' which is dual to π. Let π be the projective plane induced by the PTR $(S, \langle\ \rangle)$. Then it is easy to check that the natural duality is given by:

π	π'	π	π'
(x, y)	$\rightarrow [x, y]$	$[m, b]$	$\rightarrow (m, b)$
(m)	$\rightarrow [m]$	$[a]$	$\rightarrow (a)$
(∞)	$\rightarrow [\infty]$	$[\infty]$	$\rightarrow (\infty)$

However, π' also defines a PTR$(S, \{\ \})$ by (m, b) on $[x, y]$ if and only if $y=\langle xmb\rangle$ if and only if $b=\{mxy\}$. Thus we are led to the following definition.

$(S, \{\ \})$ is said to be the *dual ternary system* of the PTR$(S, \langle\ \rangle)$ if $y=\langle xm\{mxy\}\rangle$ or equivalently $y=\{xm\langle mxy\rangle\}$. It is easy to check that the dual is well defined for any ternary system which satisfies I.

THEOREM 7. *The dual of a PTR is a PTR. The planes induced by a PTR and its dual are dual.*

Proof. The theorem is a direct consequence of the definition and the remarks above.

THEOREM 8. *Let R be a PTR and T its dual. If R satisfies* IV, *then T satisfies* V. *If R satisfies* V, *then T satisfies* IV.

Proof. The selection of the symbols in IV and V has been chosen such that each is the dual of the other.

COROLLARY. *The dual of an ITR is an ITR.*

6. Ternary rings and isotopism. Suppose we have a projective plane coordinatized, as in Section 2, by $(S, \langle\ \rangle)$ where (x, y) on $[m, b]$ if and only if $y = \langle xmb \rangle$. Let α, ρ, σ, and τ be arbitrary permutations on S. If lines $[x]$ through A are renamed $[x^\rho]$ and lines $[y]'$ through A' are renamed $[y^\alpha]'$, then the points (x, y) are renamed (x^ρ, y^α). If the points (m) on l are renamed (m^σ) and the points $(b)'$ on l' are renamed $(b^\tau)'$, then the lines $[m, b]$ are renamed $[m^\sigma, b^\tau]$. Now the new coordinates define a new PTR$(S, \{\ \})$ by (x^ρ, y^α) on $[m^\sigma, b^\tau]$ if and only if $y^\alpha = \{x^\rho m^\sigma b^\tau\}$.

If two ternary systems have the same cardinality, we may assume, without loss of generality, that they are defined on the same set. The remarks in the last paragraph lead us to the following definition. If $(S, \{\ \})$ is a ternary system and α, ρ, σ, τ are arbitrary permutations on S, then the ternary system $(S, \langle\ \rangle)$ defined by

$$\langle xmb \rangle^\alpha = \{x^\rho m^\sigma b^\tau\}$$

for all x, m, b in S is called an *isotope* of $(S, \{\ \})$. The composite of the mappings is called the *isotopism* $(\alpha, \rho, \sigma, \tau)$. If α is the identity permutation, then $(S, \langle\ \rangle)$ is called a *principal isotope*. If $\alpha = \rho = \sigma = \tau$, then the isotopism is called an *isomorphism*.

If $(\alpha, \rho, \sigma, \tau)$ is an isotopism from ternary system R_1 onto R_2 and $(\alpha', \rho', \sigma', \tau')$ is an isotopism from R_2 onto R_3, then $(\alpha\alpha', \rho\rho', \sigma\sigma', \tau\tau')$ is the composite isotopism from R_1 onto R_3. It follows that isotopism and principal isotopism are equivalence relations. An isotopism of a ternary system onto itself is called an *autotopism*. The autotopisms of a ternary system form a group under composition.

THEOREM 9. *Every isotope of a ternary system is isomorphic to a principal isotope.*

Proof. Let $(S, \langle\ \rangle)$ be an isotope of $(S, \{\ \})$ given by $\langle xyz \rangle^\alpha = \{x^\rho y^\sigma z^\tau\}$. Define $(S, [\])$ by $[x^\alpha\ y^\alpha\ z^\alpha] = \{x^\rho y^\sigma z^\tau\}$. Then $(S, [\])$ is a principal isotope of $(S, \{\ \})$ and is isomorphic to $(S, \langle\ \rangle)$.

THEOREM 10. *An isotope of a PTR is a PTR. An isotope of an ITR is an ITR.*

The proof of this last theorem is immediate. Note that it is not true that every PTR is isotopic to some ITR.

THEOREM 11. *If two PTR are isotopic, then the projective planes which they induce are isomorphic.*

Proof. The theorem follows for ITR by the remarks at the beginning of this section. However the correspondences given there, (x, y) corresponds to (x^ρ, y^α) etc., also determine an isomorphism between two planes coordinatized by isotopic PTR.

7. Zeros in a ternary ring. Let $(S, \langle\ \rangle)$ be a ternary system. We shall say z is a *right zero* if $b = \langle xzb \rangle$ for all x, b in S, and we shall say v is a *left zero* if $b = \langle vxb \rangle$

for all x, b in S. z is a *zero* if z is both a right zero and a left zero.

The following properties are immediate in a PTR. A left zero is unique if it exists. z is a right zero if and only if z is a left zero in the dual PTR. A right zero is unique if it exists.

Examples 1, 2, and 3 in Section 4 have neither right zeros nor left zeros. Example 4 has right zero w and left zero v. Let $S=\{r, s, t\}$. Let τ be the permutation on S which interchanges s and t. Define $[xyz]=\langle xyz^{\tau}\rangle$ for x, y, z in S. If $(S, \langle\ \rangle)$ is Example 2, then $(S, [\])$ is a PTR with right zero r but no left zero. If $(S, \langle\ \rangle)$ is Example 3, then $(S, [\])$ is a PTR with left zero r but no right zero.

THEOREM 12. *An ITR has a right zero if and only if it has a left zero.*

Proof. Consider the projective plane induced by the ITR$(S, \langle\ \rangle)$. Then the following statements are each in turn equivalent: z is a right zero. $b=\langle xzb\rangle$ for all x, b. (x, b) on $[z, b]$ for all x, b. $[b]'$ is $[z, b]$ for every b. $(b)'$ is on $[b]'$ for every b where $A'=(z)$ and $l'=[v]$. $(b)'$ is (v, b) for every b. $[m, b]$ through (v, b) for every m, b. $b=\langle vmb\rangle$ for every m, b. v is a left zero.

THEOREM 13. *Let $(S, \langle\ \rangle)$ be an ITR. Let z and v be arbitrary elements of S. Then $(S, \langle\ \rangle)$ has an isotope for which z is a right zero and v is a left zero. Taking $z=v$, the isotope has a zero.*

Proof. Consider the projective plane induced by the ITR$(S, \langle\ \rangle)$. Let σ be any permutation on S which renames points on l such that A' has coordinate (z). Let ρ be any permutation on S which renames the lines through A such that l' has coordinate $[v]$. Let τ be the permutation on S which renames the points on l' such that $(b)'$ is on $[b]'$ for all b. Then, from the proof of the previous theorem, $(S, \{\ \})$ defined by $\langle abc\rangle=\{a^{\rho}b^{\sigma}c^{\tau}\}$ is the required isotope.

THEOREM 14. *Let $R=(S, \langle\ \rangle)$ be a PTR. If R has a right zero z, then $\langle xab\rangle=d$, $a\neq z$, has a unique solution x. If R has a left zero v, then $\langle axd\rangle=b$, $a\neq v$, has a unique solution x.*

Proof. The first result follows from II with $c=z$. The second result follows from III with $c=v$.

COROLLARY. *A ternary system is isotopic to a PTR with a zero if and only if it is an ITR.*

Proof. By Theorem 13, it is sufficient to show that a PTR with a zero is an ITR. In a PTR, a right zero implies IV and a left zero implies V, by Theorem 14. Thus a PTR with a zero is an ITR.

THEOREM 15. *Let $(S, \langle\ \rangle)$ be a PTR with zero 0. Then $(S^*, \cdot)$, where S^* is the set of nonzero elements of S and $\cdot$ is the binary operation defined on S^* by $a\cdot b=\langle ab0\rangle$, is a quasiloop. Also $\langle ab0\rangle=0$ if and only if $a=0$ or $b=0$.*

Proof. Take $z=v=0$ in Theorem 14.

THEOREM 16. *Let* $(S, \langle\ \rangle)$ *be a PTR with a zero. Then any isotope with a zero is isomorphic to an isotope* $(S, \{\ \})$ *defined by*

$$\langle abc\rangle = \{a^\rho b^\sigma c\},$$

where ρ *and* σ *are permutations on* S. *If* $(S, \langle\ \rangle)$ *has zero* 0, *then* $(S, \{\ \})$ *has zero* $0^\rho = 0^\sigma$.

Proof. Let $(S, \langle\ \rangle)$ be a PTR with zero 0. By Theorem 9, any isotope of $(S, \langle\ \rangle)$ is isomorphic to a principal isotope $(S, \{\ \})$ given by $\langle abc\rangle = \{a^\rho b^\sigma c^\tau\}$. In particular, for every x, y, z in S,

$$\{x^\rho 0^\sigma c^\tau\} = \langle x0c\rangle = c = \langle 0yc\rangle = \{0^\rho y^\sigma c^\tau\}.$$

Now $(S, \{\ \})$ has a zero. Setting $y=0$, we have 0^σ is a right zero. Setting $x=0$, we have 0^ρ is a left zero. So the zero of $(S, \{\ \})$ is $z=0^\rho=0^\sigma$. Hence τ is the identity permutation, since $c^\tau = c$ for all c in S.

COROLLARY. *Let* $(S, \langle\ \rangle)$ *be a PTR with zero* 0. *Then any isotope with a zero is isomorphic to an isotope* $(S, [\])$ *with zero* 0 *defined by*

$$\langle abc\rangle^\alpha = [a^\beta b^\gamma c^\alpha],$$

where $\alpha, \beta,$ *and* γ *are permutations on* S *such that* $0^\beta = 0^\gamma = 0$.

Proof. Let α be any permutation on S such that $z^\alpha = 0$, where z is the zero of the isotope $(S, \{\ \})$ given in the proof of the theorem. Let $[a^\beta b^\gamma c^\alpha] = \{a^\rho b^\sigma c\}^\alpha$ where $\beta = \rho\alpha$ and $\gamma = \sigma\alpha$.

Suppose we are given a PTR which is not an ITR and we wish to find an ITR which induces the same projective plane. We have already seen that the two algebraic systems are not isotopic. This problem has been neatly solved by J. R. Wesson. For the proof of the following theorem, see Wesson [9].

THEOREM 17. (Wesson) *Let* $(S, \langle\ \rangle)$ *be a PTR and let* 0 *be any fixed element of* S. *Define* $(S, \{\ \})$ *by*

$$\langle ba\{abc\}\rangle = \langle b0c\rangle$$

and $(S, [\])$ *by*

$$\{ba[abc]\} = \{b0c\}.$$

Then $(S, [\])$ *is a PTR with zero* 0. *The projective planes induced by* $(S, \langle\ \rangle)$ *and* $(S, [\])$ *are isomorphic, and they are dual to the projective plane induced by the PTR* $(S, \{\ \})$.

8. Isotopes of Hall ternary rings. Let $(S, \langle\ \rangle)$ be a PTR with zero 0. We say e is a *left unity* if $\langle ex0\rangle = x$ for all x in S. We say f is a *right unity* if $\langle xf0\rangle = x$ for all x in S. e is a *unity* if e is both a right unity and a left unity. Clearly, if it exists, a left unity or a right unity is unique and different from the zero. Recall that a HTR is a PTR with a zero and a unity.

The following propositions will give us a proof of the fact that a PTR is isotopic to a HTR if and only if it is an ITR.

PROPOSITION 1. *Let R be a PTR with zero 0. Let e be any nonzero element of R. Then R is isotopic to a PTR with zero 0 and left unity e.*

Proof. Let $R=(S, \langle\ \rangle)$. Let σ be the permutation on S defined by $\langle ey0\rangle = y^\sigma$. Define $\{ab^\sigma c\} = \langle abc\rangle$. Then $(S, \{\ \})$ is a PTR with zero 0 and left unity e, since $\{eb^\sigma 0\} = \langle eb0\rangle = b^\sigma$ for all b. (Geometrically σ renames the points of AA' such that the line joining (0, 0) and (e, y) intersects AA' in (y).)

PROPOSITION 2. *Let R be a PTR with zero 0. Let e be any nonzero element of R. Then R is isotopic to a PTR with zero 0 and right unity e.*

Proof. Let $R=(S, \{\ \})$. Let ρ be the permutation on S defined by $\{xe0\} = x^\rho$. Define $[a^\rho bc] = \{abc\}$. Then $(S, [\])$ is a PTR with zero 0 and right unity e, since $[a^\rho e0] = \{ae0\} = a^\rho$ for all a. (Geometrically ρ renames the lines through A in such a way that the line containing (0, 0) and (e) contains all the points (x, x) with x in S.)

PROPOSITION 3. *A PTR with a zero is isotopic to a HTR.*

Proof. Let $(S, \langle\ \rangle)$ be a PTR with zero 0. Let $(S, \{\ \})$ be the isotope given in the proof of Proposition 1, with zero 0 and left unity e. Let $(S, [\])$ be the isotope of $(S, \{\ \})$ given in the proof of Proposition 2, with zero 0 and right unity e. If is sufficient to prove that e is also a left unity of $(S, [\])$. Since $e = \{ee0\} = e^\rho$, we have $[ex0] = [e^\rho x0] = \{ex0\} = x$ for all x in S.

THEOREM 18. *A ternary system is isotopic to a HTR if and only if it is an ITR.*

Proof. The theorem follows from Proposition 3 and the Corollary to Theorem 14.

The next theorem is motivated by the standard construction of introducing a unity into an algebra without divisors of zero. See, for example, Albert [1] where isotopy for division rings is discussed.

THEOREM 19. *Let $(S, \langle\ \rangle)$ be a PTR with a zero. Let g and h be any nonzero elements of S. Let z be any arbitrary element of S. Let $e=\langle ghz\rangle$. Define permutations ρ and σ on S by*

$$x^\rho = \langle xhz\rangle \qquad \text{and} \qquad y^\sigma = \langle gyz\rangle.$$

Then $(S, \{\ \})$ defined by $\langle abc\rangle = \{a^\rho b^\sigma c\}$ is a HTR with zero z and unity e.

Conversely, let $(S, \langle\ \rangle)$ be a PTR with a zero. Then any HTR isotopic to $(S, \langle\ \rangle)$ is isomorphic to a HTR constructed this way.

Proof. First note that, since $(S, \langle\ \rangle)$ is a PTR with a zero, ρ and σ are indeed permutations on S by Theorem 14. Since $(S, \{\ \})$ is an isotope of an ITR, it is at least an ITR. Suppose $(S, \langle\ \rangle)$ has zero 0. Then $0^\rho = z = 0^\sigma$ is a zero of $(S, \{\ \})$.

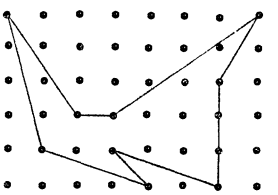
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LATTICE POINTS AND POLYGONAL AREA

IVAN NIVEN, University of Oregon, and H. S. ZUCKERMAN, University of Washington

1. By a simple polygon is meant a polygon that is topologically equivalent to a circle. By lattice points in the plane we mean those points with integral coordinates. H. Steinhaus [1, p. 76] states the result that the area of a simple polygon all of whose vertices are lattice points is equal to the number of interior lattice points, plus half the number of lattice points on the boundary, minus one. Steinhaus offers so brief a sketch of the proof that it is not easy to formulate a complete argument. Because of this, and because this elegant theorem deserves to be known more widely, we offer here a full proof that we feel may also be of interest for its own sake.



2. Triangles.

THEOREM 1. *Let* (a, b) *and* (c, d) *be lattice points such that* (a, b), (c, d) *and the origin are not collinear. Let* T *denote the triangle with vertices* $(0, 0)$, (a, b), (c, d). *Then* T *has area* $\frac{1}{2}$ *if and only if*

(I) T *contains no lattice points other than its three vertices.*

Proof. The proof we give is not the shortest one possible but it is interesting

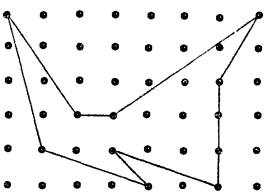
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(I) T *contains no lattice points other than its three vertices.*

Proof. The proof we give is not the shortest one possible but it is interesting

because of the various ideas that it combines. Let P be the parallelogram having vertices $(0, 0)$, (a, b), $(a+c, b+d)$, (c, d). First we prove that (I) is equivalent to:

(II) *P contains no lattice points other than its four vertices.*

It is obvious that (II) implies (I). To prove the converse suppose that P contains a lattice point (e, f) different from its vertices. If (e, f) is not in the triangle (a, b), $(a+c, b+d)$, (c, d) then it is in T and (I) is false. Therefore we can suppose that (e, f) is in that triangle. Translating P by the amount $-a-c$ in the x direction, $-b-d$ in the y direction, we see that $(e-a-c, f-b-d)$ is a lattice point in the triangle $(-c, -d)$, $(0, 0)$, $(-a, -b)$. Reflecting through the origin we find that $(-e+a+c, -f+b+d)$ is a lattice point in T, and hence (I) is false. Therefore (I) and (II) are equivalent.

Now the entire plane can be covered by replicas of P, each replica being P translated by the amount $ma+nc$ in the x direction, $mb+nd$ in the y direction, $m=0, \pm1, \pm2, \cdots, n=0, \pm1, \pm2, \cdots$. The edges of the parallelograms join to create a network N of lines, one set of lines parallel to the side $(0, 0)$ (a, b) of T, the other set parallel to $(0, 0)$ (c, d). Call the points at which these lines intersect, the vertices of N. The vertices of N are just the vertices of the replicas of P. They are all lattice points. If some lattice point is not a vertex of N it will lie in some replica of P and be distinct from the vertices of the replica of P. Translating, we would then have a lattice point in P, not a vertex of P. Thus we see that (II) is equivalent to:

(III) *All lattice points are vertices of N.*

We now get an algebraic statement equivalent to (III). It is clear that the vertices of N are just the points $(ma+nc, mb+nd)$. Therefore (III) is equivalent to

(IV) *For each pair of integers u, v there is a pair of integers m, n such that $am+cn=u$, $bm+dn=v$.*

But the solution of the equations in (IV) is

$$m = (du - cv)/\Delta, \qquad n = (av - bu)/\Delta, \qquad \Delta = ad - bc.$$

The fact that $(0, 0)$, (a, b), (c, d) are not collinear ensures that $\Delta \neq 0$. Obviously $\Delta = \pm 1$ implies (IV). Conversely (IV) implies that Δ divides both $du-cv$ and $av-bu$ for all pairs of integers u, v. In particular, taking first $u=1$, $v=0$, and then $u=0$, $v=1$, we have that (IV) implies that Δ divides each of d, b, c, a, hence Δ^2 divides $ad-bc=\Delta$. Since if Δ^2 divides Δ then $\Delta = \pm 1$, we see that (IV) is equivalent to $\Delta = \pm 1$.

Finally, it is a well-known, and easily proved, fact that the area of T is $\frac{1}{2}|\Delta|$. Therefore the theorem is proved.

3. Polygons. In order to extend the result of Section 2 we now define a certain set S. We let S consist of all simple polygons having all vertices at lattice points. Thus if U belongs to S then U is a polygon that consists of a single piece, has no holes, has all its vertices at lattice points, and is such that its boundary is a simple closed polygonal curve. If U and V belong to S and are such that the sets have in common a single connected part of the boundary of each, not just a single point, and no other points, then the union of the sets U and V forms a polygon of S. We denote this polygon by $U+V$. Notice that $U+V$ is not defined for all pairs U, V of S, but just for certain pairs.

Let $f(U)$ be a real valued function defined for all U of S. If $f(U)+f(V)=f(U+V)$ for all U, V for which $U+V$ is defined, then we say that $f(U)$ is an additive function.

THEOREM 2. *Let $I(U)$ denote the number of lattice points that are in the interior of U, not on the boundary. Let $B(U)$ denote the number of lattice points on the boundary of U. Let $f(U)=\alpha I(U)+\beta B(U)+\gamma$. Then $f(U)$ is additive if and only if $\beta=\alpha/2$, $\gamma=-\alpha$.*

Proof. Suppose that $f(U)$ is additive. Let U be the square (0, 0), (1, 0), (1, 1), (0, 1), V the square (1, 0), (2, 0), (2, 1), (1, 1), and W the rectangle (0, -1), (2, -1), (2, 0), (0, 0). Then both $U+V$ and $(U+V)+W$ are defined. Also $I(U)=0$, $B(U)=4$, $f(U)=4\beta+\gamma$. Similarly $f(V)=4\beta+\gamma$, $f(U+V)=6\beta+\gamma$, $f(W)=6\beta+\gamma$, $f((U+V)+W)=\alpha+8\beta+\gamma$. Therefore $8\beta+2\gamma=6\beta+\gamma$ and $12\beta+2\gamma=\alpha+8\beta+\gamma$, from which we find $\beta=\alpha/2$, $\gamma=-\alpha$.

Conversely, suppose that $\beta=\alpha/2$, $\gamma=-\alpha$ and that U and V are members of S such that $U+V$ is defined. Now U and V have in common just a part of the boundary of each. Let C denote the common boundary, and k the number of lattice points on C. Since C is more than just a point it contains at least two lattice points, the ends of C. Any lattice point on C other than the two endpoints is an interior point of $U+V$. Interior points of U and of V are interior points of $U+V$. We find $I(U+V)=I(U)+I(V)+k-2$. Similarly

$$B(U+V)=B(U)-k+B(V)-k+2=B(U)+B(V)-2k+2,$$

hence

$$\begin{aligned} f(U+V) &= \alpha I(U)+\beta B(U)+\alpha I(V)+\beta B(V)+\alpha(k-2)-2\beta(k-1)+\gamma \\ &= f(U)+f(V)+\alpha(k-2)-2\beta(k-1)-\gamma \\ &= f(U)+f(V) \end{aligned}$$

since $2\beta=\alpha$, $\gamma=-\alpha$. This completes the proof of the theorem.

The function $A(U)$, defined as the area of U, is also an additive function. Suppose that V is a triangle having no lattice points except its vertices. A translation by integral amounts will move V into a position with one vertex at the origin. The functions $A(U)$, $I(U)$ and $B(U)$ are invariant under such a transformation. Therefore the translated V is a T of section 2 and we have $A(V)=\frac{1}{2}$.

Also if V is any triangle in S having area $\frac{1}{2}$, then V contains no lattice points other than its three vertices.

THEOREM 3. *If W is any triangle in S then $A(W)=I(W)+\frac{1}{2}B(W)-1$.*

Proof. If $A(W)=\frac{1}{2}$ then we have just seen that $I(W)=0$, $B(W)=3$, and we see that the theorem holds in this case. For a general triangle W in S, let the vertices be (x_1, y_1), (x_2, y_2), (x_3, y_3), then

$$A(W) = \tfrac{1}{2}\left| (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1) \right|$$

which is one half an integer. Therefore the theorem will follow by induction on the size of $A(W)$ if we show that any W can be expressed in the form $W=W_1+W_2$ or $W=W_1+W_2+W_3$ where the W_i are triangles of S having area less than $A(W)$. But this is easy. If there is a lattice point, other than a vertex, on a side of W, we break W into two triangles by means of a line from that point to the opposite vertex. If W contains an interior lattice point, we connect that point to the three vertices and break W into three triangles.

COROLLARY. *If C is a convex polygon in S then $A(C)=I(C)+\frac{1}{2}B(C)-1$.*

Proof. The polygon C can be broken into pieces, each a triangle of S. Then we use additivity and the preceding theorem.

THEOREM 4. *If P is any polygon in S then $A(P)=I(P)+\frac{1}{2}B(P)-1$.*

REMARK. The proof of the preceding corollary depends on the obvious fact that a convex polygon can be dissected into suitable triangles. Theorem 4 can be proved in the same way using the fact [2, p. 238] that any polygon can be dissected into triangles of the proper sort. However this result is not really needed. For the sake of completeness we give the following independent proof of Theorem 4.

Proof. We use induction on the number of vertices of P. If $n=3$ then P is a triangle and the theorem is true for P. Suppose that $n>3$ and that the theorem is true for all polygons with $n-1$ or fewer vertices. Let P have the vertices $A_1, A_2, \cdots, A_n$ and sides $A_1A_2, A_2A_3, \cdots, A_{n-1}A_n, A_nA_1$. Let P_0 be the smallest convex polygon containing P. Then P_0 is a convex polygon in S. The vertices of P_0 form a subset of $A_1, A_2, \cdots, A_n$. Some of the A_i may be interior points of P_0 and some may be boundary points but not vertices of P_0. If all the vertices of P are vertices of P_0, then P is convex and the theorem follows from the preceding corollary.

Now we can assume that some vertex of P is not a vertex of P_0, in fact that some vertex is actually an interior point of P_0. By possibly relabelling the subscripts of $A_1, A_2, \cdots, A_n$ in a cyclical fashion, we may presume that A_1 is an interior point of P_0. Now let $s\geqq 2$ be the smallest subscript such that A_s is on the boundary of P_0 and let $r\leqq n$ be the largest subscript such that A_r is on the boundary of P_0. Since P_0 has at least three vertices, we observe that at least

and these equations are inconsistent. Thus no constants c_i exist that make (4) correct, even just for tetrahedra.

The referee has drawn to our attention that Theorem 4 does generalize to 3-dimensional space in a form containing the Euler-Poincaré characteristic from algebraic topology; see Reeve [3].

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AN EXPLICIT FORMULA FOR e^{At}

R. B. KIRCHNER, Carleton College

Let A be an $n \times n$ matrix of complex constants. In many textbooks on differential equations, e.g [1, 2, 3] the direct calculation of $e^{At} = \sum_{n=0}^{\infty} A^n t^n / n!$ is avoided except in the simplest cases. e^{At} is usually calculated by transforming A to Jordan canonical form. Although this procedure is theoretically straightforward, it is not easy to become convinced that one could actually carry out the computation for a given matrix A. E. J. Putzer has recently [see 4] given formulas for e^{At} in terms of A and solutions to an easily solved system of differential equations. We shall give an explicit formula for e^{At} in terms of A and the characteristic roots of A. The key idea is the Cayley-Hamilton theorem.

1. Notation and basic lemmas. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ denote the distinct characteristic roots of A, and put

$$p(\lambda) = \operatorname{Det}(\lambda I - A) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{s_i},$$

$$p_j(\lambda) = \prod_{\substack{i=1 \\ i \neq j}}^{k} (\lambda - \lambda_i)^{s_i},$$

$$q(\lambda) = p_1(\lambda) + \cdots + p_k(\lambda),$$

$$f_n(x) = 1 + x + x^2/2! + \cdots + x^{n-1}/(n-1)!.$$

LEMMA 1. (*Cayley-Hamilton theorem*): $p(A) = 0$.

and these equations are inconsistent. Thus no constants c_i exist that make (4) correct, even just for tetrahedra.

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$$q(\lambda) = p_1(\lambda) + \cdots + p_k(\lambda),$$

$$f_n(x) = 1 + x + x^2/2! + \cdots + x^{n-1}/(n-1)!.$$

LEMMA 1. (*Cayley-Hamilton theorem*): $p(A) = 0$.

LEMMA 2. *If $r(\lambda)$ is a polynomial such that $r(\lambda_i)\neq 0$, $i=1, \cdots, k$, then $r(A)^{-1}$ exists and is expressible as a polynomial in A.*

Proof. Since $r(\lambda)$ and $p(\lambda)$ have no common factor, there are polynomials $s(\lambda)$ and $t(\lambda)$ such that $r(\lambda)s(\lambda)+p(\lambda)t(\lambda)=1$. The result follows by substituting A for λ in this identity and using Lemma 1.

LEMMA 3. *Let B be a square matrix. For each $j=1, 2, \cdots$ there is a matrix $g_j(B)$ such that $e^B=f_j(B)+B^j g_j(B)$.*

Proof. $g_j(B)=\sum_{n=j}^{\infty} B^{n-j}/n!$.

2. Statement of the formula. Since $q(\lambda_i)=\sum_{j=1}^{k} p_j(\lambda_i)=p_i(\lambda_i)\neq 0$, $i=1, \cdots, k$, $q(A)^{-1}$ exists by Lemma 2.

Choose $q_i(\lambda)$ so that $q_i(A)=q(A)^{-1}p_i(A)$, $i=1, \cdots, k$. The formula is

$$e^{At}=\sum_{i=1}^{k} q_i(A)f_{s_i}((A-I\lambda_i)t)e^{\lambda_i t}.$$

In applications, it is usually easier to solve for the $q_i(A)$ without first computing $q(A)^{-1}$. A technique for doing this will be illustrated in examples 2 and 3.

If it is convenient to solve for initial conditions separately, note that

$$q(A)e^{At}=\sum_{i=1}^{k} p_i(A)f_{s_i}((A-I\lambda_i)t)e^{\lambda_i t}$$

can be written at once, and its columns yield n linearly independent solutions to the system $\dot{x}=Ax$.

3. Proof of the formula. Let $B_i=A-I\lambda_i$. Then,

$$\begin{aligned} e^{At} &= q(A)^{-1}q(A)e^{At}=q(A)^{-1}\sum_{i=1}^{k} p_i(A)e^{B_i t+\lambda_i I t} \\ &= q(A)^{-1}\sum_{i=1}^{k} p_i(A)e^{B_i t}e^{\lambda_i I t} \\ &= q(A)^{-1}\sum_{i=1}^{k} [p_i(A)f_{s_i}(B_i t)+p_i(A)(B_i t)^{s_i}g_{s_i}(B_i t)]e^{\lambda_i t} \\ &= q(A)^{-1}\sum_{i=1}^{k} p_i(A)f_{s_i}(B_i t)e^{\lambda_i t}. \end{aligned}$$

We have used Lemma 1, Lemma 3, and the fact that $e^{M+N}=e^M e^N$ for commuting matrices M and N. It may be noted that only $p(A)=0$ is needed for the proof. The computation of e^{At} is therefore somewhat simplified if the minimal polynomial for A is known and has degree less than n.

A second proof may be made by simply checking that the formula solves the equation $\dot{Y}=AY$ with $Y(0)=I$.

4. A derivation of the formula. There are a number of ways of showing that the components of solutions to $\dot{x}=Ax$ satisfy the differential equation $p(D)y=0$. This follows, for example, from the operator identity $[\mathrm{adj}(DI-A)](DI-A)=p(D)I$ where $\mathrm{adj}(B)$ is the adjoint of B. Therefore, there are constant matrices B_{ij} such that

$$e^{At} = F(t) = \sum_{i=1}^{k}\sum_{j=1}^{s_i} B_{ij}[t^{j-1}/(j-1)!]e^{\lambda_i t}.$$

It is found that $\dot{F}(t)=AF(t)$ if

$$B_{ij}\lambda_i + B_{i,j+1} = AB_{ij}, \qquad 1 \leqq j < s_i, \qquad \lambda_i B_{i,s_i} = AB_{i,s_i}.$$

Rewriting these conditions as

$$B_{i,j+1} = (A-\lambda_i I)B_{ij}, \qquad 1 \leqq j < s_i, \qquad 0 = (A-\lambda_i I)B_{i,s_i},$$

we see that they are satisfied if $(A-\lambda_i I)^{s_i}B_{i,1}=0$. In view of the Cayley-Hamilton theorem, a candidate for $B_{i,1}$ is $p_i(A)$. Since $F(0)=B_{1,1}+\cdots+B_{k,1}$, it follows that $e^{At}=G(0)^{-1}G(t)$ where

$$G(t) = \sum_{i=1}^{k}\sum_{j=1}^{s_i} p_i(A)(A-\lambda_i I)^{j-1}[t^{j-1}/(j-1)!]e^{\lambda_i t}.$$

This is another way of writing the formula in Section 2.

5. Examples.

Example 1. $p(\lambda)=(\lambda-\lambda_1)^n$. In this case $q(A)=p_1(A)=I$, so $q_1(A)=I$. The formula yields simply,

$$e^{At} = [I + (A-\lambda_1 I)t + \cdots + (A-\lambda_1 I)^{n-1}t^{n-1}/(n-1)!]e^{\lambda_1 t}.$$

If it is known that $(A-\lambda_1 I)^m=0$ where $m<n$, then n can of course be replaced by m in the formula.

Example 2. $p(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$, and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct. In this case the formula is

$$e^{At} = \sum_{i=1}^{n} q_i(A)e^{\lambda_i t}.$$

The $q_i(A)$ can be evaluated explicitly as follows. Assume that $q_i(A)=\sum_{j=1}^{n} c_{ij}A^{j-1}$. The Cayley-Hamilton theorem implies that $Ap_i(A)=\lambda_i p_i(A)$, and hence that $A^j p_i(A)=\lambda^j p_i(A)$, $j\geqq 1$. Therefore,

$$p_i(A) = q_i(A)q(A) = \sum_{j=1}^{n}\sum_{m=1}^{n} c_{ij}A^{j-1}p_m(A)$$

$$= \sum_{m=1}^{n} p_m(A)\left[\sum_{j=1}^{n} c_{ij}\lambda_m^{j-1}\right].$$

For example, the coefficients of $q_1(A)$ and $q_2(A)$ in example 3 are the first and third rows of

$$\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 2\\ 1 & 2 & 4\end{bmatrix}^{-1} = \begin{bmatrix}0 & 2 & -1\\ -2 & 3 & -1\\ 1 & -2 & 1\end{bmatrix}.$$

The verification of this result proceeds by substituting $\sum_{j=1}^{n} c_{ij}A^{j-1}$ for $q_i(A)$ in $q_i(A)(p_1(A)+\cdots+p_k(A))=p_i(A)$, using the fact that

$$A^j p_i(A) = (\lambda_i I + B_i)^j p_i(A) = \sum_{h=0}^{s_i-1} \binom{j}{h} \lambda^{j-h} B_i^h p_i(A),$$

and equating coefficients of $B_i^h p_i(A)$, $0 \leqq h < s_i$, $1 \leqq i \leqq k$.

The fact that Λ is always nonsingular follows by noticing that if $f_{i\beta}(t) = t^{\beta-1}e^{\lambda_i t}/(\beta-1)!$, $1 \leqq \beta \leqq s_i$, $i=1, \cdots, k$, and $g_1(t), \cdots, g_n(t)$ are the solutions to $p(D)y=0$ such that $D^{\alpha-1}g_i(0)=\delta_{\alpha i}$, $1 \leqq \alpha \leqq n$, $1 \leqq i \leqq n$, then

$$(f_{11}(t), \cdots, f_{1,s_1}(t), \cdots, f_{k,s_k}(t)) = (g_1(t), \cdots, g_n(t))\Lambda.$$

Indeed, the $f_{i\beta}(t)$ satisfy $p(D)y=0$ and have initial conditions

$$\begin{aligned} D^{\alpha-1}f_{i\beta}(0) &= e^{\lambda_i t}(D+\lambda_i)^{\alpha-1}(t^{\beta-1}/(\beta-1)!)\big|_{t=0} \\ &= \binom{\alpha-1}{\beta-1}\lambda_i^{\alpha-\beta}. \end{aligned}$$

Since the $f_{i\beta}(t)$ are independent, Λ is nonsingular.

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ANNIHILATORS OF SOME FUNCTION ALGEBRAS

JOHN LAMPERTI, Dartmouth College, and WALTER RUDIN, University of Wisconsin

Introduction. Let C be the space of all real continuous functions on the closed interval $[-1, 1]$. If $g \in C$, the *algebra generated by* g is the set A of all polynomials in g. More explicitly, $G \in A$ if and only if

$$G(x) = a_0 + a_1 g(x) + \cdots + a_n g^n(x) \qquad (-1 \leqq x \leqq 1), \tag{1}$$

For example, the coefficients of $q_1(A)$ and $q_2(A)$ in example 3 are the first and third rows of

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where $a_0, \cdots, a_n$ are real constants.

We shall say that a function f *annihilates* A, or that $f \in A^{\perp}$, if f is integrable on $[-1, 1]$ and if it is true that

$$\int_{-1}^{1} G(x)f(x)dx = 0 \quad \text{for every } G \in A. \tag{2}$$

If $g \in C$ and g is one-to-one on $[-1, 1]$, the Stone-Weierstrass Theorem implies that every $\phi \in C$ can be uniformly approximated by members of A; therefore (2) can hold only if $f(x) = 0$ a.e. Moreover, any finite signed Borel measure μ which satisfies

$$\int_{-1}^{1} G(x)d\mu(x) = 0 \quad \text{for every } G \in A \tag{2'}$$

must vanish identically.

On the other hand, suppose that $g(a) = g(b)$ for some $a \neq b$. Then $G(a) = G(b)$ for every $G \in A$, and the same equality must hold for every function in the uniform closure of A. Thus A is not dense in C, and it is obvious that there exist nonzero Borel measures μ on $[-1, 1]$ such that $\int G d\mu = 0$ for all $G \in A$. But it is not immediately clear whether there always exist nontrivial *functions* f which annihilate A if the generator g of A is not one-to-one.

In Theorem 1 we find an affirmative answer for absolutely continuous g. Theorem 2 shows that the absolute continuity of g cannot be dropped from the hypotheses of Theorem 1. Finally, Theorem 3 describes a rather striking situation that can occur.

Before we turn to these results we mention one case in which it is apparent that $A^{\perp}$ contains many functions: *If g is even, then $A^{\perp}$ contains every odd $f \in L^1$.* Thus even when the function g is locally very irregular (and not absolutely continuous) there *may* be smooth nontrivial functions in $A^{\perp}$.

The following characteristic property of absolutely continuous functions will be used in Theorem 1: If ϕ is absolutely continuous on $[a, b]$, then $\phi'(x)$ exists a.e. on $[a, b]$, $\phi' \in L^1$, and

$$\phi(x) = \phi(a) + \int_a^x \phi'(t)dt \qquad (a \leqq x \leqq b). \tag{3}$$

THEOREM 1. *Suppose g is a real-valued absolutely continuous function on $[-1, 1]$ and g is not one-to-one. In this situation, the statement "If $g' \in X$ then $A^{\perp}$ contains a nontrivial $f \in X$" is true for $X = C^{\infty}$, for $X = C^n$, for $X = C$, and for $X = L^p (1 \leqq p \leqq \infty)$.*

Our notations are the customary ones: C^{∞} is the class of all infinitely differentiable functions, and for $n = 1, 2, \cdots, f \in C^n$ provided that $f^{(n)} \in C$. For $1 \leqq p \leqq \infty$ we understand $f \in L^p$ to mean that f is defined a.e., measurable, and that $\int_{-1}^{1} |f(x)|^p dx < \infty$ when $p < \infty$, or that ess. sup. $|f(x)| < \infty$ if $p = \infty$.

Proof. By assumption there exist points a and b such that $-1 \leqq a < b \leqq 1$ and $g(a) = g(b)$. We shall assume that g is not constant on $[a, b]$, for in the contrary case it is trivial to find an f with the required properties.

Suppose first that $1 \leqq p \leqq \infty$ and $g' \in L^p$. Define $f(x) = g'(x)$ on the interval $[a, b]$ and put $f(x) = 0$ elsewhere. It follows from (3) that f does not vanish a.e., and since g^{n+1} is absolutely continuous for $n = 0, 1, 2, \cdots$, (3) also implies that

$$\int_{-1}^{1} g^n(x) f(x)\,dx = \int_a^b g^n(x) g'(x)\,dx = \frac{1}{n+1}\left[g^{n+1}(b) - g^{n+1}(a)\right] = 0.$$

Thus $f \in A^{\perp}$, and the case $X = L^p$ is settled.

If X is C or C^n or C^∞ the preceding simple proof has to be modified so as to smooth f at the points a and b. We do this as follows: Since g is not constant on $[a, b]$ there exists a $\delta > 0$ and a closed interval $J \subset [a, b]$ such that g is not constant on J and such that $|g(x) - g(a)| > \delta$ for all $x \in J$. Choose $h \in C^\infty$ so that $h(u) = 0$ if $|u - g(a)| \leqq \delta$ and $h(u) > 0$ for all other real u. For instance, put

$$h(u) = \exp\left\{-\frac{1}{|u - g(a)| - \delta}\right\} \quad \text{if } |u - g(a)| > \delta.$$

Define

$$f(x) = \begin{cases} h(g(x))g'(x) & \text{if } a \leqq x \leqq b, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Since g is not constant on J and since $h(g(x)) > 0$ for $x \in J$, we see again that f is not identically 0. Since $h(g(x))$ vanishes in neighborhoods of a and of b, we see that $g' \in X$ now implies $f \in X$, if X is C or C^n or C^∞.

To show that $f \in A^{\perp}$, let n be a nonnegative integer and choose F so that $F'(u) = u^n h(u)$. Then

$$\int_{-1}^{1} g^n(x) f(x)\,dx = \int_a^b g^n(x) h(g(x)) g'(x)\,dx = F(g(b)) - F(g(a)) = 0,$$

since $g(a) = g(b)$. This completes the proof.

THEOREM 2. *Let ψ be a strictly increasing continuous function which maps $[0, 1]$ onto $[0, 1]$, and put*

$$g(x) = \begin{cases} -x & \text{if } -1 \leqq x \leqq 0, \\ \psi(x) & \text{if } 0 \leqq x \leqq 1. \end{cases} \tag{5}$$

In this situation the following statements hold:

(i) *If $\psi'(x) = 0$ on a dense subset of $[0, 1]$, then 0 is the only continuous function which annihilates A.*

(ii) *If $\psi'(x) = 0$ a.e. then $A^{\perp}$ contains only functions which vanish a.e.*

Proof. Suppose $h \in C$ and $h(-\psi(y)) = h(y)$ for $0 \leqq y \leqq 1$. Then h is in the uni-

(b) Theorem 2(ii) gives an example of an A which is not dense in C (since g is not one-to-one) but which *is* dense in L^p for all $p < \infty$, for otherwise $A^\perp$ would contain a nontrivial $f \in L^q \subset L^1$, where $1/p + 1/q = 1$. (Also, A is weak*-dense in L^∞.)

THEOREM 3. *There exists a $g \in C$ such that $A^\perp$ contains no $f \in C$ (other than 0) although $A^\perp$ does contain a nontrivial $f \in L^\infty$.*

Proof. Let g be of the form (5) once again. If ψ is absolutely continuous and $\psi' \in L^\infty$, then by Theorem 1 $A^\perp$ contains a nontrivial member of L^∞. If in addition $\psi' = 0$ on a dense subset of $[0, 1]$, then $A^\perp \cap C$ contains only the 0 function by Theorem 2(i). We complete the proof by constructing such a function ψ.

Enumerate the rationals in $[0, 1]$ as $r_1, r_2, r_3, \cdots$ and put

$$\lambda(x) = \sum_{n=1}^{\infty} \alpha_n [(x - r_n)^{1/3} + r_n^{1/3}], \qquad (0 \leqq x \leqq 1), \tag{11}$$

where $\alpha_1, \alpha_2, \alpha_3, \cdots$ are positive numbers which are chosen so as to make $\lambda(1) = 1$. Clearly λ is continuous and strictly increasing, and so its inverse, which is to be our function ψ, is well-defined and has the same properties. Since $\lambda'(x) = +\infty$ at each rational r_n, we have $\psi'(\lambda(r_n)) = 0$ and $\{\lambda(r_n)\}$ is a dense subset of $[0, 1]$. Hence the second condition above holds.

Next, (11) shows (by the mean value theorem, applied to each summand) that

$$|\lambda(x) - \lambda(y)| \geqq c|x - y|, \qquad (x, y \in [0, 1]), \tag{12}$$

where $3c = \sum \alpha_n$. This is the same as

$$|\psi(x) - \psi(y)| \leqq c^{-1}|x - y|, \qquad (x, y \in [0, 1]), \tag{13}$$

and (13) implies that ψ is absolutely continuous and that $\psi' \in L^\infty$. Thus ψ meets all of the requirements, and Theorem 3 is proved.

This work was supported by NSF Grants GP-5193 and GP-3483 respectively.

ON CONVERGENCE PROPERTIES OF SEQUENCES OF ANALYTIC FUNCTIONS

D. A. HEJHAL, University of Chicago

In the Classroom Notes section of the May 1965 issue of this MONTHLY, O. Shisha has an article entitled "On Sequences of Power Series with Restricted Coefficients" in which he provides elementary proofs for a number of interesting theorems. While doing work in several complex variables, the author obtained independently elementary proofs for several generalizations of Shisha's theorems. We present these now.

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THEOREM 1. *Define* $A_n(z, w) = \sum_{k,l\geq 0} a_{k,l,n} z^k w^l$ *for* $|z|<1 \times |w|<1$, *z and w complex. Suppose that* $a_{k,l,n} = \xi_{k,l} b_{k,l,n}$, *where* $|b_{k,l,n}| \leq M$ *and where* $\sum_{k,l} |\xi_{k,l}| |z|^k |w|^l$ *converges for* $|z|<1 \times |w|<1$. *Suppose that there exist sequences* $\{z_m\}$ *and* $\{w_j\}$ *in* $|z|<1$ *and* $|w|<1$ *(respectively) such that* $\{z_m\}$ *and* $\{w_j\}$ *converge to points in* $|z|<1$ *and* $|w|<1$, *and such that* $\lim_{n\to\infty} A_n(z_m, w_j)$ *exists for all m and j. Then,* $a_{k,l,n} \to a_{k,l}$ *for all k and l, and throughout* $|z|<1 \times |w|<1$ *we have*

$$\lim_{n\to\infty} A_n(z, w) = \sum_{k,l} a_{k,l} z^k w^l.$$

Furthermore, the convergence is uniform on compact subsets of $|z|<1 \times |w|<1$.

Remark. A corresponding theorem holds for any finite number of variables.

COROLLARY. *The theorem holds if the coefficients are uniformly bounded—as in the case of probability generating functions.*

Proof of the theorem. Since the set of all $b_{k,l,n}$ is bounded, and since the set of all (k, l) is countable, we can apply Cantor's diagonalization process to the $b_{k,l,n}$ to get a sequence $\{n_v\}$ such that $\lim_{n_v\to\infty} b_{k,l,n_v} = b_{k,l}$, say. Thus, $\lim_{n_v\to\infty} a_{k,l,n_v} = \xi_{k,l} b_{k,l} = a_{k,l}$. It is clear that the double series $\sum_{k,l} a_{k,l} z^k w^l$ converges in $|z|<1 \times |w|<1$. Next, we consider

$$\left| \sum_{k,l} [\xi_{k,l} b_{k,l,n_v} - \xi_{k,l} b_{k,l}] z^k w^l \right| \leq \sum_{k=0}^{A} \sum_{l=0}^{B} |\xi_{k,l}| |b_{k,l,n_v} - b_{k,l}| |z|^k |w|^l + \sum_{k=A+1}^{\infty} \sum_{l=0}^{B} + \sum_{k=A+1}^{\infty} \sum_{l=B+1}^{\infty} + \sum_{k=0}^{A} \sum_{l=B+1}^{\infty}$$

Hence,

$$\leq \sum_{k=0}^{A} \sum_{l=0}^{B} |\xi_{k,l}| |b_{k,l,n_v} - b_{k,l}| + 2M \left[\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\xi_{k,l}| |z|^k |w|^l - \sum_{k=0}^{A} \sum_{l=0}^{B} |\xi_{k,l}| |z|^k |w|^l \right].$$

However, $\sum |\xi_{k,l}| |z|^k |w|^l$ is easily shown to be uniformly convergent in any $|z|<h_1<1 \times |w|<h_2<1$. We may choose the h_1 and h_2 so that $|z_m|<h_1$ and $|w_j|<h_2$ for all m and j. Choose A and B so large that

$$\left| \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\xi_{k,l}| |z_m|^k |w_j|^l - \sum_{k=0}^{A} \sum_{l=0}^{B} |\xi_{k,l}| |z_m|^k |w_j|^l \right| < \epsilon,$$

where $\epsilon > 0$ is arbitrarily chosen. Thus,

$$\left| A_{n_v}(z_m, w_j) - \sum_{k,l} \xi_{k,l} b_{k,l} z_m^k w_j^l \right| < 2M\epsilon + \sum_{k=0}^{A} \sum_{l=0}^{B} |\xi_{k,l}| \cdot |b_{k,l,n_v} - b_{k,l}|.$$

Proof. For one variable, define:

$$A_n(z) = 0,\ n \text{ odd}; \qquad A_n(z) = \sin\left(\operatorname{Log}\frac{1+z}{1-z}\right),\ n \text{ even } (|z| < 1).$$

We note that $\operatorname{Log} w = \ln|w| + i \arg w$, $-\pi < \arg w \leqq \pi$. Now, let $z_m = 1 - 2/(1+e^{m\pi})$, $m = 1, 2, 3, \cdots$. One may verify that $z_m \to 1$ from inside $|z| = 1$ and that $A_n(z_m) = 0$. If, setting

$$\sin\left(\operatorname{Log}\frac{1+z}{1-z}\right) = \sum_{k=0}^{\infty} c_k z^k, \qquad |z| < 1,$$

we let $M = 1$ and $\xi_k = c_k$, then the hypotheses of Theorem 1 are satisfied. But, $\lim_{n\to\infty} A_n(z)$ does not in general exist. To generalize to n-variables, regard the above as functions of n variables: $(z, z_2, z_3, \cdots, z_n)$.

This paper was written while the author was a student at Lane Technical High School in Chicago.

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2. O. Shisha, On sequences of power series with restricted coefficients, this MONTHLY, no. 5, 72 (1965).

CORRECTION FOR "A CHARACTERIZATION OF FINITE FIELDS"

BROTHER JOSEPH HEISLER, C.S.C., St. Edward's University, Austin, Texas

W. M. Lambert, Jr. has pointed out that the argument in the proof of Theorem 2 of my paper (this MONTHLY, 74 (1967) 537–538) is not complete. Namely, in the third full paragraph on page 538, it is possible that $u = v$ and hence there does not exist a function f with $f(u) = v$ and $f(v)$ arbitrary. The argument is valid for rings containing two distinct proper divisors of zero. If otherwise ring R has only order two nilpotent proper divisors of zero, say $uu = 0$, and contains an element v, $v \neq 0$ and $v \neq u$, then consider the mapping f: $R \to R$ with $f(0) = 0$ and $f(x) = v$ otherwise. If f is left represented by $\bar{f}(x) = a_n x^n + \cdots + a_1 x + a_0$ then

$$f(0) = 0 = a_0$$

and $f(u) = a_1 u = v$ implies $a_1 u^2 = vu$.

On the one hand, $a_1 u^2 = 0$ but $vu \neq 0$, a contradiction, so that R cannot have only order two nilpotent proper divisors of zero.

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On the one hand, $a_1 u^2 = 0$ but $vu \neq 0$, a contradiction, so that R cannot have only order two nilpotent proper divisors of zero.

In an unbounded space the situation is quite different, since the diameters of $g^n(X)$ may not tend to zero. (For example, on a Banach space $g=\frac{1}{2}I$ is contractive but $g^n(X)=X$ for all n.) In this case, for any z, let $Y(z)=\overline{B_z}(d(z, g(z))/(1-k))$, the closed ball with center z and radius $d(z, g(z))/(1-k)$. Since $g(Y(z)) \subset Y(z)$ and diam $Y(z)=\Delta \leqq 2d(z, g(z))/(1-k)$, we can apply the results of the preceding discussion to the restriction of g to $Y(z)$. After proving that the fixpoint of g, and thus of f is unique on X, we conclude that part (i) of the Theorem is proven, while part (ii) is now modified to: *If* $x_n \in f^{pn}(Y(z))$ *then* $\{x_n\} \to \xi$. Since $f^{pn+s}(Y(z))$ is not necessarily included in $g^n(Y(z))$, we can only conclude in place of (iii'): *If* $x_n \in f^{pn}(Y(z))$ *then* $d(\xi, x_n) \leqq \Delta k^n \leqq 2k^n d(z, f^p(z))/(1-k)$.

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2. I. I. Kolodner, Fixed Points, this MONTHLY, 71 (1964) 906.

ON QUADRILATERAL SPACES

DONALD WATSON, RAAF Academy, Point Cook, Australia

D. M. Smiley and M. F. Smiley [1] have defined a quadrilateral space as a real normed linear space for which the quadrilateral inequality

$$|x| + |y| + |z| + |x+y+z| \geqq |x+y| + |y+z| + |z+x|$$

holds. They showed that in a quadrilateral space both the triangle and polygonal inequalities are true; that is,

$$|x| + |y| - |x+y| \geqq 0 \tag{1}$$

and

$$C(n-2, k-1)S_1 + C(n-2, k-2)S_n \geqq S_k \qquad (1 < k < n), \tag{2}$$

where

$$S_k = \sum (|x_{i_1} + x_{i_2} + \cdots + x_{i_k}| : 1 \leqq i_1 < i_2 < \cdots < i_k \leqq n)$$

and $\{x_1, x_2, \cdots, x_n\}$ is a set N of any n vectors in the space.

The purpose of this note is to interpret (1) and (2) in terms of a sequence of means associated with the set N, and to extend (2).

We construct a mean, μ_k, in the following way. Select k vectors from N and form the norm of their arithmetic mean. The number of such selections, and norms, is $C(n, k)$; and μ_k is the arithmetic mean of these norms. That is,

$$\mu_k = \{kC(n, k)\}^{-1} S_k \qquad (k = 1, \cdots, n).$$

The means have a simple geometrical interpretation when N comprises three vectors proceeding from an origin O. The ends of the vectors and the origin are the vertices of a tetrahedron: μ_1 is the mean length of the edges through O;

In an unbounded space the situation is quite different, since the diameters of $g^n(X)$ may not tend to zero. (For example, on a Banach space $g=\frac{1}{2}I$ is contractive but $g^n(X)=X$ for all n.) In this case, for any z, let $Y(z)=\overline{B_z}(d(z, g(z))/(1-k))$, the closed ball with center z and radius $d(z, g(z))/(1-k)$. Since $g(Y(z)) \subset Y(z)$ and diam $Y(z)=\Delta \leqq 2d(z, g(z))/(1-k)$, we can apply the results of the preceding discussion to the restriction of g to $Y(z)$. After proving that the fixpoint of g, and thus of f is unique on X, we conclude that part (i) of the Theorem is proven, while part (ii) is now modified to: *If* $x_n \in f^{pn}(Y(z))$ *then* $\{x_n\} \to \xi$. Since $f^{pn+s}(Y(z))$ is not necessarily included in $g^n(Y(z))$, we can only conclude in place of (iii'): *If* $x_n \in f^{pn}(Y(z))$ *then* $d(\xi, x_n) \leqq \Delta k^n \leqq 2k^n d(z, f^p(z))/(1-k)$.

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The means have a simple geometrical interpretation when N comprises three vectors proceeding from an origin O. The ends of the vectors and the origin are the vertices of a tetrahedron: μ_1 is the mean length of the edges through O;

I close with an observation concerning equality in (5). The necessary and sufficient conditions on a set N for equality to hold in (2) have been given in [1]. Equality will clearly hold in (5) if these conditions are satisfied by every subset M comprising m vectors of N, and the possibility of equality in (5) under other conditions is shown by an example supplied—along with other helpful comments—by a referee. If $n=5$, $x_1=1$, $x_2=2$, $x_3=3$, $x_4=4$, $x_5=-9$ there is equality in (5) for $m=4$, but not in (2).

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ON AN nth ORDER DIFFERENTIAL EQUATION

M. S. KLAMKIN, Ford Scientific Laboratory, Dearborn, Michigan

The differential equations

$$[D^3 + 3axD^2 + 3a^2x^2D + a^3x^3]y = 0, \tag{1}$$

$$[D^4 + 4axD^3 + 6a^2x^2D^2 + 4a^3x^3D + a^4x^4]y = 0, \tag{2}$$

have been solved by Craig [1] by transforming them into linear equations with constant coefficients. In this note, we effect the reduction of the corresponding nth order equation, i.e.,

$$\left[D^n - \binom{n}{1}xD^{n-1} + \binom{n}{2}x^2D^{n-2} - \cdots (-1)^n\binom{n}{n}x^n\right] = 0. \tag{3}$$

(By letting $x=kx'$, we can obtain the forms (1) and (2).)

The reduction follows immediately from the identity

$$(i\sqrt{2})^n \sum_{s=0}^{n} (-1)^s \binom{n}{s} x^s D^{n-s}y \equiv e^{x^2/2}H_n(iD/\sqrt{2})ye^{-x^2/2}, \tag{4}$$

where $H_n(x)$ is the Hermite polynomial

$$H_n(x) = \sum_{r=0}^{[n/2]} \frac{(-1)^r(n!)(2x)^{n-2r}}{(n-2r)!\,r!}.$$

To establish the identity, note that

$$e^{t^2}H_n\left(\frac{i}{2}\frac{d}{dt}\right)ye^{-t^2} = e^{t^2}\sum_{r=0}^{[n/2]}\frac{(-1)^r n!}{(n-2r)!\,r!}\left(i\frac{d}{dt}\right)^{n-2r}ye^{-t^2}$$

$$= \sum_{r=0}^{[n/2]}\frac{i^n n!}{(n-2r)!\,r!}\sum_{s=0}^{n-2r}\binom{n-2r}{s}(D^s y)e^{t^2}D^{n-2r-s}e^{-t^2}.$$

Since $H_n(t) = (-1)^n e^{t^2} D^n e^{-t^2}$, we have, after interchanging the order of summation, that

$$e^{t^2} H_n\left(\frac{i}{2}\frac{d}{dt}\right) y e^{-t^2} = \sum_{s=0}^{n} \sum_{r=0}^{[s/2]} \frac{i^n (-1)^{s-2r} n!}{(n-2r)!\, r!} \binom{n-2r}{n-s} H_{s-2r}(t) D^{n-s} y.$$

Now letting $t = x/\sqrt{2}$ and using the expansion [2]

$$(2x)^s = \sum_{r=0}^{[s/2]} \frac{s! H_{s-2r}(x)}{(s-2r)!\, r!},$$

we finally obtain the identity (4).

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A GENERALIZED ARITHMETIC COMPOSITION

MARGARET D. GESSLEY, Temple, Texas

DEFINITION. *Let f and g be two arithmetic functions. Then*

$$(f \times g)(n) = \sum_{ab=n} f(a) g(b) K(a, b),$$

where the kernel $K(a, b)$ is a nonvanishing function of two variables satisfying

(1) $K(a, b)K(ab, c) = K(b, c)K(a, bc)$

(2) $K(a, b)$ *is multiplicative in both variables, i.e., if* $(ac, bd) = 1$, *then* $K(ab, cd) = K(a, c)K(b, d)$.

This composition is associative, and has the function $E_0(n) = [1/n]$ as an identity. To establish commutativity, it is first necessary to prove the following

THEOREM 1. *For any two positive integers a and b, $K(a, b) = K(b, a)$.*

Proof. We shall use induction.

By condition (1), $K(a, b)K(ab, c) = K(b, c)K(a, bc)$. Let $a = 1$, then $K(1, b)K(b, c) = K(b, c)K(1, bc)$ which implies $K(1, b) = K(1, bc)$ for all c and b. It certainly holds for $b = 1$, which implies $K(1, 1) = K(1, c)$ for all c. Therefore $K(1, c)$ is constant.

Let $b = 1$, then $K(a, 1)K(a, c) = K(1, c)K(a, c)$ which implies $K(a, 1) = K(1, c)$ for all a and c. Therefore $K(a, 1)$ is constant. Since K is multiplicative in both variables, if $(ac, bd) = 1$, $K(ab, cd) = K(a, c)K(b, d)$. Take $a = b = 1$, $(c, d) = 1$. Then $(ac, bd) = 1$, so that $K(1, cd) = K(1, c)K(1, d)$. But $K(1, x)$ is constant, say N. Therefore $N = N^2$. Since $K(a, b) \neq 0$, $N = 1$ so $K(1, a) = 1 = K(b, 1)$ for all a, b.

We next shall show that $K(p^\alpha, p^\beta) = K(p^\beta, p^\alpha)$, where p is a prime, $\alpha, \beta > 0$.

By condition (1) taking $a = p^x$, $b = p^y$, $c = p^z$, we have

$$K(p^x, p^y)K(p^{x+y}, p^z) = K(p^y, p^z)K(p^x, p^{y+z}). \tag{3}$$

With $x=y=z=1$ in (3) we get $K(p^2, p)=K(p, p^2)$. Assume $K(p^\gamma, p)=K(p, p^\gamma)$, and in (3) take $x=z=1$, $y=\gamma$; we get $K(p^{\gamma+1}, p)=K(p, p^{\gamma+1})$. Hence $K(p^\alpha, p^\beta)=K(p^\beta, p^\alpha)$ is true for all $\beta \geqq 0$ if $\alpha=1$. Now assume this statement is also true for all β if $\alpha<t$, $(t \geqq 1)$. Hence for $\alpha=t$, we have the result immediately if $\beta<\alpha$ by the induction hypothesis. Also there is nothing to prove if $\alpha=\beta$.

Taking $x=z=t$ and $y=1$ in (3) we get $K(p^t, p)K(p^{t+1}, p^t)=K(p^t, p^{t+1}) \cdot K(p, p^t)$, which implies $K(p^{t+1}, p^t)=K(p^t, p^{t+1})$. Assume $K(p^t, p^\beta)=K(p^\beta, p^t)$ for $\beta=t, \cdots, \delta+t(\delta \geqq 0)$ and in (3) take $x=z=t$ and $y=\delta+1$ to get

$$K(p^t, p^{\delta+1})K(p^{t+\delta+1}, p^t) = K(p^t, p^{t+\delta+1})K(p^{\delta+1}, p^t). \tag{4}$$

Now if $\delta+1 \leqq t$, we have already proved that $K(p^t, p^{\delta+1})=K(p^{\delta+1}, p^t)$, so from (4) the induction is complete. If $t<\delta+1$, we have $\delta+1 \leqq t+\delta$, so the induction hypothesis applies and we again cancel in (4) to get $K(p^{t+\delta+1}, p^t)=K(p^t, p^{t+\delta+1})$. This completes the proof that $K(p^\alpha, p^\beta)=K(p^\beta, p^\alpha)$ for all α, β.

By induction on the number of primes in a and b, the proof is complete.

Further, by generalizations of the proofs for the Dirichlet composition, it can be shown that the composition of two multiplicative functions is multiplicative and the inverses with respect to $E_0(n)$ of multiplicative functions exist and are multiplicative.

Acknowledgment. The author wishes to thank Dr. A. A. Gioia for his assistance in the preparation of this paper.

Reference

1. A. A. Gioia, The K-product of arithmetic functions, Canad. J. Math., 17 (1965) 970–976.

THE EQUI-PAPPIAN PROPERTY

J. GARFUNKEL, Forest Hills High School and S. GARFUNKEL, Cornell University

An article on concentric polygons appeared in this MONTHLY [3]. We have arrived at some additional results which we believe to be of sufficient interest to be added to the literature of this subject.

Our investigation consists of examining a certain property of three directed line segments, a property which we call "equi-Pappian". In order to give a rigorous definition of this property we first introduce the notion of a *directed polygon.*

Suppose that we are given a polygon with consecutive vertices A_1, A_2, A_3, $\cdots$, A_n. Such a polygon is usually described by listing the vertices in order, starting with A_1; thus, $A_1A_2A_3 \cdots A_n$ describes this polygon. At the same time, $A_1A_nA_{n-1} \cdots A_2$ describes this polygon equally well, unless we wish to assign a consistent sense of orientation to the sides of the polygon. Such a polygon whose sides are directed line segments yielding such a consistent sense of orientation is

$$K(p^x, p^y)K(p^{x+y}, p^z) = K(p^y, p^z)K(p^x, p^{y+z}). \tag{3}$$

With $x=y=z=1$ in (3) we get $K(p^2, p)=K(p, p^2)$. Assume $K(p^\gamma, p)=K(p, p^\gamma)$, and in (3) take $x=z=1$, $y=\gamma$; we get $K(p^{\gamma+1}, p)=K(p, p^{\gamma+1})$. Hence $K(p^\alpha, p^\beta)=K(p^\beta, p^\alpha)$ is true for all $\beta \geqq 0$ if $\alpha=1$. Now assume this statement is also true for all β if $\alpha<t$, $(t \geqq 1)$. Hence for $\alpha=t$, we have the result immediately if $\beta<\alpha$ by the induction hypothesis. Also there is nothing to prove if $\alpha=\beta$.

Taking $x=z=t$ and $y=1$ in (3) we get $K(p^t, p)K(p^{t+1}, p^t)=K(p^t, p^{t+1}) \cdot K(p, p^t)$, which implies $K(p^{t+1}, p^t)=K(p^t, p^{t+1})$. Assume $K(p^t, p^\beta)=K(p^\beta, p^t)$ for $\beta=t, \cdots, \delta+t(\delta \geqq 0)$ and in (3) take $x=z=t$ and $y=\delta+1$ to get

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called a *directed polygon*, and is denoted by $A_1A_2A_3 \cdots A_n$ or by $A_1A_nA_{n-1} \cdots A_2$ (depending on the specific orientation). Thus, for example, any three noncollinear points X, Y, Z are the vertices of two directed triangles XYZ and XZY, [4].

DEFINITION. *Let* $\overrightarrow{A_1B_1}$, $\overrightarrow{A_2B_2}$, $\overrightarrow{A_3B_3}$ *be three directed line segments. On each of these erect directed equilateral triangles* $A_iB_iC_i$ $(i=1, 2, 3)$ *so that all three triangles have the same sense of orientation, then the segments* $\overrightarrow{A_iB_i}$ $(i=1, 2, 3)$ *are said to be equi-Pappian if and only if triangle* $C_1C_2C_3$ *is equilateral.*

Thus, according to the above definition, the three segments consisting of the middle third of each of the sides of any directed scalene triangle [2] are equi-Pappian. We will now prove a few theorems involving this property. The method used will be that of considering vectors in the complex plane. We will make extensive use of the vectors ω, ω^2 to describe rotations of 120° and 240°, employing the fact that $\omega^3=1$, and $1+\omega+\omega^2=0$.

THEOREM 1. *If* ABC *and* $A'B'C'$ *are two directed equilateral triangles, then the joins of corresponding vertices* $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$ *are equi-Pappian.*

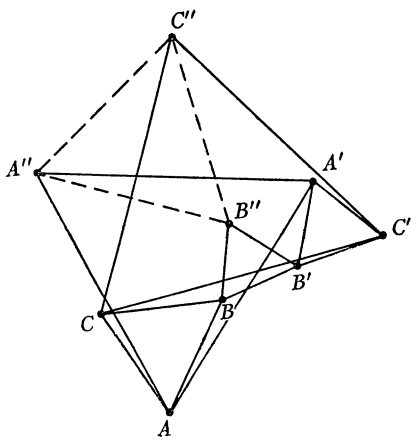


FIG. 1

Proof. Let $\overrightarrow{CB}=p$, $\overrightarrow{CC'}=q$, $\overrightarrow{B'C'}=r$. Then $\overrightarrow{CA}=-p\omega$, $\overrightarrow{AB}=-p\omega^2$. We have

$$\overrightarrow{CB'} = q - r, \quad \overrightarrow{BB'} = q - (r + p), \quad \overrightarrow{B'B''} = q\omega - (r + p)\omega,$$

$$\overrightarrow{B''B} = q\omega^2 - (r + p)\omega^2, \quad \overrightarrow{BA'} = q - (r + p) - r\omega^2,$$

$$\overrightarrow{AA'} = - p\omega^2 + q - (r + p) - r\omega^2 = - (p + r)\omega^2 + q - (p + r),$$

$$\overrightarrow{A'A''} = - (p + r) + q\omega - (p + r)\omega, \quad \overrightarrow{A''A} = - (p + r)\omega + q\omega^2 - (p + r)\omega^2.$$

Hence

$$\overrightarrow{B''C''} = \overrightarrow{B''B'} + \overrightarrow{B'C'} + \overrightarrow{C'C''} = p\omega - r\omega^2,$$

$$\overrightarrow{C''A''} = \overrightarrow{C''C'} + \overrightarrow{C'C} + \overrightarrow{CA} + \overrightarrow{AA''} = -q\omega - q - p\omega + (p+r)\omega - q\omega^2 + (p+r)\omega^2 = p\omega^2 - r = (\overrightarrow{B''C''})\omega,$$

which proves the theorem.

COROLLARY. *Three directed line segments emanating from a point are equi-Pappian if and only if the joins of their endpoints form an equilateral triangle.*

We omit the proof of this corollary.

THEOREM 2. *Quadrilateral $ABCD$ is a parallelogram if and only if $\overrightarrow{AD}$, $\overrightarrow{AC}$, $\overrightarrow{BC}$ are equi-Pappian.*

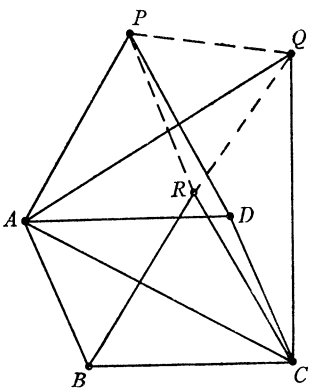


FIG. 2

Proof. Let $\overrightarrow{AB}=p$, $\overrightarrow{BC}=q$, and $\overrightarrow{AD}=r$. Let P, Q, and R be points such that the directed triangles ADP, ACQ, and BCR with the same orientation are equilateral. Assume that triangle PQR is equilateral. $\overrightarrow{RQ}=\overrightarrow{RC}+\overrightarrow{CQ}=-q\omega+(p+q)\omega=p\omega$.

Now, $\overrightarrow{QP}=\overrightarrow{QA}+\overrightarrow{AP}=(p+q)\omega^2-r\omega^2$. $(\overrightarrow{RQ})\omega=\overrightarrow{QP}$ by hypothesis, i.e., $p\omega^2=p\omega^2+q\omega^2-r\omega^2$, or $(q-r)\omega^2=0$. Hence $q=r$ and $ABCD$ is a parallelogram. Assume $ABCD$ is a parallelogram, i.e. $\overrightarrow{AD}=q$. As before, $\overrightarrow{RQ}=p\omega$. $\overrightarrow{QP}=(p+q)\omega^2-q\omega^2=p\omega^2$, therefore $(\overrightarrow{RQ})\omega=\overrightarrow{QP}$ and triangle PQR is equilateral.

In 1956 Beth and Tarski [1] proved that in the plane, all of Euclidean geometry cannot be formalized solely on the basis of the notion of equilateral triangles. This theorem shows that the concept of parallelism is one notion that can be so formalized.

THEOREM 3. *If three consecutive sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ of a directed quadrilateral*

ABCD are equi-Pappian, then $\overrightarrow{AD}, \overrightarrow{AC}, \overrightarrow{BD}$ *(the fourth side and the two diagonals) are equi-Pappian.*

Proof. In Figure 3, let $\overrightarrow{AB}=q$, $\overrightarrow{BC}=r$, $\overrightarrow{CD}=p$ and proceed as before to show that $(\overrightarrow{Q'P'})\omega=\overrightarrow{P'R'}$

THEOREM 4. *If the set of three alternate sides of a directed hexagon (not necessarily convex) is equi-Pappian, then the remaining three sides are equi-Pappian.*

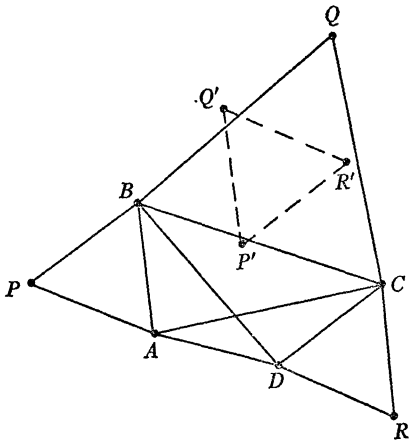


FIG. 3

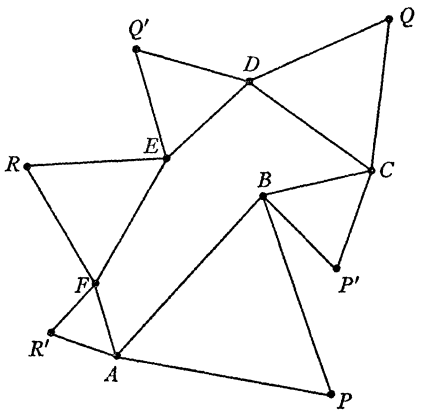


FIG. 4

Proof. In Figure 4, let $\overrightarrow{AB}=p$, $\overrightarrow{BC}=q$, $\overrightarrow{CD}=r$, $\overrightarrow{DE}=s$, $\overrightarrow{EF}=t$, $\overrightarrow{FA}=u$, and proceed exactly along the same lines as in previous theorems to show that $(\overrightarrow{Q'R'})=(P'Q')\omega$.

The following corollary is interesting as it yields an easy proof of Theorem 7 of [2].

COROLLARY. *If the set of alternate segments* $\overrightarrow{AN}, \overrightarrow{BL}, \overrightarrow{CM}$ *of a directed triangle ABC is equi-Pappian, then the set of segments* $\overrightarrow{NB}, \overrightarrow{LC}, \overrightarrow{MA}$ *is also equi-Pappian.*

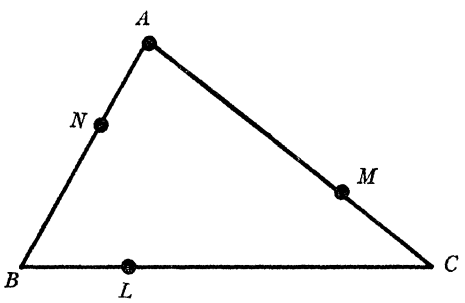


FIG. 4.1

Proof. Consider triangle ABC with L, M, N, points on the sides as directed hexagon $ANBLCM$ with straight angles at L, M, N and the corollary follows from the previous theorem (Figure 4.1).

No attempt has been made to exhaust the results that can be obtained by the investigation of this so called equi-Pappian property of three directed line segments. We feel sure that many additional results involving this property can be found.

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PRODUCT OF ULTRASPHERICAL POLYNOMIALS

RICHARD ASKEY, The University of Wisconsin, Madison

As is well known $\cos nx \cos mx = \frac{1}{2}[\cos(n+m)x + \cos(n-m)x]$ and $\sin nx \cos mx = \frac{1}{2}[\sin(n+m)x + \sin(n-m)x]$. The first of these formulas has been generalized to ultraspherical polynomials and we give here a generalization of the second one.

Let $P_n^\lambda(x)$ be defined by $(1-2rx+r^2)^{-\lambda} = \sum_{n=0}^\infty P_n^\lambda(x) r^n$, $\lambda > 0$. Then Dougall and Hsü [2] have obtained the formula

$$P_n^\lambda(x) P_m^\lambda(x) = \sum_{k=0}^{\min(n,m)} \frac{\alpha_{m-k}^\lambda \alpha_{n-k}^\lambda}{\alpha_{n+m-k}^\lambda} \frac{\alpha_k^\lambda h_{n+m-k}^\lambda}{h_{n+m-2k}^\lambda} P_{n+m-2k}^\lambda(x). \tag{1}$$

Here

$$\alpha_n^\lambda = \binom{n+\lambda-1}{n} \quad \text{and} \quad h_n^\lambda = \frac{2^{1-2\lambda}\pi}{[\Gamma(\lambda)]^2} \frac{\Gamma(n+2\lambda)}{(n+\lambda)n!}.$$

We define $\alpha_n^\lambda = 0$ if n is a negative integer. Observe that the coefficient of $P_r^\lambda(x)$ is positive in (1).

Since $\lim_{\lambda \to 0^+} [P_n^\lambda(\cos\theta)]/P_n^\lambda(1) = \cos n\theta$ it is easy to see that (1) generalizes the above formula for $\cos nx \cos mx$. In trying to generalize the second recall that $P_n^1(\cos\theta) = \sin(n+1)\theta/\sin\theta$. Thus we should consider the product $P_n^{\lambda+1}(x) P_m^\lambda(x)$. We show that

$$P_n^{\lambda+1}(x) P_m^\lambda(x) = \sum_{k=0}^{\min(n+1,m)} \beta_k P_{n+m-2k}^{\lambda+1}(x),$$

where β_k is given explicitly and that $\beta_k \geqq 0$ if $n \geqq m-1$. Recall that $(m+\lambda)P_m^\lambda(x) = \lambda[P_m^{\lambda+1}(x) - P_{m-2}^{\lambda+1}(x)]$, [1, p. 178, (36)]. In what follows $\alpha_k = \alpha_k^{\lambda+1}$ and $h_k = h_k^{\lambda+1}$. Using (1) we see that

A DIRECT PROOF OF STIRLING'S FORMULA

WILLIAM FELLER, Princeton University

1. Introduction. The main excuse for returning once more to the much belabored subject of Stirling's formula is not the simplicity and directness of the following approach, but the circumstance that it provides good illustrations for several topics and techniques of importance. It seems to me that for teaching purposes the discussion is of considerably greater interest than Stirling's formula as such.

The following derivation is self-contained and conceptual. The only formula of calculus used explicitly is

$$I(x) = \int_0^x \log y \, dy = x \log x - x, \tag{1.1}$$

and this can be avoided by expressing the exponent in (2.1) in terms of $I(n)$. Wallis' formula is not used, but the argument leads without artifice to an integral involving the function ϕ defined by

$$\phi(t) = t \prod_{k=1}^{\infty} \left(1 - \frac{t^2}{k^2}\right). \tag{1.2}$$

It is a lucky circumstance that this is the well-known product expansion for the sine:

$$\phi(t) = \frac{\sin \pi t}{\pi} \cdot \tag{1.3}$$

This classical identity is usually proved by relatively deep complex variable methods, but in Section 3 it is shown how the identity can be made plausible, and then proved, by simple elementary methods. The proof merely paraphrases the elegant argument used by E. Artin in his lectures to derive the partial fraction expansion for the cotangents. The whole note is in the spirit of E. Artin to whose cherished memory it is dedicated.

2. Stirling's formula states that

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}. \tag{2.1}$$

The sign $\sim$ indicates that the ratio of the two sides tends to unity. To see how this relation arises naturally we start from the beginning: The problem is to find some estimate for $\log n!$

It is natural to interpret the sum $\log 1 + \cdots + \log n$ as an integral of the step function L which equals $\log k$ in the unit interval centered at k. Our problem is then to appraise the integral of the difference $L(x) - \log x$. In order to deal only with positive integrands we consider separately the several intervals of length $\frac{1}{2}$ in which this difference has a constant sign. Accordingly we put

$$a_k = \tfrac{1}{2}\log k - \int_{k-\frac{1}{2}}^{k} \log x\,dx = \int_{k-\frac{1}{2}}^{k} \log (k/x)dx \tag{2.2}$$

$$b_k = \int_{k}^{k+\frac{1}{2}} \log x\,dx - \tfrac{1}{2}\log k = \int_{k}^{k+\frac{1}{2}} \log (x/k)dx. \tag{2.3}$$

With the abreviation (1.1) we have then

$$a_1 - b_1 + a_2 - b_2 + - \cdots + a_n = \log n! - \tfrac{1}{2}\log n + I(n) - I(\tfrac{1}{2}). \tag{2.4}$$

But

$$a_k = \int_0^{\frac{1}{2}} \log \frac{1}{1-(t/k)}\,dt, \qquad b_k = \int_0^{\frac{1}{2}} \log\left(1+\frac{t}{k}\right)dt, \tag{2.5}$$

and it is seen trivially that $a_k > b_k > a_{k+1} > 0$. The left side in (2.4) is therefore a partial sum of an alternating series $\Sigma(-1)^k c_k$ with c_k decreasing monotonically to 0. The series therefore converges to a finite sum S, and (2.4) shows that

$$\log n! - (n+\tfrac{1}{2})\log n + n \to S - I(\tfrac{1}{2}). \tag{2.6}$$

This is Stirling's formula, except that the coefficient $\sqrt{2\pi}$ is replaced by the constant e^C where

$$C = \sum_{k=1}^{\infty} (a_k - b_k) - I(\tfrac{1}{2}). \tag{2.7}$$

[From this C could be calculated to any number of decimals, and in principle our problem is solved. It is also easy to obtain bounds on the error τ_n defined as the difference between the left side and the right side in (2.6). This τ_n is the remainder of our alternating series and hence $0 < \tau_n < b_n$. Since $\log(1+x) \leqq x$ this implies $0 < \tau_n < 1/8n$. This argument can be sharpened, but it is preferable to use the simple and elegant argument of H. E. Robbins which shows that $1/(12n+1) < \tau_n < 1/12n$. (See this MONTHLY, 62 (1955) 26–29)].

It is just a lucky circumstance that C reduces to a familiar constant. Indeed,

$$a_k - b_k = -\int_0^{\frac{1}{2}} \log\left(1 - \frac{t^2}{k^2}\right)dt \tag{2.8}$$

and so (2.7) expresses C as an integral of $-\log \phi(t)$, with ϕ defined in (1.2). In order not to interrupt the argument let us assume (1.3) as known. Then

$$C = -\int_0^{\frac{1}{2}} \log \frac{\sin \pi t}{\pi}\,dt = \frac{1}{2}\log \pi - \int_0^{\frac{1}{2}} \log \sin \pi t\,dt. \tag{2.9}$$

The last integral can be evaluated by considering the familiar symmetries of the sine and cosine. Obviously the value of the integral remains unchanged if $\sin \pi t$ is replaced by $\cos \pi t$, and hence, by the double angle formula

$$\int_0^{\frac{1}{2}} \sin 2\pi t\, dt = \tfrac{1}{2}\log 2 + 2\int_0^{\frac{1}{2}} \sin \pi t\, dt. \tag{2.10}$$

Again, the two integrals in (2.10) are trivially equal, and hence $= -\frac{1}{2}\log 2$. Accordingly $C = \frac{1}{2}\log 2\pi$, as asserted.

3. Proof of the identity (1.3). When the product for ϕ is decomposed into linear factors one notices immediately that $\phi(x+1) = -\phi(x)$. Thus ϕ is an odd periodic function having the same period and same zeros as $\pi^{-1}\sin \pi x$. One may therefore suspect that the two functions are the same. This suspicion becomes virtual certainty when one notices that in the product for $\phi(2x)$ the factor corresponding to $k = 2\nu$ coincides with the νth factor in (1.2), and so

$$\phi(2x) = 2\phi(x)\psi(x) \tag{3.1}$$

with

$$\psi(x) = \prod_{\nu=1}^{\infty}\left(1 - \frac{x^2}{(\nu - \frac{1}{2})^2}\right). \tag{3.2}$$

The decomposition (3.1) resembles the double angle formula for $\sin \pi x$ and leads to the suspicion that $\psi(x) = \cos \pi x$. Now it is easily verified that

$$\psi(x) = \frac{\phi(x + \frac{1}{2})}{\phi(\frac{1}{2})} \tag{3.3}$$

and this is exactly the relation between $\cos \pi x$ and $\pi^{-1}\sin \pi x$. Thus (3.1) reduces to the form of the double angle formula.

After these preparations it is easy to prove the identity (1.3) using Artin's elegant argument. Put

$$f(x) = \log\frac{\pi\phi(x)}{\sin \pi x}. \tag{3.4}$$

The fraction on the right is defined for nonintegral x, but it is easily seen that f is continuous at the origin if we put $f(0) = 0$. Thus f becomes a continuous periodic function. The double angle formulas for ϕ and $\pi^{-1}\sin \pi x$ imply that

$$f(2x) = \frac{f(x) + f(x + \frac{1}{2})}{2}. \tag{3.5}$$

Let m be the maximum of f. It follows from (3.5) that $f(2x) = m$ implies $f(x) = m$, and hence the maximum is assumed at a sequence of points converging to 0. But $f(0) = 0$, and hence f vanishes identically.

This paper is connected with research sponsored by the U. S. Army Research Office (Durham) at Princeton University.

in our sense with $\boldsymbol{S}$ the class of all subsets of X. Such a weight, if nonvacuous, is just the dual of a filter in $\boldsymbol{S}$.

Following Kimber, we call a point b of X a *barypoint* for a set I whenever UI belongs to $\boldsymbol{H}$ for every neighborhood U of b.

KIMBER'S THEOREM. *Every topologically bounded, heavy subset I of X has a barypoint in X.*

Proof. The Boolean algebra $\boldsymbol{A}$ generated by $\boldsymbol{S}$ contains a neighborhood base for every point in X since $\boldsymbol{A}$ is closed under finite intersection and $\boldsymbol{S}$ contains subbasic neighborhoods. Let $\boldsymbol{B}$ be the Boolean algebra with unit I consisting of all AI with A in $\boldsymbol{A}$. We contend that the heavy members of $\boldsymbol{B}$ form the dual $\boldsymbol{F}^*$ of some filter $\boldsymbol{F}$ in $\boldsymbol{B}$. Indeed, (i)* and (ii)* are clear. (iii)* follows from the identities $AI \cup BI = AI + (B - AB)I = BI + (A - BA)I$ and $AI = ABI + (A - AB)I$ for A, B in $\boldsymbol{A}$ since the terms joined by plus signs are $\boldsymbol{A}$-separated and, by our lemma, $\boldsymbol{H}$ is $\boldsymbol{A}$-conservative. Since the members of $\boldsymbol{F}$ lie in a compact set they must, by (i), (ii) and (iii), have a common cluster point b. That is, every neighborhood of b meets every member of $\boldsymbol{F}$. Now an arbitrary neighborhood U of b contains some neighborhood A of b belonging to $\boldsymbol{A}$. Thus AI belongs to $\boldsymbol{B}$ and meets every member of $\boldsymbol{F}$. That is, AI belongs to $\boldsymbol{F}^*$, hence to $\boldsymbol{H}$. Since $UI = AI + (U - A)I$ with the right-hand terms $\boldsymbol{A}$-separated, our lemma implies that UI is heavy; hence b is a barypoint for I.

THE FUNCTIONAL EQUATION $f(a+b, b) = f(a, a+b)$

S. K. STEIN, University of California at Davis

In this note we solve the functional equation

$$f(a + b, b) = f(a, a + b). \tag{1}$$

More precisely, we will use the Galois theory to exhibit three polynomials that generate all rational functions f with real coefficients that satisfy the equation.

Let F denote the field of real numbers, $F[x, y]$ the ring of polynomials in x and y over F, and $F(x, y)$ the field of rational functions in x and y over F. Consider a rational function $f \in F(x, y)$ that satisfies (1), or equivalently,

$$f(x, y) = f(x - y, x). \tag{2}$$

Define the automorphism T: $F(x, y) \to F(x, y)$ by setting $T(x) = x - y$ and $T(y) = x$. We are interested in the field F^* of elements fixed under T.

Straightforward computations show that

$$T(x) = x - y, \quad T^2(x) = -y, \quad T^3(x) = -x,$$
$$T^4(x) = -x + y, \quad T^5(x) = y, \quad T^6(x) = x.$$

From this it follows that $T^6(y) = y$ and hence $T^6 = I$, the identity automorphism of $F(x, y)$. Thus we have $[F(x, y): F^*] = 6$.

It might be conjectured that the six elementary symmetric functions of $T(x)$, $T^2(x)$, $\cdots$, $T^6(x)$ would generate F^*. However, consider

$$\prod_{i=1}^{6} [w - T^i(x)] = (w^2 - x^2)(w^2 - y^2)(w^2 - (y - x)^2). \tag{3}$$

Letting $S_j (j=1, 2, \cdots, 6)$ be the jth elementary symmetric function of $T(x)$, $T^2(x)$, $\cdots$, $T^6(x)$, we see from (3) that $S_1 = S_3 = S_5 = 0$ and

$$S_2 = 2x^2 - 2xy + 2y^2$$
$$S_4 = x^2y^2 + x^2(x - y)^2 + y^2(x - y)^2$$
$$S_6 = x^2y^2(x - y)^2.$$

We note that $S_4 = (1/4)(S_2)^2$. Thus $F(S_1, S_2, S_3, S_4, S_5, S_6) = F(S_2, S_6) \subseteq F^* \subset F(x, y)$. To determine whether $F(S_2, S_6) = F^*$ we compute $[F(x, y): F(S_2, S_6)]$. Since x is a root of (3), which is an equation of degree 6 having coefficients in $F(S_2, S_6)$, and y is a root of an equation of degree 2 having coefficients in $F(S_2, S_6, x)$ (since $2x^2 - 2xy + 2y^2 - S_2 = 0$), we have

$$[F(x, y): F(S_2, S_6)] \leqq 12.$$

On the other hand, $[F(x, y): F(S_2, S_6)] \geqq 12$, since the group of automorphisms of $F(x, y)$ over $F(S_2, S_6)$ includes not only the group of order 6 generated by T, but also the automorphism C defined by $C(x) = y$, $C(y) = x$. Thus $[F(x, y): F(S_2, S_6)] = 12$ and consequently $[F^*: F(S_2, S_6)] = 2$. Thus S_2 and S_6 do not generate F^*. Now, the polynomial

$$P = xy(x - y)(x^3 - x^2y - 2xy^2 + y^3)$$

is fixed under T but not under C. Thus $F^* = F(S_2, S_6, P)$ and the problem is solved. The extra generator P is needed, since S_2 and S_6 have excessive symmetry, being fixed not only under T, used in their construction, but also under C.

ON FINITE NEAR-RINGS WITH IDENTITY

C. J. Maxson, State University College, Fredonia, N. Y.

A near-ring $(N, +, *)$ is a mathematical system having the following properties:

(1) $(N, +)$ is a group
(2) $(N, *)$ is a semi-group
(3) $(a+b) * c = a * c + b * c$ $\qquad a, b, c \in N.$

In [1], J. R. Clay showed how to construct all near-rings whose additive groups are finite and cyclic. We show here that if such a near-ring has a multiplicative identity e then this near-ring is a ring. We also show (Proposition 3) that if the cardinality of the set N is the product of distinct primes then the near-ring $(N, +, *)$ with identity e is a ring.

It might be conjectured that the six elementary symmetric functions of $T(x)$, $T^2(x)$, $\cdots$, $T^6(x)$ would generate F^*. However, consider

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is a sub-near-ring of $(N, +, *)$. In fact $(N', +, *)$ is a commutative ring with identity $1 \cdot e$ and $(N', +, *) \cong (Z_{n_e}, +, \cdot)$ where $n_e = O(e)$. This gives us

PROPOSITION 4. *If $(N, +, *)$ is a near-ring with identity e then $(N, +, *)$ contains a nontrivial commutative ring. If $O(e)$ is not finite then $(Z, +, \cdot)$ is a sub-ring of $(N, +, *)$ where Z denotes the set of integers.*

COROLLARY 5. *If $(N, +, *)$ is a near-ring with identity e and $\|N\| = P$ where P is a prime then $(N, +, *)$ is a field.*

Proof. From Lemma 2, $O(e) = P$. Therefore $(N', +, *) \cong (Z_p, +, \cdot)$ which shows that $\|N'\| = P$. Thus $(N', +, *) = (N, +, *)$.

We conclude with a result in which $\|N\|$ is not finite. Let $\{(N_i, +, *)\}$, $i = 1, 2, 3, \cdots$ be a family of near-rings such that $(N_i, +)$ is a cyclic group of order p^i and $(N_i, +, *)$ is a sub-near-ring of $(N_{i+1}, +, *) i = 1, 2, 3, \cdots$. Let $(N, +, *) = \bigcup_{i=1}^{\infty} (N_i, +, *)$. It is clear that $(N, +, *)$ is a near-ring.

PROPOSITION 6. *The near-ring $(N, +, *)$ defined above is a ring if it contains an identity e.*

Proof. Since $e \in N$, $e \in N_{i_0}$ for some i_0. Whence N_{i_0} is a ring. Thus for $j \leqq i_0$, N_j is also a ring. Furthermore since e is an identity for N, e is an identity for N_k, $k \geqq i_0$, and so these N_k are rings. This shows that $(N_i, +, *)$ is a ring for $i = 1, 2, 3, \cdots$ and whence $(N, +, *)$ is a ring.

This work was done while the author was an NSF fellow at SUNY at Buffalo.

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2. Marshall Hall, The Theory of Groups, Macmillan, New York, 1959.

A CHARACTERIZATION OF ALMOST UNIFORM CONVERGENCE

N. Y. LUTHER, Washington State University

The author is unable to find the following result explicitly in print (notation and terminology as in the reference [1]):

THEOREM. *Let f_n, f be real valued measurable functions on a measure space (X, S, μ). In order that $f_n \to f$ almost uniformly, it is necessary and sufficient that for each $\epsilon > 0$,*

$$\lim_{n \to \infty} \mu\left(\bigcup_{m \geqq n} \{x; |f_m(x) - f(x)| \geqq \epsilon\} \right) = 0.$$

Proof: Let $E_n(\epsilon) = \{x; |f_n(x) - f(x)| \geqq \epsilon\}$ $(\epsilon > 0, n = 1, 2, \cdots)$. To prove necessity, we fix $\epsilon > 0$ and show that $\lim_{n \to \infty} \mu(\bigcup_{m \geqq n} E_m(\epsilon)) = 0$: Let $\delta > 0$. By hypothesis, there is a measurable set F such that $\mu(F) < \delta$ and f_n converges to f uniformly on F'. Hence there is a positive integer N such that $F' \subset \bigcap_{m \geqq N} E_m(\epsilon)'$; i.e., $\bigcup_{m \geqq N} E_m(\epsilon) \subset F$. It follows that $\mu(\bigcup_{m \geqq n} E_m(\epsilon)) < \delta$ for all $n \geqq N$.

To show sufficiency, we let $\delta>0$ and exhibit a measurable set F such that $\mu(F)<\delta$ and f_n converges to f uniformly on F': By hypothesis, we have for each positive integer p a positive integer $N(p)$ such that $\mu(\bigcup_{m\geq N(p)} E_m(1/p))<\delta/2^p$. Let $F=\bigcup_{p\geq 1}\bigcup_{m\geq N(p)} E_m(1/p)$. Then $\mu(F)<\delta$ and clearly f_n converges to f uniformly on $F'=\bigcap_{p\geq 1}\bigcap_{m\geq N(p)} E_m(1/p)'$.

An interesting feature of this characterization is that it fully exposes the relationship between almost uniform convergence, convergence in measure, and almost everywhere convergence since the last mentioned is characterized by the condition that for every $\epsilon>0$,

$$\mu\left(\bigcap_{n\geq 1}\bigcup_{m\geq n}\{x;\ |f_m(x)-f(x)|\geq\epsilon\}\right)=0.$$

As an example, one by-product of this approach is Egoroff's Theorem [1, 21.A].

Reference

1. P. R. Halmos, Measure theory, Van Nostrand, Princeton, N. J., 1950.

CORRECTION FOR "THE INTERVAL TOPOLOGY AND ORDER CONVERGENCE AS DUAL CONVERGENCE STRUCTURES"

D. C. KENT, Washington State University

It has been called to my attention by Professor Jorg Mayer that the proof of Theorem 2 in my paper, (Vol. 74, No. 4 (1967) 426–427) is incorrect. The corrected proof is given below.

THEOREM 2. (As stated.)

Proof. (1) Let $\mathfrak{F}$ be an ultrafilter on S which converges to x with respect to the topology t described above, and $y\notin x^+$. Then the complement of y^* is in $\mathfrak{F}$, which implies that y^* is not in $\mathfrak{F}$, and hence $y\notin\mathfrak{F}^+$. Thus $x^+\supset\mathfrak{F}^+$, and $x^{+*}=x^*\subset\mathfrak{F}^{+*}$. By a similar argument, $x^+\subset\mathfrak{F}^{*+}$, and $\mathfrak{F}$ p-converges to x. Conversely, assume that an ultrafilter $\mathfrak{F}$ p-converges to x. Then $x^*\subset\mathfrak{F}^{+*}$ implies that $x^+\supset\mathfrak{F}^{+*+}$. Assume that $y\notin x^+$; then $y\notin\mathfrak{F}^{+*+}$, and $y^*\supset\mathfrak{F}^{+*}$. On the other hand, if $y^*\in\mathfrak{F}$, then $y\in\mathfrak{F}^+$, and $y^*\supset\mathfrak{F}^{+*}$. This contradiction establishes that the complement of y^* is in $\mathfrak{F}$, and we conclude that $\mathfrak{F}$ t-converges to x.

Erratum: (March 1967 issue, page 297 "A Lower Bound for the Number of Vertices of a Graph" by M. E. Watkins) Theorem 1 should read:

If G is a graph on n vertices with connectivity $\lambda\geq 1$ and diameter $\delta\geq 1$, then $n\geq\lambda(\delta-1)+2$.

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NOTE ON A THEOREM OF BLAKE AND AARON

IRVING KATZ, George Washington University

THEOREM. *Let the Farey sequence F_n be written $f_0=x_0/y_0$, $f_1=x_1/y_1$, $\cdots$, $f_m=x_m/y_m$ in ascending order (so that $x_0/y_0=0/1$ and $x_m/y_m=1/1$). Then f_0, f_1, $\cdots$, f_t satisfy $\sum_{i=0}^{t} x_i/\sum_{i=0}^{t} y_i=1/2$ if and only if $t=m$.*

Proof. $\sum_{i=0}^{t} x_i/\sum_{i=0}^{t} y_i=1/2$ if and only if

$$0 = \sum_{i=0}^{t} (y_i - 2x_i) = \sum_{i=0}^{t} 2y_i(1/2 - x_i/y_i).$$

Moreover, using Lemmas 3 and 1 of [1], we note

$$y_i = y_{m-i}, \qquad 1/2 - x_i/y_i = x_{m-i}/y_{m-i} - 1/2.$$

If $i<m/2$, then $1/2-x_i/y_i>0$ and so it follows that $\sum_{i=0}^{t} 2y_i(1/2-x_i/y_i)=0$ if and only if $t=m$.

The sufficiency was first established by Blake. We may prove the sufficiency independently of Blake's argument by noting that $\phi(i)$ terms of F_n have denominator i and that

$$\sum_{\substack{1\le j\le i \\ (i,j)=1}} j = i\phi(i)/2$$

and so

$$\sum_{i=0}^{m} x_i \Big/ \sum_{i=0}^{m} y_i = \frac{1 + 2\phi(2)/2 + \cdots + n\phi(n)/2}{2 + 2\phi(2) + \cdots + n\phi(n)} = 1/2.$$

Mail Address: Mathematics Department, The George Washington University, Washington, D. C. 20006.

Reference

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DIRECTED GRAPHS WITH TRANSITIVE ABELIAN GROUPS

R. L. HEMMINGER, Vanderbilt University

Chao [Proc. Amer. Math. Soc., 15 (1964) 291–292] proved that if P is a transitive, abelian, nonelementary 2-permutation group then P does not have an undirected graph (i.e., there does not exist an undirected graph X such that $G(X)=P$ where $G(X)$ is the group of automorphisms of X). In this paper we prove

THEOREM 1. *If P is a transitive abelian permutation group abstractly isomorphic to $\oplus \sum C(n_{ij})$ where $1\le j\le m_i$, $i=1, \cdots, k$, $n_1<n_2< \cdots <n_k$, $C(n_{ij})$ are the integers* mod n_i, *$m_1\neq 2$, 3, or 4 if $n_1=2$, and $m_i\neq 2$ if $n_i=3$, then P has a directed graph.*

It is also noted that the results of Sabidussi [Proc. Amer. Math. Soc., 9 (1958) 800–804] hold for directed graphs. In particular we get

THEOREM 2. *If $G(X)$ is a regular permutation group for a directed graph X, X different from the null graph on two vertices, then X is connected and isomorphic to $X_{G(X),H}$ for some set H of generators for $G(X)$. $X_{G(X),H}$ is the graph with vertex set $G(X)$ and edge set $\{(g, g+h) \mid g \in G(X), h \in H\}$.*

The proof of Theorem 1 involves showing that the set of left additions, $L(G)$, of G equals $G(X_{G,H})$, where $G = \oplus \sum C(n_{ij})$, the $C(n_{ij})$ as in the theorem, and $H = \bigcup_{u=1}^{k} H_u$ where $H_u = \{1_{ui} - 1_{uj},\ 1_{ui} + 1_{uj},\ 1_{up} \mid 1 \leqq i < j \leqq m_u,\ 1 \leqq p \leqq m_u\}$. ($1_{ij}$ is the element of G that equals 1 in the ijth component and zero elsewhere). The theorem follows since P is regular; for then P and $L(G)$ are isomorphic as permutation groups.

Theorem 1 has the following corollary which is pertinent in the elementary 2-group case in Chao's paper mentioned above.

COROLLARY. *Let H be a minimal set of generators for a finite abelian group G. Then $L(G) = G(X_{G,H})$ if and only if distinct elements of H have distinct orders.*

Mail Address: Department of Mathematics, Vanderbilt University, Nashville, TN 37203.

A NOTE ON THE NUMBER OF IDEMPOTENT ELEMENTS IN SYMMETRIC SEMIGROUPS

BERNARD HARRIS, University of Wisconsin

The set of all mappings of a finite set into itself is a semigroup under composition called the symmetric semigroup on n elements and denoted by T_n. This note gives a derivation of the exponential generating function of the number of idempotent elements U_n in T_n. For every idempotent mapping, there is a partition of the n points into components; let k_i, $i = 1, 2, \cdots, n$ be the number of components with i elements, and let $N(k_1, k_2, \cdots, k_n)$ be the number of idempotent elements of T_n with the indicated partition, where $\sum ik_i = n$. It can be shown that the enumeration of idempotent elements is equivalent to the enumeration of forests of rooted labeled trees of height at most one. $r = \sum k_i$ is the number of components (rooted labeled trees). From this, we have

$$N(k_1, k_2, \cdots, k_n)/n! = \frac{1^{k_1}2^{k_2} \cdots n^{k_n}}{1!^{k_1}2!^{k_2} \cdots n!^{k_n}k_1!k_2! \cdots k_n!}, \qquad \sum ik_i = n.$$

Defining,

$$\psi(x_1, x_2, \cdots) = \exp\left(\sum_{i=1}^{\infty} x_i/(i-1)!\right)$$

it follows that $N(k_1, k_2, \cdots, k_n)$ is the coefficient of $(1/n!)x^n t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n}$ in $\psi(t_1x, t_2x^2, \cdots, t_nx^n, 0, 0, \cdots)$. The number of idempotents of rank r in T_n is the coefficient of

$$\frac{1}{n!}\, x^n t^r \text{ in } \psi(tx, tx^2, \cdots, tx^n, 0, 0, \cdots) = e^{txe^x}.$$

Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin, under Contract No.: DA-31-124-ARO-D-462.

Mail Address: Mathematics Research Center, University of Wisconsin, Madison, WI 53706.

A NOTE ON COMMUTATORS AND QUOTIENT GROUPS

L. C. GROVE, University of Oregon

If G is a group and $n>1$ is any cardinal number, let us denote by G^n the weak direct product of n copies of G. Thus for some set I with cardinality n, G^n is the subgroup of the cartesian product $\prod\{G: \alpha\in I\}$ consisting of all elements g such that $g(\alpha)=1$ for all but finitely many $\alpha\in I$. Denote by H the subgroup of G^n generated by the set of all elements g for which there are $\alpha, \beta\in I$, with $\alpha\neq\beta$, and $x\in G$ such that $g(\alpha)=x$, $g(\beta)=x^{-1}$, and $g(\gamma)=1$ if $\gamma\neq\alpha, \beta$.

THEOREM. *If G, n, and H are as above, then H is a normal subgroup of G^n and the quotient group G^n/H is independent of n. In fact, G^n/H is isomorphic with G/G', where G' is the commutator subgroup of G.*

Proof. For each $g\in G^n$ set $f(g)=\prod_\alpha g(\alpha)G'$. Then f is a homomorphism on G^n onto G/G', and it is clear that $H\subseteq\ker f$. If $g\in\ker f$, with $g(\alpha_i)=y_i$, $i=1, \cdots, k$, and $g(\alpha)=1$ otherwise, then by [1], p. 25, there exist $x_1, \cdots, x_m\in G$ such that

$$\prod_{i=1}^{k} y_i = \prod_{i=1}^{m} x_i \prod_{i=1}^{m} x_i^{-1}.$$

Choose $\beta\neq\alpha_k$ and define $g_i(\alpha_k)=x_i$, $g_i(\beta)=x_i^{-1}$, and $g_i(\alpha)=1$ otherwise, $1\leqq i\leqq m$. Define $g_{m+1}(\alpha_k)=\prod_{i=1}^{m} x_i^{-1}$, $g_{m+1}(\beta)=\prod_{i=0}^{m-1} x_{m-i}$, and $g_{m+1}(\alpha)=1$ otherwise. Set $h=\prod_{i=1}^{m+1} g_i$. Next define $h_i(\alpha_i)=y_i$, $h_i(\alpha_k)=y_i^{-1}$, and $h_i(\alpha)=1$ otherwise, $1\leqq i\leqq k-1$. It is not hard to see that h, $h_i\in H$ and that $g=h_{k-1}\cdots h_1 h$. Thus $H=\ker f$ and the theorem is proved.

Mail Address: Department of Mathematics, University of Oregon, Eugene, OR 97403.

Reference

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A NOTE ON UNCONDITIONALLY CONVERGENT SERIES

H. P. YAP, University of Singapore

In this note we shall prove the following

THEOREM. *Let $\sum_{i=0}^{\infty} a_i$ be an absolutely convergent numerical series which converges to a, and $\sum_{i=0}^{\infty} x_i$ be an unconditionally convergent series in a Banach space B such that $\sum_{i=0}^{\infty} x_i = x$. Then*

(1) $a_0x_0 + a_0x_1 + a_1x_0 + a_0x_2 + a_1x_1 + a_2x_0 + a_0x_3 + a_1x_2 + a_2x_1 + a_3x_0 + \cdots$

converges to ax unconditionally.

Proof. Let $\sum_{i=0}^{\infty}|a_i| = A$. Since $\sum_{i=0}^{\infty} x_i$ is unconditionally convergent, therefore by applying a theorem in [1] we can prove that there exists a constant $K>0$ such that $\sum_{i=0}^{\infty}|f^*(x_i)| \leqq K$ for every $f^* \in B^*$ (the conjugate space of B) satisfying $\|f^*\| = 1$. Now, for any $\epsilon>0$, there exist positive integers N_1, N_2 such that $\sum_{i=p}^{\infty}|a_i| < \epsilon/2K$ for all $p \geqq N_1$ and whenever $\|f^*\| = 1$, $\sum_{j=q}^{\infty}|f^*(x_j)| < \epsilon/2A$ for all $q \geqq N_2$.

Let $N_3 = \max(N_1, N_2)$. Then for every f^* satisfying $\|f^*\| = 1$, we have

$$\sum_{p+q \geq N_3}^{\infty} |f^*(a_p x_q)| \leqq \left(\sum_{i=0}^{\infty} |a_i|\right)\left(\sum_{j=N_3}^{\infty} |f^*(x_j)|\right) + \left(\sum_{i=N_3}^{\infty} |a_i|\right)\left(\sum_{j=0}^{\infty} |f^*(x_j)|\right) < \epsilon.$$

This shows that (1) is unconditionally convergent [1]. Since (1) is convergent, therefore

(2) $$a_0x_0 + (a_0x_1 + a_1x_0) + (a_0x_2 + a_1x_1 + a_2x_0) + \cdots$$

is also convergent.

By [2], (2) converges to ax.

Finally, it is well known that if (1) is convergent, then (2) converges to the same sum as (1). Hence (1) converges to ax.

Mail Address: Department of Mathematics, University of Singapore, Singapore, 10.

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ON THE TRIANGULAR FORM OF A LINEAR TRANSFORMATION

C. L. SUFFEL, Polytechnic Institute of Brooklyn

Every linear transformation on a finite dimensional vector space defined over an algebraically closed field has a triangular form. Equivalent conditions for the existence of a triangular form, when the field is arbitrary, are given in the following theorem:

THEOREM. *If V is a finite dimensional vector space over the field F and the linear transformation, A, has characteristic polynomial $p(\lambda)$ then*:

(1) $$p(\lambda) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{r_i}$$

i.e., $p(\lambda)$ factors completely in $F[\lambda] \Leftrightarrow$(2) A has a triangular form $\Leftrightarrow$(3) $\{\alpha \in F \mid \alpha$ is an eigenvalue of $f(A)\} = \{f(\beta) \in F \mid \beta$ is an eigenvalue of $A\}$ for all $f \in F[\lambda]$.

Proof. (1)$\Rightarrow$(2): With minor modifications the proof for the algebraically closed case applies here. (2)$\Rightarrow$(3): see [1].

(3)$\Rightarrow$(1) If $r(\lambda)$ is any nonscalar, monic, irreducible factor of $m(\lambda)$, the minimal polynomial of A, then there exists a polynomial, $h(\lambda)$, with degree strictly less than $m(\lambda)$, such that $m(\lambda) = r(\lambda)h(\lambda)$. Since $m(\lambda)$ is the minimal polynomial of A, there exists a nonzero vector y, such that $h(A)(y) \neq 0$. Hence "0" is an eigenvalue of $r(A)$ and by (3) an eigenvalue of A exists such that $r(\beta) = 0$. Since $r(\lambda)$ is monic and irreducible in $F[\lambda]$, $r(\lambda) = \lambda - \beta$. Finally (1) follows because every irreducible factor of $p(\lambda)$ is a factor of $m(\lambda)$; see [2].

Mail Address: Polytechnic Institute of Brooklyn, 333 Jay Street, Brooklyn, New York 11201.

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ANOTHER CHARACTERIZATION OF THE JACOBSON RADICAL

T. L. JENKINS, University of Wyoming

Let A be an arbitrary ring and I a proper (left, right) ideal of A. If $A^n \subseteq I$ for some integer $n \geq 2$, I is called a *special* (left, right) *ideal.* If no such integer exists, I is called *nonspecial.* A is called a *left m-ring* if A has no nonspecial maximal left ideals (i.e. a maximal left ideal which is nonspecial). A left ideal I of A is a *left m-ideal* if, as a subring of A, I is a left m-ring. If a ring has no maximal left ideals it is clearly a left m-ring. Every nilpotent ring is a left m-ring.

THEOREM. *If I is a left m-ideal of the ring A then $I \subseteq J(A)$, the Jacobson radical of A. Furthermore, if $Q(A)$ denotes the ideal generated by the sum of all the left m-ideals of A then $Q(A) = J(A)$.*

COROLLARY 1. *A left m-ring is a right m-ring and conversely.*

COROLLARY 2. *Let A be a ring in which $J(A) \neq A$. Then there exists a maximal left ideal I of A such that for each positive integer n there exists an $x_n \in A^n$ such that $A = I + (x_n)_1$ where $(x_n)_1$ is the left ideal in A generated by x_n.*

Relative to the Kurosh-Amitsur terminology, the Jacobson radical class is characterized as the class of rings which are groups under the multiplication $a \circ b = a + b - ab$. From the theorem, this class can also be characterized as the class of all left m-rings.

Mail Address: Mathematics Department, 322 Hoyt Hall, University of Wyoming, Laramie, WY 82070.

A NOTE ON LEVI'S THEOREM

E. W. WALLACE, Leeds University, England

1. Introduction. We use the following lemma to show that there is an anti-commutative algebra for which Levi's theorem fails. The reader is referred to [1] for definitions and results.

LEMMA. *If R is an n-dimensional algebra with constants of structure C_{jk}^{i}, then a linear functional λ defines a subalgebra if, and only if, there is a linear functional μ such that*

$$C_{jk}^{i}\lambda_i = \lambda_j\mu_k - \lambda_k\mu_j \qquad (i, j, k = 1, \cdots, n).$$

(This is the most useful form of the lemma for calculations.)

2. An example for which Levi's theorem fails. Consider the 5-dimensional algebra P defined in Section 6 of [1]. It can be verified that $((e_5))$ is both the radical and centre of P. Using the lemma, we find that $((e_2, e_3, e_4, e_5))$ is the only subalgebra of dimension 4, and this is not semisimple, being solvable. Hence Levi's theorem fails for P.

3. Two examples for which Levi's theorem holds. On the other hand, the 4-dimensional algebra P defined in Section 7 of [1] has radical $\Gamma = ((e_1))$, and a simple subalgebra $S = ((e_2, e_3, e_4))$. It follows that Levi's theorem holds for P, although P is not a Lie algebra. In fact, the same is true of S, and it should be observed that S is not an ideal, so P is not merely the direct sum of Γ and S.

Mail Address: Department of Mathematics, The University, Leeds, 2, England.

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COMPACTNESS AND SEQUENTIAL COMPACTNESS IN METRIC SPACES

L. M. SIMON, University of Adelaide, Australia

The purpose of this article is to give a simple and fairly short proof of the fact that a sequentially compact subset of a metric space is compact. The procedure is straightforward enough to enable one to derive the Heine Borel Theorem for Euclidean space from the Bolzano-Weierstrass theorem.

Henceforth let X be a space with metric d, C a sequentially compact subset of X, and for each $x \in X$ and $\lambda > 0$ put $N(x, \lambda) = \{y; y \in X \text{ and } d(x, y) < \lambda\}$.

First the following lemma is needed.

LEMMA 1. *For each $\lambda > 0$ there exists a positive integer $n(\lambda)$ and a finite set $S_\lambda = \{x_1, \cdots, x_{n(\lambda)}\}$ with $S_\lambda \subset C$ and $C \subset \bigcup_{i=1}^{n(\lambda)} N(x_i, \lambda)$.*

This lemma is standard (see e.g. Baum: Elements of Point Set Topology, Prentice Hall, 1964 p. 122).

Next observe that some of the proofs of the Lebesgue covering lemma (Lemma 2 below) are valid for sequentially compact subsets of a metric space. (See e.g. Hocking and Young, Topology, Addison Wesley, 1961 p. 30; but note that their hypothesis of finite open covers is not used in the proof.) Thus we have

LEMMA 2. *Let* $\mathfrak{U}$ *be a collection of open sets covering* C. *Then there exists* $\lambda>0$ *such that, for every* $x\in C$, *there is a set* $U\in\mathfrak{U}$ *with* $N(x, \lambda)\subset U$.

We can now prove that C is compact.

Proof: Let $\mathfrak{U}$ be any collection of open sets covering C. By Lemma 2 there exists $\lambda'>0$ such that, for each $x\in C$, there is a $U\in\mathfrak{U}$ with $N(x, \lambda')\subset U$. But, by Lemma 1, there now exists a finite subset $\{x_1, \cdots, x_n\}$ of C such that $C\subset\bigcup_{i=1}^{n} N(x_i, \lambda')$. Then for each $i=1, \cdots, n$ let $U_i\in\mathfrak{U}$ be such that $N(x_i, \lambda')\subset U_i$.

It now follows that $C\subset\bigcup_{i=1}^{n} U_i$.

Mail Address: c/o J. H. Michael, Mathematics Department, University of Adelaide, South Australia.

A GENERAL FORMULA FOR THE nth DERIVATIVE OF $1/f(x)$

P. SHIEH AND K. VERGHESE, North Carolina State University

A convenient recursion formula which we have derived for finding the nth derivative of $1/f(x)$ is shown below:

$$(1)\qquad D_n\left[\frac{1}{f(x)}\right]=\frac{d^n}{dx^n}\left[\frac{1}{f(x)}\right]=-\frac{n!}{f}\sum_{i=0}^{n-1}\frac{f_{n-i}D_i}{(n-i)!\,i!},\qquad n=1, 2, 3, \cdots,$$

where D_0 is defined to be $1/f(x)$ and f_n stands for the nth derivative of $f(x)$.

The proof by mathematical induction is straightforward. Equation (1) requires the determination of all the $(n-1)$ lower order derivatives of $1/f(x)$ in evaluating D_n. In order to remove this requirement, it may be written in the expanded form.

$$(2)\qquad D_n+\frac{f_n}{f^2}=\sum_{r=2}^{n}\frac{n!}{f^{r+1}}(-1)^r\left[\sum_{r_1=1}^{n-1}\frac{f_{n-r_1}}{(n-r_1)!}\sum_{r_2=1}^{r_1-1}\frac{f_{r_1-r_2}}{(r_1-r_2)!}\right.$$
$$\left.\sum_{r_3=1}^{r_2-1}\frac{f_{r_2-r_3}}{(r_2-r_3)!}\cdots\sum_{r_{r-1}=1}^{r_{r-2}-1}\frac{f_{r_{r-2}-r_{r-1}}}{(r_{r-2}-r_{r-1})!}\frac{f_{r_{r-1}}}{r_{r-1}!}\right]$$

for $n\geqq 1$. In using this formula it must be borne in mind that

$$\sum_{r_k=1}^{r_{k-1}-1}X_{r_k}=0\quad\text{if } r_{k-1}\leqq 1.$$

Equation (2) when used along with Leibnitz formula for the nth derivative of a product, yields the nth derivative of a quotient $g(x)/f(x)$. Then,

$$\frac{d^n}{dx^n}\frac{g(x)}{f(x)} = \frac{g_n}{f} + \sum_{r=1}^{n} \frac{n!}{(n-r)!\,r!} g_{n-r} D_r,$$

where g_n is the nth derivative of $g(x)$ and D_r is given by equation (2) or (1).

Mail Address: Department of Nuclear Engineering, North Carolina State University, Raleigh, North Carolina 27607.

THE UPPER RADICAL DETERMINED BY REGULAR RINGS

T. L. JENKINS, University of Wyoming

An element a of a ring A is called *regular* if there exists an element $b \in A$ such that $a = aba$. A ring is called regular if all its elements are regular. The definition of a *radical class* is given in [1, p. 4] and of a *hereditary* radical class in [1, p. 125].

LEMMA 1. *The class R of all regular rings is a hereditary radical class.*

LEMMA 2. *Let J be the Jacobson radical class. Then $J \cap R = \varnothing$.*

LEMMA 3. *If M is the Brown-McCoy radical class then $M \not\subseteq R$, $R \not\subseteq M$, and $M \cap R \neq \varnothing$.*

LEMMA 4. *R is a radical class which does not coincide with the classical Wedderburn radical on rings having the D.C.C. on right ideals.*

In a manner similar to the one used by N. J. Divinsky [1, p. 5], the upper radical class V determined by R is constructed.

LEMMA 5. $J \subseteq V$.

LEMMA 6. *If $V \subseteq M$, then every simple ring with unit is regular.*

LEMMA 7. *Let A be a ring with unit which is not a division ring and in which the only idempotents are* 0 *and* 1. *Then A is not regular.*

That $V \not\subseteq M$ is established by exhibiting a simple ring with unit which is not a division ring and in which the only idempotents are 0 and 1. This then is the contrapositive of Lemma 6. N. Jacobson [3, p. 211] constructs such a ring and another may be found in [2, p. 237].

THEOREM: *$M \not\subseteq V$, $V \not\subseteq M$, and $M \cap V \neq \varnothing$.*

Mail Address: Mathematics Department, 322 Hoyt Hall, University of Wyoming, Laramie, WY 82070.

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EDITORIAL NOTE ON "BETWEEN T_1 AND T_2"

Albert Wilansky asked in [1] whether a locally compact US space must be T_2. Many answers have been supplied in the form of letters, and manuscripts submitted for publication. There have been constructions by C. E. Aull, S. P. Franklin, R. Jones and G. Miller, Dennis G. King and George D. Poole, Mark Mandelker, Paul R. Meyer, John W. Taylor.

One of the examples is presented here. (Other constructions use sets of ordinals with the order topology and compactifications thereof.) Let X be a compact T_2 space which is not discrete but is such that no sequence of distinct points can converge. For example we might take $X=\beta N$. (See [1], examples 5, 6.) Let x be a nonisolated point of X and let Y be the same as X, but with x "doubled." (This means that an object $y\notin X$ is chosen; $Y=X\cup\{y\}$ so topologized that x, y have exactly the same deleted neighborhoods and the relative topology of X is its original topology.)

See also Problem 5444 in this MONTHLY, 74 (1967).

Reference

1. Albert Wilansky, Between T_1 and T_2, this MONTHLY, March (1967) 261–266.

CLASSROOM NOTES

EDITED BY GEORGE RANEY, University of Connecticut

Material for this department should be sent to George Raney, Department of Mathematics, University of Connecticut, Storrs, CT 06268.

ON THE RING OF INTEGERS

A. HANNA, American University of Beirut

1. Introduction. The ring of integers is usually characterized as an integral domain A whose set of positive elements is well-ordered. A set of positive elements of a ring A is defined to be a subset P of A which is closed under addition and multiplication, and satisfies the following *trichotomy* condition:

(T) For each $a\in A$, exactly one of the following holds:

$$a \in P, \qquad a = 0, \qquad \text{or} \qquad -a \in P.$$

We will show that if a ring A has a subset P satisfying (T) and the *well-ordering* condition:

(W) Every nonempty subset S of P contains a *least element*, that is an element a such that, for all $x\in S$,

$$x = a \qquad \text{or} \qquad x - a \in P,$$

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(W) Every nonempty subset S of P contains a *least element*, that is an element a such that, for all $x\in S$,

$$x = a \qquad \text{or} \qquad x - a \in P,$$

then strong restrictions are imposed upon the structure of A. In particular, if A is an infinite ring with identity e and if e is the least element of P, then A is isomorphic to the ring of integers.

2. Notations. Let $\mathbb{Z}$ denote the ring of integers with the usual addition and multiplication, and let $\mathbb{Z}^-$ denote the ring whose elements are the integers with the usual addition, but with the product $a \cdot b$ of two integers a and b equal to $-ab$, where ab is the usual product. Although $\mathbb{Z}$ and $\mathbb{Z}^-$ are isomorphic rings, they will be distinguished from each other. Let $\mathbb{Z}^\circ$ denote the ring whose elements are the integers with the usual addition, but with the product $a \cdot b$ of two integers a and b equal to zero, and let $\mathbb{Z}_k$, $\mathbb{Z}_k^-$ and $\mathbb{Z}_k^\circ$ denote respectively the subrings of $\mathbb{Z}$, $\mathbb{Z}^-$ and $\mathbb{Z}^\circ$ consisting of all integers mk, $m \in \mathbb{Z}$.

3. A characterization of the ring of integers. Throughout this section the ring A is supposed to consist of more than one element.

LEMMA 1. *A subset P of a ring A satisfies* (T) *if and only if the following two conditions hold*: (T_1) *The sum of any two elements of P is unequal to zero.* (T_2) *The sum of any two nonzero elements not belonging to P is unequal to zero.*

Proof. If P satisfies (T) then it is obvious that (T_1) and (T_2) hold.

Suppose conversely that (T_1) and (T_2) hold. By (T_1), $0 \notin P$ for otherwise $0+0 \neq 0$. Hence for each $a \in A$ at most one of the following holds: $a \in P$, $a=0$ or $-a \in P$. By (T_2), at least one of the following holds: $a \in P$, $a=0$ or $-a \in P$. Hence (T_1) and (T_2) imply (T).

LEMMA 2. *If a ring A has a subset P satisfying* (T), *then*:

(i) *Either A is infinite or A consists of an odd number of elements.*

(ii) *If A is finite, then for each $a \in P$, there exists $b \in P$ such that $a+b \notin P$. Consequently*

(iii) *if P contains an element a such that $a+b \in P$ for all $b \in P$, then A is infinite.*

Proof. (i) If A consists of an even number of elements, then there exists an element $a \neq 0$ such that $a=-a$, and hence $a \in P$ and $-a \in P$.

(ii) Since A is finite, there exists a least integer $m>1$ such that $ma=0$. Since $a \in P$, we have, by (T_1), $-a=(m-1)a \notin P$. Hence there exists a least positive integer $n \geqq 1$ such that $na \notin P$. Then $b=(n-1)a \in P$ is such that $a+b \notin P$.

The *converse* of the statement (ii) and (iii) of Lemma 2 is not necessarily true. This is shown by the following example.

Let P be the subset of $\mathbb{Z}$ consisting of all odd positive integers and all even negative integers. Then P satisfies (T), but for each $a \in P$ there exists $b \in P$ such that $a+b \notin P$.

LEMMA 3. *If a ring A has a subset P which satisfies* (T) *and* (W) *then the additive group of A is cyclic.*

Conversely, if the additive group of a ring A is cyclic and is infinite or consists of an odd number of elements, then there exists a subset P of A satisfying (T) *and* (W).

Proof. Let a be the least element of P. It will be shown that $A=\{ma: m\in Z\}$. Suppose this is false, and let U be the nonempty subset of A consisting of all elements of A which are not of the form ma. Then U contains elements of P, for $0\notin U$ and if $x\in U$, then $-x\in U$. Let b be the least element of U such that $b\in P$. Since $a=1a$, $b\neq a$ and hence $b-a\in P$. Since $b-(b-a)=a\in P$, we have $b-a\notin U$. It follows that $b-a=ma$ for some $m\in Z$, and therefore $b=(m+1)a$ which is a contradiction.

If the additive group of A is cyclic with the generator a and consists of the $(2n+1)$ elements $0, a, \cdots, 2na$, let P be the subset $\{ma: 1\leqq m\leqq n\}$ of A. Then P satisfies (T) and (W).

If the additive group of A is infinite and cyclic with the generator a, let P be the subset $\{ma: m\geqq 1\}$ of A. Then P satisfies (T) and (W).

THEOREM. *Suppose that a ring A has a subset P which satisfies* (T) *and* (W) *and let a be the least element of P.*

(i) *If $a^2=0$, then either A is isomorphic to any of the isomorphic rings Z_k° $(k=1, 2, \cdots)$ or A is isomorphic to a quotient ring $Z_k^\circ/Z_{k^2}^\circ$ for some positive integer k.*

(ii) *If $a^2\neq 0$, then there exist nonnegative integers h and k such that A is isomorphic to one of the rings Z_k/Z_{hk} or Z_k^-/Z_{hk}^-.*

Proof. (i) If $ka=0$ implies $k=0$, then the mapping $mk\to ma$ is an isomorphism from Z_k° onto A for all $k=1, 2, \cdots$; otherwise there exists a least positive integer k such that $ka=0$. The above mapping is then a homomorphism from Z_k° onto A with the kernel $Z_{k^2}^\circ$. Hence $A\cong Z_k^\circ/Z_{k^2}^\circ$.

(ii) There exists an integer k' such that $a^2=k'a$. Choose k' so that $|k'|=k$ is least. If k' is positive, then the mapping $mk\to ma$ is a homomorphism from Z_k onto A with the kernel Z_{hk}, where $h=0$ if A is infinite and h is the least positive integer such that $ha=0$ if A is finite. Hence $A\cong Z_k/Z_{hk}$.

If k' is negative, then the same mapping is a homomorphism from Z_k^- onto A such that $A\cong Z_k^-/Z_{hk}^-$.

COROLLARY. *If an infinite ring A with identity e has a subset P which satisfies the trichotomy condition* (T) *and the well-ordering condition* (W), *and if e is the least element of P, then A is isomorphic to the ring of integers.*

Proof. Z_k/Z_{hk} or Z_k^-/Z_{hk}^- has an identity element if and only if h and k are relatively prime. Since A is infinite, $h=0$ and hence $k=1$. That is, A is isomorphic to Z or Z^-. Since e is the least element of P, the mapping $m\to me$ is an isomorphism from Z onto A.

A NOTE ON REDUNDANCIES IN THE AXIOM SYSTEM FOR THE REAL NUMBERS

J. A. DYER, Iowa State University

Eves and Newsom in [1] have noted that in the usual axioms for a field the axiom of commutativity of addition is redundant. One can, however, give a perhaps simpler proof of this redundancy than that presented in [1]. This redundancy is in fact a consequence of the well-known result that a group G is Abelian if and only if the mapping, $\mathfrak{N}$: $G \to G$, defined by $\mathfrak{N}(x) = x^{-1}$, is an isomorphism.

It seems not, however, to have been noted by authors giving axioms for the real numbers, that in the axiom system for the real numbers, the commutative axioms for both addition and multiplication are redundant, if one extends the distributive axiom to include right distributivity of multiplication over addition. The purpose of this note is to outline a proof of this redundancy.

As an axiom system we take the following:

Suppose that R is a set, $+$ and $\cdot$ are mappings on $R \times R$ into R, and $<$ is a relation on R. The statement that $(R, +, \cdot, <)$ is a real number system means that

A1. $\exists 0 \in R$ such that $(R, +)$ is a group with 0 as identity.
A2. $\forall a, b, c \in R$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
A3. $\exists 1 \in R$ such that $1 \neq 0$ and $\forall a \in R - \{0\}$, $1 \cdot a = a$.
A4. $\forall a \in R - \{0\}$, $\exists a^{-1} \in R$ such that $a^{-1} \cdot a = 1$.
A5. $\forall a, b, c \in R$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.
A6. $<$ is a linear order on R such that $\forall a \in R$, $(a, a) \notin <$.
A7. $\forall a, b, c \in R$, $a < b \Rightarrow a + c < b + c$.
A8. $\forall a, b, c \in R$, $a < b$ and $0 < c \Rightarrow a \cdot c < b \cdot c$.
A9. Every subset of R which is bounded above has a least upper bound.

By standard techniques one can show:

THEOREM 1. $\forall a \in R$, $0 \cdot a = 0$.

THEOREM 2. $\forall a \in R$, $(-1) \cdot a = -a$.

As a consequence of Theorem 2 and A5 it follows that the mapping, $\mathfrak{N}: R \to R$, defined by $\mathfrak{N}(a) = -a$, $\forall a \in R$, is an isomorphism which implies:

THEOREM 3. $\forall a, b \in R$, $a + b = b + a$.

By the same proofs used in group theory one may show:

THEOREM 4. $\forall a \in R - \{0\}$, $a \cdot a^{-1} = 1$.

THEOREM 5. $\forall a \in R - \{0\}$, $a \cdot 1 = a$.

As an immediate consequence of Theorems 1 through 5 one has:

THEOREM 6. $\forall a \in R$, $a \cdot (-1) = -a$.

since a_n, $n=1, 2, 3, \cdots$, is rational, $a_n \cdot b = b \cdot a_n$, $n=1, 2, 3, \cdots$. Then, since R with the order topology is clearly a Hausdorff space, we obtain:

THEOREM 19. $\forall a, b \in R$, $a \cdot b = b \cdot a$.

Since A1 through A9 together with Theorems 3 and 19 constitute the usual axiom system for the real numbers, it follows that $(R, +, \cdot, <)$ is a real number system.

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KEPLER'S SECOND LAW AND THE SPEED OF A PLANET

S. K. STEIN, University of California, Davis

As is well known, the farther a given planet is from the sun the slower it moves. (We shall call this the "speed property.") Sometimes Kepler's second law, which asserts that the radius vector from the sun to the planet sweeps out area at a constant rate, is cited as an explanation of the speed property. But this law, equivalent to the equation

$$\frac{1}{2}r^2 \frac{d\theta}{dt} = h, \qquad h \text{ constant}, \tag{1}$$

assures us that the farther the planet is from the sun, the smaller is its *angular speed* as viewed from the sun. On the other hand, the second law is also equivalent to the equation,

$$\frac{1}{2}p \frac{ds}{dt} = h, \qquad h \text{ constant}, \tag{2}$$

where $|ds/dt|$ is the speed of the planet at a typical point P and p is the distance from the sun to the tangent line to the orbit at P. Now (2) tells us that the farther the "instantaneous line of motion" of the planet is from the sun, the slower the planet moves. Thus neither (1) nor (2) immediately implies the speed property.

As we will show below, the speed property is *not* a consequence of the second law alone, but of that law together with a small part of the first law, which asserts that the orbit is an ellipse and the sun is at one focus. (What is needed is that the orbit is made up of congruent arcs.) The explanation, which makes use of the support function of a smooth plane convex curve, is summarized in Theorems 1 and 2. We will consider only convex curves that have a continuously turning tangent and only a finite number of normals through the pole of the coordinate system. We conclude with a remark on the physicist's explanation of the speed property in terms of energy.

since a_n, $n=1, 2, 3, \cdots$, is rational, $a_n \cdot b = b \cdot a_n$, $n=1, 2, 3, \cdots$. Then, since R with the order topology is clearly a Hausdorff space, we obtain:

THEOREM 19. $\forall a, b \in R$, $a \cdot b = b \cdot a$.

Since A1 through A9 together with Theorems 3 and 19 constitute the usual axiom system for the real numbers, it follows that $(R, +, \cdot, <)$ is a real number system.

Reference

1. H. Eves and C. V. Newsom, An introduction to the foundations and fundamental concepts of mathematics, Holt, Rinehart and Winston, New York, 1965.

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Consider a plane convex curve whose interior contains the origin of an xy-coordinate system, which coincides with the pole of the polar coordinate system whose polar axis is the positive x axis. The tangent line perpendicular to the ray of angle θ has the normal form

$$x \cos \theta + y \sin \theta = p(\theta).$$

The function p is the *support function* of the curve.

Now let the curve have the polar equation $r=f(\theta)$. The line $x \cos \theta+y \sin \theta = p(\theta)$ meets the curve at a unique point $(r^*, \theta^*) = (fh(\theta), h(\theta))$. It is known that

$$[p(\theta)]^2 + [p'(\theta)]^2 = (r^*)^2. \tag{3}$$

(To establish (3), note that the curve is the envelope of its tangent lines; hence (r^*, θ^*) is the intersection of $x \cos \theta+y \sin \theta=p$ and $x(-\sin \theta)+y \cos \theta=p'$.) In the proof of Theorem 2, where we refer to two convex curves, we will use the symbols $P(\phi)$, $R=F(\phi)$, ϕ^*, R^*, and H analogously.

THEOREM 1. *Let $r=f(\theta)$ describe a smooth convex curve and let α, β be two numbers such that $0 \leqq \alpha < \beta < 2\pi$. Assume that $f'(\alpha)=0=f'(\beta)$, and $f'(\theta) \neq 0$ for any θ in (α, β). Then not only is f monotonic for θ in $[\alpha, \beta]$, but so is the support function p.*

Proof. Let us assume that f is increasing for θ in $[\alpha, \beta]$. We will show that $p'(\theta)>0$ for θ in (α, β). Note that $p(\alpha)=f(\alpha)<f(\beta)=p(\beta)$. If $p'(\theta)=0$ then by (3) $fh(\theta)=p(\theta)$, whence $h(\theta)=\theta$ and $r^*=f(\theta)$. Thus the ray of angle θ is perpendicular to the curve at the point where it crosses the curve. Hence $f'(\theta)=0$, a contradiction. The theorem is proved.

The theorem assures us that on the arc stretching from a minimum of the function f to an (adjacent) maximum, not only f but also p is an increasing function. On this arc, therefore, the speed property holds: for the larger $f(\theta)$ is, the larger is $p(\theta)$, and, according to (2), the smaller is the speed of the planet.

If now we make use of Kepler's first law, then the speed property follows easily for the whole orbit. We may assume that the ellipse has the equation $r=f(\theta)=pe/(1+e \cos \theta)$. Hence f is increasing for θ in $[0, \pi]$. On this "top" arc Theorem 1 assures us that the speed property holds. Similarly the property holds on the "bottom" arc, corresponding to θ in $[\pi, 2\pi]$. Moreover, because of the congruence of these two arcs, at any two points on the ellipse that are equidistant from the sun, the tangent lines are also equidistant from the sun, hence by (2) the planet is moving at the same speed. This congruence is all we need from Kepler's first law, but, as the next theorem shows, it must be referred to. That is, the speed property is a consequence of Kepler's second law together with a certain geometric property of the orbit. Actually, it suffices that the ellipse is the union of a set of congruent arcs, each arc joining a minimum of f to an "adjacent" maximum linked in such a way that the union of two adjacent arcs is symmetric with respect to the line joining the sun to their common endpoint.

where C is a constant.

In view of (5) and (b) it follows that (a) is equivalent to the assertion "P depends only on the distance from the origin," or, equivalently, $\mathbf{F}=\nabla P$ is a central force; the assertion that $\mathbf{F}$ is a central force is a restatement of Kepler's second law. The speed property demands not only that P depend only on r but that it be a monotonically decreasing function of r.

Thus, in either approach Kepler's second law by itself does not imply the speed property.

ON THE DEFINITION OF EVEN AND ODD PERMUTATIONS

W. PHILLIPS, University of Colorado

Introduction. All of the usual definitions of even and odd permutations involve the polynomial $\prod_{i<j}(x_i-x_j)$ either as part of the definition (DEF: an even permutation is one that sends the polynomial $\prod_{i<j}(x_i-x_j)$ into itself) or in justifying the definition (DEF: an even permutation is one that can be written as the product of an even number of transpositions).

Of course there is nothing mathematically incorrect with either of these definitions, but students often find its use confusing as it seems to involve something totally irrelevant to the subject matter at hand, namely, permutations and permutation groups.

It is the purpose of this note to introduce an alternative to the above definitions. A proof is given (without the use of the above polynomial) that by this new definition an even permutation is, indeed, one that can be written as the product of an even number of transpositions. The definition and proof presuppose only the most basic knowledge of permutations, including the following elementary theorems:

THEOREM. *Any $x\in S_N$ can be written as the product of disjoint cycles. This decomposition is unique except for order.*

THEOREM. *The symbols appearing in a k-cycle may be permuted cyclicly without changing the permutation.*

THEOREM. *Any $x\in S_N$ $(N\geqq 2)$ can be written (in at least one way) as the product of transpositions.*

Even—Odd

DEFINITION. *Let $x\in S_N$. Define the (uniquely determined) integer $n(x)$ to be the number of cycles occurring in the unique decomposition of x into disjoint cycles including the 1-cycles.*

DEFINITION. *In S_N define a permutation x to be even or odd as the (uniquely determined) integer $N-n(x)$ is even or odd.*

where C is a constant.

In view of (5) and (b) it follows that (a) is equivalent to the assertion "P depends only on the distance from the origin," or, equivalently, $\mathbf{F}=\nabla P$ is a central force; the assertion that $\mathbf{F}$ is a central force is a restatement of Kepler's second law. The speed property demands not only that P depend only on r but that it be a monotonically decreasing function of r.

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Proof. Write the permutation as a product of transpositions and starting from the *right* begin "collapsing" this expression by repeatedly multiplying the last two cycles together. At each step we have the product of a transposition and some other cycle. By the previous theorem the evenness or oddness always changes at each step however the evenness or oddness of the final result is fixed. Denying the conclusion, then, leads to a contradiction.

NOTE: From this point we can derive all the usual corollaries, for example:

COROLLARY. *The product of two permutations is even if either both are even or both odd. Otherwise the product is odd.*

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

Material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D.C. 20005.

THE DITCHLEY CONFERENCE SCHOOL MATHEMATICS OF TWO COUNTRIES

E. L. LOMON, Massachusetts Institute of Technology and C.C.S.M.

The first wave of mathematics curriculum reform has crested in both the United Kingdom and the United States. Those initial efforts were sensitive to the traditional curriculum, school organization and teacher preparation of each country. New curricula had to be separately tested on a small scale before being committed to large scale use within each country. Until these results were seen, it was unlikely that either party could help the other significantly. Published texts and reports allowed the monitoring of the ideas and progress of the other party. Now that both countries have had experience on a wide scale with the "new math," a second wave of new curricula is forming. At this stage it seems appropriate to capitalize on the corroborative, contradictory or complementary experience of the two countries.

The initiative was taken by Professor Bryan Thwaites and Professor W. T. Martin. The former is director of the School Mathematics Project (SMP), the most extensive project of its type in England. The latter is Chairman of the Cambridge Conference on School Mathematics (CCSM). SMP undertook to invite the U.K. delegation and to obtain the conference facilities. The American delegates were invited by CCSM, using a Carnegie Corporation grant and NSF funds. The excellent representation of major curriculum projects indicates the interest aroused in both countries by the topic. The participants are listed at the end of this article.

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The meeting was held September 9–11, 1966, at Ditchley Park Estate, Oxfordshire, operated as an "Anglo-American Conference Centre." This note relates my personal impressions. A report for distribution in the U.K. has been prepared by Professor Thwaites.

Professor Thwaites and Professor Martin, the co-chairmen, skillfully preserved a relaxed atmosphere. The topics of the plenary and group sessions are appended. I shall make no attempt to relate my remarks to particular sessions.

In order to profit from the different curriculum histories in the two countries the conferees were alert to variations of accomplishment and attitude. A difference affecting the whole of the secondary school programs is in the amount of integration of geometry with algebra or analysis. SMSG and other widely used new American texts largely maintain the traditional separation of these materials, with exceptions, as in the presentation of such topics as graphing. In the texts of SMP and of the Midlands Mathematical Experiment, these subjects are interlaced in each term, with frequent cross-referencing. A related divergence is the larger degree of systematization and formality in the American courses. Historical reasons for these differences are discernible, but it is more interesting to enquire about the present reactions and future intentions of the two groups.

I digress here to make a point that I believe essential. There is a wide divergence of attitudes among those active in curriculum development on each side of the Atlantic. The spread of opinion on either side is greater than the difference of the average opinion between the two countries. Any opinion or attitude concerning school mathematics that has substantial support in one country will have important advocates in the other. Present differences of substance are mostly due to the pressure of historical and accidental circumstance. The actual diversity of opinion of the U.S. is well known to many. The meeting displayed a similar diversity among the initiators of curriculum reform in the U.K.

The existence of reform curricula in both countries has effected a separation from the historical requirements. This allows the succeeding reforms in each country to approach a common result. In my opinion the major circumstance causing the present difference of emphasis in the two countries is the relative brevity of experience in the U.K. with new mathematics in the elementary school. The first large scale elementary school project was introduced only last September, by the Nuffield Mathematics Project. This project is sophisticated, and well in advance of the first such American attempts. Its impact, however, has not yet been felt. For most of the British curriculum developers, this leaves the early secondary school mathematics as the beginning of the "spiral" of mathematics ideas and applications. An intuitive approach at this level is thus prescribed. The dominantly intuitive development of many topics is now expected by Americans to take place in elementary school. It follows naturally that a larger degree of systematization and deductive reasoning is planned for secondary school courses. I would expect a similar evolution in the U.K. to take place between the coming revision and the one succeeding it.

On the other hand, the impact of the first reform has made the American schools more flexible with respect to the organization of topics. The U.S. projects will probably take the next opportunity to use several specific effective examples, developed in the U.K. and here, of combining geometric and algebraic material.

An important example of complementary curriculum investigations exists in the experience of each country with motion (or transformation) geometry. The SMP program uses symmetry as the organizing principle for much of its geometry. We can here observe on a large scale the efficacy of the approach in the context of a related program. In the U.S. the experiments have been directed towards finding an effective intuitional development of symmetry motions in elementary school, developing through the junior high school into a systematic basis for geometry and modern algebra in high school. These experiments have been on a small scale, and not in the context of a related mathematics course. The U.K. experience can be taken as strong encouragement for the large scale development of the American materials.

One of the claims most often made for the "new math" is that it teaches unifying concepts which help the student generalize and transfer from one problem to another. The introduction of "sets" at an early primary grade is usually justified in that way. It is thus salutary that delegates from both countries felt that we are only coming to grips with a unifying language and imagery. The role of "set theory" or "functions" in concisely stating the structure of different branches of mathematics was said to require more experience and sophistication in mathematics than the student would have through his school years. A corollary is that the introduction of such topics as sets and functions must avoid being pretentious or formal. There was more hope and intention among Americans than among Britons that the improvement of the curriculum would eventually permit some revelation of structurally unifying significance of concepts.

Both groups stressed the importance of developing and continually presenting good applications of the mathematical ideas and tools. The best applications use the appropriate mathematics to study situations of importance. They elucidate the modeling process, giving the student the power to devise his own applications. In both countries collections of good applications have been made.

Good applications tend to go deeply into the field of the application, so that more than the mathematics curriculum is involved. In this country we are beginning explorations of the correlation of the mathematics with the science curricula. The absence of science material in the U.K. elementary schools has left the matter of correlation less urgent.

In the light of the need for good applications, the teaching of some probability and a little statistics is very appealing. All of arithmetic is applied in obtaining distributions from data, and in calculating theoretical distributions. The mathematics of probability is in turn applied to problems in almost every scientific and sociological endeavour. Experimental probability can be introduced in the earliest years, and the theory developed as facility with sets, alge-

bra and analysis is successively developed. The subject shows what so few laymen realize, that mathematics can deal with imprecise information and indeterminate models. I expect that probability and perhaps statistics will soon have a major role in the curricula of both countries.

In present practice, the British give more emphasis to cardinal numbers and discrete algebraic systems, and in the United States there has been a tendency to develop the concept of real numbers at an early stage, and later treat fields and analysis. Present opinions in both countries are widely divergent.

The program of the conference directed us to discuss matters of teaching style and method, as well as content. It is perhaps surprising that the U.K. emphasis on informality in the presentation of mathematics is not correlated with strong emphasis on an "open-ended discovery" classroom approach. We viewed a film in which a deeply involved Madison Project class was purposefully nondirected by Professor Robert Davis. Most, if not all, of the British present felt that the students could not extract useful results from the emotional discussion, or they questioned that the usual teacher could handle the situation. Lest one is still tempted to oversimplify the contrast of U.S. and U.K. attitudes, it should be noted that Professor Davis has an interest in the teaching of axiomatic structures and some level of logic to the very young!

Consideration was given to the results of behavioral science and learning studies of psychologists. In both countries there have been only minor attempts at devising curricula oriented about approaches suggested by these studies. Their results on perception and mental skills at different ages have been interpreted as indicating important but not insurmountable obstacles to the goals of the mathematics curriculum. The major programs in both countries take the findings of psychologists into account by, for instance, a careful development of spatial perception at the age they think it relevant. A more direct reaction to those findings would be to ignore specific types of perception until they develop "naturally." Very little curriculum development has been based on the latter reaction.

Professor A. Gleason suggested a direct use of the educational process that is perhaps best understood—the conditioned reflex. There are many good reasons to concentrate on more interesting mathematics in elementary school than the traditional arithmetic drill. Professor E. Begle had reported on a very satisfying SMSG experiment in which seventh grade children behind grade level in computation were relieved of all computation for a year. At the end of the year they had, on the average, gained two and a half years in arithmetic reasoning and also one and a half years in computation! But there will be children who are poor computers at the end of several grades of stimulating mathematical ideas, just as there are at the end of several grades of boring drill. Professor Gleason pointed out that the skill of the behavioral scientist at inculcating responses may provide a comparatively painless and rapid way of remedying the situation for these children.

The effect of technical revolutions on the teaching of mathematics received

much attention. The British delegates were impressed by the appearance of computer consoles in many American secondary schools. Although there was little enthusiasm for courses in programming as such, integrated use of a computer in the mathematical course had wide appeal. Many advantages of the availability of a computer were cited. In problem solving additional insight would be given into the procedure of solution, even for analytically soluble problems. Applications could be introduced, for which the comparison of computed numbers with measured numbers would not have otherwise been possible. The programming itself was regarded as a training in careful, precise thought, and in logic. On another level the use of a computer provides motivation to the student, confidence through checking and a release from boring computation.

The reaction was very different when attention was directed towards computers as teaching machines; that is, programmed to teach the student rather than programmed by the student. The expectation was expressed that the programming of sufficiently flexible and subtle teaching would be a formidable task. Very costly investments would be required in the highly competitive computer industry. It was feared that the resultant industry pressure may lead to early adoption preceding the proper development and testing of teaching programs. This would leave schools with the burden of expensive and perhaps harmful computer-based teaching machines.

The recommendation was that the mathematical community should devote serious study to computer-based instruction with the object of producing an acceptable product or at least of being an effective critic. The British felt that there was little immediate danger of funds being available in their country for an over-commitment.

The need for large scale testing was a matter of concern to the British. This is needed in the relative evaluation of curricula (as in the National Longitudinal Study of Mathematical Abilities of SMSG) if not for the grading of increasing numbers of students. Professor R. P. Dilworth presented impressive evidence of the reliability of carefully prepared objective machine-scored tests. It would appear that they are more reliable predictors (of something!) than the conventional "essay type" responses that require careful reading. As Dilworth pointed out, an advantage of large scale machine-scored testing is the possibility of statistically analyzing pretests. The modified test prepared on this basis contains a higher percentage of meaningful questions. The British expressed the desire for the help of American testing experts in devising tests of their own.

The growing exchange of information between the resulting curriculum development in both countries will tend to bring them closer together. However, differences in organization and attitude, noted at the conference, will condition the relative direction of progress for some time to come. There is an existing difference in the standards for teachers (masters) and in the content of their training. Standards are more uniform in the U.K. and on the high side of the U.S. levels. The smaller percentage of college bound students in the U.K. will be a factor in determining the content of secondary school instruction. The

presence in the USSR of schools specializing in mathematics or science was discussed and the few American parallels noted. This has impact on only a small part of the student population. In the U.K. some specialization takes place in secondary schools prior to college entrance, as it does in differing forms in the United States.

I detected a difference of attitude that may lead to long term differences in the curricula of the two countries. The American attitude is dominantly (remember the individual diversity!) that some ability to use the subtler mathematical ideas associated with symmetry, continuity, probability, and modelling should be widely diffused through the population. In the U.K. the desire is, it seems to me, to present these more esoteric topics to the general population only at the conceptual level, reserving any formal command for the more "practical" skills. Calculus, for example, is considered relevant only for students going on to be scientists, engineers or mathematicians. There is clearly a division on this issue in this country, and this is not the place to present the arguments for either approach. But the next "wave" here is likely to present a program designed to enable the bulk of the population to make some headway in modelling and analyzing real situations involving, for instance, probability. In the U.K. the more formal material may be designed for the smaller population that is specially motivated and able.

The direction of the reform movements may also be affected by the relatively smaller participation of university mathematicians in the U.K. In the U.S.A., university mathematicians direct curriculum development groups, are on the textbook writing teams, and are occasionally in the experimental classrooms. It was noted that some internationally known Soviet mathematicians teach classes of young children on a regular basis. In the U.K. the university participation has mostly been on the level of general formulation and advice. The writing, directing and participation in experimental work is by schoolmasters, some of whom have been on University faculties. There is a handful of active research mathematicians participating actively in the curriculum reform.

Most of the American delegates arrived at the conference knowing little of the scope of U.K. accomplishments and intentions in mathematics curriculum reform. It is my impression that on the whole the U.K. contingent was better informed of American accomplishments, but as poorly informed of the present direction of our thinking and effort as we were of theirs. Only a few delegates from either country had previously had sufficient contacts, such as in the African Program, to reveal their orientation to each other.

On leaving we felt that we had only scratched the surface. The Americans had learned of the important scale on which new and very enjoyable courses have been introduced in the U.K. We had become familiar with the organizations and some of the key people involved. Their special interests and their publications are now largely known to us. This is a critical first step in being able to gather further information by correspondence and in looking forward to working contacts. Effective liaison may expand rapidly from this beginning.

G.3 Unifying concepts; sets and functions (*Reporter: Mr. D. A. Quadling*).
P.4 Relative emphasis on discrete and continuous systems (*Reporter: Professor P. J. Hilton*).
G.5 Aims and content for geometry, with special reference to its study through transformations (*Reporter: Dr. A. J. Weir*).
G.6 Examining Procedures (*Reporter: Professor R. P. Dilworth*).
G.7 The impact of computers on mathematics and on the curriculum (*Reporter: Dr. H. Pollak*).
G.8 Statistics (*Reporter: Dr. J. M. Hammersley*).
G.9 Teachers training and shortage of teachers (*Reporter: Mr. C. S. Hope*).
P.10 Special treatment for highly gifted mathematical children (*Reporter: Dr. J. H. Hlavaty*).
G.11 Acquisition of computational skill; arithmetical operations (*Reporter: Professor A. Gleason*).
G.12 Communication with the sciences at school level (*Reporter: Professor E. L. Lomon*).
P.14 Report of conference.

SOME REMARKS ON "A DILEMMA IN DEFINITION"

R. B. DEAL, Oklahoma State University

These comments were conceived in preparation for a discussion of the article "A Dilemma in Definition" by C. P. Nicholas, U. S. Military Academy, in this MONTHLY, Volume 73, number 7, scheduled for one of the pedagogical discussion periods in the seminar for graduate teaching assistants at Oklahoma State University.

Mr. Nicholas discusses ambiguities and concomitant student frustrations arising from some of the current attempts to define functions and to distinguish between functions and function values. The points are important, and his criticism merits serious consideration.

It should first be remarked that this is one of a variety of related problems which makes teachers necessary. Mathematics teachers should make the meaning of definitions and theorems intuitively clear. They must distinguish between the informal intuitionistic plausible procedures often used to develop mathematical results and the strictly logical processes with precise definitions which insure the validity of their conclusions.

The fundamental basis for mathematical structures today is that of "naive" set theory. It is simply a matter of experience that being able to go back to these fundamental structures is a safeguard in case of ambiguities or difficulties in understanding. It is just as foolhardy to assume that such a return to basic principles is always necessary as to insist that in doing arithmetic with integers one always should return to Peano's postulates.

The old calculus books were replete with examples where difficulty in understanding arose from a lack of adequate definitions. There would seem to be little point in a return to one of these concepts, namely, "variable," especially since a good understanding of functions and their compositions is so important to the understanding of limits, derivatives, integrals, and the chain rule.

The recent widespread use of categories as a synthesizing concept in mathematics illustrates very well the importance of emphasizing the domain and range, as well as the procedure for assigning to each member in the domain an

element in the range, and the basic properties of composition of functions.

After having taught the "classical" calculus many times several years ago, it was a pleasant surprise last year to find that many of the modern innovations add considerably to depth of understanding. It is not difficult to impress upon one's students that whatever formal definition is used for function, it must be such that the domain is specified, a range is specified, and that to each element in the domain there is assigned one and only one element of the range. Furthermore, it is easy to see that giving a definition as a subset of a Cartesian product of sets is an easy way of accomplishing this purpose. Just as one uses theorems to build further mathematical structures instead of always reverting to axioms, one uses the properties of functions, once they have been established, instead of always reverting to the definition.

The simple device of

$$f: \begin{cases} A \to B \\ x \to f(x) \end{cases}$$

avoids many ambiguities and in making notes for a calculus class it was found that often only function values were used in a discussion and it was not necessary to mention the functions per se.

It seems that there should be no difficulty with good definitions in referring to real-valued functions on some subset of the reals as being increasing or decreasing. It may be important to have this concept independent of the physical idea of the point moving up or down. From a sound modern point of view on the "scientific method" it is necessary to have available in the mathematical models a definition of function in order to express the physical concept of variable or the statistical concept of random variable.

As for the function f: $x \to \log \sin x$, it is important to specify the domain. In elementary calculus, this must be a subset of $\{x \mid \exists k \in Z \ni 0 < x - 2k\pi < \pi\}$ where Z is the set of integers. This point is also worth laboring.

It is quite likely that in 1905 a sound argument could have been made that the horse could do everything a car could, and better, and without all the expense of building special trails. A transition to better concepts should not be influenced by a few mistakes along the way.

A REPLY TO "A DILEMMA IN DEFINITION"

H. A. THURSTON, University of British Columbia

C. P. Nicholas' [1] first sentence is true: many text-books of calculus do place the student in a quandary over the word "function." But his third sentence—"The result is an ambiguity which even the most careful writers are unable to avoid"—is false. The most careful writers can and do avoid it. Would that more of them (or, rather, fewer of the others) wrote calculus texts!

Part of Mr. Nicholas' trouble is this: the logical analysis of a familiar and

element in the range, and the basic properties of composition of functions.

After having taught the "classical" calculus many times several years ago, it was a pleasant surprise last year to find that many of the modern innovations add considerably to depth of understanding. It is not difficult to impress upon one's students that whatever formal definition is used for function, it must be such that the domain is specified, a range is specified, and that to each element in the domain there is assigned one and only one element of the range. Furthermore, it is easy to see that giving a definition as a subset of a Cartesian product of sets is an easy way of accomplishing this purpose. Just as one uses theorems to build further mathematical structures instead of always reverting to axioms, one uses the properties of functions, once they have been established, instead of always reverting to the definition.

The simple device of

$$f: \begin{cases} A \to B \\ x \to f(x) \end{cases}$$

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and 3, $2\times3=3\times2$." This is because they are not merely numerical variables; they are *bound* numerical variables.) In the preceding paragraph, τ, τ_1, and τ_2 are numerical variables.

It is not rare for a word to be used differently in different branches of mathematics (for example: "normal" in solid geometry and in the theory of groups). The different uses of "variable" in physics and practical calculus on the one hand and in mathematical logic on the other should give no trouble, but unfortunately these topics overlap in elementary analysis (or "text-book calculus") where the two concepts are sometimes confused. One certain cure for the confusion is to use separate alphabets, just as in this note I have used the Roman alphabet for "variable quantities" or "function variables" and the Greek alphabet for "pro-numerals" or "numerical variables." On this convention it is quite clear that x^2 and $\sin x$ denote functions, whereas ξ^2 and $\sin \xi$ do not. Without it, one has to know what x denotes: if x denotes a function, then x^2 denotes a function; if x denotes a number, then x^2 denotes a number.

Reference

1. C. P. Nicholas, A dilemma in definition, this MONTHLY, 73 (1966) 762–768.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE

COLLABORATING EDITORS: J. BARLAZ, Rutgers—The State University; L. CARLITZ, Duke University; HASKELL COHEN, University of Massachusetts; H. EVES, University of Maine; M. S. KLAMKIN, Ford Scientific Laboratory; R. C. LYNDON, University of Michigan; and A. WILANSKY, Lehigh University.

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N.J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to M. S. Klamkin, Ford Scientific Laboratory, P.O. Box 2053, Dearborn, Mich. 48121. To facilitate their consideration, solutions for Elementary Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before April 30, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2035. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Can the Euler line of a nonisosceles triangle pass through the Fermat point

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SOLUTIONS OF ELEMENTARY PROBLEMS

Conditions for a Real Inequality

E 1885 [1966, 537]. *Proposed by D. E. Daykin and C. J. Eliezer, University of Malaya, Kuala Lumpur*

For which continuous functions $f(x)$ do we have

$$a_1^{f(a_1)}\, a_2^{f(a_2)} \cdots a_n^{f(a_n)} \geqq 1, \tag{1}$$

for all positive real numbers $a_1, \cdots, a_n$ with product $a_1a_2 \cdots a_n = 1$?

Solution by the proposers. The inequality holds if and only if

$$f(a) \leqq f(1) \leqq f(b) \qquad \text{for } 0 < a \leqq 1 \leqq b. \tag{2}$$

Since

$$a_1^{f(a_1)} \cdots a_n^{f(a_n)} = (a_1^{f(a_1)-f(1)}) \cdots (a_n^{f(a_n)-f(1)}),$$

(1) is true and the condition is sufficient. To see that it is necessary, let (1) be given and let $a \leqq 1 \leqq b$. Then there are α, β as close as we like to a, b, respectively, such that $1 < \beta$ and $\alpha^r\beta^s = 1$ for positive integers r, s. Let $n = r+s$ and $a_1 = \cdots = a_r = \alpha$ and $a_{r+1} = \cdots = a_n = \beta$. Then

$$1 \leqq a_1^{f(a_1)} \cdots a_n^{f(a_n)} = \alpha^{rf(\alpha)}\beta^{sf(\beta)} = \beta^{s(f(\beta)-f(\alpha))},$$

so that $f(\alpha) \leqq f(\beta)$, and by continuity of $f(x)$ we have $f(a) \leqq f(b)$ giving (2).

Also solved by P. R. Chernoff, M. A. Ettrick, G. A. Heuer, E. S. Langford, and Jürg Rätz (Switzerland).

Special Quadrics

E 1886 [1966, 538]. *Proposed by W. M. Sanders, Lawrence University, Appleton, Wis.*

Show that if $ax^2+by^2+cz^2+2fxy+2gzx+2hyz+2rx+2sy+2tz+d=0$ represents two distinct planes, then the matrix

$$E = \begin{pmatrix} a & f & g & r \\ f & b & h & s \\ g & h & c & t \\ r & s & t & d \end{pmatrix}$$

is not the adjoint of any matrix.

I. *Solution by Yu Chang and Sidney Spital, California State Polytechnic College.* The conclusion is true also for cones (real or imaginary) and cylinders (elliptic, hyperbolic or parabolic). A proof uses the following theorems applied to 4×4 matrices:

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where D is diagonal, nonsingular, and at least 2×2. Since E is singular, it follows that $EA=AE=$ the zero matrix, so that A must be of the following form:

$$A=\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & X \end{array}\right).$$

If we now compute A^*, we see that the northwest corner is at least 2×2, so that the cofactors A_{11} and A_{22} are both zero. This contradicts the fact that D is nonsingular.

Also solved by M. G. Greening (Australia).

Sums of 2^p Squares

E 1889 [1966, 538]. *Proposed by Joseph Arkin, Suffern, New York*

Prove: The sum of 2^p squares $(\sum_{j=1}^{2^p} y_j^2)$ can always be expressed (in an infinite number of ways) as the quotient $(\sum_{j=1}^{2^p} A_j^2/\sum_{j=1}^{2^p} x_j^2)$ of two numbers, each of which is a sum of 2^p squares $(p=0, 1, 2, \cdots)$.

Solution by Leon Bankoff, Los Angeles, Cal. The following proof is for a more general proposition in which 2^p is replaced by any positive integer n.

Starting with any Pythagorean triangle, we may construct a succession of other Pythagorean triangles, each having an arm on the hypotenuse of its predecessor. (See W. Sierpinski, *Pythagorean Triangles*, Yeshiva University, N. Y., 1962, p. 26). Hence there exist an infinite number of squares of natural numbers that can be expressed as the sum of any finite number of other squares.

Now let $k^2=x_1^2+\cdots+x_n^2$. Then there are an infinite number of solutions of

$$k^2\left(\sum_{j=1}^{n} y_j^2\right)=\sum_{j=1}^{n} k^2y_j^2=\sum_{j=1}^{n} A_j^2.$$

Also solved by Anders Bager (Denmark), G. A. Heuer, E. S. Langford & E. B. Keeler' Norman Miller, D. L. Silverman, Sidney Spital, and the proposer.

Editorial Note. Some of our readers tried to show that the product of two sums of 2^p squares can always be written as the sum of 2^p squares. This is indeed true for $p=1, 2, 3$, but not for $p>3$. This was first proved by A. Hurwitz. For references and a detailed discussion see a paper by Dickson: *On quaternions and their generalization and the history of the eight square theorem.* Ann. Math., 20 (1919) 155–171.

A Sequence of Square Roots

E 1890 [1966, 538]. *Proposed by Emanuel Vegh, Naval Research Laboratory*

If m and n are given integers greater than one, show that there exist positive integers $N_1, N_2, \cdots, N_{m-1}$ such that

$$\sqrt{m}=1+\sum_{i=1}^{m-1}(\sqrt{N_i}-\sqrt{N_i-1})^{1/n}.$$

Solution by Anders Bager, Hjørring, Denmark. Let i be one of the numbers $1, 2, \cdots, m-1$. By the binomial theorem:

$$(\sqrt{i+1} \pm \sqrt{i})^n = x_i \pm y_i\sqrt{i(i+1)},$$
$$(\sqrt{i+1} \pm \sqrt{i})^n = u_i\sqrt{i+1} \pm v_i\sqrt{i},$$

according as n is even or odd, respectively. In both cases we therefore have two equations:

$$(\sqrt{i+1} + \sqrt{i})^n = \sqrt{N_i} + \sqrt{M_i}, \tag{1}$$

$$(\sqrt{i+1} - \sqrt{i})^n = \sqrt{N_i} - \sqrt{M_i}. \tag{2}$$

By multiplication of (1) and (2) we get $1 = N_i - M_i$, so that

$$\sqrt{i+1} - \sqrt{i} = (\sqrt{N_i} - \sqrt{N_i - 1})^{1/n}. \tag{3}$$

Summing (3) through all values of i we get the desired formula.

Also solved by Robert Bart, N. J. Fine, Donald Jeffords, Carey Jensen, E. S. Langford, D. C. B. Marsh, Norman Miller, Wanda J. Mourant, Jürg Rätz (Switzerland), and Simeon Reich (Israel).

Multiplicative Function

E 1891 [1966, 538]. *Proposed by A. E. Livingston, Oregon State University*

Let f be a number-theoretic function, and set

$$F(n) = \sum_{d|n} f(d) f\left(\frac{n}{d}\right).$$

If $f(1) = 1$, and F is multiplicative, then f is multiplicative.

I. *Solution by D. A. Smith, Duke University.* We prove $f(mn) = f(m)f(n)$ for $(m, n) = 1$ by induction on mn. For $mn = 1$, this is clear. Write

$$F(mn) = \sum_{d|mn} f(d) f\left(\frac{mn}{d}\right) = 2f(mn) + \sum_{\substack{d|mn \\ d \neq 1, mn}} f(d) f\left(\frac{mn}{d}\right). \tag{1}$$

By hypothesis, this is also equal to

$$\begin{aligned} F(m)F(n) &= \left[\sum_{d_1|m} f(d_1) f\left(\frac{m}{d_1}\right)\right]\left[\sum_{d_2|n} f(d_2) f\left(\frac{n}{d_2}\right)\right] \\ &= \sum_{\substack{d_1|m \\ d_2|n}} f(d_1) f(d_2) f\left(\frac{m}{d_1}\right) f\left(\frac{n}{d_2}\right). \end{aligned} \tag{2}$$

We have $(d_1, d_2) = 1$ and $(m/d_1, n/d_2) = 1$ for each term, so the induction hypothesis applies to all terms in the sum except the first and last, hence (2) can

be rewritten

$$F(m)F(n) = 2f(m)f(n) + \sum f(d_1d_2)f\left(\frac{mn}{d_1d_2}\right), \tag{3}$$

where the sum is over pairs $d_1 \mid m$, $d_2 \mid n$ such that $d_1d_2 \neq 1$, mn. If we set $d = d_1d_2$, again using $(m, n) = 1$, the sum in (3) is the same as the sum in (1), and it follows that $f(mn) = f(m)f(n)$, completing the induction step.

II. *Solution by Sidney Spital, California State Polytechnic College.* Inversion via the Moebius function yields

$$f(1)f(n) = \sum_{d \mid n} \mu(d)F(n/d).$$

With μ and F multiplicative (and $f(1) = 1$), it is easily seen that f is also multiplicative.

Also solved by Anders Bager (Denmark), P. R. Chernoff, M. G. Greening (Australia), Donald Jeffords, E. S. Langford, Al Somayajulu, C. S. Venkataraman (India), and M. M. Whiteside.

A Generalized Coupon Collector's Problem

E 1892 [1966, 539]. *Proposed by Melvin Hausner and Paul Magriel, New York University*

An experiment has n possible outcomes σ_i, each with probability $p_i > 0$, $\sum p_i = 1$. The experiment is repeated as many times (N) as necessary until every outcome has occurred. What is the expected value of N?

Solution by P. J. Burke, Middletown, N. J. Let σ_{i_j} be the jth outcome to appear and let $E_{i_1 \cdots i_m}$ be the event that the first m outcomes to appear are $\sigma_{i_1}, \cdots, \sigma_{i_m}$. Let N_m be the number of trials between the first appearance of σ_{i_m} and that of $\sigma_{i_{m+1}}$, inclusive of the latter. Then, since σ_{i_1} always appears on the first trial, $N = 1 + \sum_{m=1}^{n-1} N_m$. Define

$$q_{i_1 \cdots i_m} = 1 - (p_{i_1} + \cdots + p_{i_m}).$$

Then the probability of $E_{i_1 \cdots i_m}$ is

$$p_{i_1} \frac{p_{i_2}}{q_{i_1}} \cdots \frac{p_{i_m}}{q_{i_1 \cdots i_{m-1}}}.$$

The conditional expectation of N_m given $E_{i_1 \cdots i_m}$ is $1/q_{i_1 \cdots i_m}$. Hence the expectation of N_m is

$$S_m = \sum_{i_1 \neq \cdots \neq i_m} \frac{p_{i_1} \cdots p_{i_m}}{q_{i_1} \cdots q_{i_1 \cdots i_m}},$$

and the expectation of N is $1 + \sum_{m=1}^{n-1} S_m$.

Also solved by Shirley Braverman & P. A. Scheinok, P. R. Chernoff, E. S. Keeping, and E. S. Langford.

Several noted that the present problem is a direct generalization of one in Feller, *Probability Theory*, v. I, pp. 210–211.

Functions for which $f^{-1} = 1/f$

E 1894 [1966, 539]. *Proposed by K. E. Whipple, Auburn University*

Characterize those real valued functions, continuous everywhere except zero, for which $f^{-1} = 1/f$.

Solution by F. G. Gustavson, IBM, Yorktown Heights, N. Y. The functions f are characterized by the following properties:

(a) $f(x)f(1/x) = 1$, (b) $f(1) = -1$,
(c) $f(x)$ is monotonic in $(0, 1)$ (d) $\lim_{x\to 0+} f(x) = 0$ or $-\infty$,
(e) $f(x)$ is defined in $(-\infty, 0)$ by reflection of $1/f(x)$ on $(0, \infty)$ in the line $y = x$.

Example:

$$f(x) = \begin{cases} -1/x & x < 0 \\ -x & x \geqq 0. \end{cases}$$

Proof: (a) From $f^{-1}(f(x)) \equiv x$ and $f^{-1}(x) = 1/f(x)$ we find $f(f(x)) = 1/x$. Thus $f(x) = f^{-1}(1/x) = 1/f(1/x)$ or $f(x)f(1/x) = 1$.

(b) Put $x = 1$ in (a) and find $f(1) = \pm 1$. If $f(1) = 1$, then no function f exists, for f and f^{-1} have the same monotonicity property in a neighborhood of $x = 1$ while $1/f$ has the opposite property.

(c) If $f(x)$ were not monotonic in $(0, 1)$, then $f^{-1}(x)$ would not be single-valued for all negative x.

(d) Suppose that $f(x)$ is monotonic increasing in $(0, 1)$. Then if $\lim_{x\to 0+} f(x) \neq -\infty$, we have $\lim_{x\to 0+} f(x) = k$, $k < -1$. Using (a) we find $\lim_{x\to 0+} f^{-1}(x) = 1/k$ and $\lim_{x\to\infty} f^{-1}(x) = k$. However, this implies that the domain of $f(x)$ for negative x is $(k, 1/k)$ which contradicts the fact that f must be defined for all nonzero real values. A similar proof shows that $\lim_{x\to 0+} f(x) = 0$ whenever f is monotonic decreasing in $(0, 1)$.

(e) If $f(x)$ has properties (a) to (d) and $f^{-1}(x) = 1/f(x)$ on $(0, \infty)$, then it is necessary that $f(x)$ be defined for negative x as in (e). Furthermore, $f^{-1}(x)$ must also be defined in $(-\infty, 0)$ by reflecting $f(x)$ on $(0, \infty)$ in the line $y = x$. Using property (a) we see that under these reflections the property $f^{-1}(x) = 1/f(x)$ is preserved for negative x.

Also solved by P. R. Chernoff, Michael Goldberg, Jerome Harrison, G. A. Heuer, H. E. Lahmann (Germany), and Allan Wachs. Several contributors cited some of the characterizing properties and/or gave the example noted above.

Acknowledgment. The editors wish to acknowledge the following solutions of elementary problems, received in good time but misplaced (see Notice 1967,199) and not re-

trieved until too late to be acknowledged with other solutions of the various problems:

J. C. Abad, E 1851, E 1958; Leon Bankoff, E 1861; D. E. Daykin & T. K. Sheng (Malaysia), E 1844; A. H. Durfee, E 1858; Michael Goldberg, E 1840; M. G. Greening (Australia), E 1846; Stephen Hoffman, E 1840, E 1842; Erwin Just, E 1859; R. S. Kaluzniacki, E 1858, E 1859; B. W. King, E 1842; P. G. Kirmser, E 1859; E. S. Langford, E 1829, E 1830, E 1831, E 1833; E. L. Magnuson, E 1842; Andrłej Makowski (Poland), E 1899; D. C. B. Marsh, E 1867; Norman Miller, E 1858; P. L. Montgomery, E 1904; Larry Ortloff & John Knox, E 1867; R. E. Pippert & L. W. Beineke, E 1852; J. M. Quoniam (France), E 1839, E 1840; Simeon Reich (Israel), E 1820, E 1821, E 1823, E 1824, E 1839, E 1840, E 1846; Arun Sanyal, E 1870; Lie Ann Tan, E 1839; P. D. Thomas, E 1863.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before June 30, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5540. *Proposed by G. F. Schumm, University of Chicago*

If X and Y are chains, then X is said to be *dense* in Y if for every $A, B \in Y$, $A \neq B$, there exists $F \in X$ such that $A \subset F \subset B$ (where $\subset$ denotes a proper subset). Prove or disprove that for an uncountable chain Z there exists a countable chain dense in it if and only if there exists a countable set T such that $(B-A) \cap T \neq \varnothing$ for all $A, B \in Z$, $A \subset B$. Does there exist for every cardinal number $\aleph$, a chain of cardinality $2^{\aleph}$ having a chain of cardinality $\aleph$ dense in it?

5541. *Proposed by Robert Breusch, Amherst College*

Find

$$\lim_{m\to\infty} \frac{m!}{m^m}\left(2\sum_{k=0}^{m}\frac{m^k}{k!} - e^m\right).$$

5542. *Proposed by Robert Breusch, Amherst College*

The first edition of Herstein, *Topics in Algebra* contains (p. 174) the following problem: "If $a, b \in K$ are algebraic over F of degree m and n respectively, and if m and n are relatively prime, prove that ab is algebraic over F of degree mn." [Here K is an extension field over the field F.] The problem has been deleted from the third edition because, as stated, it is incorrect.

Find counterexamples.

5543. *Proposed by U. Güntzer, Göttingen, and J. J. Hirschfelder, University of Notre Dame*

Let A be a commutative ring with 1, M an A-module, L a submodule of M.

trieved until too late to be acknowledged with other solutions of the various problems:

J. C. Abad, E 1851, E 1958; Leon Bankoff, E 1861; D. E. Daykin & T. K. Sheng (Malaysia), E 1844; A. H. Durfee, E 1858; Michael Goldberg, E 1840; M. G. Greening (Australia), E 1846; Stephen Hoffman, E 1840, E 1842; Erwin Just, E 1859; R. S. Kaluzniacki, E 1858, E 1859; B. W. King, E 1842; P. G. Kirmser, E 1859; E. S. Langford, E 1829, E 1830, E 1831, E 1833; E. L. Magnuson, E 1842; Andrłej Makowski (Poland), E 1899; D. C. B. Marsh, E 1867; Norman Miller, E 1858; P. L. Montgomery, E 1904; Larry Ortloff & John Knox, E 1867; R. E. Pippert & L. W. Beineke, E 1852; J. M. Quoniam (France), E 1839, E 1840; Simeon Reich (Israel), E 1820, E 1821, E 1823, E 1824, E 1839, E 1840, E 1846; Arun Sanyal, E 1870; Lie Ann Tan, E 1839; P. D. Thomas, E 1863.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed or written legibly on separate, signed sheets and should be mailed before June 30, 1968. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5540. *Proposed by G. F. Schumm, University of Chicago*

If X and Y are chains, then X is said to be *dense* in Y if for every $A, B \in Y$, $A \neq B$, there exists $F \in X$ such that $A \subset F \subset B$ (where $\subset$ denotes a proper subset). Prove or disprove that for an uncountable chain Z there exists a countable chain dense in it if and only if there exists a countable set T such that $(B-A) \cap T \neq \varnothing$ for all $A, B \in Z$, $A \subset B$. Does there exist for every cardinal number $\aleph$, a chain of cardinality $2^{\aleph}$ having a chain of cardinality $\aleph$ dense in it?

5541. *Proposed by Robert Breusch, Amherst College*

Find

$$\lim_{m\to\infty} \frac{m!}{m^m}\left(2\sum_{k=0}^{m}\frac{m^k}{k!} - e^m\right).$$

5542. *Proposed by Robert Breusch, Amherst College*

The first edition of Herstein, *Topics in Algebra* contains (p. 174) the following problem: "If $a, b \in K$ are algebraic over F of degree m and n respectively, and if m and n are relatively prime, prove that ab is algebraic over F of degree mn." [Here K is an extension field over the field F.] The problem has been deleted from the third edition because, as stated, it is incorrect.

Find counterexamples.

5543. *Proposed by U. Güntzer, Göttingen, and J. J. Hirschfelder, University of Notre Dame*

Let A be a commutative ring with 1, M an A-module, L a submodule of M.

5547. *Proposed by Anders Bager, Hjørring, Denmark*

Let $\rho(n)$ (resp. $\nu(n)$) be the number of prime divisors of the positive integer n counted with (resp. without) multiplicity. Prove that $\sum_{d|n}(-1)^{\rho(d)}2^{\nu(n/d)}=1$ for all n.

5548. *Proposed by E. J. Barbeau, The University of Western Ontario*

Let p be a prime, congruent to 1 (mod 4). Show that $\frac{1}{4}(p-1)$ and -4 are quartic residues (mod p), and that the fourth roots of $\frac{1}{4}(p-1)$ consist of two pairs of consecutive integers.

5549. *Proposed by Steve Ligh, University of Houston*

Let G be an additively written torsion free abelian group. H is any subgroup of G. If the automorphism γ of G/H that takes x onto $-x$ can be lifted to an automorphism α of G that is the identity on H, then G/H splits.

SOLUTIONS OF ADVANCED PROBLEMS

Points of Continuity

5436 [1966, 1019]. *Proposed by Seymour Metz, Computer Associates, Arlington, Va.*

Does there exist a real function on the real line whose points of continuity form a dense set of measure zero?

Solution by E. S. Langford, Naval Postgraduate School, Monterey, Calif. Yes. It is known that if X is any G_δ set, then there exists a function whose set of points of continuity is precisely X. (B. R. Gelbaum and J. M. H. Olmsted, *Counterexamples in Analysis*, Ex. 2.23, p. 30). However, a dense G_δ set of measure zero can be constructed via Cantor sets. (Ibid., Ex. 8.20, p. 99.)

Also solved by D. T. Adams, Duane Bailey, J. P. Celenza, Edgar Cohen, Jr., Carlos Contour-Carrère (Argentina), Leonard Gillman, Michael Goldberg, Robert Goldstein (England), W. L. Griffith, D. A. Hejhal, Ellen Hertz, G. A. Heuer, C. M. Joiner, Jr. & S. J. Pierce, M. D. Mavinkurve (India), P. R. Meyer, J. C. Morgan, Jr., Jürg Rätz (Switzerland), Charles Riley, G. F. Schumm, Al Somayajulu, G. W. Soules, Charles Widger, Albert Wilansky, Dieter Wolke (Germany), and the proposer.

Contour-Carrère proves the result of the problem for R^n, and proves that the collection of points of continuity may contain any prescribed countable set.

Goldstein, using the procedure in the solution above, proves the result: Let m be any nonnegative number. Then there is a real valued function $f(x)$ of the real variable x, whose set of continuity A is a dense set of measure m. Further, any such set A is of the second category.

Although the Editors realize that readers of the MONTHLY wait anxiously to see the solutions of problems, it is not the normal practice to offer a solution in the same issue which proposes the problem. When this situation does occur it merely serves as evidence of the independence of the individual editors in trying to bring out interesting material. We acknowledge, while reeling from the blows, the information from our readers that the solution for this problem was printed just twenty pages earlier in the article by D. B. Goodner, *On points of continuity*, p. 1000.

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Also solved by J. P. Celenza, Robert Goldstein (England), D. A. Hejhal, Albert Wilansky, and the proposer.

Several contributors point out that the result is contained implicitly in Kelley, *General Topology*; see p. 208.

See also the paper by Yu-Lee Lee and C. J. Mozzochi, *Topological spaces with large uniformities*, contributed to the American Mathematical Society, Abstract 66T-269 in the Notices.

Infinitely Differentiable Functions

5442 [1966, 1124]. *Proposed by J. T. Renfrow, California Institute of Technology*

Let $f(x)$ be an infinitely differentiable real-valued function defined on an open interval (a, b). If $f(x)$ vanishes infinitely often on a closed bounded subinterval $[c, d]$, $c \neq d$, and $\sup_{x\in(a,b)}|f^{(n)}(x)| = O(n!)$ as $n\to\infty$, then $f(x)$ vanishes identically on an open subinterval of (a, b).

I. *Solution by W. G. Dotson, Jr., North Carolina State University.* By the Bolzano-Weierstrass theorem, the set of zeros of f has a limit point p in $[c, d]$. Suppose k is a nonnegative integer such that the set of zeros of $f^{(k)}$ has p as a limit point. Then, since $f^{(k)}$ is continuous for all x [because $f^{(k+1)}$ exists for all x], $f^{(k)}(p) = 0$. By Rolle's theorem, $f^{(k+1)}$ must have at least one zero between two zeros of $f^{(k)}$; and so p is a limit point of the set of zeros of $f^{(k+1)}$. It follows that $f^{(k)}(p) = 0$ for all nonnegative integers k. Taylor's formula (with Lagrange's form of the remainder) yields

$$f(x) = \sum_{k=0}^{n-1} \{f^{(k)}(p)/k!\}(x - p)^k + \{f^{(n)}(\xi)/n!\}(x - p)^n,$$

x in (a, b), ξ between p and x. There exist $M>0$, $N>0$ such that if $n>N$ then $|f^{(n)}(t)/n!| \leqq M$ for all $t\in(a, b)$; and so for all $x\in(a, b)$ and all $n>N$ we have $|f(x) \leqq M\cdot|x-p|^n$. Thus, if $|x-p|<1$ then $f(x)=0$; and so f vanishes identically on the open subinterval $(\max\{a, p-1\}, \min\{b, p+1\})$.

II. *Solution by R. P. Boas, Northwestern University.* Under the hypothesis $\sup_{x\in(a,b)}|f^{(n)}(x)| = O(n!)$, direct estimation of the remainder in Taylor's series shows that $f(x)$ is the restriction to (a, b) of a function that is analytic on $(a-1, b+1)$. Hence if $f(x)$ vanishes infinitely often on (a, b) (not necessarily on a closed subinterval), it vanishes identically on (a, b). The hypothesis can be weakened to

$$\limsup_{n\to\infty} |f^{(n)}(x)/n!|^{1/n} \leqq 1$$

for each x, i.e. the radius of convergence of the Taylor series about each point in (a, b) is at least 1. (See Boas, Bull. Amer. Math. Soc., 41 (1935), 233–236, or Salzmann and Zeller, Math. Z., 62 (1955), 354–367).

Also solved by D. A. Hejhal, M. D. Mavinkurve (India), Amram Meir, Břetislav Novák, (Czechoslovakia), Stanton Philipp, Charles Riley, C. H. Sampson, Chang Sung-sheng (Taiwan, China), R. J. Weinacht, and the proposer.

The Intersection of Compact Sets in a T_1 Space

5444 [1966, 1124]. *Proposed by T. S. Frank, Le Moyne College, and W. A. Lopez, University of Wisconsin*

It is well known that the intersection of compact sets in a Hausdorff space is again compact. Prove or disprove the corresponding statement for T_1 spaces.

Solution by N. J. Kuenzi, Iowa City, Iowa. In a T_1 space the intersection of compact sets need not be compact. Let $[0, 1]$ be the unit interval and $1'$ a point not in $[0, 1]$. Let $I' = [0, 1] \cup \{1'\}$. Extend the ordering of $[0, 1]$ to a partial ordering of I' by defining $a < 1'$ for each $a \in [0, 1)$ and observe that 1 and $1'$ are not related by this ordering. Extend the usual topology of $[0, 1]$ to I' by defining basic neighborhoods of $1'$ to be sets of the form $(a, 1']$ for $a \in [0, 1)$. This space is easily seen to be T_1 (and not T_2), $[0, 1]$ and $[0, 1']$ are compact sets, but $[0, 1] \cap [0, 1'] = [0, 1)$ is not compact. Clearly I' is not Hausdorff.

Also solved by R. A. Alo, Stephen Baron, John Beidler, Andreas Blass, Sam Cox, Jr., S. W. Dharmadhikari, D. Ž. Djoković (Yugoslavia), Jerry Fischer, S. P. Franklin, R. C. Freiwald, C. F. Gibson, Neil Gray, D. M. Hancasky, G. A. Heuer, W. B. Johnson, J. E. Joseph, L. L. Krajewski, D. A. Marcus, D. A. Mattson, M. D. Mavinkurve (India), D. C. Mayne, W. G. McArthur, P. R. Meyer, Tassos Nakassis (Greece), Brětislav Novák (Czechoslovakia), F. J. Papp, D. R. Read, Charles Riley, S. M. Robinson, D. M. Rodriguez, L. E. Rogers, C. A. Ruiz (Argentina), P. S. Schnare, A. W. Schurle, Francis Siwiec, J. P. Smith, Al Somayajulu, R. H. Sorgenfrey, R. C. Steinlage & T. E. Gantner, P. R. Thie, J. P. Thomas, J. L. Tollefson, James Washenberger, D. A. Wick, J. B. Wilker, and the proposers.

Solvers cited the following references to similar counter-examples in the literature:

(1) E. Čech, Topologické Prostory, Prague, 1959, p. 211, ex. 8.5.11.
(2) A. Wilansky, Functional Analysis, p. 164, problem 31.
(3) N. Bourbaki, Topology, Part I (English translation) p. 142, Problem 5.
(4) J. L. Kelley, General Topology, p. 161, p. 47.
(5) J. Greever, Theory and Examples of Point Set Topology, Brooks-Cole, 1967.

See also *Editorial Note on "Between T_1 and T_2"*, this issue, page 1241.

A Number-theoretic Identity

5446 [1966, 1125]. *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India*

Let $\lambda(n)$ denote the number of distinct prime divisors of n and let $\Omega(n)$ be the total number of prime divisors of n. Prove that

$$\sum_{d|n} (-1)^{\Omega(d)} \phi(d) \sigma(n/d) = \sum_{k^2|n} k^2 \cdot 2^{\lambda(n/k^2)},$$

where $\phi(n)$ is the Euler function and $\sigma(n)$ is the sum of the divisors of n.

I. *Solution by Al Somayajulu, State University of New York, Buffalo.* Since both the left and right sides of the proposed identity are multiplicative functions we need only consider $n = p^\alpha$, p a prime, α a positive integer. If α is even

$$\sum_{d|p^\alpha} (-1)^{\Omega(d)} \phi(d) \sigma(n/d) = (1 + p + \cdots + p^\alpha) - (p-1)(1 + p + \cdots + p^{\alpha-1})$$

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where $\phi(n)$ is the Euler function and $\sigma(n)$ is the sum of the divisors of n.

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$$\sum_{d|p^\alpha} (-1)^{\Omega(d)} \phi(d) \sigma(n/d) = (1 + p + \cdots + p^\alpha) - (p-1)(1 + p + \cdots + p^{\alpha-1})$$

whose coefficients are generated by multiplicative functions:

$$G_p(x, f) = \sum_0^\infty f(p^k)x^k$$

where p is a prime. The set $\{G_p(x, f)\}$ as p ranges over the primes determines f; furthermore $G_p(x, f * g) = G_p(x, f)G_p(x, g)$.

Let μ represent the Möbius function, $n_\alpha(k) = k^\alpha$. We have

$$G_p(x, (-1)^\Omega f) = G_p(-x, f);$$

$$G_p(x, \phi) = G_p(x, \mu * n_1) = G_p(x, \mu)G_p(x, n_1) = \frac{1-x}{1-px};$$

$$G_p(x, (-1)^\Omega \phi) = \frac{1+x}{1+px};$$

$$G_p(x, \sigma) = G_p(x, n_0)G_p(x, n_1) = \frac{1}{1-x}\,\frac{1}{1-px};$$

$$G_p(x, \text{left side}) = \frac{1}{1-p^2x^2}\,\frac{1+x}{1-x}.$$

However, $G_p(x, 2^\lambda) = (1+x)/(1-x)$ and $f(k)$ [$=0$ or k according as k is nonsquare or square] clearly satisfies $G_p(x, f) = 1/[1-(px)^2]$. This completes the proof.

Note that this method was used by Richard Stanley in the solution of 5293 [1966, 553]. Also, in 5290 [1966, 548], the first part is verified by showing

$$G_p(x, t) = \left(\frac{1}{1-x}\right)^2 = \frac{1+x}{1-x}\,\frac{1}{1-x^2} = G_p(x, 2^\lambda)G_p(x, f),$$

where $f(n) = 0$ or $\sqrt{n}$ according as n is nonsquare or square.

Also solved by J. Galambos (Ghana), Ray Glenn, M. G. Greening (Australia), Torleiv Kløve (Norway), S. D. Lawn, Břetislav Novák (Czechoslovakia), Simeon Reich (Israel), Dieter Wolke (Germany), and the proposer.

Property of a One Line Reflection

5447 [1966, 1125]. *Proposed by H. A. Smith, Institute for Defense Analysis*

Let K be a compact subset of the plane having a nonvacuous interior. Let A be a linear transformation on the plane which maps K onto itself. Show that if A does not preserve orientation, then A^2 is the identity.

I. *Solution by Alan Berger, Student, Rutgers—The State University.* Denote $\{Ax \mid x \in K\}$ by $A(K)$, where A is the matrix of the given linear transformation relative to $\binom{1}{0}$ and $\binom{0}{1}$. Since A is real and $|A| < 0$, the eigenvalues λ_1 and λ_2 of A are real and of opposite sign, say $\lambda_1 < 0 < \lambda_2$, and there exists a real nonsingular matrix P such that $PAP^{-1} = J$ where

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Now J^n maps $P(K)$ onto $P(K)$, where n is a positive integer, because $(PAP^{-1})^n = PA^nP^{-1}$ and $PA^nP^{-1}[P(K)] = PA^n(K) = P(K)$. Also $P(K)$ is bounded. Now if $|\lambda_1|$ or $|\lambda_2| > 1$, we obtain a contradiction to $P(K)$ being bounded by taking n sufficiently large, noting K, and thus $P(K)$, contains a basis of $R \times R$. If both $|\lambda_1|$ and $|\lambda_2|$ are less than 1, we look at

$$PA^{-1}P^{-1} = J^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{bmatrix}$$

and obtain the same contradiction, for $PA^{-n}P^{-1}$ also maps $P(K)$ onto $P(K)$. Thus

$$J = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

so $PA^2P^{-1} = J^2 = I$ and $A^2 = I$.

II. *Solution by D. Ž. Djoković, University of Belgrade, Yugoslavia.* From $AK = K$ we get $m(K) = m(AK) = \pm \det A\, m(K)$ where m is the measure. Since K has interior points we have $m(K) > 0$, so that $|\det A| = 1$. Noting that A does not preserve the orientation we get $\det A = -1$. The eigenvalues λ_1, λ_2 of A are real and different, and $\lambda_1\lambda_2 = -1$. We take $\lambda_1 < 0$, $\lambda_2 > 0$. If $\lambda_2 \neq 1$ then either λ_1 or λ_2 has absolute value greater than 1 and the set $K = AK = A^2K = A^3K = \cdots$ cannot be compact. Hence $\lambda_1 = -1$, $\lambda_2 = 1$, which implies that A is a reflection and $A^2 = I$.

III. *Solution by H. Guggenheimer, University of Minnesota.* Since A is linear, $A(K) = K$ implies that A also is an automorphism of the convex hull of K. Hence, by Brouwer's fixed point theorem, A has a fixed point O. By corollary 3.1, p. 297, of Harry Levy, *Projective and Related Geometries*, A is the product of three affine line reflections. Since the lines are concurrent at O, the product can be reduced to one line reflection.

Also solved by E. R. Gentile (Argentina), L. F. Meyers, Charles Riley, and Peter Scherk.

Berger, Guggenheimer and Riley observe that the theorem is true if K is assumed merely to be bounded and have three noncollinear points.

The Trace of the Matrix A, $A^2 = I$

5448 [1966, 1125]. *Proposed by H. A. Smith, Institute for Defense Analysis*

Let A be an $n \times n$ matrix, $A^2 = I$, the identity, and $A \neq \pm I$. Show

$$(1)\ \mathrm{Tr}(A) \equiv n \pmod 2, \qquad (2)\ |\mathrm{Tr}(A)| \leqq n - 2,$$

where $\mathrm{Tr}(A)$ is the trace of A.

Solution by J. E. Vinson, The Boeing Company. Let λ be an eigenvalue of A with eigenvector u. We have $u = Iu = A^2u = A\lambda u = \lambda^2 u$ so that $\lambda = \pm 1$. Suppose exactly m of the n eigenvalues, λ_i $(i = 1, 2, \cdots, n)$, of A are -1. Then

$$|\operatorname{Tr}(A)| = \left|\sum_{i=1}^{n} \lambda_i\right| = |(n-m) - m| = |n - 2m|,$$

which gives (1). Since $A \neq \pm I$, the λ_i are not all of the same sign, i.e. $1 \leqq m \leqq n-1$. It follows that $|\operatorname{Tr}(A)| = |n-2m| \leqq n-2$.

Also solved by Alan Berger, Martin Billik & Eldon Hansen, W. J. Blundon, J. P. Celenza, Richard Citron, W. M. Conner, D. E. Crabtree, C. G. Cullen, D. Ž. Djoković (Yugoslavia), W. G. Dotson, Jr., Mary R. Embry, Peter Enis, J. D. Fulton, Philip Fung, Kenneth Gehner, E. R. Gentile (Argentina), M. G. Greening (Australia), Richard Guilfoyle, J. R. Kuttler & Nathan Rubinstein, T. L. Markham, D. A. Marcus, M. D. Mavinkurve (India), R. K. Meany, L. F. Meyers, G. L. Musser, Walter Noll, Stanton Philipp, S. J. Pierce, E. J. F. Primrose (England), P. V. Subba Rao & B. Ramachandra Rao (India), Charles Riley, R. F. Rinehart, C. C. A. Sastri, Perry Scheinok, Peter Scherk, Stanisław Sedziwy (Poland), Sidney Spital, Arthur Steger, Zoran Stojaković (Yugoslavia), J. E. Vinson, and Jože Vrečko (Yugoslavia).

Uniform Continuity on a Product Space

5449 [1966, 1125]. *Proposed by G. R. Sell, University of Minnesota*

Let $f: K \times R \to R$ be a bounded, continuous function, where R denotes the real numbers and K is a compact set in R. It can be easily shown that if f is uniformly continuous, then f satisfies the following condition:

(U): For every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x, t) - f(x, \tau + t)| \leqq \epsilon, \qquad ((x, t) \in K \times R),$$

whenever $|\tau| \leqq \delta$. Is the converse true? That is, does condition (U) imply that f is uniformly continuous?

Solution by J. R. Isbell, Case-Western Reserve University. No. For instance, let $K = [0, 1]$, let f by 1 above $xt = 1$, let f be 0 on the axes, and take f linear on vertical segments below the hyperbola. We have $f \equiv xt$ if $xt \leqq 1$, and $|f(x, t) - f(x, t+\tau)| \leqq |\tau|$. Finally $f(1/t, t) - f(0, t) = 1$ and f is not uniformly continuous on $K \times R$.

Condition (U) is equiuniform continuity of the functions f_x, where $f_x(t) = f(x, t)$. The general condition for uniform continuity on a product space is this and equiuniform continuity of the similarly defined f^t (III. 21 of *Uniform Spaces*, Amer. Math. Soc. Math. Surveys, no. 12, Providence, 1964). Compactness of K makes uniform continuity of the f_x and equiuniform continuity of the f^t suffice (*ibid.*, III. 24). Neither suffices alone, because of t^2 and the function above.

Also solved by W. M. Causey, D. A. Hejhal, Jan Hejeman (Czechoslovakia), G. A. Heuer, H. E. Lahmann (Germany), M. D. Mavinkurve (India), Amram Meir, Tassos Nakassis (Greece), Břetislav Novák (Czechoslovakia), Charles Riley, Peter Scherk, F. E. Siwiec, and J. B. Wilker.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

Materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto, 5, Canada. Correspondence about reviews is welcomed.

Algebra. By Saunders MacLane (University of Chicago) and Garrett Birkhoff (Harvard University). Macmillan, New York, 1967. xix+598 pp. $11.95.

It is natural to compare this book with the authors' *A Survey of Modern Algebra* since it is directed to the same audience. Most of the topics covered in the earlier book appear in some form in the present book; there is one notable exception: Galois Theory is omitted. The following chapters are primarily concerned with topics not included in *Survey:* VI Modules, XV Categories and Adjoint Functors, XVI Multilinear Algebra. Chapter I Sets, Functions, and Universal Elements has no real counterpart in the earlier book.

Despite the large intersection of topics covered in *Survey* and in *Algebra*, the new book is not just an updating of the old. The treatment of many topics and the general tone differ greatly from that of *Survey*. The notation and terminology are "categorical," as are many proofs. Universal properties and duality are introduced in the first chapter and play an essential role throughout most of the book. Many concepts are defined in terms of, or are related to, universal mapping properties; and these properties are used when the concepts are applied.

The linear algebra is distinctly module oriented, in marked contrast to *Survey*. For example, the rational and Jordan canonical forms of a matrix are deduced from the decomposition theorem for modules of finite type over a principal ideal domain.

The level of abstraction is much greater than that of *Survey*, and in the reviewer's opinion students will find it more difficult. But a course from this book should provide the talented and well-motivated student an excellent introduction to the study of algebra.

A. C. MEWBORN, University of North Carolina at Chapel Hill

Introduction to Contemporary Mathematics. By Sze-Tsen Hu. Holden-Day, San Francisco, 1966. viii+202 pp. $7.95.

Awareness of the existence of the infinitesimal calculus is not a prerequisite for reading and understanding this book. However, some mathematical maturity is required for completion of these tasks. Moreover, this requirement is accentuated by the numerous typographical and other errors which appear in almost all parts of the book.

For example, the first entry in the TABLE OF CONTENTS (Page vii) is "Special Symbols and Abbreviations . . . ix." Unfortunately, the review copy contained no page numbered ix, and such a list of symbols and abbreviations is sorely needed. The five chapters are titled Sets and Functions, Number Systems, Algebraic Structures, Geometries, and Elementary Topology, respectively. There is faithful observance of this list and its suggested coverage. On page 20 appears the definition: "Any two functions f and g are said to be *composable* [to form $h=g\circ f$] iff the range of f is equal to the domain of g; . . ." Later, on the same page, appears the example: ". . . let f, g: $R\to R$ denote the functions defined by $f(x)=2x+3$, $g(x)=x^2$ for every $x\in R$. Then the compositions $h=g\circ f$ and $k=f\circ g$ are both defined; · · · ."

After reading this book, one may have the impression that Mr. Hu's "exciting success" in using the manuscript as a text "in a very heterogeneous class of about forty students" constitutes a bright compliment to his effectiveness as a teacher.

WADE ELLIS, University of Michigan

A New Look at Geometry. By Irving Adler, with diagrams by Ruth Adler. John Day, New York, 1966. 414 pp. $7.95.

The aim of this book is to present the reader with a broad view of geometry. It is directed at the layman, or the high school teacher, or perhaps the bright high school student. The author proceeds to his task with a generous sprinkling of historical, philosophical, and scientific allusions. (How one wishes for a Society of Zeno Followers which would exist only to refute Zeno's refuters!) Much pre-Euclidean, Euclidean, and non-Euclidean geometry is presented. Along the way, analytic and vector geometry are developed, and there is a neat, group-theoretic chapter on the algebraic-geometric properties of isometric involutions of the plane. Although no calculus background is expected, discussions of differential geometry and of topology are attempted in separate chapters. Synthetic projective geometry is extensively treated, even to the extent of constructing, in nine unmotivated pages, the underlying field of scalars for an abstract projective plane.

The result is a strange brew. So much undigested, new material is introduced that the reader for whom this book was apparently intended will inevitably be overwhelmed and confused. Included are groups and fields, the infinite series for the hyperbolic and ordinary sine and cosine functions, infinite cardinals, differential and integral calculus, and the axioms for a topological space. In all cases, the motivation and the uses of these topics are quite limited, and frustration would seem to be the inevitable consequence.

In brief, this is an interesting attempt to present geometry in depth. It fails, however, because it does not succeed in being both elementary and comprehensive.

MELVIN HAUSNER, New York University

Lectures on the Foundations of Geometry. By A. V. Pogorelov (Kharkov State University). Translated from the second Russian edition by Leo F. Boron with the collaboration of Ward D. Bouwsma. Noordhoff, Groningen, Netherlands, 1966. 137 pp. $6.90.

After a brief historical sketch (Chapter I), an axiomatization of three-dimensional euclidean geometry is presented in Chapter II, based on five groups of axioms (incidence, order, motion, continuity, and parallelism). In contrast with the five axiom groups of Hilbert, the order axioms are in terms of an order relation for pointpairs, Hilbert's congruence axioms are replaced by seven axioms for motion, and the fourth group contains only one axiom of continuity (Dedekind). The author is able to reach important theorems (e.g., Pythagoras) with rapidity and ease. The axioms themselves are studied in Chapter III. Their relative consistency follows from their cartesian realization, which is established in all detail, and the completeness of the axiom system is proved by showing that every model is isomorphic with the cartesian realization. Independence of the continuity and the parallel axioms is established. On the other hand, three of the seven axioms of motion are derived as theorems (and hence those axioms may be deleted from the system). Hyperbolic geometry (Chapter IV) is dealt with in a fresh way by deriving the planar linear element in absolute geometry (i.e., without making any assumptions concerning parallels). The axioms for hyperbolic geometry are shown to be complete. The concluding Chapter V presents a foundation for projective geometry, with order axioms (in terms of a three-point relation) that permit a rigorous basis for the theory of vectors. Relative consistency and completeness of the axiom system are established.

This small book manages to develop quite rigorously a large amount of material. It is very clearly written, the figures are well-drawn, and no typographical errors were noted. The book is especially well adapted for Summer Institute courses on the foundations of geometry; it is suitable reading for all interested in increasing their general mathematical culture. We interpret as humor the author's warning (p. 30) that a group of axioms is not a group in the mathematical sense!

L. M. BLUMENTHAL, University of Missouri

Linear Algebra. By Serge Lang. Addison-Wesley, Reading, Mass., 1966. x+294 pp. $8.95.

This text was used by each reviewer in his undergraduate class in linear algebra. Its most notable feature is the neatness and elegance of the presentation. An effort is made to prove each theorem as efficiently as possible; computational proofs are avoided in favor of conceptual proofs. This feature is perhaps more appreciated by the instructor than by the students, who did not seem to be able to learn more from this text than from the one it replaced.

The first eleven chapters of the book treat the material usually included in an undergraduate course in linear algebra and can just about be covered in a single semester with an average class. Lang considers only vector spaces over subfields of the complex numbers; the great majority of the proofs carry over to vector spaces over general fields.

The reviewers found Chapter VI on determinants to be unsatisfactory. Lang utilizes Artin's elegant approach and develops determinants from their properties. The student needs an explanation and outline of this approach. Lang attempts to summarize everything in one main theorem (Theorem 1) at the start of the chapter. The proof of this main theorem is actually not completed until Section 6 of the chapter; however, this is not made clear to the student.

Proofs of the theorems concerning the rank of a matrix are exceptionally well done. These theorems are treated as corollaries of theorems on the rank of a linear transformation and the dimension of the orthogonal complement of a subspace under a nondegenerate scalar product.

The book contains a nice combination of routine, computational problems and exercises requiring more ingenuity. While the book is not free from errors, there are not an unusual number for a first edition. The reviewers consider this to be an above average linear algebra text, which, however, could be improved.

R. D. KOPPERMAN AND J. B. FRALEIGH, University of Rhode Island.

La Régularité Moyenne dans la Théorie Métrique. By W. J. Trjitzinsky. Mémor. Sc. Math., fasc. 157, Gauthier-Villars, Paris, 1965. 88 pp. NF26.

The origin of this research monograph lies in the generalization by A. Denjoy of the classical Vitali covering theorem. The hypothesis of Vitali's theorem involves the topology of the space while the conclusion is purely measure-theoretic. Denjoy approached the problem by removing the topology; as a simple example, instead of a sequence of sets with diameter approaching zero, there is substituted a sequence with measure approaching zero. This direction was pursued in a series of papers and further generalized by Trjitzinsky [Théorie métrique . . . , Mémor. Sc. math., fasc. 143, 1960]. The volume under review continues this generalization.

The first third of the monograph involves proofs of covering theorems under less restrictive hypotheses than previously assumed in the literature and constructions of examples to show the new assumptions are indeed weaker. Then these theorems are applied to obtain generalizations of classical results in the Lebesgue theory. Thus under suitable assumptions, the density (relative to a family of covering sets) of a measurable covered set is almost everywhere equal to the characteristic function. Further results deal with differentiation of set functions, the Lebesgue decomposition theorem, and analogues of the Vitali-Carathéodory and Lusin theorems. A final section deals with cellular divisions of an abstract space; these divisions lead to a metric for the space but the present volume does not exploit the idea any further.

Since many proofs are only indicated as modifications of proofs that appear in the author's previous monograph, it is recommended that both volumes be read together. As may be inferred, this book is not meant for the casual reader, but for the specialist it represents an interesting approach to a problem of intrinsic interest.

GERALD FREILICH, The City College, New York

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GERALD FREILICH, The City College, New York

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 15 (junior level)—18 (second graduate year).

Séminaire Bourbaki. Volume 1965/1966. Exposés 295–312. Benjamin, New York, 1966. 311 pp. The thirteenth (and first current) volume of the series covering the papers presented at the Paris seminars presided over by the redoubtable N. Bourbaki. L.

Romanian-English Dictionary and Grammar for the Mathematical Sciences. By S. H. Gould (American Mathematical Society) and P. E. Obreanu (Queen's University, Kingston). American Mathematical Society, 1967. v+51 pp. $7.10 ($5.33 to members of AMS). Designed for English-speaking mathematicians, it includes a basic Romanian grammar. P, L.

Dictionary of Physics and Mathematics Abbreviations, Signs and Symbols. Edited by David D. Polon. Odyssey, New York, 1965. xvi+333 pp. $20.00. Includes much related material such as abbreviations for learned societies. L.

Analysis

Introduction to the Operational Calculus. By Lothar Berg (University of Halle). North-Holland, Amsterdam (Wiley, New York, in U.S.A. and Canada), 1967. x+294 pp. $12.00. A translation of the second German edition with the insertion of exercises by the author. The modern Mikusinski approach is linked with the traditions of Heaviside and Laplace transforms. T (15–16), S, P.

Constructive Real Analysis. By Allen A. Goldstein (University of Washington). Harper, New York, 1967. xii+178 pp. $9.25. "My object in this book is to introduce students of mathematics, science, and technology to the methods of applied functional analysis and applied convexity. To demonstrate and illustrate these techniques I have applied them to the related problems of finding zeros and extremals of functions and operators, with and without side conditions. For students of mathematics the book can perhaps be regarded as an initiation to 'hard analysis,' while for students of science, an initiation to 'soft analysis'." Prerequisites are courses in modern advanced calculus and linear algebra. T(16–17), S, P.

Differential Operators of Mathematical Physics. By Günter Hellwig (Technische Hochschule Aachen). Translated from the German by Birgitta Hellwig. Addison-Wesley, Reading, Mass., 1967. viii+296 pp. $12.50. An introduction to differential operators for students with advanced calculus and ordinary differential equations. The main topics are Hilbert space, linear operators, spectral theory of completely continuous operators, and the Weyl-Stone Eigenvalue Problem. There are very few problems. T(16), S, P.

Elements of Real Analysis. By Sze-Tsen Hu (University of California, Los Angeles). Holden-Day, San Francisco, 1967. xii+365 pp. $10.95. With this book the author keeps up his amazing output of one advanced level book per year since 1959. This one covers most of the material recommended by CUPM: sets, functions and relations, the real line, topological spaces, metric spaces, topological linear spaces, measures and integrals (including the Lebesgue and Daniell approach), differentiation. T (16–17).

Multiple Integrals. By Walter Ledermann. Dover, New York, 1966. vi+106 pp. $1.00 (paper). One of a series of inexpensive paperbacks "primarily intended for readers who study mathematics as a tool" and aiming "to cover the topics which are usually included in courses of mathematics for scientists, engineers and statisticians . . . " It includes the traditional material on line, double, surface, and volume integrals. Existence theorems are stated but not proved. Other related volumes in the series, all priced the same, are *Differential Calculus* by P. J. Hilton, *Integral Calculus* by W. Ledermann, *Partial Derivatives* by P. J. Hilton, *Elementary Differential Equations and Operators* by G. E. H. Reuter, *Sequences and Series* by J. A. Green, *Differential Geometry* by K. L. Wardle, *Fourier Series* by I. N. Sneddon, and several others. A selection might be used to cover a course with greater convenience and lower cost than by using the usual massive tomes. T (13–14), S.

Theory of Functions of a Complex Variable. Volume III. By A. I. Markushevich (Moscow State University). Revised English edition translated and edited by R. A. Silverman. Prentice-Hall, Englewood Cliffs, N. J., 1967. xi+360 pp. $12.95. The last volume of this very substantial course in complex variables, which includes many examples, problems, illustrations, and bibliographic references. Topics here covered are conformal mapping, approximation theory, periodic and elliptic functions, Riemann surfaces, analytic continuation. T (17), S, T.

Commutation Properties of Hilbert Space Operators and Related Topics. By C. R. Putnam (Purdue University). Springer-Verlag, New York, 1967. xi+167 pp. $7.00. Bounded operators, spectral theory, semi-normal operators, quantum mechanics, wave operators, unitary equivalence, Laurent and Toeplitz operators, singular integral operators and Jacobi matrices, substantial bibliography and indexes. P.

Modern Nonlinear Equations. By Thomas L. Saaty (Conference Board of the Mathematical Sciences). McGraw-Hill, New York, 1967. xv+473 pp. $15.50. Primarily a reference book, covering operator, functional, difference, delay-differential, integral, integrodifferential, and stochastic differential equations. There are two chapters on basic concepts and techniques, bibliographies at the end of each chapter, but no problems. Together with *Nonlinear Mathematics* by Saaty and Joseph Bram (McGraw-Hill, 1964, reviewed in this MONTHLY 73, 323), which covered algebraic and differential equations, this volume completes the coverage of "all the major types of classical equations except partial differential equations." T (17–18), S, P, L.

An Introduction to Analysis. By Wilson M. Zaring (University of Illinois). Macmillan New York, 1967. xi+364 pp. $9.95. This text "was designed specifically for the prospective teacher of calculus," and has been used for several years in Academic Year Institutes for high school teachers at the University of Illinois. It consists of Part I on logic and set theory followed by Part II, an introduction to analysis. T (13–15), TT.

Applications

The Theory of Splines and Their Applications. By J. H. Ahlberg (United Aircraft Research Laboratories), E. N. Nilson (Pratt and Whitney Aircraft Company), and J. L. Walsh (University of Maryland). Academic Press, New York, 1967. xi+284 pp. $13.50. A treatise on a field that dates from 1946. "The mathematical spline is the result of replacing the draftsman's spline by its elastica and then approximating the latter by a piecewise cubic. . . . " P.

Mathematical Reasoning in Economics and Management Science, Twelve Topics. By John C. G. Boot (SUNY at Buffalo). Prentice-Hall, Englewood Cliffs, N. J., 1967. xii+178

pp. $7.95. The twelve topics are characteristic roots and vectors, difference equations, assorted problems in probability theory, Markov chains, decision criteria and utility curves, game theory, strategies, dynamic programming, sensitivity analysis, elementary inventory models, growth models, and Leontief models. In each the mathematical theory and applications are given. Coverage is not intended to be deep, but there are references for further study. S, P, L.

Decision Rules for Inventory Management. By Robert Goodell Brown (Arthur D. Little, Inc.). Holt, Rinehart, and Winston, New York, 1967. x+398 pp. $9.95. A mathematical treatise in the form of technical memoranda written by Roger ("a new type of business student . . . with an engineering degree and a strong interest in mathematics . . . ") for his superiors in the Warmdot Company. S, P.

Foundations of Mechanics. A mathematical exposition of classical mechanics with an introduction to the qualitative theory of dynamic systems and applications to the three-body problem. By Ralph Abraham (Princeton University) with assistance of Jerrold E. Marsden (Princeton University). Benjamin, New York, 1967. ix+296 pp. $14.75. This unusually attractive and interesting looking book starts with a "Museum" of twenty-nine portraits, including several contemporaries who are very far from being relegated to any museum, continues with a brief historical introduction to the main body consisting of revised notes on a series of lectures "aimed at recent mathematical results in mechanics, especially the work of Kolmogorov, Arnold and Moser and its applications to Laplace's question of the stability of the solar system," and concludes with appendices by Marsden, Al Kelly, and Kolmogorov. There are exercises and several bibliographies. The pre-requisites are "the mathematical background of a physics graduate student." Anyone who imagines that mathematics has cut itself off from the traditional interaction with physics and astronomy should look at this treatment of classical mechanics in the language of modern differential geometry. T (16–17), S, P, L.

Prague Studies in Mathematical Linguistics, Vol. I. Edited by Lubomir Dolezel, Petr Sgall, and Josef Vachek. University of Alabama Press, 1966. 240 pp. $8.50. This is the first of a series published under the auspices of the Czechoslovak Academy of Sciences. It contains nineteen papers on statistical linguistics, algebraic linguistics, and machine translation, fourteen in English and five in Russian. P, L.

Gyroscopic Theory. By Sir G. Greenhill. Chelsea, Bronx, New York, 1966. viii+275 pp. $9.50. Reprint of the original report (London, 1914) "intended to serve as a collection in one place of the various methods of the theoretical explanation of the motion of a spinning body." P, L.

Advanced Engineering Mathematics. By Erwin Kreyszig (Ohio State University). 2nd ed. Wiley, New York, 1967. xvii+898 pp. $11.95. The usual topics including probability and statistics, special functions, complex variables, partial differential equations and the Laplace transformation. T (15).

Scattering Theory. By Peter D. Lax (New York University) and Ralph S. Phillips (Stanford University). Academic Press, New York, 1967. xii+276 pp. $12.00. "Scattering theory compares the asymptotic behavior of an evolving system as t tends to $-\infty$ with its asymptotic behavior as t tends to $+\infty$." This monograph presents a new approach to scattering theory for hyperbolic differential equations. Tools used are spectral theory, harmonic analysis, semi-groups, and various aspects of partial differential equations. T (17–18), S, P.

Blanch Anniversary Volume. A series of papers presented on the occasion of her retirement by the friends of Gertrude Blanch. Edited by Bertram Mond. (Aerospace Research Laboratories, Wright Patterson AFB, Ohio) United States Air Force, 1967. In an anniversary volume of this sort one has a right to expect some biographical information, a bibliography, possibly a portrait. Except for less than a page of rather vague praise for Dr. Blanch, this volume consists of a miscellaneous collection of papers whose relationship to the person being honored is not indicated. P.

Finite Deformation of an Elastic Solid. By Francis D. Murnaghan. Dover, New York, 1967. 140 pp. $1.85 (paper). An unaltered reprint of the original 1951 publication, dedicated by the author "to the Glory of God, Honor of Ireland, and solidarity of the Americas." T (16), S, P.

An Introduction to Magnetohydrodynamics. By P. H. Roberts (School of Mathematics, University of Newcastle upon Tyne). American Elsevier, New York, 1967. x+264 pp. $11.00. Devoted to the mathematics of the subject but with some attention to problems of application and significance. P.

Introduction to the Theory of Operational Research. By B. van der Veen. PhilipsTechnical Library and Springer-Verlag, New York, 1967. vii+204 pp. $9.20. A non-technical introduction to "the mathematical and operational aspects" of queueing problems, stocks, industrial dynamics, multi-stage decisions, dynamic programming, linear programming, game theory, network planning, operations research and decisions. S (15), P.

Education

An Introduction to Fibonacci Discovery. By Brother U. Alfred (St. Mary's College, California). The Fibonacci Association, San Jose State College, San Jose, Calif., 1965. 56 pp. $1.50 (paper). A reprint of the pamphlet in response to its wide use as a device for stimulating student activity in mathematics. It deals briefly with many topics, including mathematical induction, shift formulas, the Fibonacci and Lucas sequences, division properties, generalized Fibonacci sequences, Pascal's triangle, the golden section, matrices, and continued fractions. It may be ordered from Brother U. Alfred, St. Mary's College, California 94575, at bulk prices of $1.25 each for from 4–9 copies, $1.00 each from 10–24 copies, and $0.75 each for 25 or more copies.

Number Systems: An Elementary Approach. By J. Richard Byrne (Portland State College). McGraw-Hill, New York, 1967. xiii+291 pp. $6.95. Primarily for future elementary teachers, this book ends by defining a real number "as an infinite sequence of rational numbers which can be represented by an infinite decimal," states that the set of real numbers is complete, but omits the proof "because it is beyond the scope of this book." TT.

Teaching Modern Mathematics in the Elementary School. By Howard F. Fehr (Teacher's College, Columbia University) and Jo McKeeby Phillips (University of Illinois Curriculum Laboratory). Addison-Wesley, Reading, Mass., 1967. xvi+448 pp. $7.95. Beginning and ending with chapters on general pedagogical questions, this book deals with arithmetic and geometry in the spirit of the "new math." Included are chapters on teaching ratio, proportion, and per cent, problem solving and enrichment topics. TT.

Mathematics for Elementary School Teachers. By Helen L. Garstens and Stanley B. Jackson (University of Maryland). Macmillan, New York, 1967. xii+500 pp. $9.95.

Mathematical reasoning, whole numbers, mathematical structure, sets and geometry, congruence in geometry, the integers, the rationals, linear measurement and coordinates, angle, area, volume, coordinates and similarity, plus three appendices on the pyramid volume formula, spherical geometry, geographical maps, and "a geometrical fantasy" (a universe on a spherical surface). TT, P.

Exploring University Mathematics 1, Lectures Given at Bedford College, London. By P. Chadwick, J. H. E. Cohn, H. G. Eggleston, J. R. Ellis, G. T. Kneebone, M. Levison, S. J. Taylor. Edited by N. J. Hardiman. Pergamon, New York, 1967. x+120 pp. $2.95 (paper), $4.50 (cloth). Though addressed to secondary school students, these lectures contain interesting observations on such matters as the history of foundations, relativity, digital computers, and the isoperimetric problem. S, P, L.

International Study of Achievement in Mathematics, a comparison of twelve countries. Edited by Torsten Husen. Wiley, New York, 1967. Two volumes, 303+368 pp. $19.95. This is a report on phase one of the International Project for the Evaluation of Educational Achievement. It has already received considerable publicity (The U.S. ranks tenth in mathematics achievements of 13-year olds.) and deserves more thoughtful analysis and discussion than can be given in the book review section. See *Science* 30 June 1967 for a critical review. P, L.

Report of the International Clearinghouse on Science and Mathematics Curricular Developments, 1967. Edited by J. David Lockard. A joint project of the American Association for the Advancement of Science and the Science Teaching Center, University of Maryland, College Park, Md., 1967. xxxvii+413 pp. Single copies free to individuals on request to the editor, International Clearinghouse on Science and Mathematics Curriculum Developments, Science Teaching Center, University of Maryland, College Park, Md. 20740. This describes in detail a large number of projects and has indexes by geographical location, name of director, level (including college and miscellaneous) and subject matter. P, L.

Research in Mathematics Education. National Council of Teachers of Mathematics, Washington, D. C., 1967. vi+125 pp., (paper) $2.00. This publication, which might be the precursor of a journal for research in mathematics education, contains eleven papers, three of which survey the field and argue for its importance, and eight that report on recent research. The lead article by Patrick Suppes is an eloquent plea for basic research in mathematics education in order "to understand how students learn mathematics, and to use this understanding to outline more effective ways of organizing the curriculum." P, L.

In-Service Education in Elementary School Mathematics. The National Council of Teachers of Mathematics, Washington, D.C., 1967. iv+55 pp. $.90 (paper). This informative booklet for all those interested in in-service programs for teachers includes a bibliography and a list of addresses in each state from which information can be obtained. TT, L.

Theory of Arithmetic. By John A. Peterson (University of Montana) and Joseph Hashisaki (Western Washington State College). 2nd ed. Wiley, New York, 1967. xiv+337 pp. $7.50. This revision of a book first published in 1963 is motivated by the fact that future teachers are coming to mathematics courses with a more adequate background. It begins with a brief history of numeration and ends with the real numbers and topics from geometry. TT.

mation on numerical methods and statistics. A basic knowledge of computer programming is assumed. P.

Multivariate Analysis. Proceedings of an International Symposium held in Dayton, Ohio, June 14–19, 1965. Edited by Paruchuri R. Krishnaiah (Wright-Patterson Air Force Base). Academic Press, New York, 1966. xix+592 pp. $19.50. Contributions by thirty-nine authors covering both methodology and applications in a very wide range. P, L.

Combinatorial Methods in the Theory of Stochastic Processes. By Lajos Takacs (Case Institute of Technology). Wiley, New York, 1967. xi+262 pp. $12.75. This treatise contains largely original work of the author. It aims to show how a generalization of the "classical ballot theorem" can be used to solve a very wide class of problems. P.

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Acknowledgement. The following have generously helped in evaluating books: T. M. COVER, T. EVANS, R. C. N. HOURSTON, W. LOUD, S. M. ROBINSON, D. ROSEN, P. SCHERK, R. SINGLETON, D. H. SKYPEK, E. F. WILDE.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor E. F. Myers, University of Pittsburgh, represented the Association at the inauguration of President A. A. Blum of Point Park College on June 18, 1967.

Memphis State University: Mr. M. L. Heen has been promoted to Assistant Professor; Associate Professors E. B. McBride and D. T. Walker have been promoted to Professors.

SUNY at Buffalo: Associate Professor K. D. Magill, Jr., has been promoted to Professor and appointed Chairman of the Mathematics Department for 1967–68; Assistant Professor S. R. Cavior has been promoted to Associate Professor; Dr. R. G. Bauer, University of Massachusetts, and Dr. Margaret G. Piech, Cornell University, have been appointed Lecturers; Mr. C. K. Chui, University of California, San Diego, and Dr. Vladimir Drobot, University of Illinois, have been appointed Assistant Professors; Dr. V. L. N. Sarma, Hindu University, has been appointed Visiting Associate Professor; Professor Dov Tamari will be on leave at the Institute for Advanced Study at Princeton.

Associate Professor F. T. Birtel, Tulane University, has been promoted to Professor.

Dr. H. P. Edmundson, University of Maryland, has been appointed Professor of Computer Science and Mathematics at the University of California, Los Angeles.

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MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MARCH MEETING OF THE METROPOLITAN NEW YORK SECTION

The annual meeting of the Metropolitan New York Section of the MAA was held at Long Island University, Zeckendorf Campus, on March 18, 1967. Dr. Meyer Jordan, Vice-Chairman of the section for Senior Colleges, presided. There were approximately 120 persons in attendance.

At the business meeting, the following officers were elected: Chairman, Professor Meyer Jordan, Brooklyn College; Vice-Chairman for Senior Colleges, Professor Joseph Houle, Pace College; Vice-Chairman for Junior and Community Colleges, Professor Norman Schaumberger, Bronx Community College; Vice-Chairman for High Schools, Dr. Ira Ewen, Springfield Gardens High School; Secretary, Professor Theresa J. Barz, St. John's University; Treasurer, Dr. Aaron Shapiro, Midwood High School.

Also at the business meeting, Dr. Abraham Schwartz reported on the meetings of the Board of Governors, Dr. C. T. Salkind and Dr. N. Schaumberger reported on the activities of the Contest Committee and the Speakers' Bureau, respectively, while Dr. A. Shapiro presented the treasurer's report.

The following papers were presented:

1. *Probability methods in various branches of mathematics*, by Mark Kac, The Rockefeller University (by invitation).

2. *One hundred years to square a logarithm*, by Harold Shapiro, New York University.

3. *Calculus for secondary school teachers*, by Lester Gavurin, Brooklyn College.

4. *Some reactions to CUPM proposals*, by J. E. Houle, Pace College.

5. *An experimental program in freshman calculus*, by Howard Levi, Hunter College.

6. *The effectiveness of a programmed text for mathematics fundamentals—an experiment at Pace College*, by Maurice Nadler, Pace College.

7. *Freshman experimental course in modern mathematics*, by Maurice Richter, Staten Island Community College.

8. *Mathematics for prospective elementary school teachers*, by Margaret Woodbridge, Brooklyn College.

THERESA J. BARZ, *Secretary*

JUNE MEETING OF THE NORTHEASTERN SECTION

The spring meeting of the Northeastern Section of the Mathematical Association of America was held at Mount Allison University, Sackville, New Brunswick on June 24, 1967. Thirty-five people attended the meeting including twenty members of the Association. Professor Robin Robinson, Chairman of the Section, presided at both the morning and afternoon sessions.

The following program was presented:

1. *Random packing of spheres and applications*, by E. M. Tory, Mt. Allison University.

2. *What is a calculation?* by Joachim Lambek, McGill University.

3. *The Sylvester Theorem and some of its applications*, by Michael Edelstein, Dalhousie University.

4. *Knotting, nylon, and nucleic acids*, by E. M. Brown, Dartmouth College.

G. W. BEST, *Secretary-Treasurer*

JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The annual meeting of the Pacific Northwest Section of the MAA was held at the University of Montana, Missoula, Montana on June 16 and 17, 1967. There were 67 members of the Association in attendance. Professor W. R. Ballard, Chairman of the Section, presided over the Friday sessions which were held jointly with the Northwest Section of SIAM. Mr. John Knutson presided over the Junior and Community College Sessions.

At the business meeting the following officers were elected: Chairman, C. T. Long, Washington State University; Vice-Chairman, Ronald Harrop, Simon Fraser University; Second Vice-Chairman, Theodore White, Everett Junior College; Secretary-Treasurer, E. A. Maier, University of Oregon. The position of Second-Vice-Chairman is a new office in the Section. The duties of the office include serving as representative of the two-year colleges to the Executive Committee, arranging for the program of the Junior and Community College Session at the annual Section meeting and presiding over these sessions.

The program of the joint sessions with SIAM was as follows:

1. *Bounds on the eigenvalues of nonhomogeneous strings*, by D. C. Barnes, Washington State University (by invitation).

2. *Report of the ad hoc subcommittee of CUPM on the qualifications of college teachers of mathematics*, by L. J. Paige, University of California at Los Angeles.

3. *Structure theorems for Banach Algebras*, by Bertram Yood, University of Oregon (by invitation).

4. *The application of biharmonic eigenfunctions to a plate problem*, J. W. Weber, Washington State University (by invitation).

The program of the Junior and Community College session was as follows:

1. *Report on the ad hoc committee of the NCTM on two-year college mathematics*, by Theodore White, Everett Junior College.

2. *Practical approach to trigonometry*, by David Fine, Everett Junior College (by invitation).

3. *How to put a number scale on the Euclidean line?* by S. A. Jennings, University of Victoria (by invitation).

4. *Notes in the margin of a calculus book*, by H. E. Reinhardt, University of Montana (by invitation).

5. *The use of audio-visual materials in the teaching of mathematics*, by Norman Barton, Vancouver City College (by invitation).

6. *Circular functions from a square viewpoint*, by James Van Dyke, Portland Community College (by invitation).

E. A. MAIER, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-first Annual Meeting, San Francisco Hilton, California, January 25–27, 1968.

Forty-ninth Summer Meeting, University of Wisconsin, Madison, Wisconsin, August 26–28, 1968.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Indiana University of Pennsylvania, Indiana, April 27, 1968.

FLORIDA, Miami-Dade Junior College, South Campus, Miami, March 22–23, 1968.

ILLINOIS, Southern Illinois University, Edwardsville Campus, May 10–11, 1968.

INDIANA

IOWA, Wartburg College, Waverly, April 19, 1968.

KANSAS, Marymount College, Salina, March 23, 1968.

KENTUCKY, University of Kentucky, Lexington, April 20, 1968.

LOUISIANA-MISSISSIPPI, Broadwater Beach Hotel, Biloxi, Mississippi, February 16–17, 1968.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Staten Island Community College, Staten Island, March 16, 1968.

MICHIGAN

MINNESOTA

MISSOURI, Lindenwood College, St. Charles, April 27, 1968.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 26–27, 1968.

NEW JERSEY, Rider College, Trenton, May 4, 1968.

NORTHEASTERN

NORTHERN CALIFORNIA, San Francisco Hilton, January 27, 1968.

OHIO

OKLAHOMA-ARKANSAS, Federal Aviation Agency, Oklahoma City, March 29–30, 1968.

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 14–15, 1968.

PHILADELPHIA

ROCKY MOUNTAIN, University of Denver, Colorado, May 10–11, 1968.

SOUTHEASTERN, East Carolina College, Greenville, North Carolina, March 29–30, 1968.

SOUTHERN CALIFORNIA, Loyola University of Los Angeles, March 9, 1968.

SOUTHWESTERN, New Mexico State University, University Park, Spring 1968.

TEXAS, Texas A and M University, College Station, April 19–20, 1968.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 11, 1968.

WISCONSIN, Wisconsin State University, La Crosse, May 4, 1968.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE: Dallas, Texas, December 26–31, 1968.

AMERICAN MATHEMATICAL SOCIETY, San Francisco Hilton, January 23–26, 1968.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of California, Los Angeles, June 17–20, 1968.

ASSOCIATION FOR COMPUTING MACHINERY, Chicago, Illinois, August 20–22, 1968.

ASSOCIATION FOR SYMBOLIC LOGIC, University of California, Los Angeles, March 22, 1968.

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, St. Louis, November 28–30, 1968.

INSTITUTE OF MATHEMATICAL STATISTICS

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Convention Hall, Philadelphia, Pennsylvania, April 17–20, 1968.

OPERATIONS RESEARCH SOCIETY OF AMERICA, St. Francis Hotel, San Francisco, May 1–3, 1968.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, King Edward Sheraton Hotel, Toronto, Canada, June 11–14, 1968. (Symposium on Optimization.)

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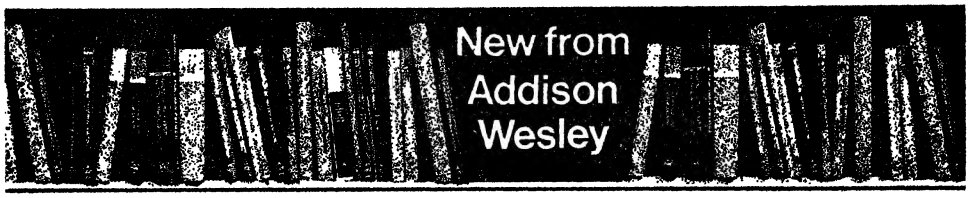
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